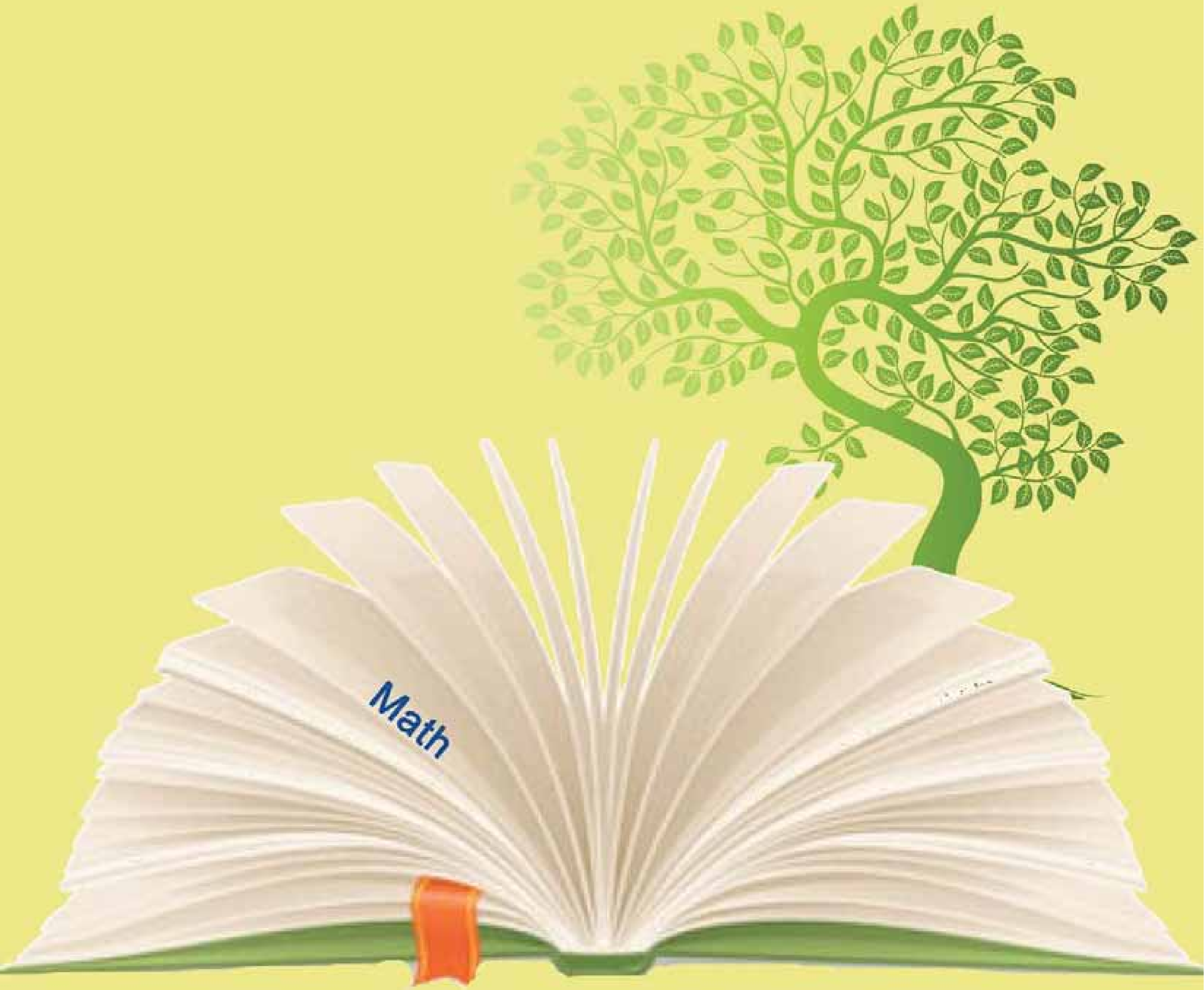

OPEN EDUCATIONAL RESOURCES 4 OPEN SCHOOLS

Taking Education to the People

Open Educational Resources (OER) for Open Schooling

The Commonwealth of Learning (COL) Open Schools Initiative launched an Open Educational Resources (OER) Project to provide materials under the Creative Commons license agreement to support independent study in 17 specially selected secondary school subjects. Funded by the William and Flora Hewlett Foundation its aim is to broaden access to secondary education through the development of high quality Open Distance Learning (ODL) or self-study materials.

These specially selected OER subjects include:

1. Commerce 11
2. Coordinated Science 10 (Biology, Chemistry and Physics)
3. English 12
4. English Second Language 10
5. Entrepreneurship 10
6. Food & Nutrition
7. Geography 10
8. Geography 12
9. Human Social Biology 12
10. Life Science 10
11. Life Skills
12. Mathematics 11
13. Mathematics 12
14. Physical Science 10
15. Physical Science 12
16. Principles of Business
17. Spanish

Open Educational Resources are free to use and increase accessibility to education. These materials are accessible for use in six countries: Botswana, India, Lesotho, Namibia, Seychelles and Trinidad & Tobago. Other interested parties are invited to use the materials, but some contextual adaptation might be needed to maximise their benefits in different countries.

The *OER for Open Schooling Teachers' Guide* has been developed to guide teachers/instructors on how to use the Open Educational Resources (OER) in five of these courses.

1. English
2. Entrepreneurship
3. Geography
4. Life Science
5. Physical Science

The aim of this teachers' guide is to help all teachers/instructors make best use of the OER materials. This guide is generic, but focuses on Namibian examples.

Print-based versions are available on CD-ROM and can be downloaded from www.col.org/CourseMaterials. The CD-ROM contains the module and folders with additional resources, multimedia resources and/or teacher resources. Note that not all subjects have multimedia resources.

Acknowledgements:

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Ministry of Education, Seychelles: www.education.gov.sc
Ministry of Education and Training, Lesotho: www.gov.ls/education
Ministry of Education Trinidad& Tobago: www.moe.gov.tt
National Open School of Trinidad & Tobago (NOSTT): www.moe.gov.tt/NOSTT
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The Commonwealth of Learning (COL) is an intergovernmental organisation created by Commonwealth Heads of Government to encourage the development and sharing of open learning and distance education knowledge, resources and technologies.

Mathematics

Grade 12

COL Open Schools Initiative
Lesotho

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About this Error! No text of specified style in document.

These materials for Grade 12 MathematicsGrade 12 have been produced by the Commonwealth of Learning's COL Open Schools Initiative. All **Error! No text of specified style in document.**s produced by the COL COL Open Schools Initiative are structured in the same way, as outlined below.

How This Error! No text of specified style in document. **is Structured**

Course Overview

The course overview gives you a general introduction to the course. Information contained in the course overview will help you determine:

- If the course is suitable for you.
- What you will need to know before starting the course.
- What you can expect from the course.
- How much time you will need to invest to complete the course.

The overview also provides guidance on:

- How to improve your study skills.
- Where you can get help.
- How to complete course assignments and assessments.
- How to find your way around the course using the margin icons.
- What is included in each unit.

We strongly recommend that you read the overview *carefully* before starting your study.

Course Content

The course is broken down into units. Each unit comprises:

- An introduction to the unit content.
- Unit outcomes.
- New terminology.
- Core content of the unit with a variety of learning activities.
- A unit summary.

- Assignments and/or assessments, as applicable.

Resources

For those interested in learning more on this subject, we provide you with a list of additional resources at the end of this **Error! No text of specified style in document.**. These may be books, articles or websites.

Your comments

After completing this course material for Grade 12 Mathematics we would appreciate it if you would take a few moments to give us your feedback on any aspect of this course. Your feedback might include comments on things such as:

- Content and Structure
- Reading Materials and Resources
- Assignments
- Assessments
- Duration
- Support (assigned tutors, technical help, etc.)

Your constructive feedback will help us to improve and enhance this course.

Course overview

Welcome to Grade 12 Mathematics

These materials constitute a course in Mathematics to prepare students for the Cambridge Overseas School Certificate examination. They are designed to meet the requirements of the national Grade 12 syllabi in many nations across the Commonwealth.

The course materials are intended to equip learners with the knowledge and skills to solve problems in real life situations and handle advanced mathematics courses with ease. They are suitable for use by those enrolled for distance education, but will also be a valuable resource for students in conventional classrooms. They should thus provide a sound foundation for learners who wish to further their education in any field of study for which Mathematics is a prerequisite.

Is this course for you?

This course is intended for anyone who wants to extend their knowledge of Mathematics beyond the basics.

In order to cope with the course materials, you should have successfully completed a Mathematics course at the Junior Secondary Certificate or Grade 10 level.

Course outcomes

Upon completion of this Mathematics course, you will be able to:

Outcomes

- *recognise, describe and represent* numbers and their relationships;
- count, estimate, calculate and solve problems;
- *recognise, describe and represent* patterns and relationships;
- solve problems using algebra;
- *describe, represent analyse and explain* properties of shapes, including the characteristics and relationships between two-dimensional shapes and three-dimensional objects in a variety of orientations and positions, with reasoned justification for their answers;
- *use* appropriate measuring units, instruments and formulae in a variety of contexts;
- *collect, summarise, display and critically analyse* data in order to establish statistical and probability models to solve related problems.

Timeframe

How long?

Students at conventional schools in Lesotho normally take two years to prepare themselves for the Cambridge Overseas School Certificate examination in Mathematics. This is equivalent to six or seven hours per week for 40 weeks per year over two years, a total of roughly 500 study hours.

However, those who register with an open school often have to make time for their studies while fulfilling other responsibilities at home or at work. For this reason, a more flexible study schedule may be necessary. Nevertheless, it is essential that you draw up a timetable for yourself. It you do not set aside time on a regular basis to work through these materials, you may not be able to complete the course.

Study skills

As an adult learner your approach to learning will be different to that from your school days. You can choose what you want to study, you may be motivated by personal and/or professional goals and you will have to find time for your studies while coping with other responsibilities at home or at work.

Essentially you will need to take control of your learning environment. You will should consider a number of issues that can affect your performance, including how to manage your time, how to set goals or how to manage stress. You may also need to re-acquaint yourself with areas such as essay planning, coping with exams and using the web as a learning resource.

Your most significant considerations will be *time* and *space*, that is the time you dedicate to your learning and the environment in which you engage in that learning.

We recommend that you take time now—before starting your self-study—to familiarize yourself with these issues. There are a number of excellent resources on the web. A few suggested links are:

- http://www.how-to-study.com/

  The "How-to-study" website is dedicated to information on study skills. You will find links to resources on study preparation (a list of nine essentials for a good study place), taking notes, and strategies for reading text books, using reference sources and coping with test anxiety.

- http://www.ucc.vt.edu/stdysk/stdyhlp.html

  This is the website of the Virginia Tech Division of Student Affairs. You will find links to resources on time scheduling (including a “where does time go?” link), a study skills checklist, basic concentration techniques, control of the study environment, note taking, how to read essays for analysis and memory skills (“remembering”).

- http://www.howtostudy.org/resources.php

  This is another “how-to-study” website with useful links to time management, efficient reading, questioning/listening/observing skills, getting the most out of doing (“hands-on” learning), memory building, tips for staying motivated and developing a learning plan.

The above links are our suggestions to start you on your way. At the time of writing these web links were active. If you want to look for more go to www.google.com and type “self-study basics”, “self-study tips”, “self-study skills” or something similar in the address pane of your web browser.

Need help?

Help

As you study this Grade 12 Mathematics course you may encounter academic and social issues that hinder your progress. For example, you may have questions or need help to make sense of the materials. To help you overcome some of these problems, the Lesotho Distance Teaching Centre (LDTC) established a Student Advice Section, whose main objective is to offer you counselling before, during and after the course. Whenever you come across an issue that impedes your studies, you should write or call the LDTC office, where someone will assist you:

Address: The Director

Lesotho Distance Teaching Centre

P.O. Box 781

Maseru

Tel: + 266 22316961

Fax: +266 22310245

Email ldtc@ymail.com

LDTC also has offices in five other districts. These offices are housed in the Ministry of Education and Training's District Resource Centres and are manned by LDTC officers. All services provided by the head office in Maseru are available in the districts. In this way if you live in one of these districts you do not have to travel to Maseru to get study materials and other services. Please feel free to drop into these Resource Centres any time between 8.00 and 4.30 p.m. to get advice or help with any problems relating to your studies, assignments, face-to-face sessions or end of course examinations. The districts are:

Qacha's-Nek (Tel. +266 22950702)

Mokhotlong (Tel. + 266 22920396)

Thaba-Tseka (Tel. + 266 22900492)

Quthing (Tel. +266 22751459)

Leribe (Tel. +266 22400022)

Since LDTC uses print as its main medium of instruction, you will receive information through letters, leaflets, pamphlets and booklets. But you too must write whenever you have information or news to share.

From time to time tutorials will also be arranged so that you can get advice and assistance on any points of confusion. A schedule listing the dates of these tutorials will be given to you when you register.

Assignments

Assignments

This Grade 12 Mathematics course has 34 units that are going to be bound together into workbooks. At the end of each workbook you will be expected to do an assignment or worksheet as it is sometimes called. The aim of these assignments is to test how well you have studied and understood the contents of each workbook.

Some of the assignments and activity questions you will mark yourself, and the Student Advice Section will let you know which ones these are. After completing such assignments, check your work against the model answers provided. If you did not get at least 80% of them correct, go back through the unit and study the material carefully before trying the assignment again.

The Student Advice Section will also indicate which assignments are for your tutor to mark. As soon as you complete each workbook, you should complete the Tutor-Marked Assignment. Please read your workbooks and answer the assignment questions in the set order. Following the logical order will enhance your comprehension and help you to complete your work on time.

When you finish each assignment, please submit it to the Student Advice Section at LDTC Head Office. You may do this through the post or drop it at the office if you live nearby.

LDTC engages teachers (tutors) on a part time basis to mark these assignments and assist you. Each assignment has a cover sheet on which you can seek an explanation of any part you could not understand. Since the tutors provide advice as they mark, they will answer all your queries and give you feedback on your assignment. When your assignments are marked, you can collect them from the office or have them posted to you.

In addition to marking your assignments, the tutors hold face-to-face tutorials. During these sessions you are given the chance to meet your tutors and to get immediate answers to all your questions. The tutorials also afford you a chance to meet and interact with other learners. If you still have some difficulty understanding the material, ask your tutor at the next scheduled meeting.

Assessments

Assessments

In addition to the assignment, each unit also includes an assessment to test your understanding of the content. The time you should take to complete the assessment is shown at the top of each paper, as well as the total marks available and the marks for each question.

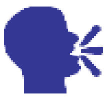
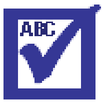
After completing the assessments, you should submit them to officials at your nearest study centre, where they will be marked by your tutor. The Student Advice Section will advise you on the way this should be done. The deadlines for submitting these assignments will be provided by your tutor at the first meeting each year. Feedback will be provided to you about two weeks after the submission dates.

Getting around this Error! No text of specified style in document.

Margin icons

While working through this **Error! No text of specified style in document.** you will notice that little pictures or icons appear frequently in the left-hand margin. These icons serve to "signpost" a particular piece of text, a new task or change in activity. They have been included to help you to find your way around this **Error! No text of specified style in document.**.

A complete set of these icons is shown below. We suggest that you familiarize yourself with the icons and their meaning before starting your studies.

 Activity	 Assessment	 Assignment	 Case study
 Discussion	 Group activity	 Help	 Note it!
 Outcomes	 Reading	 Reflection	 Study skills
 Summary	 Terminology	 Time	 Tip

Unit Contents

Unit 1

Sets

Introduction

Welcome to the “world” of classification or groups!

You may be surprised to know that groups are all around you. Your family is a group. Your classmates are also a group. Similarly, sets are also all around you. Your cooking utensils are a set. Your clothes can also be considered a set.

Things naturally form sets or groups; and therefore, things are easily dealt with when they are grouped or are in their respective groups or sets.

You learned about notation of sets such as using capital letters for sets and subsets. You also used symbols of union, intersection, compliment and number of elements in a set. You used Venn diagrams with not more than two sets in the **universal** set. In this unit, you are also going to use Venn diagrams with more than two sets in the **universal** set.

This unit consists of 69 pages. It covers approximately 3% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 30 hours to work on this unit, 20 hours for formal study and 10 hours for self-study and completing assessments/assignments.

When reading the following learning outcomes, think about them as a guide to what you should focus on while studying this unit.

This Unit is Comprised of Four Lessons:

Lesson 1 Describing Sets Using Set Builder Notation and Listing

Lesson 2 Expressing Relationships in Sets Using Set Notation

Lesson 3 Interpreting Information That Is Represented with Venn Diagrams

Lesson 4 Drawing Venn Diagrams in Problem Solving

Upon completion of this unit you will be able to:

Outcomes

- *describe* sets using **set builder notation** and **listing**.
- *express* relationships in sets using set notation.
- *interpret* information that is represented with Venn diagrams.
- *draw* Venn diagrams in problem solving.

Terminology

Complement of a Set: A set of **all members not** in a set, but in the universal set.

Disjoint Sets: Sets which do not intersect.

Element or Member: An object in a set; the symbol $\in$ is used to show that an object is a member or an element of a set, while the symbol $\notin$ is used to show that an object is not a member or an element of a set.

Empty/ Null Set: A set which has **no members or elements**; the symbol Φ or $\{\ \}$ is used to denote an empty or null set.

Finite Set: A set which has a **countable** number of elements or members.

Infinite Set: A set which has an **uncountable** number of elements or members.

Intersection of Sets: A set which is formed by putting together **all common** elements or members of the sets; the symbol $\cap$ is used for the **intersection** of sets.

Listing a Set: Writing the actual elements or members within the braces of a set.

Proper Subset: A set that contains **no** elements or members of another set or contains **some** elements or members of another set.

Set Builder Notation: Writing a description or a mathematical expression(s) within the braces of a set.

Subset: A set that contains **no** elements or members of another set; or contains **some or all** elements or members of another set.

Union of Sets: A set which is formed by putting together **all** elements or members of the sets without repeating any element or member; the symbol $\cup$ is used for the **union** of sets.

Universal Set: A set of **all elements or members** under discussion; the symbol $\mathcal{E}$ is normally used to denote the **universal** set.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Describing Sets Using Set Builder Notation and Listing

Introduction

By the end of this subunit, you should be able to:

- list the members of a set from set builder notation.
- use set builder notation to describe a listed set.

This subunit is about 5 pages in length.

Ways of Describing Sets

A set, which can be *finite* –meaning that it has countable elements, or *infinite* – which means the list of elements is endless, is a collection of objects which can defined by either using set builder notation **or** listing.

Look at table 1.1, in which set builder notation and listing of some sets are shown side by side:

Set builder notation	**Listing**
{ square numbers less than 10 }	{ 1, 4, 9 }
{ y | y is a counting number }	{ 1, 2, 3, 4, 5, 6, 7, . . . }
{ z | z is an odd number, 6 ≤ z ≤ 15 }	{ 7, 9, 11, 13,15 }

Table 1.1

Example 1

Take the set { square numbers less than 10 } from table 1.1.

It is read "The set of square numbers less than ten."

Notice that the actual elements are not written inside the braces, but only the statement 'square numbers less than 10'.

Now think about which square numbers are less than 10; remember that you get a square number from multiplying a counting number by itself.

List the square numbers that are less than 10:

____

____

Compare your list with the following**:**

1, 4, 9.

Therefore, the set of square numbers less than ten can be listed as:

$\{ \ 1, 4, 9 \ \}$.

Example 2

Look at the set $\{ \ 1, 2, 3, 4, 5, 6, 7 \ldots \ \}$ from the same table 1.1.

You can find it under the **Listing** column.

The actual elements, which are numbers in this case, are listed inside the braces.

The set can be read "The set of numbers, one, two, three, four, five, six, seven, and so on". The three dots after seven are read 'and so on'; and they mean there are other numbers after seven, making the list endless.

Is this set finite or infinite? Give a reason for your answer.

____

____

Compare your answer with the following:

It is infinite because it has endless list of elements.

Look closely at the elements in this set and find a way of describing them:

What is the smallest number in the list?

____

Compare your answer with the following:

It is: 1

How can you get the next number from the one before it?

____

Compare your answer with:

Add 1 to the number before in order to get the next.

<u>Think about these:</u> The smallest number in the endless list is 1and to get the next number in the list you always add 1.

What list of numbers is this?

____

Compare your answer with:

This is the list of counting numbers. (Counting numbers are defined in the unit entitled 'Types of Numbers'.)

Therefore, $\{\ 1, 2, 3, 4, 5, 6, 7, \ \}$ is a set of counting numbers.

That is $\{\ 1, 2, 3, 4, 5, 6, 7, \ \} = \{$ Counting numbers $\}$.

So, if y stands for any counting number, the set can also be written as

$\{\ y \mid y$ is a counting number $\}$, which is read "The set of all y such that y is a counting number".

Example 3

The set $\{\ z \mid z$ is an odd number, $6 \le z \le 15\ \}$ from table 1.1 has been described in set builder notation.

Write down the way you read the set.

____

____

Compare your answer with:

The set of all z such that z is an odd number, z is not less than 6 and z is not greater than 15.

Notice that this set is finite since $6 \le z \le 15$ means that z has lower limit as 6 and upper limit as 15.

What is the smallest odd number which is not less than 6?

____

Compare your answer with:

7

What is the greatest odd number which is not greater than 15?

____

Compare your answer with:

15

So, $\{ z \mid z$ is an odd number, $6 \leq z \leq 15 \}$, is a set of odd numbers from 7 to 15.

That is $\{ z \mid z$ is an odd number, $6 \leq z \leq 15 \} = \{ 7, 9, 11, 13, 15 \}$.

Activity 1

Activity 1

1. a) Let A = { y | y is a prime number }. List the elements of set A.

   b) Let B = { Types of triangle }. List the elements of set B.

   c) Let C = { Multiples of 5 up to 25 }. List the elements of set C.

   d) Let D = { y | y is a factor of 24, y is greater than 10 }. List the elements of set D.

2. a) Let E = { 2, 4, 6, 8, . . . }. Use set builder notation to describe set E.

   b) Let F = { vertebrates, invertebrates, flowering plants, non-flowering plants }. Use set builder notation to describe set F.

   c) Let G = { Mercury, Venus, Earth, Mars, Jupiter, Uranus, Saturn, Neptune, Pluto }. Use set builder notation to describe set G.

   d) Let H = { 1, 8, 27, 64, 125, . . . }. Use set builder notation to describe set H.

Compare your answers with those at the end of the subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content and try the activity again.

Note it!

Remember:

A set, which can be *finite* or *infinite*, is a collection of objects that can define by using *set builder notation* **or** *listing*.

Now, before you learn about expressing relationships in sets using set notation think about what you learnt in describing sets using set builder notation and listing.

Solutions to Activity 1

1. a) A = { y | y is a prime number }.

   "Set A is the set of all y such that y is a prime number".

   A prime number is a counting number which has only two factors, 1 and itself.

   The smallest prime number is 2. The next is 3; then 5, 7, 11. After 11 it is 13. Since there is an infinite number of counting numbers, the set of prime numbers is infinite.

   Therefore, A = { 2, 3, 5, 7, 11, 13, . . . }.

   b) B = { Types of triangle }.

   A triangle has three sides. Equilateral triangle has three sides equal; isosceles triangle has at least two sides equal; and scalene triangle has no equal sides.

   Therefore, B = { equilateral triangle, isosceles triangle, scalene triangle }.

   c) C = { Multiples of 5 up to 25 }.

   $5\times1=5$

   $5\times2=10$

   $5\times3=15$

   $5\times4=20$

   $5\times5=25$

   C = { 5, 10, 15, 20, 25 }.

   d) D = { y | y is a factor of 24, y is greater than 10 }.

   Factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.

   Now, factors of 24 which are also greater than 10 are 12 and 24.

Therefore, D = $\{$ 12, 24 $\}$.

2. a) E = $\{$ 2, 4, 6, 8, . . . $\}$.

These are even numbers, and therefore:

E = $\{$ Even numbers $\}$.

b) F = $\{$ vertebrates, invertebrates, flowering plants, non-flowering plants $\}$.

F = $\{$ Living things $\}$.

c) G = $\{$ Mercury, Venus, Earth, Mars, Jupiter, Uranus, Saturn, Neptune, Pluto $\}$.

G = $\{$ Planets $\}$.

d) H = $\{$ 1, 8, 27, 64, 125, . . . $\}$.

$1 \times 1 \times 1 = 1^3 = 1$

$2 \times 2 \times 2 = 2^3 = 8$

$3 \times 3 \times 3 = 3^3 = 27$

$4 \times 4 \times 4 = 4^3 = 64$

$5 \times 5 \times 5 = 5^3 = 125$

Therefore, H = $\{$ cube numbers $\}$.

Lesson 2 Expressing Relationships in Sets Using Set Notation

Introduction

By the end of this subunit, you should be able to correctly use the proper symbol for:

- **element or member** in a set.
- **number** of members or elements of a set.
- **equal sets**.
- a **subset** of a set.
- an **intersection** of sets.

- a **union** of sets.
- a **complement** of a set.

This subunit is about 6 pages in length.

Set Symbols

Now, that you can list and even define sets, in this subunit you will learn to use set symbols in various relationships in sets.

An element or member is an object in a set.

The symbol $\in$ is used to show that an object is a member or an element of a set.

The symbol $\notin$ is used to show that an object is not a member or an element of a set.

For example:

(i) If set A is a set of counting numbers, and you want to write that 2 is an element of A, then you write:

$2 \in A$.

(ii) If $C = \{ 1, 4, 9 \}$, then

$3 \notin C$, because 3 is not there in set C.

Number of members or elements of a set: The symbol n (A) is used to denote the **number** of members or elements of a set A.

For example:

(i) $C = \{ x \mid x$ is a square number less than 100 $\}$.

List set C.

____

Compare your answer with the following:

$C = \{ 1, 4, 9, 16, 25, 36, 49, 64, 81 \}$.

How many elements are set C?

____

Compare your answer with the following:

There are 9 elements in set C.

In set symbols, to write "There are 9 elements in set C.", you write:

n (C) = 9.

(ii) If F = { **Selekane, Setŝabelo** , **Tŝoarelo** }, then
n (F) = 3.

Equal sets: Two sets are **equal** if they contain exactly the same elements. If set A is equal to set B, then you write A = B.

For example:

{2, 4, 6}, {2, 6, 4}, {4, 2, 6}, {4, 6, 2}, {6, 2, 4} and {6, 4, 2} are equal sets.

Subset: If **every** member of set A also belongs to another set B, then A is a subset of B. Again, if there are **no members** in A which are not there in B, then A is a subset of B.

In set symbols it is written **A** $\subset$ **B**. Sometimes **A** $\subseteq$ **B** is written when emphasis is made on the fact that if A is a subset of B, generally, then A is either a proper subset of B, or A = B.

A $\subset$ B can also be written as B $\supset$ A and

A $\subseteq$ B can also be written as B $\supseteq$ A.

For example:

(i) If C = $\{$ x | x is a square number less than 10 $\}$ and D = $\{$ y | y is a counting number $\}$, then C $\subset$ D or D $\supset$ C .

(ii) If E = Φ and F = { Selekane, **Setŝabelo** , **Tŝoarelo** }, then E $\subset$ F or F $\supset$ E. This is because there are **no members** in an empty set which are not there in F. ***As a matter of fact, an empty set does not have elements!***

The symbol $\not\subset$ or $\nsubseteq$ is used when a set is not a subset of another set.

Intersection: If **some** member(s) of set A also belong(s) to another set B, then A and B are intersecting sets, written A $\cap$ B.

For example:

(i) If C = $\{$ 1, 4, 9 $\}$ and D = $\{$ 1, 2, 3, 4, 5, . . . $\}$, then
C $\cap$ D = { 1, 4, 9 }.

(ii) If E ={ Names that begin with the letter 's'} and F = { Selekane, **Setŝabelo** , **Tŝoarelo** }, then

E $\cap$ F = {Selekane, **Setŝabelo** }.

Union: If A and B are two sets, a set which is formed by putting together **all** elements or members of these sets without repeating any element or member is the union, written A $\cup$ B.

For example:

(i) If I = { north, south } and D = { east, west }, then

I $\cup$ D = { north, south, east, west }.

(ii) If P = { red, green, blue } and S = { red, orange, yellow, green, blue, indigo, violet }, then

P $\cup$ S = {red, orange, yellow, green, blue, indigo, violet}.

Complement: A complement of set A is a set of **all members not** in set A but in the universal set, **written A$^{/}$**.

For example:

(i) Let ε = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

If set A = {1, 3, 5, 7, 9 }, which elements are not found in A, but are in ε?

___

Compare your answer with the following:

The elements are 2, 4, 6, 8 and 10.

The complement of set A is formed by 2, 4, 6, 8 and 10.

Therefore, A$^{/}$ = {2, 4, 6, 8, 10}

(ii) Let ε = {Tumisang, Nthatisi, Tumelo, **Tŝepo**, Jack, Betty, Selekane, **Setŝabelo**, **Tŝoarelo** }

If F = { Selekane, **Setŝabelo** , **Tŝoarelo** }, then

$F' = \{$Tumisang, Nthatisi, Tumelo, **Tŝepo**, Jack, Betty $\}$;

Activity 2

Activity 2

Let $\varepsilon = \{3, 5, 7, 9, 11, 13, 15\}$;

$A = \{3, 5, 7, 9, 11, 13\}$;

$B = \{3, 5, 9, 11\}$;

$C = \{5, 11, 13\}$;

$D = \{3, 11\}$.

Write *true* or *false* for each of the following in table1.2:

Set notation	*true* or *false*
(a) $\varepsilon = A$	
(b) $C \subset D$	
(c) $A \cap B = A$	
(d) $A \cap B = B$	
(e) $A \cup B = A$	
(f) $A \cup B = B$	
(g) $C \supset D$	
(h) $C \subseteq D$	
(i) $D' = \varepsilon$	
(j) $A' = \Phi$	

Table 2.2

Compare your answers with those at the end of the subunit. Continue on if you scored at least 80%. If not, review this topic again.

Now, make your own summary of this subunit in space below. This will help you remember the information.

Note it!

Remember:

An **element or member** is an object in a set. The symbol $\in$ is used for "is an element of " while $\notin$ means "is not an element of ".

The symbol **n (A)** is used to denote the **number** of members or elements of a set A.

Sets are **equal** if they contain exactly the same elements. If set A is equal to set B, then A = B is true.

An **empty set** is a subset of any set. A subset is also a set containing **some or all** elements or members of another set.

Intersection of sets is a set which is formed by putting together **all common** elements or members of the sets; the symbol $\cap$ is used for the intersection of sets.

If A and B are two sets, a set which is formed by putting together **all** elements or members of these sets without repeating any element or member is the **union,** written $A \cup B$.

A **complement of set A** is a set of **all members not** in set A but in the universal set, **written** A'.

The next subunit is making use the various symbols which you have now learnt, to interpret information that is represented with Venn diagrams.

Answers to Activity 2

Set Notation	*true* or *false*
(a) $\mathcal{E} = A$	*false*
(b) $C \subset D$	*false*
(c) $A \cap B = A$	*false*
(d) $A \cap B = B$	*true*
(e) $A \cup B = A$	*true*
(f) $A \cup B = B$	*false*
(g) $C \supset D$	*true*
(h) $C \subseteq D$	*false*
(i) $D' = \mathcal{E}$	*false*
(j) $A' = \Phi$	*false*

Lesson 3 Interpreting Information That Is Represented with Venn Diagrams

Introduction

By the end of this subunit, you should be able to:

- use set symbols you learnt in '**Expressing relationships in sets using set notation**', to interpret the information from Venn diagrams.
- shade proper regions in Venn diagrams.

This subunit is about 27 pages in length.

Statements about Venn diagrams and shading regions.

Statements can be made about the data that is shown in Venn diagrams. Shading can also be used to highlight some data with certain things in common in the Venn diagrams.

Making statements about Venn diagrams:

Various statements can be made, using set notation or set symbols, about the information that is shown in Venn diagrams.

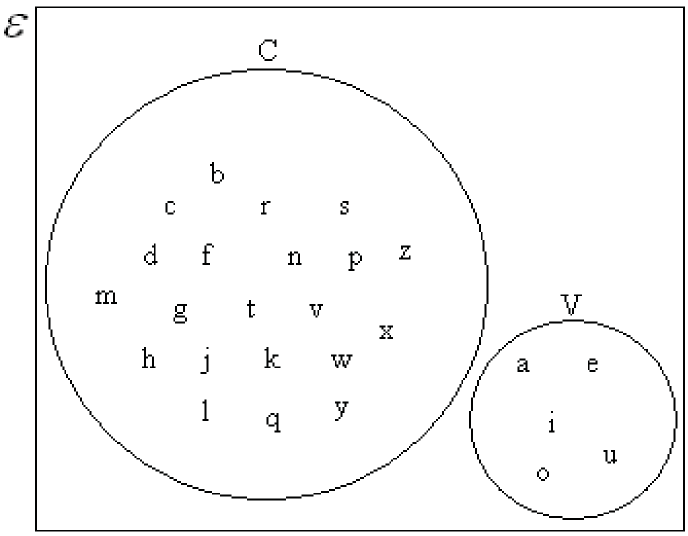
Example 1:

Let $\mathcal{E}$ = {letters of English alphabet};

C = {consonants};

V = {vowels}.

The Venn diagram is shown below:



The following set notation can be used to interpret some of the information that is represented with the above Venn diagram:

(a) $C \cup V = \mathcal{E}$. This means that the universal set consists of elements of C, together with those of V.

(b) $C \cap V = \{\ \}$. The Venn diagram shows that no vowels are consonants; therefore, their intersection is an empty set, as they are disjoint sets.

(c) $e \in V$. The element 'e' is an element of set V. Notice that 'e' is not written in braces since { e } means a set containing the element 'e'.

(d) $o \notin C$. The element 'o' is not an element of set C.

(e) $\{b, c, d\} \subset C$. The set of elements b, c and d, is a subset of set C.

(f) $o \not\subset V$. The element 'o' is not a subset of set V. An element cannot be a subset!

(g) $n(C \cup V) = n(\mathcal{E}) = 26$. The number of elements in the union of C and V is equal to the number of elements in $\mathcal{E}$, which is 26.

Example 2:

Let $\mathcal{E}$ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15};

F = {Factors of fifteen};

V = {Factors of forty};

T = {Factors of twelve}.

List the factors of fifteen which are in $\mathcal{E}$.

____

Compare your list with the following:

{ 1, 3, 5, 15 }.

Now, list the factors of forty which are in $\mathcal{E}$.

____

Compare your list with the following:

{ 1, 2, 4, 5, 8, 10 }.

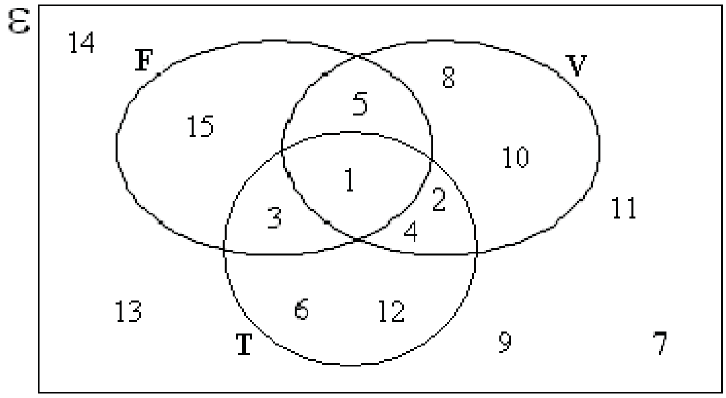
Also, list set T.

____

Compare your list with the following:

{ 1, 2, 3, 4, 6, 12 }.

The Venn diagram for the data is as follows:



The following set notation can be used to interpret some of the information that is represented with the above Venn diagram:

(a) $F' \cup V = \{1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ is a true statement from the Venn diagram.

Proof:

Well, from list of set V earlier, $V = \{1, 2, 4, 5, 8, 10\}$.

The elements which are not in F but in ε are 2, 4, 6, 7, 8, 9, 10, 11, 12, 13 and 14. These are elements of compliment of F.

Therefore, $F' = \{2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$.

$F' \cup V$ means the elements of set F' and those of set V.

When collecting the elements of set F' and those of set V together, without repeating elements, the set $\{1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ is formed.

(b) $T \cap V = \{1, 2, 4\}$.This is correct, as the numbers 1, 2 and 4 are found in set T, and they are there in set V.

(c) $(T \cap V) \notin (F' \cup V)$. This means the set $(T \cap V)$ is not an element of set $(F' \cup V)$. A set cannot be 'an element' of another set; it can only be a subset or not a subset of another one!

So, the statement $(T \cap V) \notin (F' \cup V)$ is true.

(d) $\{6\} \subset T$. This means that the set $\{6\}$ is a subset of set T, which is true.

(e) $6 \not\subset T$. A number '6' cannot be a subset, but an element. Therefore, it is true that $6 \not\subset T$.

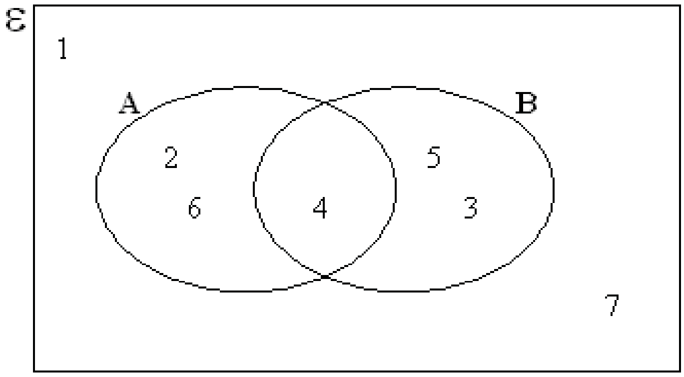
Activity 3

(i) Let $\mathcal{E}$ = { 1, 2, 3, 4, 5, 6, 7};

A = { 2, 4, 6};

B = {3, 4, 5}.

Below, is a Venn diagram representing the sets?



Activity 3

Write *true* or *false* for each of the following statements:

Set notation	*true* or *false*
(a) n (A) =2	
(b) n (B) =3	
(c) $A \subseteq B$	
(d) $A \cup B = \mathcal{E}$	
(e) $A \cap B = \{ 4 \}$	
(f) $n(A \cap B) = 4$	
(g) $A^{/} = \{1, 3, 5, 7 \}$	
(h) $\{1, 3, 5, 7 \} \not\subseteq \mathcal{E}$	

(i) $\{1, 3, 5, 7\} \cap B = \{3, 5\}$	
(j) $A \not\subset B$	

Table 3.3

Compare your answers with those at the end of the subunit. Continue on if you scored at least 80%. If not, review this content again.

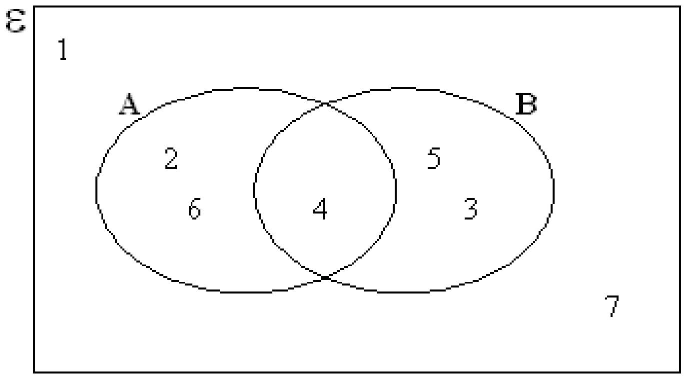
Shading Venn Diagrams – Part 1

This can done by first listing the members of the set which is to be shaded, and then shading the region or regions, in which those members are found.

For example:

(i) Let $\mathcal{E} = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathbf{A} = \{2, 4, 6\}$ and $\mathbf{B} = \{3, 4, 5\}$:

Below, is a Venn diagram representing the sets.

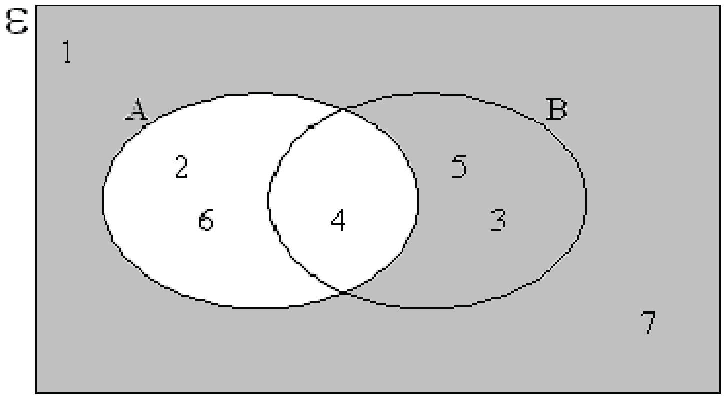


(a) Shading the region representing $A^{/}$ in the Venn diagram.

First, list the set $A^{/}$:

$A^{/} = \{1, 3, 5, 7\}$

Now, shade the regions in which the members of $A^{/}$ are found in the Venn diagram:



(b) Shading the region representing A ∩ B in the Venn diagram.

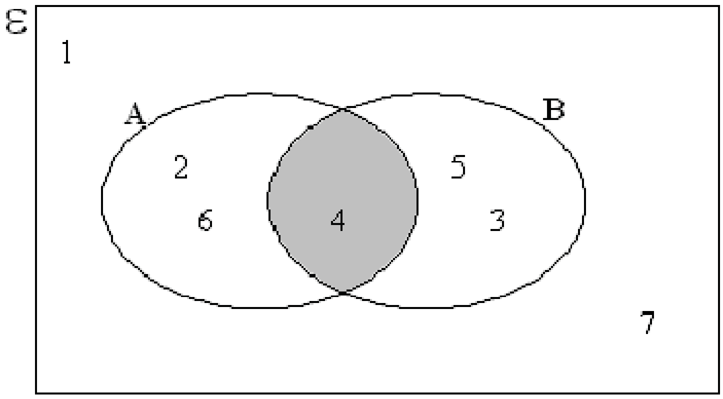
List the set A ∩ B:

A = { 2, <u>4</u>, 6}

B = { 3, <u>4</u>, 5}

Therefore, A ∩ B = {4}

Shade the region in which the member 4 is found in the Venn diagram:



(c) Shading the region representing B ∪ A in the Venn diagram.

List the set B ∪ A:

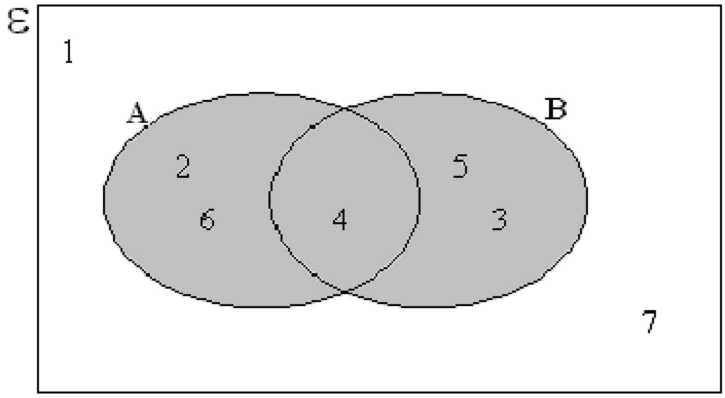
A = { 2, 4, 6}

B = { 3, 4, 5}

Remember that a member of a set is never written more than once.

Therefore, B ∪ A = {2, 3, 4, 5, 6}

Shade the regions in which the members of B ∪ A are found in the Venn diagram:

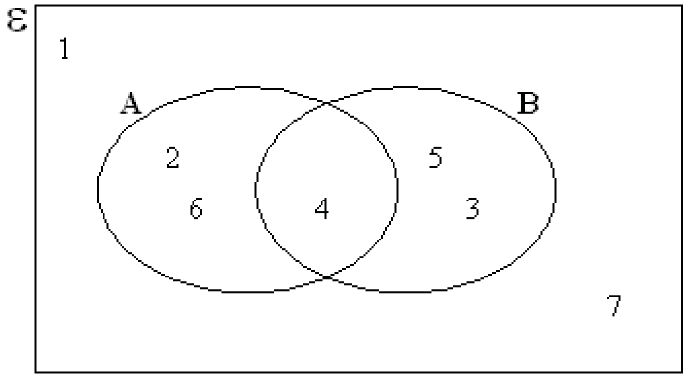

ε
1
A
B
2
6
4
5
3
7

Activity 4

Activity 4

Use the Venn diagram below to answer the questions that follow.

1. Based on the Venn diagram:



(a) list the set $B^{/} \cap A^{/}$

____

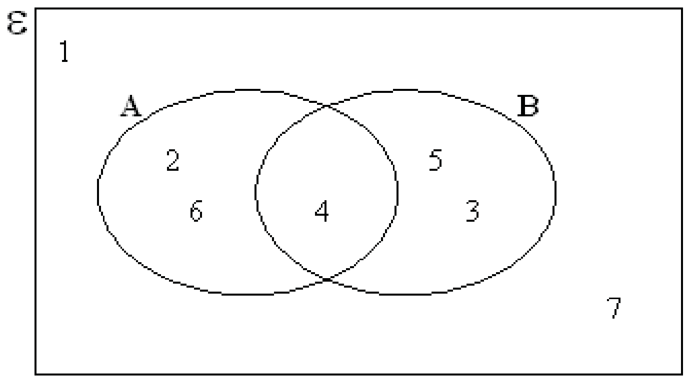
____

____

____

(b) shade the set in the above diagram

2. Based on the Venn diagram:



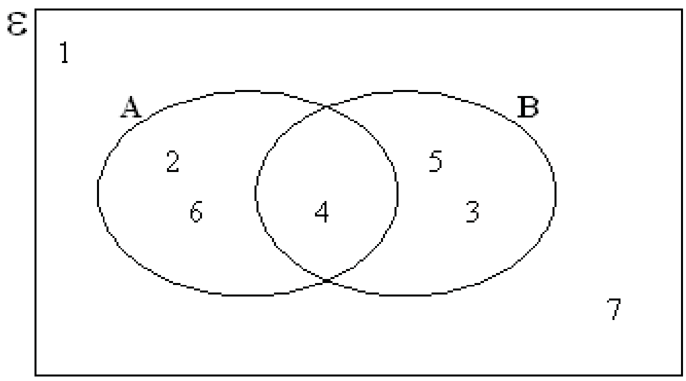
(a) list the set $(B \cup A$

____

____

(b) shade the set in the above diagram

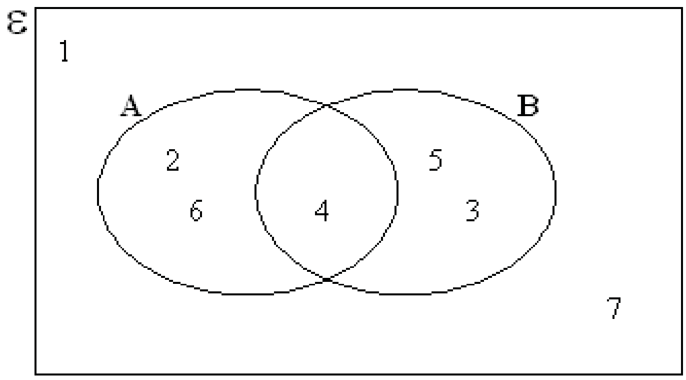
3. Based on the Venn diagram:



(a) list the set $A^{/} \cup B$

(b) shade the set in the above diagram

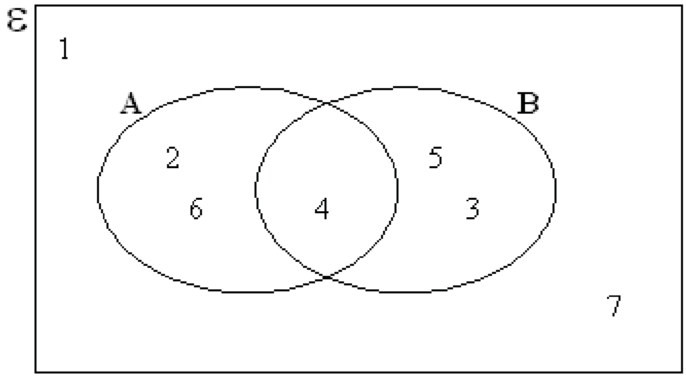
4. Based on the Venn diagram:



(a) list the set $(A \cap B)^{/}$

(b) shade the set in the above diagram

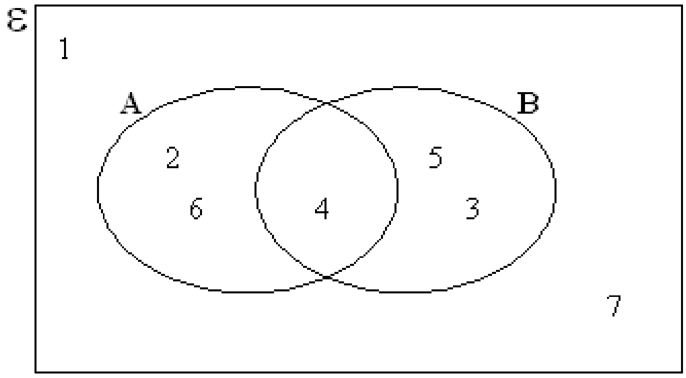
5. Based on the Venn diagram:



(a) list the set B′

(b) shade the set in the above diagram

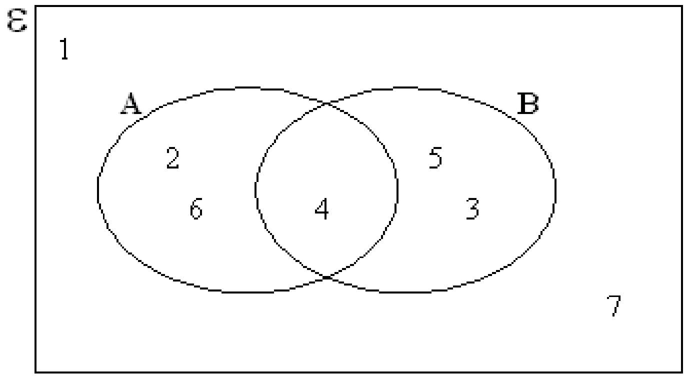
6. Based on the Venn diagram:



(a) list the set $A' \cap B$

(b) shade the set in the above diagram

7. Based on the Venn diagram:



(a) list the set $A' \cup B$

(b) shade the set in the above diagram.

Compare your answers with those at the end of the subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content again.

Shading Venn Diagrams – Part 2

Example 1

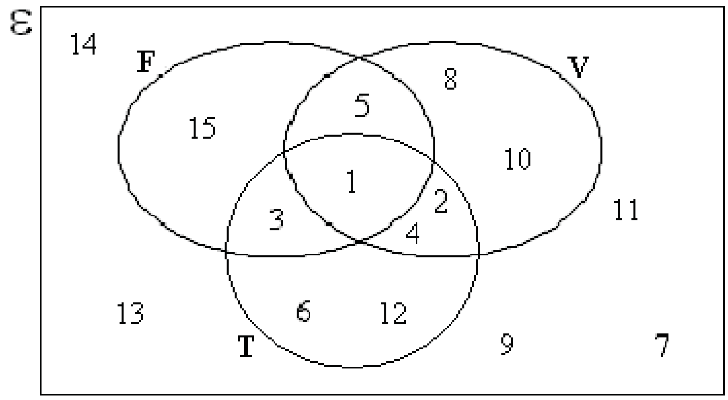
Let ε = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15};

F = {Factors of fifteen};

V = {Factors of forty};

T = {Factors of twelve}.

From example 2 under the subunit '**Interpreting information that is represented with Venn diagrams**' the Venn diagram is like this:

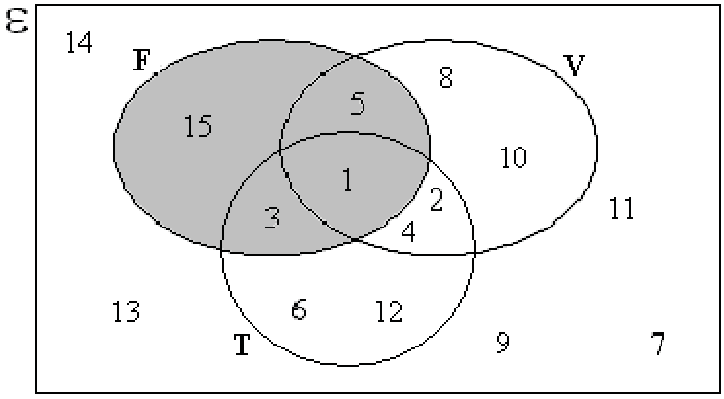


(a) Shading the region representing F in the Venn diagram.

List the set F:

F= {1, 3, 5, 15}

Shade the regions in which the members of F are found in the Venn diagram:



(b) Shading the region representing F ∩ V in the Venn diagram.

List F ∩ V

First, list the set F.

Compare your list with the following:

F= {1, 3, 5, 15}

Also, list the set V.

Compare your list with the following:

V= {1, 2, 4, 5, 8, 10}

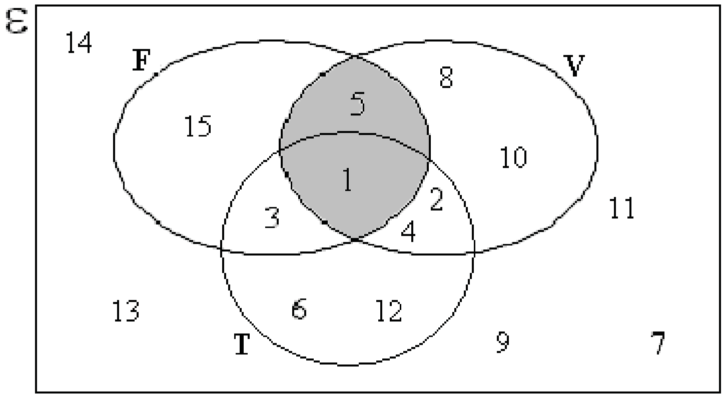
Now, pick out the common elements to the two sets.

Compare your list with the following:

1 and 5 appear in both sets.

Therefore, F ∩ V = {1, 5}

Shade the regions in which 1 and 5 are found in the Venn diagram:



(c) Shading the region representing $(F \cup V \cup T)'$ in the Venn diagram.

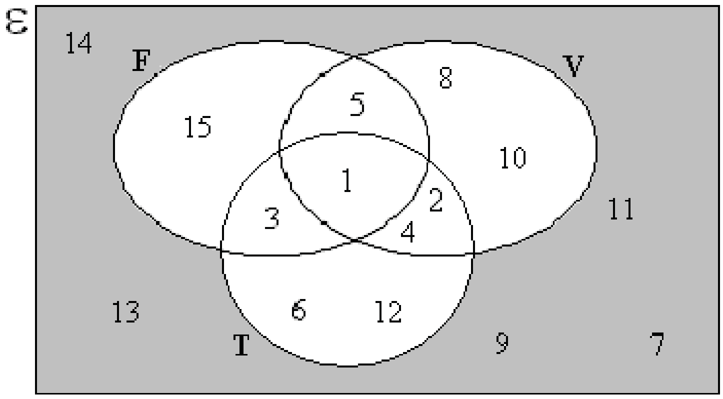
List $(F \cup V \cup T)'$.

Notice that this is the complement of $F \cup V \cup T$.

$F \cup V \cup T = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15\}$

Therefore, $(F \cup V \cup T)' = \{7, 9, 11, 13, 14\}$

Shade the regions in which the members of $(F \cup V \cup T)'$ are found in the Venn diagram:



Example 2

Let $\mathcal{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$;

F = {Factors of fifteen};

V = {Multiples of seven};

T = {Multiples of two}.

Draw the Venn diagram, and shade:

a) $(F \cap T) \cup (T \cap V)$

b) $(F \cup V)' \cap T$

c) $T' \cap V$.

First, list the subsets F, V, and T.

___

___

___

Compare your subsets with these:

F = {1, 2, 4, 5, 8, 10};

V = {7, 14};

T = {2, 4, 6, 8, 10, 12, 14}.

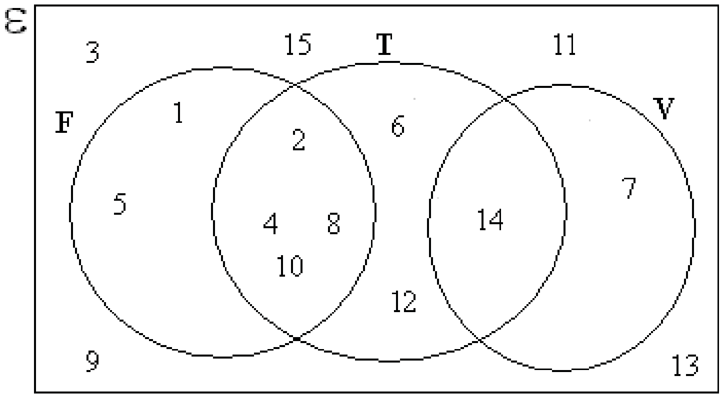
Which of these subsets are disjoint?

___

Compare your answer with:

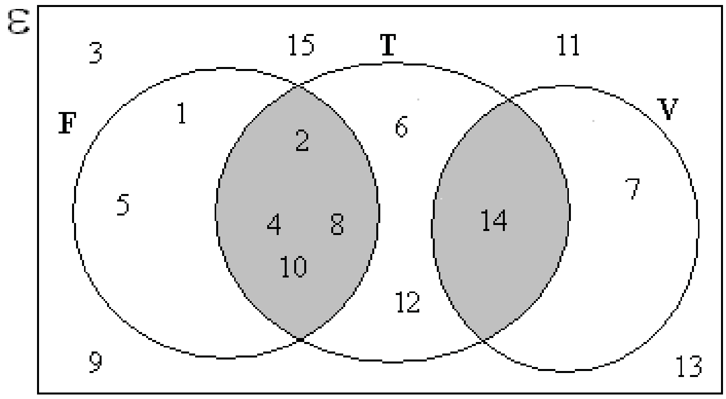
F and V are disjoint sets.

Now, the Venn diagram looks thus:



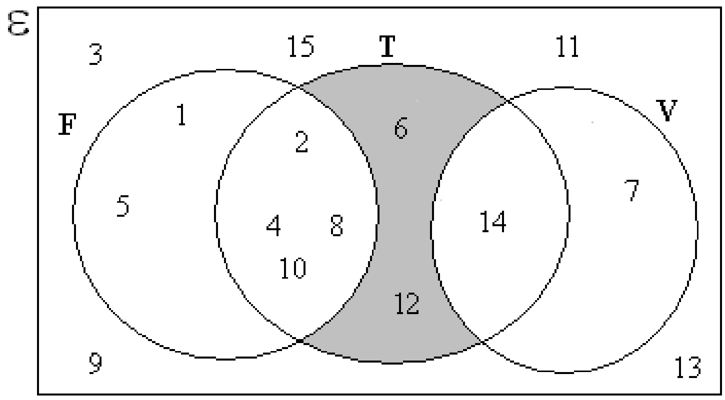
a) Shading $(F \cap T) \cup (T \cap V)$:

$(F \cap T) = \{2, 4, 8, 10\}$ and $(T \cap V) = \{14\}$. Therefore, $(F \cap T) \cup (T \cap V) = \{2, 4, 8, 10, 14\}$. So the diagram looks thus:



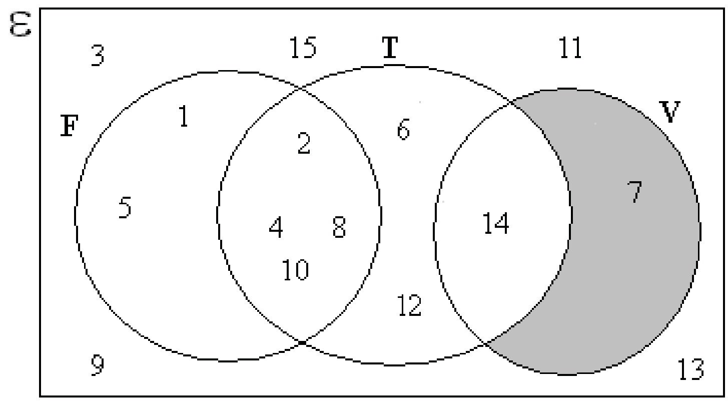
b) Shading $(F \cup V)' \cap T$

$(F \cup V)' = \{3, 6, 9, 11, 12, 13, 15\}$ and $T = \{2, 4, 6, 8, 10, 12, 14\}$. Therefore, $(F \cup V)' \cap T = \{6, 12\}$. So the diagram looks thus:



c) Shading $T' \cap V$

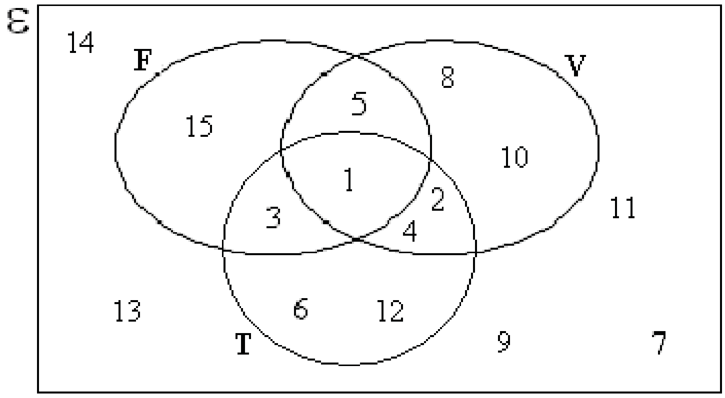
$T' = \{1, 3, 5, 7, 9, 11, 13, 15\}$ and $V = \{7, 14\}$. Therefore, $T' \cap V = \{7\}$. So the diagram looks thus:



Activity 5

Activity 5

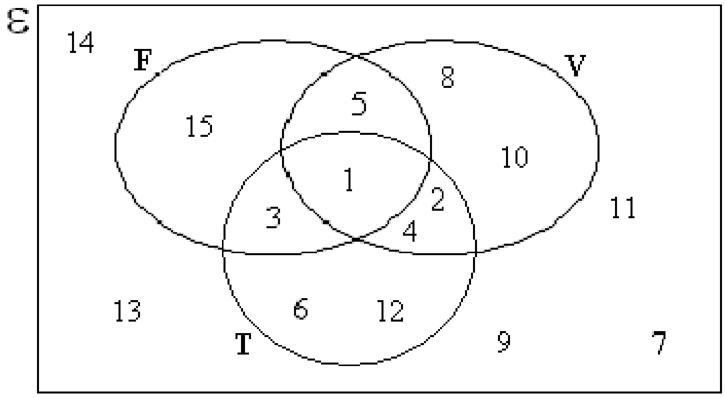
1. Based on the Venn diagram:



(a) list the set $(F \cap V) \cap F'$

(b) shade the set in the above diagram

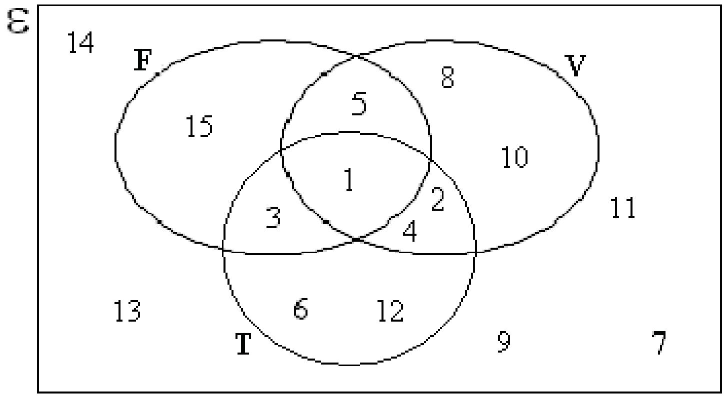
2. Based on the Venn diagram:



(a) list the set V

(b) shade the set in the above diagram

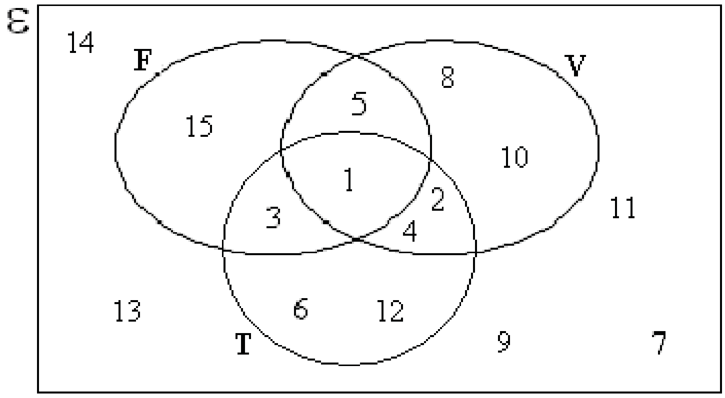
3. Based on the Venn diagram:



(a) list the set T

(b) shade the set in the above diagram

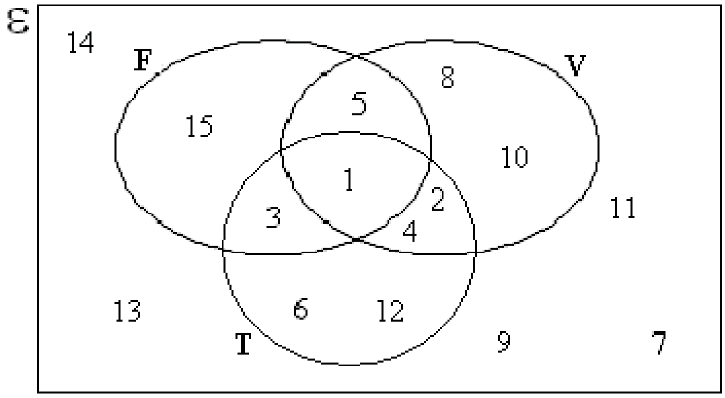
4. Based on the Venn diagram:



(a) list the set $F^{/}$

(b) shade the set in the above diagram

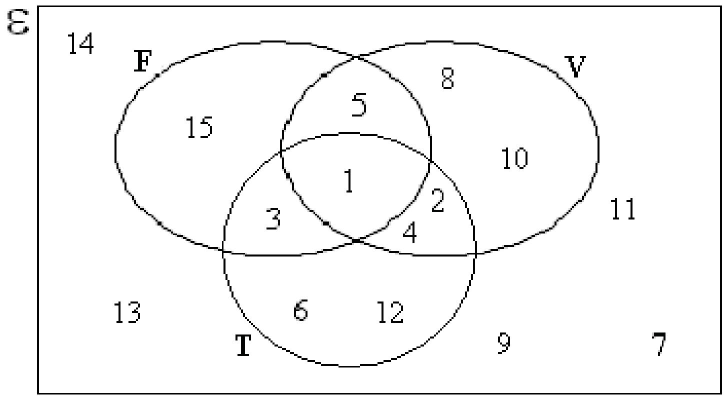
5. Based on the Venn diagram:



(a) list the set $V^{/}$

(b) shade the set in the above diagram

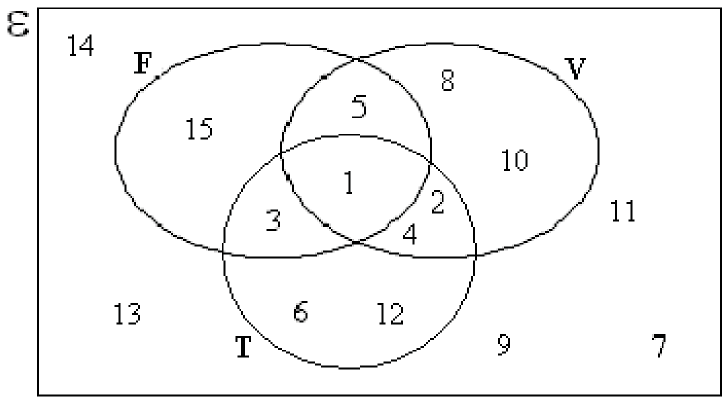
6. Based on the Venn diagram:



(a) list the set T'

(b) shade the set in the above diagram

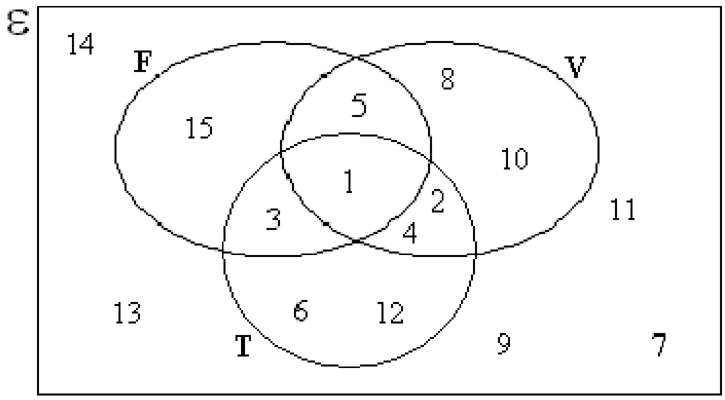
7. Based on the Venn diagram:



(a) list the set $F' \cap V'$

(b) shade the set in the above diagram

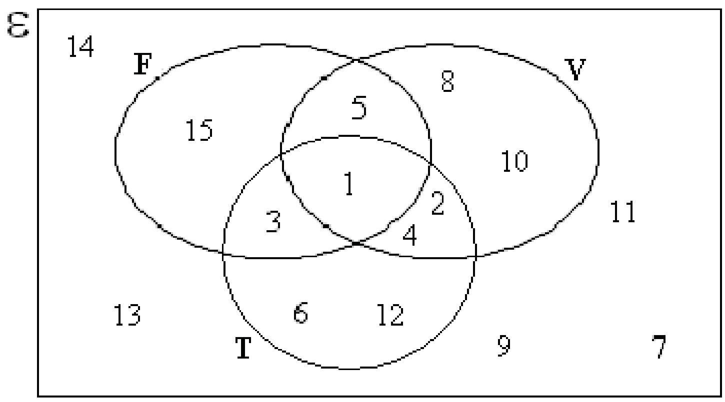
8. Based on the Venn diagram:



(a) list the set $F' \cap V' \cap T'$

(b) shade the set in the above diagram

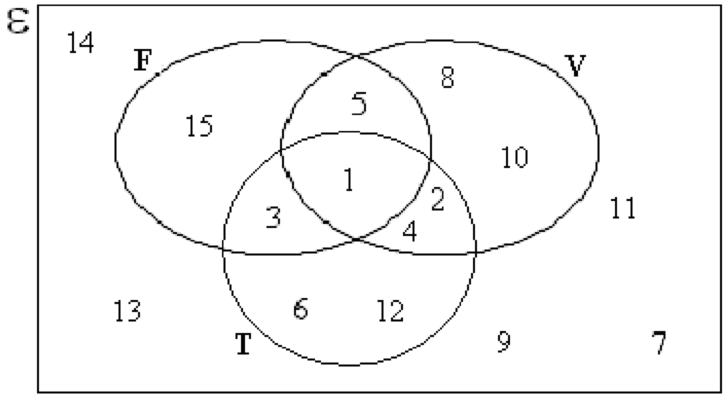
9. Based on the Venn diagram:



(a) list the set $V \cup T$

(b) shade the set in the above diagram

10. Based on the Venn diagram:



(a) list the set $T' \cup V'$

(b) shade the set in the above diagram

Check your performance against the given solutions at the end of the subunit. Continue if you are satisfied with your score. If not, review this content again.

Note it!

Remember:

Various statements can be made, using **set notation**, about the information that is shown in Venn diagrams.

When shading the regions of Venn diagrams, list the members of the set which are to be shaded, and then shade the region or regions, in which those members are found.

Solutions to the Subunit Activities

Answers to Activity 3

Set notation	*true* or *false*
(a) n (A) =2	*false*
(b) n (B) =3	*true*
(c) A $\subseteq$ B	*false*
(d) A $\cup$ B = ε	*false*
(e) A $\cap$ B = { 4 }	*true*
(f) n(A $\cap$ B) = 4	*false*
(g) $A^{/}$ = {1, 3, 5, 7 }	*true*
(h) {1, 3, 5, 7 } $\not\in \varepsilon$	*true*
(i) {1, 3, 5, 7 } $\cap$ B = { 3, 5 }	*true*
(j) A $\not\subset$ B	*true*

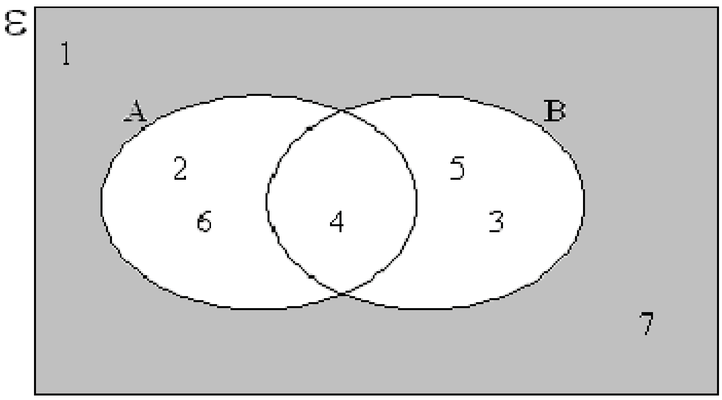
Answers to Activity 4

1. $B^{/} \cap A^{/}$

$A^{/}$ = {<u>1</u>, 3, 5, <u>7</u>}

$B^{/}$ = {<u>1</u>, 2, 6, <u>7</u>}

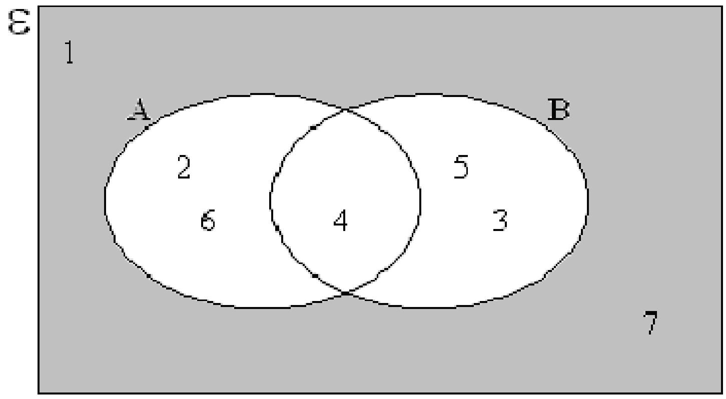
Therefore, $B^{/} \cap A^{/}$ = {1, 7}



2. $(B \cup A)^{/}$

$B \cup A = \{2, 3, 4, 5, 6\}$

Therefore, $(B \cup A)^{/} = \{1, 7\}$



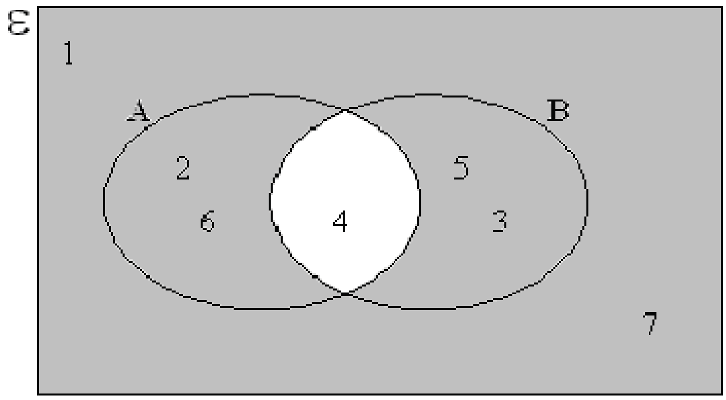
3. $A^{/} \cup B^{/}$

$A^{/} = \{1, 3, 5, 7\}$

$B^{/} = \{1, 2, 6, 7\}$

Remember that a member of a set is never written more than once.

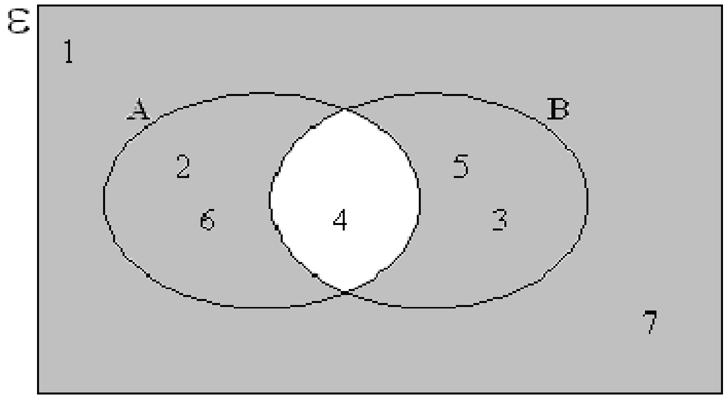
Therefore, $A^{/} \cup B^{/} = \{1, 2, 3, 5, 6, 7\}$



4. $(A \cap B)^{/}$

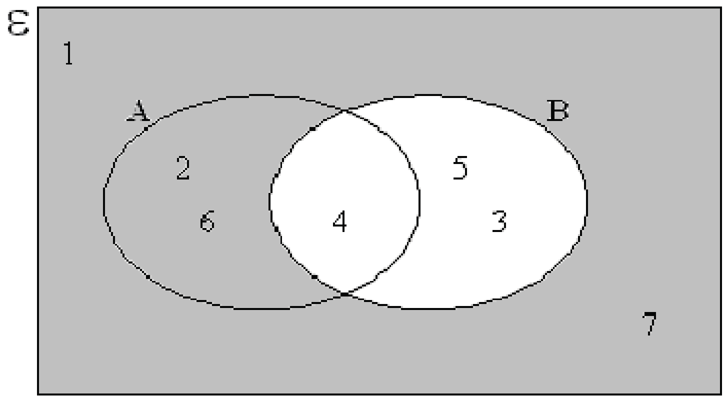
$A \cap B = \{4\}$

Therefore, $(A \cap B)^{/} = \{1, 2, 3, 5, 6, 7\}$



5. B'

$B' = \{1, 2, 6, 7\}$

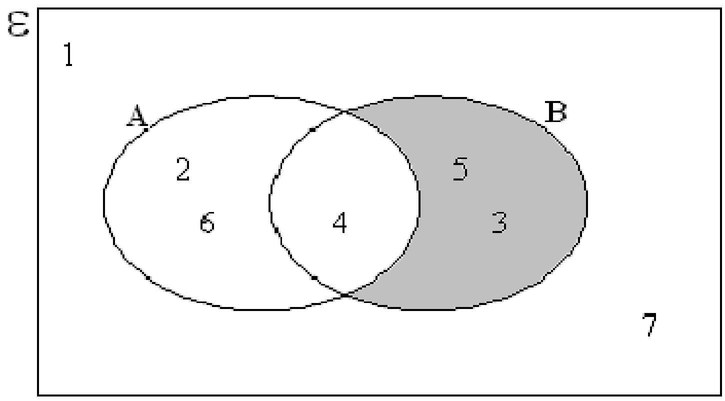


6. $A' \cap B$

$A' = \{1, \underline{3}, \underline{5}, 7\}$

B = $\{\underline{3}, 4, \underline{5}\}$

Therefore, $A' \cap B = \{3, 5\}$



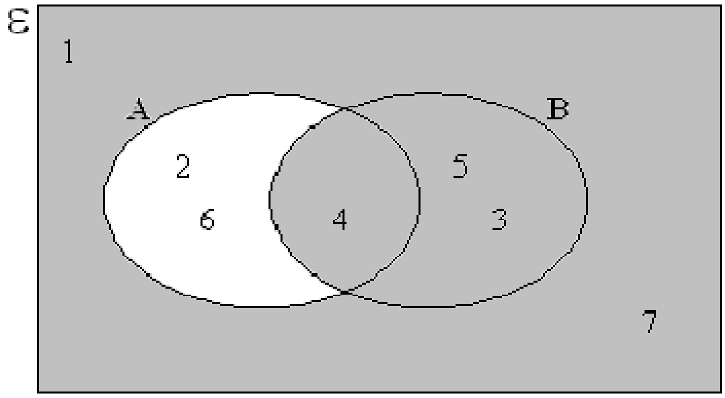
7. $A' \cup B$

$A^{/} = \{1, 3, 5, 7\}$

B = {3, 4, 5}

Remember that a member of a set is never written more than once.

Therefore, $A^{/} \cup B = \{1, 3, 4, 5, 7\}$



Answers to Activity 5

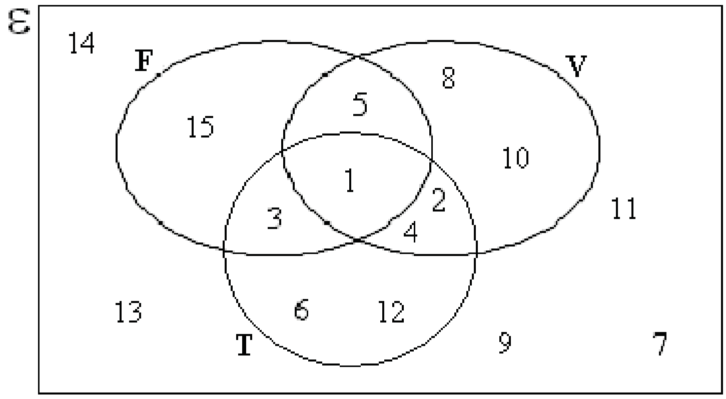
1. $(F \cap V) \cap F^{/}$

$F \cap V = \{1, 5\}$

$F^{/} = \{2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$

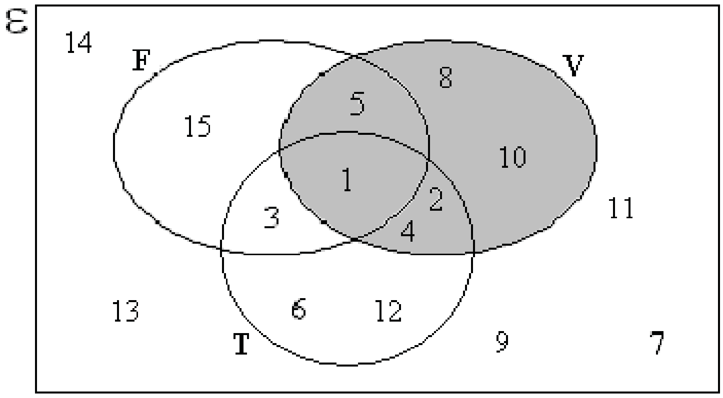
Therefore, $(F \cap V) \cap F^{/} = \{\ \}$

No shading in the Venn diagram.



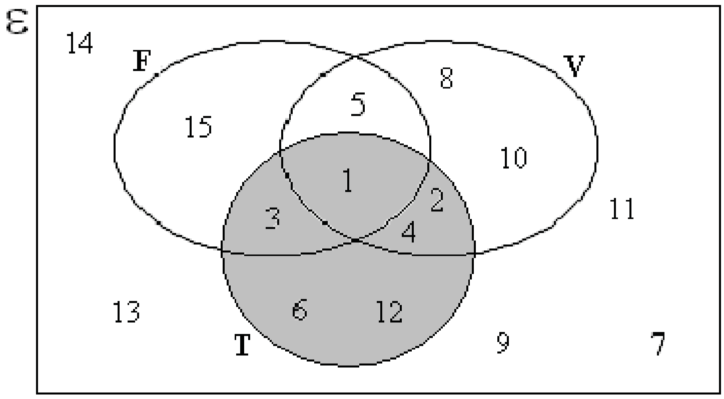
2. V

V= {1, 2, 4, 5, 8, 10}



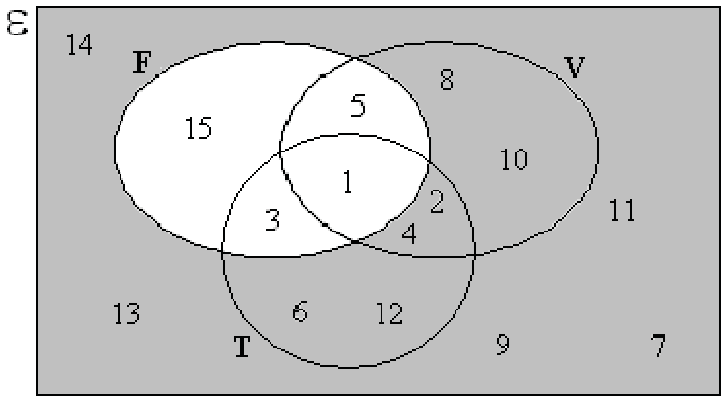
3. T

T= {1, 2, 3, 4, 6, 12}



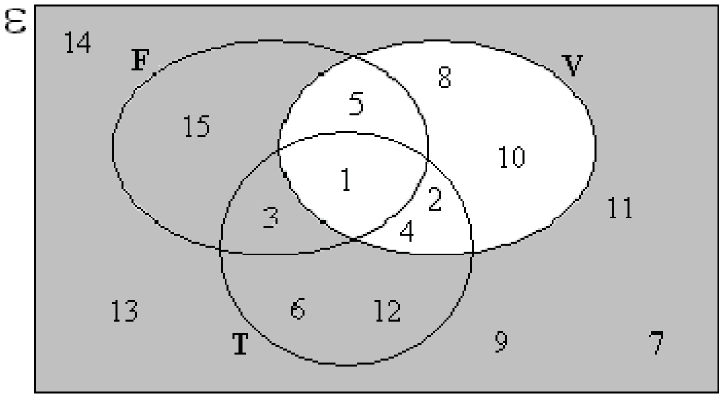
4. F′

F′ = {2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14}



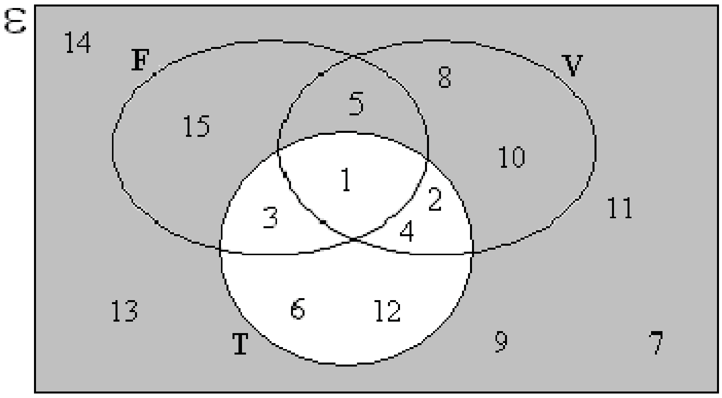
5. V′

$V' = \{3, 6, 7, 9, 11, 12, 13, 14, 15\}$



6. T'

$T' = \{5, 7, 8, 9, 10, 11, 13, 14, 15\}$

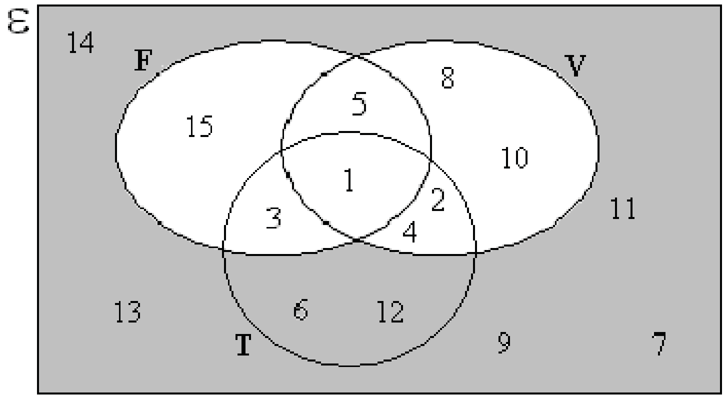


7. $F' \cap V'$

$F' = \{2, 4, \underline{6}, \underline{7}, 8, \underline{9}, 10, \underline{11}, \underline{12}, \underline{13}, \underline{14}\}$

$V' = \{3, \underline{6}, \underline{7}, \underline{9}, \underline{11}, \underline{12}, \underline{13}, \underline{14}, 15\}$

Therefore, $F' \cap V' = \{6, 7, 9, 11, 12, 13, 14\}$



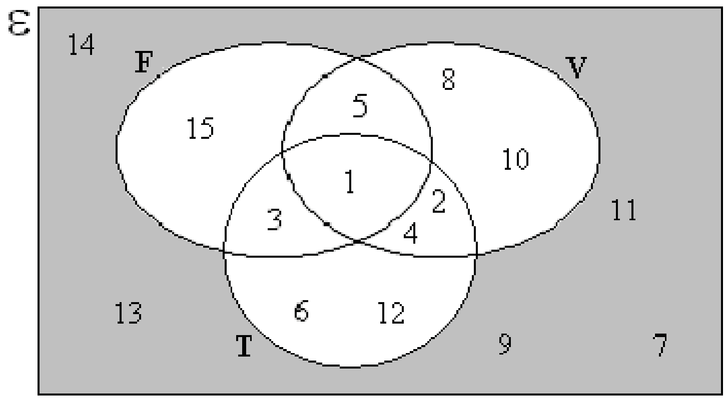
8. $F' \cap V' \cap T'$

F' = {2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14}

V' = {3, 6, 7, 9, 11, 12, 13, 14, 15}

T' = {5, 7, 8, 9, 10, 11, 13, 14, 15}

Therefore, $F' \cap V' \cap T'$ = {7, 9, 11, 13, 14}



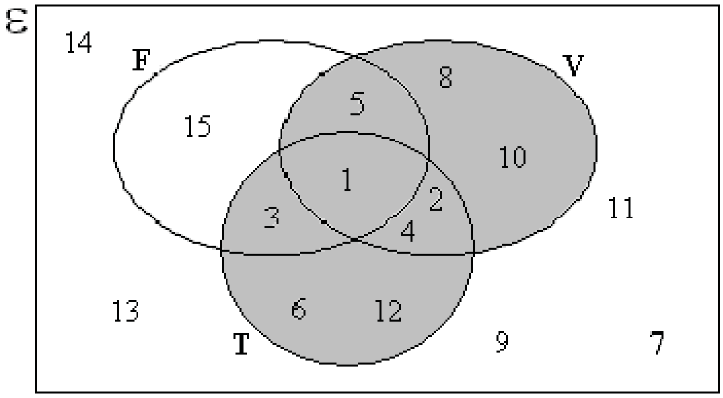
9. $V \cup T$

V= {1, 2, 4, 5, 8, 10}

T= {1, 2, 3, 4, 6, 12}

Remember that a member of a set is never written more than once.

Therefore, $V \cup T$ = {1, 2, 3, 4, 5, 6, 8, 10, 12}



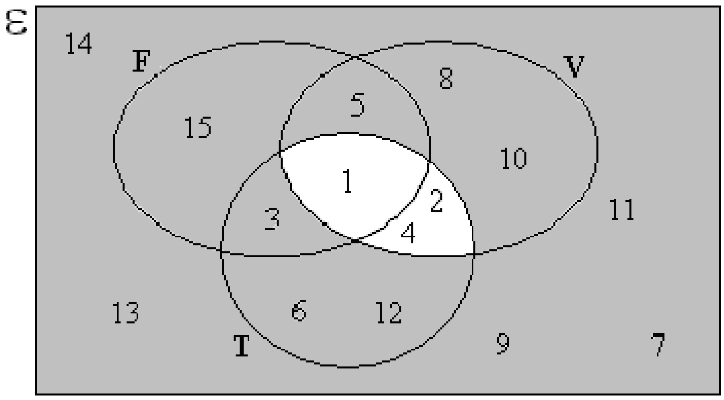
10. $T' \cup V'$

T' = {5, 7, 8, 9, 10, 11, 13, 14, 15}

V' = {3, 6, 7, 9, 11, 12, 13, 14, 15}

Remember that a member of a set is never written more than once.

Therefore, $T' \cup V'$ = {3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}



Lesson 4 Drawing Venn Diagrams in Problem Solving

Introduction

By the end of this subunit, you should be able to:

- apply the knowledge of sets in problem solving.

This subunit is about 14 pages in length.

Sets in Problem Solving

Venn diagrams can be used in practical situations as already stated that naturally things form sets or groups. In actual fact, any data are easier to interpret and understand when they are classified. Instead of showing the actual elements, the number of elements can be written in the respective regions of the Venn diagram.

Example 1:

142 people were interviewed to find out what they used for cooking. The following results were obtained:

58 used gas cookers;

58 used microwave ovens;

63 used electric cookers;

4 used both gas cookers and electric cookers;

17 used both microwave ovens and gas cookers;

19 used both microwave ovens and electric cookers;

1 used gas cookers, microwave ovens and electric cookers;

2 cooked with solar energy only.

Use the given data to answer the questions that follow.

(a) How many people used gas cookers only?

(b) How many people used microwave ovens and electric cookers only?

(c) How many people used microwave ovens or electric cookers?

(d) How many people used microwave ovens or electric cookers, but did not use gas cookers?

(e) How many people used microwave ovens or solar energy?

You can answer the questions by first constructing your own Venn diagram:

How many people were interviewed altogether?

____

Compare your answer with:

They are 142.

So, the universal set consists of 142 people. Let ε = {142 people interviewed}.

Find the subsets of the universal set that you can form from the given data.

____

____

Compare your work with the following:

Three subsets, namely, {people who used microwave ovens} , {people who used electric cookers}and {people who used gas cookers}.

Let M = {people who used microwave ovens} , E = {people who used electric cookers} and G = {people who used gas cookers}.

Before drawing the Venn diagram, look for more information in the given data which will help you to decide whether the three subsets intersect or not.

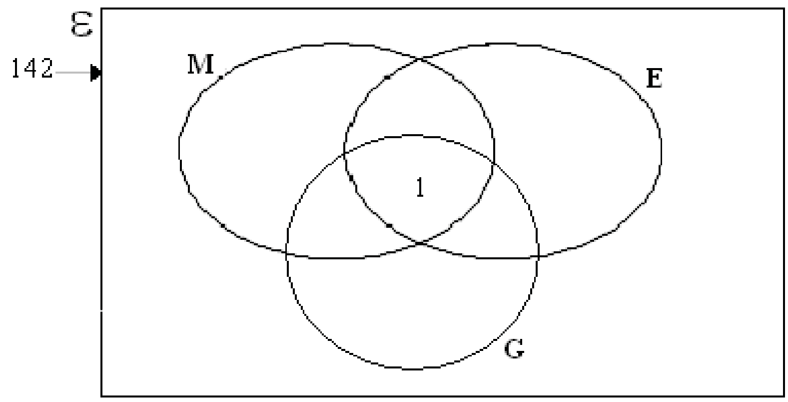
Are there any people who used the **all three ways** of cooking?

____

Compare your answer with the following:

Yes; from the data '1 used gas cookers, microwave ovens and electric cookers'.

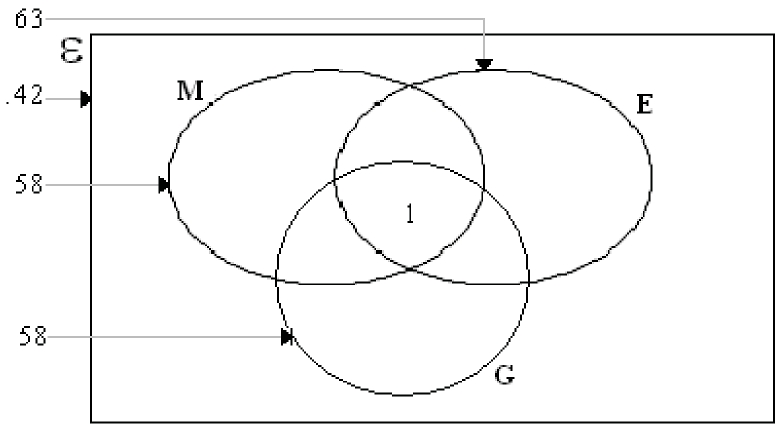
Therefore, the three subsets intersect. So far, the Venn diagram looks thus:



"There are 142 people in the universal set; and 1 used gas cookers, microwave ovens and electric cookers".

Now, '58 used gas cookers' means that n (G) = 58 ; '58 used microwave ovens' means that n (M) = 58; and '63 used electric cookers' means that n (E) = 63.

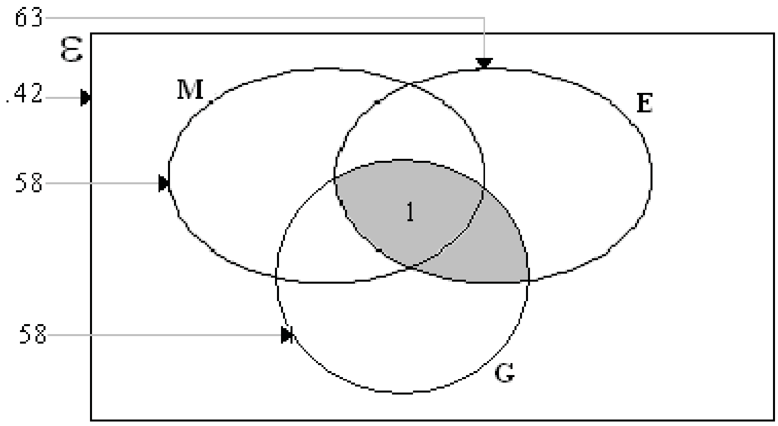
Label the Venn diagram some more:



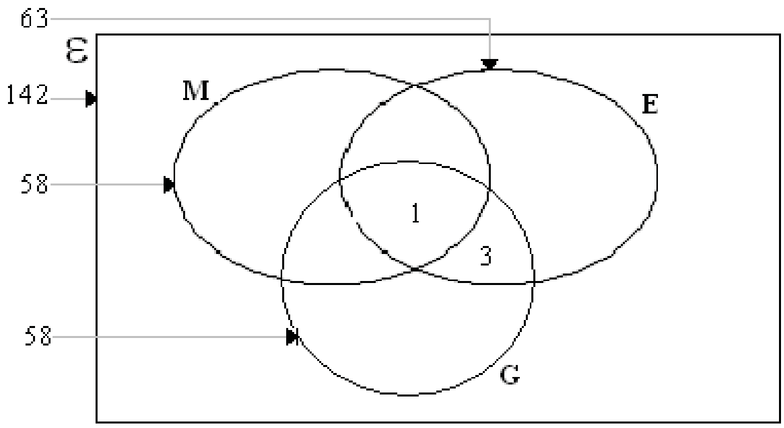
You are told that '4 used both gas cookers and electric cookers'.

That means n (G $\cap$ E) = 4.

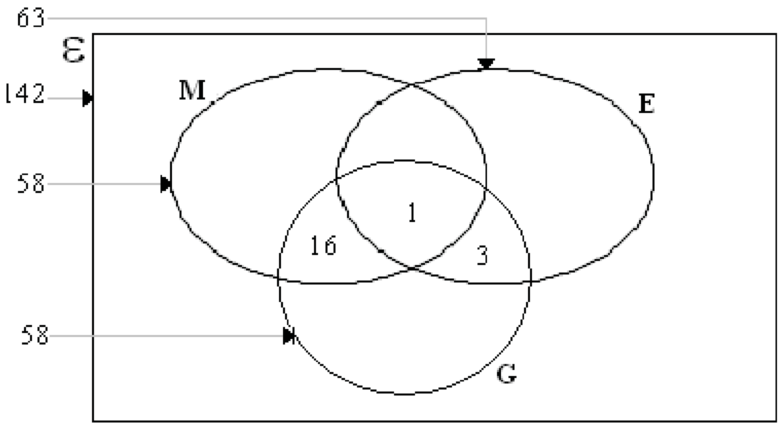
The shaded region is for G $\cap$ E in the Venn diagram that follows.



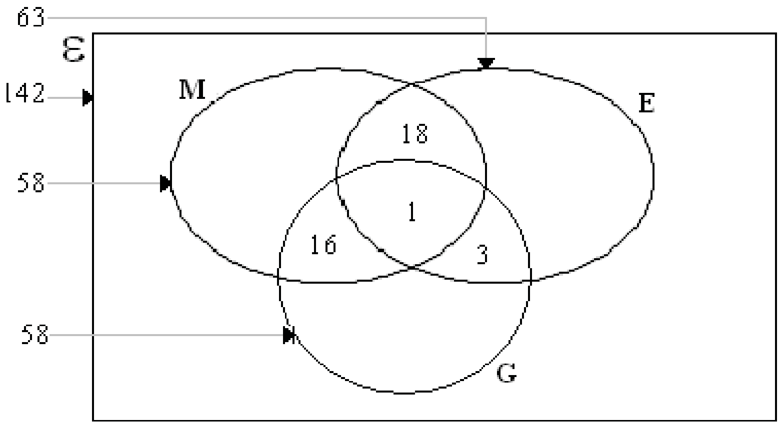
Since 1 is already put in one part, then put 3 in the other part to make up 4, so that the Venn diagram looks thus:



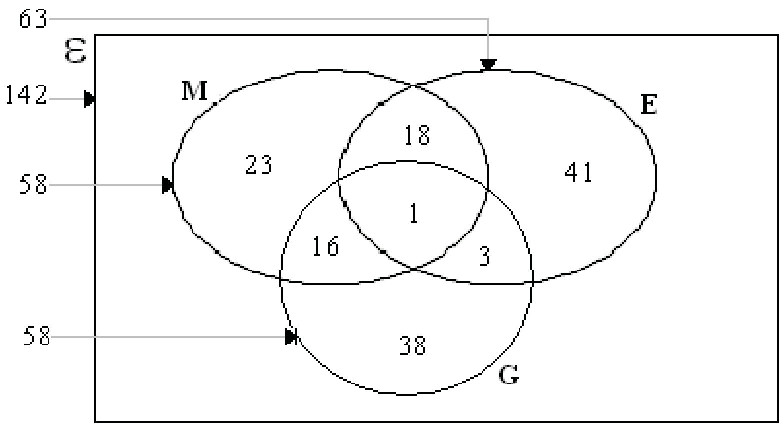
From '17 used both microwave ovens and gas cookers', the Venn diagram looks thus:



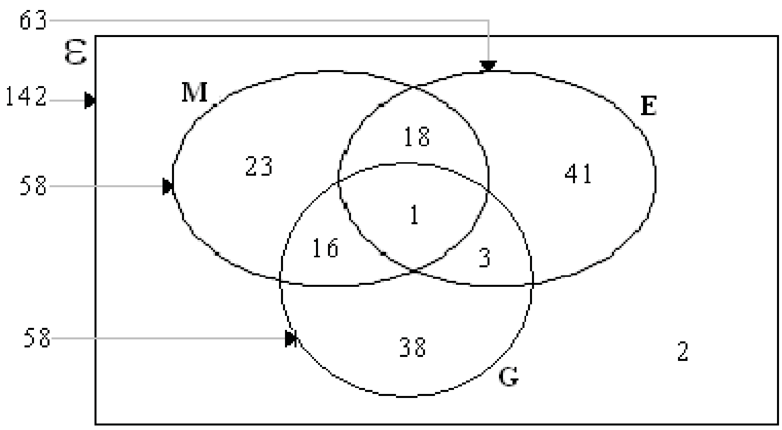
Similarly, from '19 used both microwave ovens and electric cookers', the Venn diagram looks thus:



Now, fill in the missing numbers in the subsets to equal the totals indicated by the arrowed lines:



Finally, use '2 cooked with solar energy only', to complete the Venn diagram:



This Venn diagram, which shows 'the number of elements in different regions', can now be used to answer the questions:

(a) How many people used gas cookers only?

____

Compare your answer with:

38

(b) How many people used microwave ovens and electric cookers only?

____

Compare your answer with:

18 + 1 = 19

(c) How many people used microwave ovens or electric cookers?

____

Compare your answer with:

23 + 18 + 16 + 1 + 41 + 3 = 102

(d) How many people used microwave ovens or electric cookers, but did not use gas cookers?

____

Compare your answer with:

23 + 18 + 41 = 82

(e) How many people used microwave ovens or solar energy?

____

Compare your answer with:

23 + 18 + 1 + 16 + 2 = 60

Example 2:

A group of 80 pupils were asked which mode of transport they used to go to school in the year 2000, and the following results were obtained:

36 pupils walked to school;

32 used taxis;

32 used family cars;

16 used family cars and taxis;

16 used taxis and walked to school;

14 used family cars and walked to school;

6 used all the three modes of transport.

(a) How many pupils walked to school only?

(b) How many pupils used none of the three modes of transport?

(c) How many pupils used taxi only?

(d) How many pupils used family cars and taxis, but not walked to school?

(e) How many pupils did not use family cars?

Let:

ε = {the group 80 pupils}

T = {pupils who used taxis}

F = {pupils who used family cars}

W = {pupils who walked to school}

Now, use the given data to construct your own Venn diagram.

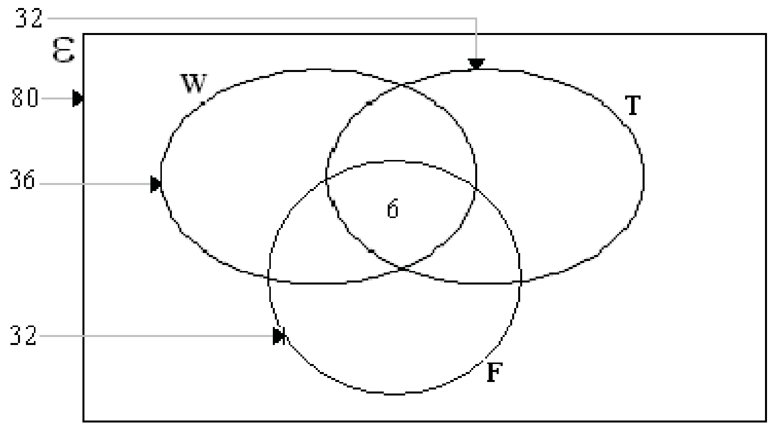
What can you conclude about the subsets T, F and W, from the data ‘6 used all the three modes of transport.’?

____

Compare your answer with the following:

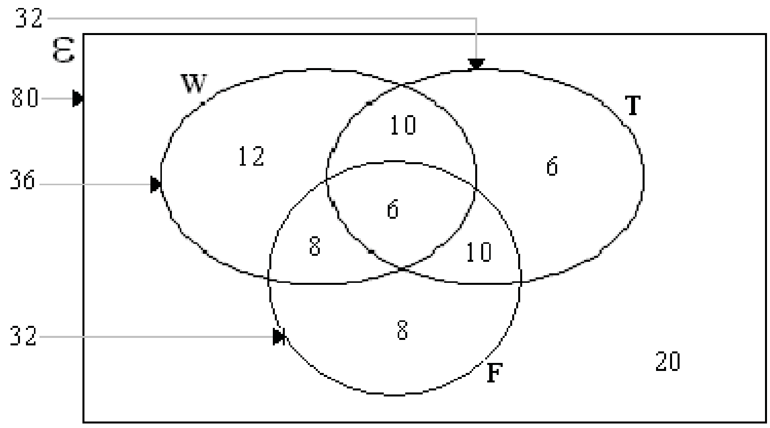
The subsets T, F and W have a common intersection, and therefore, they overlap.

Considering the data ‘36 pupils walked to school; 32 used taxis; 32 used family cars;’ the Venn diagram should, thus far, look:



Use the other given data to complete the diagram.

Compare your Venn diagram with the following:



Now, use the Venn diagram to answer the questions:

(a) How many pupils walked to school only?

Compare your answer with:

12

(b) How many pupils used none of the three modes of transport?

Compare your answer with:

20

(c) How many pupils used taxi only?

Compare your answer with:

6

(d) How many pupils used family cars and taxis, but not walked to school?

Compare your answer with:

10

(e) How many pupils did not use family cars?

Compare your answer with:

12 + 10 + 6 + 20 = 48

Activity 6

Activity 6

1.

(a) Draw the Venn diagram for the information below.

Let $\mathcal{E}$ = {the group 50 boys}, S = {softball players} F = {football players} and V = {volleyball players}.

50 boys were asked about the games they played:

30 played football, 24 played softball and 11 played volleyball.

12 played softball and football.

6 played football and volleyball.

5 played all three games and 2 played volleyball only.

(b) Use your Venn diagram to answer the questions below.

(i) How many boys played softball and football but not volleyball?

(ii) How many boys played volleyball and football but not softball?

(iii) How many boys played football only?

(iv) How many boys played softball only?

(v) How many boys played none of these games?

2.

(a) 100 people were interviewed at a library, on the novels they liked.

65 people liked adventure novels.

43 people liked detective novels.

38 people liked romantic novels.

28 people liked both adventure novels and detective novels, and of these, 10 also

liked romantic novels.

22 people liked both detective novels and romantic novels.

32 people liked adventure novels only.

Let $\mathcal{E}$ = {the group 100 people interviewed at a library}, A = {people who like adventure novels} and D = {people who like detective novels} and R = {people who like romantic novels}.

Draw the Venn diagram to illustrate the given data.

(b) Use your Venn diagram to answer these questions.

(i) How many people liked all three types of novels?

(ii) How many people liked romantic novels only?

(iii) How many people liked none of the novels?

Compare your answers with those at the end of the subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content again.

Note it!

Remember:

From the given information, you should:

- draw the appropriate Venn diagram.
- identify appropriate region(s) with the solution(s).

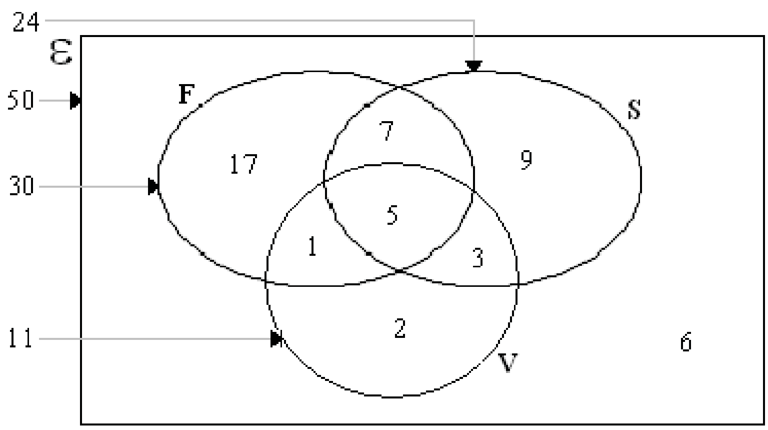
You have now completed the last subunit of this unit on sets. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Answers to Activity 6

1.

(a) Let $\mathcal{E}$ = {the group 50 boys}, S = {softball players} F = {football players} and V = { volleyball players }.

Venn diagram:



(b) Using the Venn diagram to answer the questions:

(i) How many boys played softball and football but not volleyball?

7

(ii) How many boys played volleyball and football but not softball?

1

(iii) How many boys played football only?

17

(iv) How many boys played softball only?

9

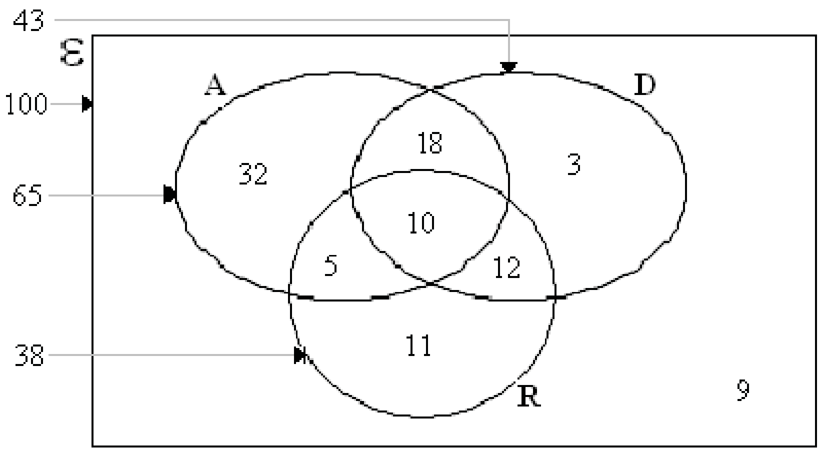
(v) How many boys played none of these games?

6

2.

(a) Let $\mathcal{E}$ = {the group 100 people interviewed at a library}, A = {people who like adventure novels} and D = {people who like detective novels} and R = {people who like romantic novels}.

Venn diagram:



(b) Using the Venn diagram to answer the questions:

(i) How many people liked all three types of novels?

10

(ii) How many people liked romantic novels only?

11

(iii) How many people liked none of the novels?

9

Unit Summary

Summary

In this unit you learned that a set, which can be *finite* or *infinite*, is a collection of objects that can defined by using *set builder notation* **or** *listing*.

An **element or member** is an object in a set. The symbol $\in$ is used for "is an element of" while $\notin$ means "is not an element of".

The symbol **n (A)** is used to denote the **number** of members or elements of a set A.

Sets are **equal** if they contain exactly the same elements. If set A is equal to set B, then A = B is true.

An **empty set** is a subset of any set. A subset is also a set containing **some or all** elements or members of another set.

Intersection of sets is a set which is formed by putting together **all common** elements or members of the sets; the symbol $\cap$ is used for the intersection of sets.

If A and B are two sets, a set which is formed by putting together **all** elements or members of these sets without repeating any element or member is the **union,** written $A \cup B$.

A **complement of set A** is a set of **all members not** in set A but in the universal set, **written A**$^{/}$.

Various statements can be made, using **set notation**, about the information that is shown in Venn diagrams.

When shading the regions of Venn diagrams, list the members of the set which is to be shaded, and then shade the region or regions, in which those members are found.

Venn diagrams can be used in practical situations as already stated that naturally things form sets or groups. Instead of showing the actual elements, the number of elements can sometimes be written in the respective regions of the Venn diagram.

You have completed the material for this unit on sets. You should now spend some time reviewing the content. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

When you work on this assignment, please observe the time allocated and show your work or reason for each answer.

TOTAL MARKS: 35

TIME: 40 minutes

1. Complete the table.

Set builder notation	Listing
{ even numbers not less than 6 }	
	{ 1, 2, 4, 8, 16 }
{ t | t is a multiple of 3, $6 < t \leq 12$ }	

(6 marks)

2.

Let $\mathcal{E}$ = {2, 6, 7, 8, 10, 12, 16};

A = {2, 6, 7, 8, 10, 12};

B = {2, 6, 8, 10};

C = {6, 10, 12}.

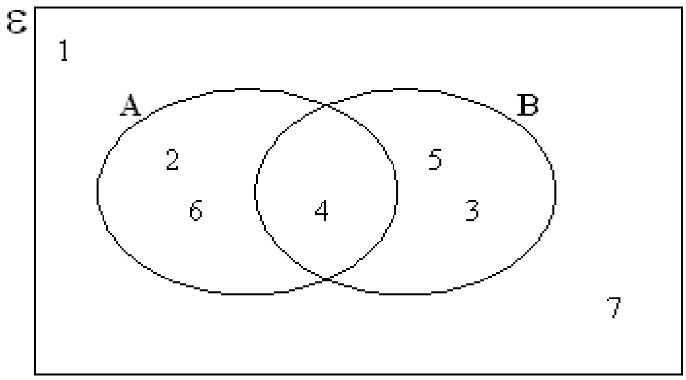
Write *true* or *false* for each of the following:

Set notation	*true* or *false*
(a) $n(\mathcal{E}) = 7$	
(b) $C \subset B$	
(c) $\{2, 6, 8, 10\} \cap \{6, 10, 12\} = \{6, 10\}$	
(d) $\{6\} \in B$	

(e) $A \cup B = A$	

(5 marks)

3. Based on the Venn diagram:



(a) list the set $(A' \cap B) \cup (B' \cap A)$

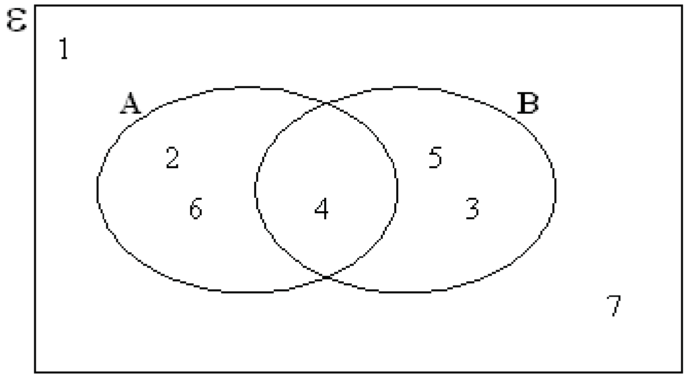
____

____

____

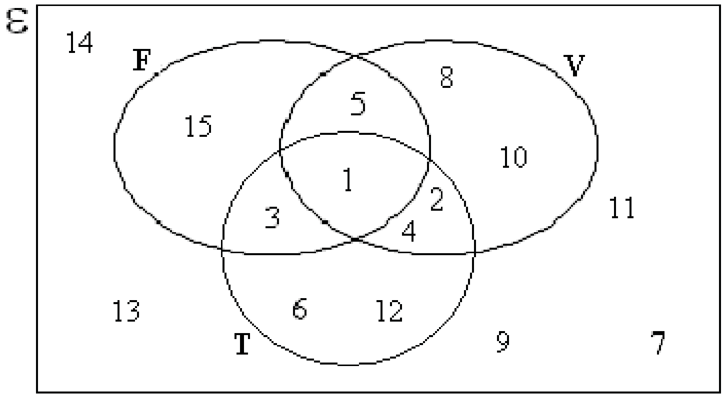
____(2 marks)

(b) shade the set $(A' \cap B) \cup (B' \cap A)$.



(2 marks)

4. Based on the Venn diagram:



(a) list the set (F ∩ V) ∪ (F ∩ T) ∪ (T ∩ V)

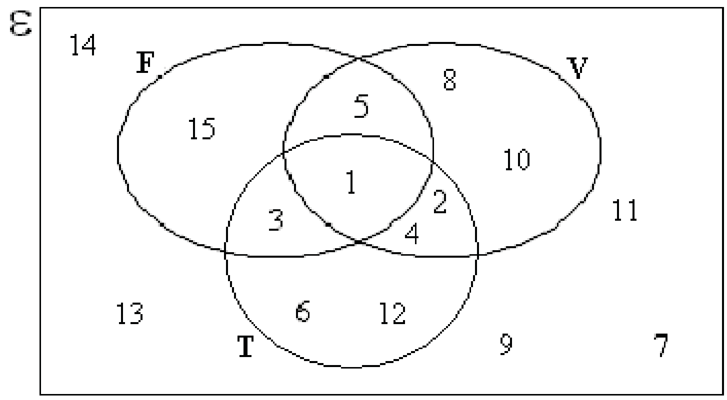
____

____

____

____(2 marks)

(b) shade the set (F ∩ V) ∪ (F ∩ T) ∪ (T ∩ V) .



(2 marks)

5. Let $\mathcal{E} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{ n \mid n$ is a counting number, $5 \leq n \leq 9 \}$, B = {even numbers } and C = {multiples of 4 bigger than 5 }.

a) Draw the Venn diagram for the given information.

(5 marks)

b) Find $n(A \cup B)$.

____(2 marks)

c) Shade $A \cap B \cap C'$.

(2 marks)

d) Shade $(B \cap A^{/}) \cup (A \cap B^{/}) \cup C$.

(2 marks)

6. Let $\mathcal{E}$ = {the group 50 learners in Mathematics grade 12}, B = {male learners}, G = {female learners } and N = { learners whose surnames begin with the letter 'm'}.

There are 39 female learners in Mathematics grade 12.

There are 8 male learners in Mathematics grade 12 whose surnames do not begin with the letter 'm'.

There are 28 female learners in Mathematics grade 12 whose surnames begin with the letter 'm'.

Draw the Venn diagram to illustrate these data, writing the number of elements in the proper regions.

(5 marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Solutions to the ASSIGNMENT:

1. Completed table:

Set builder notation	Listing
{ even numbers not less than 6 }	{ 6, 8,10, 12, 14, **...** }
{ factors of 16 }	{ 1, 2, 4, 8, 16 }
$\{ t \mid t$ is a multiple of $3, 6 < t \leq 12 \}$	{ 9, 12 }

2.

Set notation	*true* or *false*

(a) n(ε) = 7	*true*
(b) C $\subset$ B	*false*
(c) {2, 6, 8, 10} $\cap$ {6, 10, 12}={6, 10}	*true*
(d) {6} $\in$ B	*false*
(e) A $\cup$ B = A	*true*

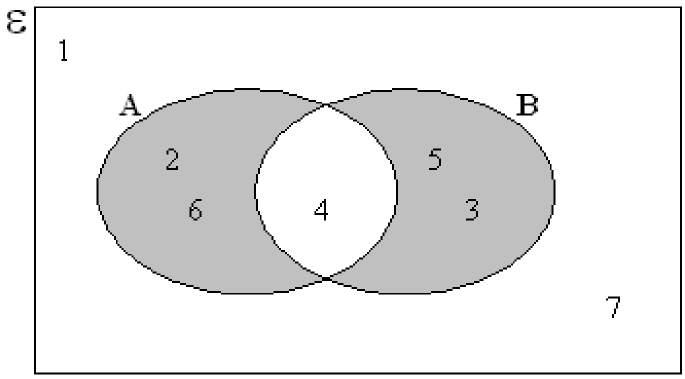
3. Listing $(A' \cap B) \cup (B' \cap A)$:

$(A' \cap B) = \{3, 5\}$

$(B' \cap A) = \{2, 6\}$

Therefore, $(A' \cap B) \cup (B' \cap A) = \{2, 3, 5, 6\}$

Shaded $(A' \cap B) \cup (B' \cap A)$ in the Venn diagram:



4. Listing $(F \cap V) \cup (F \cap T) \cup (T \cap V)$:

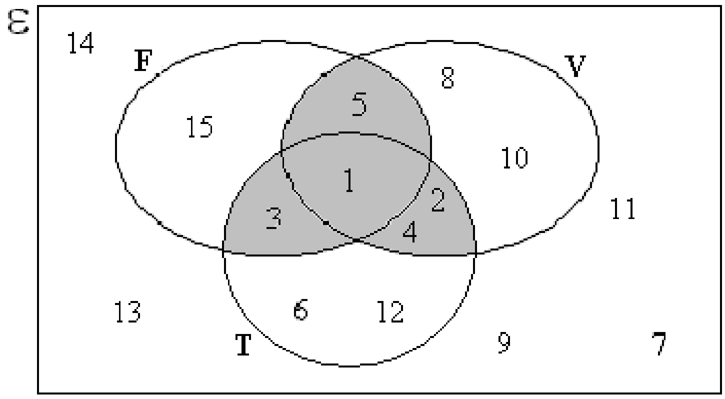
$(F \cap V) = \{1, 5\}$

$(F \cap T) = \{1, 3\}$

$(T \cap V)$ = {1, 2, and 4}

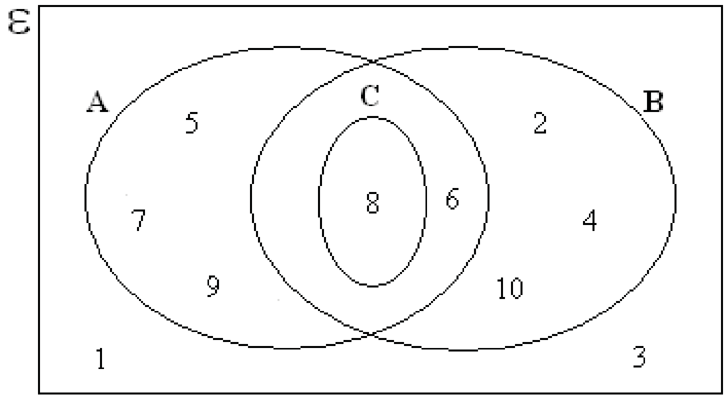
Therefore, $(F \cap V) \cup (F \cap T) \cup (T \cap V) = \{1, 2, 3, 4, 5\}$

Shaded $(F \cap V) \cup (F \cap T) \cup (T \cap V)$ in the Venn diagram:



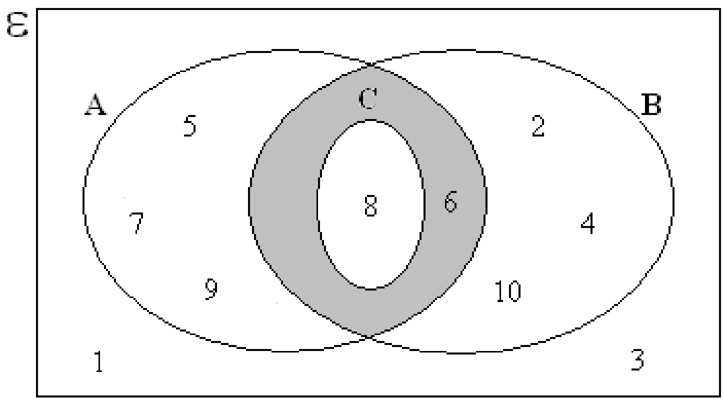
5. Let ε = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, A = { n | n is a counting number, $5 \leq n \leq 9$ }, B = {even numbers } and C = {multiples of 4 bigger than 5 }.

a) The Venn diagram for the given information :

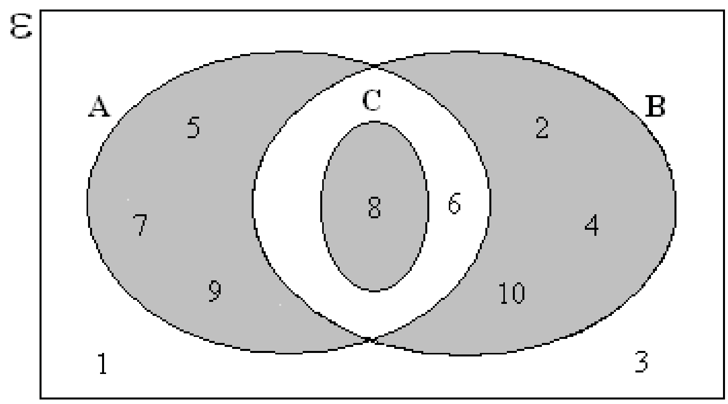


b) $n(A \cup B) = 8$.

c) Shaded $A \cap B \cap C'$:



d) Shaded $(B \cap A^{/}) \cup (A \cap B^{/}) \cup C$:



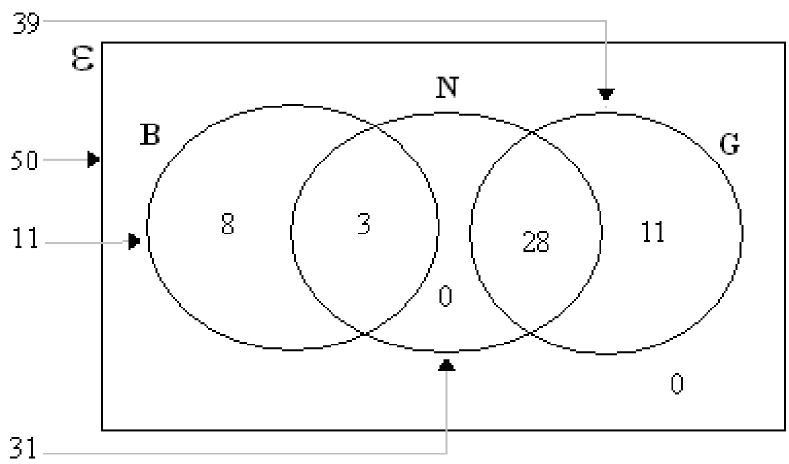
6. Let ℇ = {the group 50 learners in Mathematics grade 12}, B = {male learners}, G = {female learners } and N = { learners whose surnames begin with the letter 'm'}.

There are 39 female learners in Mathematics grade 12.

There are 8 male learners in Mathematics grade 12 whose surnames do not begin with the letter 'm'.

There are 28 female learners in Mathematics grade 12 whose surnames begin with the letter 'm'.

From the data, this is the Venn diagram:



Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

When you work on this assessment, please observe the time allocated and show your work or reason for each answer.

TOTAL MARKS: 20

TIME: 25 minutes

1) A = { 7, 14, 21, 28, 35, . . . }. Describe the given set in set builder notation.

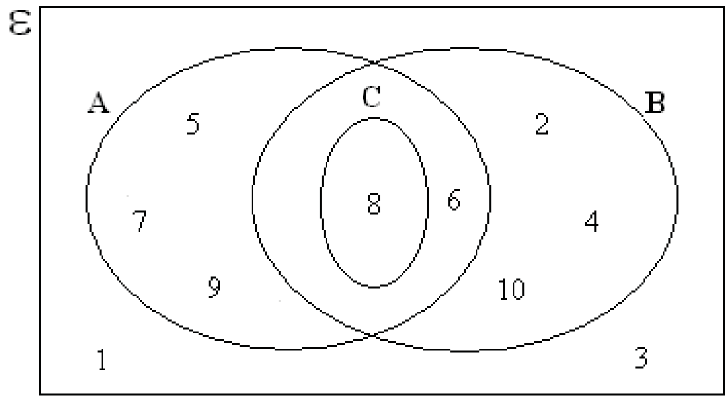
____

____(2 marks)

2) List the following set: F = { x | x^2 is a multiple of 2, 3 < x ≤ 40 }.

____(2 marks)

3) Let ℰ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }, A = { n | n is a counting number, 5 ≤ n ≤ 9 }, B = {even numbers } and C = {multiples of 4 bigger than 5 }. The Venn diagram for the given information is as follows :



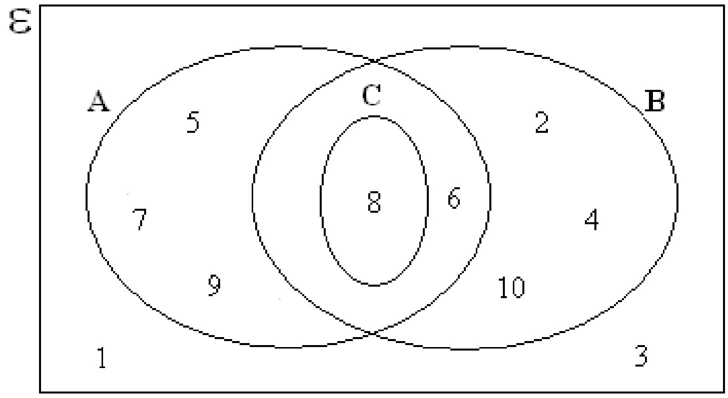
(a) Find $n((A \cup B)' \cup C)$.

____(2 marks)

(b) List $A' \cup C$.

____(2 marks)

(c) Shade $(A \cup B)' \cup C$.



(2 marks)

4) A group of 60 pupils in grade 12 were asked which sport they liked to watch on the television. 40 pupils liked to watch soccer; 33 pupils liked to watch rugby; 17 pupils liked to watch netball and rugby; 21 pupils liked to watch rugby and soccer; 19 pupils liked to watch netball and soccer; 29 pupils liked to watch netball; 5 pupils liked to watch rugby only.

Let ε = {60 pupils in grade 12}, S = { pupils who liked to watch soccer}, N = { pupils who liked to watch netball pupils} and R = { pupils who liked to watch rugby}.

Construct the Venn diagram for the given data.

(4 marks)

(a) How many pupils liked to watch netball only?

____(1 mark)

(b) How many pupils liked none of the three sports?

____(1 mark)

(c) How many pupils liked to watch netball and soccer, but not rugby?

____(2 marks)

(d) How many pupils liked at least two sports?

____(2 marks)

Solutions to assessment questions.

1) A = { multiples of 7 }.

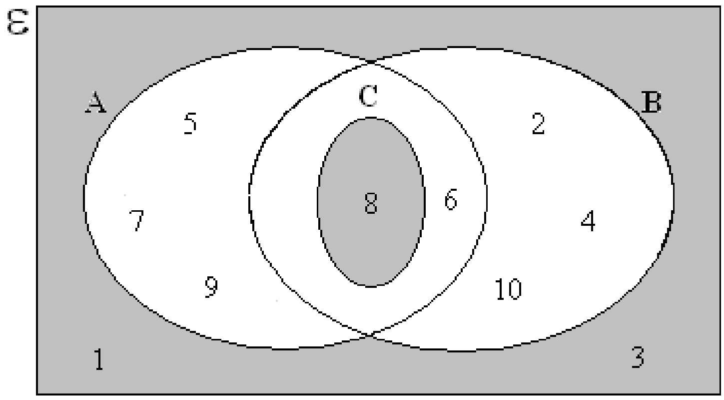
2) F = { 4, 6 }.

3)

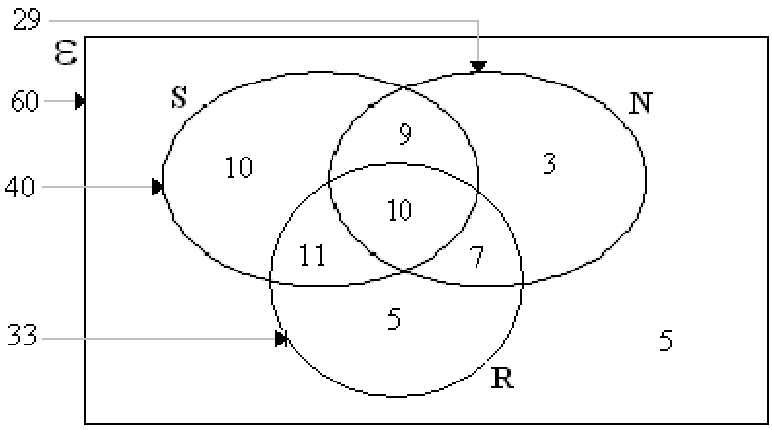
(a) n($(A \cup B)^{/} \cup C$) = 3.

(b) $A^{/} \cup C$ = { 1, 2, 3, 4, 8, 10 }

(c) Shaded $(A \cup B)^{/} \cup C$.



4) The Venn diagram for the given data:



(a) 3 pupils liked to watch netball only.

(b) 5 pupils liked none of the three sports.

(c) 9 pupils liked to watch netball and soccer, but not rugby.

37 pupils liked at least two sports

Unit Contents

Unit 2

Unit 2

Types of Numbers

Introduction

Real Numbers! *Overly Imaginary*, And **Yet So Real**.

You use numbers when counting the members of your family. Fractions are involved when sharing a cake with your friends.

You should have already used numbers in different situations, like in addition, subtraction, division and multiplication. In this unit, you are going to classify numbers and review the mathematical operations with rational numbers.

This unit consists of 74 pages. It covers approximately 3% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 30 hours to work on this unit, 20 hours for formal study and 10 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Seven Lessons:

Lesson 1 Identifying Natural Numbers, Whole numbers, Integers, Rational Numbers, Irrational Numbers and Real numbers
Lesson 2 Adding Rational Numbers
Lesson 3 Subtracting Rational Numbers
Lesson 4 Multiplying Rational Numbers
Lesson 5 Dividing Rational Numbers
Lesson 6 Demonstrating an Understanding of Classes of Numbers Like Odd, Even, Polygonal, Cube and Prime Numbers in a Variety of Situations
Lesson 7 Solving Practical Situations Involving Directed Numbers

Take a moment to read the following learning outcomes. They are a guide to what you should focus on while studying this unit.

Upon completion of this unit you will be able to:

Outcomes

- *identify* natural (counting) numbers, whole numbers, integers, rational numbers, irrational numbers, and real numbers.
- *add* rational numbers.
- *subtract* rational numbers.
- *multiply* rational numbers.
- *divide* rational numbers.
- *demonstrate* an understanding of classes of numbers like odd, even, polygonal (triangular, rectangular or composite and square), cube and prime numbers in a variety of situations.
- *solve* practical situations involving directed numbers.

Terminology

|x|:	The distance of the number x from zero on a number line.
Composite/ rectangular number:	A composite (rectangular) number is a positive integer which has more than two factors.
Cube number :	A cube number_ is a whole number which can be written as a power, with 3 as the index.
Difference:	The result of subtracting a lesser number from a greater number.
Directed numbers:	Directed numbers are positive and negative real numbers.
Even number :	An even number_is a whole number which leaves a remainder of 0 when divided by 2.
Odd number :	An odd number_is a whole number which leaves a remainder of 1when divided by 2.
Opposites:	Two numbers which are the same distance from zero but on the opposite side of zero.
Polygonal	Polygonal numbers are sequential numbers which form second-order arithmetic sequences, in which the second

numbers: term of the sequence shows the number of sides in the *polygonal structures* that can be produced from these sequences.

Prime number: A prime number is a whole number which has only *two factors, 1 and itself.*

Product: The result of multiplication.

Quotient: The result of dividing one number by another.

Reciprocal: Reciprocal of number x is the number obtained by dividing 1 by x.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Identifying Natural (counting) Numbers, Whole numbers, Integers, Rational Numbers, Irrational Numbers and Real numbers

Introduction

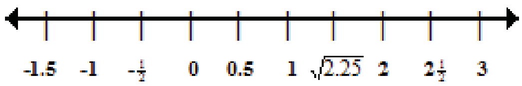
By the end of this subunit, you should be able to:

- identify natural or counting numbers from other numbers.
- identify whole numbers from other numbers.
- identify integer numbers from other numbers.
- identify rational numbers from other numbers.
- identify irrational numbers from other numbers.
- classify all real numbers into natural numbers, whole numbers, integer numbers, rational numbers and irrational numbers.

This subunit is about 6 pages in length.

Different types of ***real numbers.***

Numbers are the basic building blocks of mathematics which can be shown on a real number line.



Numbers match with other numbers which are called their **opposites**, which are the same distance from zero but on the opposite side of zero, such as 4 and -4 or -2 and 2.

Numbers become greater on the number line as we move from left to right. For example, 4 is greater than 2 since 4 is farther to the right on the number line.

Activity 1

Identifying natural (counting) numbers.

Write down the number of letters in the following words:

Activity 1

(a) **I**

____

(b) **AM**

____

(c) **NOT**

____

(d) **WELL**

____

(e) **TODAY**

____

Check your answers with those at the end of the subunit.

Counting numbers

The numbers which you have written are called **counting** numbers and are used when counting objects or things. Counting numbers are also called **natural** numbers.

On the real number line, **natural** numbers are equally spaced; with **1 as the smallest natural number and *there is no largest natural number*.**

Identifying whole numbers

Suppose you buy 5 apples from a market; and you eat 3 apples and give the other 2 to your friend.

Write down the number of apples that you are left with thereafter.

___

You should have:

0 apples

Counting (natural) numbers and zero are named **whole** numbers.

On the real number line, **whole** numbers are equally spaced; with **0 as the smallest whole number and *there is no largest whole number*.**

Identifying integers

The opposites of counting numbers *on a number line* are called negative integers while counting numbers are positive integers.

Zero is neither negative nor positive.

Whole numbers and their opposites are called **integers**.

On the real number line, **integers** are equally spaced; with **no smallest integer and *there is no largest integer too*.**

Identifying rational numbers

These are *quotients* of integers.

Numbers that can be written as ratios of two integers, **x/y**, with **y** not equal to zero (you cannot divide by zero) are called **rational** numbers.

For example, 1/3, 2/3, 3/3, 1, 4/3, $1\frac{1}{3}$ are rational numbers.

On the real number line, **rational** numbers have **no smallest rational number and *there is no largest rational number too*.**

Activity 2

Complete the table below by filling in YES or NO where appropriate:

Activity 2

Number	Counting number	Whole number	Integer	Rational number
-2	**NO**			
0		**YES**		
20			**YES**	
$\frac{3}{5}$		**NO**		
$-\frac{4}{7}$				
$\sqrt{9}$				
$-\sqrt{10}$				
1.33				

Compare your completed table with the table at the end of the subunit. Be sure that you understand the differences between counting numbers, whole numbers, integers, and rational numbers before continuing.

Identifying Irrational Numbers.

Numbers that *cannot be written as ratios* of two integers, x/y with y not equal to zero are called **irrational** numbers.

For example, $\sqrt{10}$, $-\sqrt{3}$, π (pie) and $\sqrt[3]{2}$ are irrational numbers, since they cannot be written as ratios of two integers.

On the real number line, **there is no smallest and no largest irrational number.**

Identifying Real Numbers

A continuous real number line consists of natural (counting) numbers, whole numbers, integers, rational numbers and irrational numbers.

A Venn diagram for all types of numbers on a real number line is shown in figure 1.1.

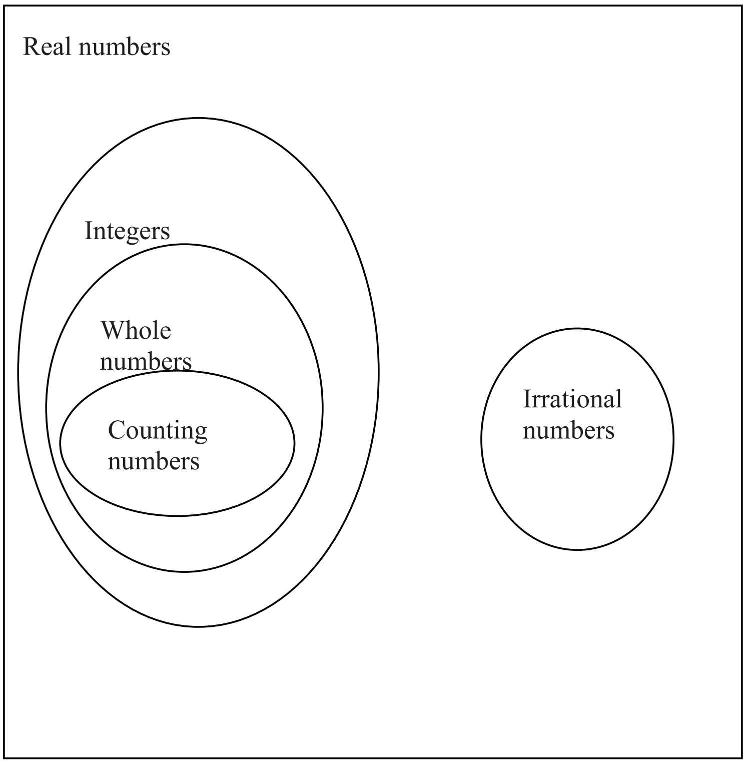


Figure 1.1 – Types of Numbers

Absolute Value

The distance of any number from zero on a number line is called its **absolute value.** The absolute value is positive or zero because distance is never negative. The absolute values of opposites are the same: For example, the absolute value of -7 and 7 is 7. Absolute value of x is denoted by $|x|$ and $|x| \geq 0$.

Activity 3

Complete the table below:

Activity 3

X	Notation of the absolute value of the number	Absolute Value
-2	| -2	
0		0
20		
$\frac{3}{5}$		
$-\frac{4}{7}$		
$\sqrt{9}=3$		
$-\sqrt{10}$		$\sqrt{10}$
1.33		

Compare your completed table with the table at the end of the subunit. Be sure that you can determine the absolute value of numbers before continuing.

Note it!

Remember:

natural (counting) numbers	1, 2, 3 ,4, 5, . . .
whole numbers	zero and all natural (counting) numbers
integers	whole numbers and their opposites
rational numbers	integers and all numbers that can be written as ratios of two integers, **x/y**, with **y** not equal to zero

irrational numbers	real numbers that are *not rational*
ALL ABOVE NUMBERS TOGETHER FORM REAL NUMBERS	

Solutions to the subunit activities

Solutions to ACTIVITY 1:

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

Solutions to ACTIVITY 2:

Number	Counting number	Whole number	Integer	Rational number
$-2=-\frac{2}{1}$	NO	NO	YES	YES
$0=\frac{0}{1}$	NO	YES	YES	YES
$20=\frac{20}{1}$	YES	YES	YES	YES
$\frac{3}{5}$	NO	NO	NO	YES
$-\frac{4}{7}$	NO	NO	NO	YES

$\sqrt{9}=3$	YES	YES	YES	YES
$-\sqrt{10}$	NO	NO	NO	NO
$1.33=\frac{133}{100}$	NO	NO	NO	YES

Solutions to ACTIVITY 3:

X	Notation of the absolute value of the number	Absolute Value
-2	| -2 |	2
0	| 0 |	0
20	| 20 |	20
$\frac{3}{5}$	$\lvert \frac{3}{5} \rvert$	$\frac{3}{5}$
$-\frac{4}{7}$	$\lvert -\frac{4}{7} \rvert$	$\frac{4}{7}$
$\sqrt{9}=3$	$\lvert \sqrt{9} \rvert$	3
$-\sqrt{10}$	$\lvert -\sqrt{10} \rvert$	$\sqrt{10}$
1.33	| 1.33 |	1.33

Lesson 2 Adding Rational Numbers

Introduction

By the end of this subunit, you should be able to:

- add any two rational numbers with the same sign.
- add any two rational numbers with different signs.

This subunit is about 10 pages in length.

Addition of rational numbers.

You can add any two rational numbers basically in three steps.

Adding any two rational numbers with the same sign:

- Write the addition of their absolute values in brackets and their sign outside the brackets;
- Work out the sum in the brackets;
- Remove the brackets.

Example 1:

$^{-}\mathbf{5} + {}^{-}\mathbf{7}$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{-}\mathbf{5} + {}^{-}\mathbf{7}$

$= {}^{-}(\mathbf{5} + \mathbf{7})$

Work out the sum in the brackets **.**

$^{-}(\mathbf{5} + \mathbf{7})$

$= {}^{-}(\mathbf{12})$

Remove the brackets.

$^{-}(\mathbf{12})$

$= {}^{-}\mathbf{12}$

Let us do Example 2 together:

$^{+}\frac{2}{5} + {}^{+}\frac{5}{7}$

Write the addition of their absolute values in brackets and their sign outside the brackets.

____

____

Compare your answer with:

$^{+}\frac{2}{5} + {}^{+}\frac{5}{7}$

$$= {}^{+}\left(\frac{2}{5} + \frac{5}{7}\right)$$

Work out the sum in the brackets.

Compare your answer with:

$${}^{+}\left(\frac{2}{5} + \frac{5}{7}\right)$$

$$= {}^{+}\left(\frac{39}{35}\right)$$

Remove the brackets and simplify.

Compare your answer with:

$${}^{+}\left(\frac{39}{35}\right)$$

$${}^{+}\frac{39}{35}$$

$$= {}^{+}\mathbf{1}\,\frac{4}{35}$$

Activity 4

Work out the addition questions below:

Activity 4

(a) ${}^{-}6 + {}^{-}8$

(b) ${}^{+}6 + {}^{+}8$

(c) $\frac{^{-}2}{3} + {}^{-}8$

(d) ${}^{+}6 + {}^{+}8\frac{1}{3}$

(e) ${}^{-}6.3 + {}^{-}8.9$

Compare your answers with those at the end of the subunit. If you scored at least 80%, continue on. If not, review *adding any two rational numbers with the same sign.*

Adding any two rational numbers with different signs:

- Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets;
- Work out the difference in the brackets;
- Remove the brackets.

Example1:

$^{-}4 + {}^{+}8$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$^{-}4 + {}^{+}8$

$= {}^{+}(8 - 4)$

Work out the difference in the brackets.

$^{+}(8 - 4)$

$= {}^{+}(4)$

Remove the brackets.

$^{+}(4)$

$= {}^{+}4$

Let us do Example 2 together:

$^{-}9.4 + {}^{+}2.5$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

____

____

Compare your answer with:

$^{-}9.4 + {}^{+}2.5$

$= {}^{-}(9.4 - 2.5)$

Work out the difference in the brackets.

____

____

Compare your answer with:

$^{-}(9.4 - 2.5)$

$= {}^{-}(6.9)$

Remove the brackets.

____

Compare your answer with:

$^{-}(\mathbf{6.9})$

$= {}^{-}\mathbf{6.9}$

<u>Activity 5</u>

Work out the additions below:

Activity 5

(a) $^{+}6 + {}^{-}8$

(b) $^{-}6 + {}^{+}8$

(c) $^{-}\frac{2}{3} + {}^{+}8$

(d) $^{-}6 + {}^{+}8\frac{1}{3}$

(e) $^{+}6.3+^{-}8.9$

Check your performance against the given solutions at the end of the subunit. Continue if you are satisfied with your score. If not, review *adding any two rational numbers with different signs.*

Note it!

Remember:

When adding any two rational numbers with the same sign:

1. Write the addition of their absolute values in brackets and their sign outside the brackets;
2. Work out the sum in the brackets;
3. Remove the brackets.

When adding any two rational numbers with different signs:

1. Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets;
2. Work out the difference in the brackets;
3. Remove the brackets.

Solutions to the subunit activities

Solutions to ACTIVITY 4:

(a) $^{-}6+^{-}8$

$^{-}\mathbf{6} + {}^{-}\mathbf{8}$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{-}6 + {}^{-}8$

$= {}^{-}(6+8)$

Work out the sum in the brackets.

$^{-}(6+8)$

$= {}^{-}(14)$

Remove the brackets.

$^{-}(14)$

$= {}^{-}14$

(b) $^{+}6+{}^{+}8$

$^{+}6+{}^{+}8$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{+}6+{}^{+}8$

$= {}^{+}(6+8)$

Work out the sum in the brackets.

$^{+}(6+8)$

$= {}^{+}(14)$

Remove the brackets.

$^{+}(14)$

$= {}^{+}14$

(c) $^{-}\frac{2}{3}+{}^{-}8$

$^{-}\frac{2}{3}+{}^{-}8$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{-}\frac{2}{3}+{}^{-}8$

$= {}^{-}\left(\frac{2}{3}+8\right)$

Work out the sum in the brackets.

$^{-}\left(\frac{2}{3}+8\right)$

$= {}^{-}\left(\frac{26}{3}\right)$

Remove the brackets and simplify.

$^{-}\left(\frac{26}{3}\right)$

$= -\frac{26}{3}$

$= -8\frac{2}{3}$

(d) $^{+}6 + {}^{+}8\frac{1}{3}$

$^{+}6 + {}^{+}8\frac{1}{3}$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{+}6 + {}^{+}8\frac{1}{3}$

$= {}^{+}(6 + 8\frac{1}{3})$

Work out the sum in the brackets .

$^{+}(6 + 8\frac{1}{3})$

$= {}^{+}(14\frac{1}{3})$

Remove the brackets.

$^{+}(14\frac{1}{3})$

$= {}^{+}14\frac{1}{3}$

(e) $^{-}6.3 + {}^{-}8.9$

$^{-}6.3 + {}^{-}8.9$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{-}6.3 + {}^{-}8.9$

$= {}^{-}(6.3 + 8.9)$

Work out the sum in the brackets .

$^{-}(6.3 + 8.9)$

$= {}^{-}(15.2)$
Remove the brackets.

${}^{-}(15.2)$

$= {}^{-}15.2$

Solutions to ACTIVITY 5:

(a) ${}^{+}6 + {}^{-}8$

${}^{+}6 + {}^{-}8$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

${}^{+}6 + {}^{-}8$

$= {}^{-}(\mathbf{8 - 6})$

Work out the difference in the brackets.

${}^{-}(\mathbf{8 - 6})$

$= {}^{-}(\mathbf{2})$
Remove the brackets.

${}^{-}(\mathbf{2})$

$= {}^{-}\mathbf{2}$

(b) ${}^{-}6 + {}^{+}8$

${}^{-}6 + {}^{+}8$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

${}^{-}6 + {}^{+}8$

$= {}^{+}(\mathbf{8 - 6})$

Work out the difference in the brackets.

${}^{+}(\mathbf{8 - 6})$

$= {}^{+}(\mathbf{2})$
Remove the brackets.

${}^{+}(\mathbf{2})$

$= {}^{+}\mathbf{2}$

(c) $^{-}\frac{2}{3} + {}^{+}8$

$^{-}\frac{2}{3} + {}^{+}8$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$^{-}\frac{2}{3} + {}^{+}8$

$= {}^{+}\left(8 - \frac{2}{3}\right)$

Work out the difference in the brackets.

${}^{+}\left(8 - \frac{2}{3}\right)$

$= {}^{+}\left(\frac{22}{3}\right)$

Remove the brackets.

${}^{+}\left(\frac{22}{3}\right)$

$= \frac{^{+}22}{3}$

(d) $^{-}6 + {}^{+}8\frac{1}{3}$

$^{-}6 + {}^{+}8\frac{1}{3}$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$^{-}6 + {}^{+}8\frac{1}{3}$

$= {}^{+}(8\frac{1}{3} - 6)$

Work out the difference in the brackets.

${}^{+}(8\frac{1}{3} - 6)$

$= {}^{+}(2\frac{1}{3})$

Remove the brackets.

$^{+}(2\frac{1}{3})$

$= {}^{+}2\frac{1}{3}$

(e) $^{+}6.3 + {}^{-}8.9$

$^{+}6.3 + {}^{-}8.9$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$^{+}6.3 + {}^{-}8.9$

$= {}^{-}(8.9 - 6.3)$

Work out the difference in the brackets**.**

$^{-}(8.9 - 6.3)$

$= {}^{-}(2.6)$

Remove the brackets.

$^{-}(2.6)$

$= {}^{-}2.6$

Lesson 3 Subtracting Rational Numbers

Introduction

By the end of this subunit, you should be able to:

- subtract a rational number from another, with the same sign.
- subtract a rational number from another, with a different sign.

This subunit is about 6 pages in length.

Subtraction in rational numbers.

Subtracting a rational number is equivalent to adding the opposite of the rational number. When subtracting -4, for example, it is the same as adding +4; and vice versa. After this, you follow the three steps which you learned in '**adding rational numbers**'.

Subtracting a rational number from another rational number:

- ➢ Change subtraction to addition of the opposite of the rational number which is to be subtracted.

- Use the rules for adding any two rational numbers which have been studied.

Example 1:

$^{-}4 - {}^{+}5$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$^{-}4 - {}^{+}5$

$= {}^{-}4 + {}^{-}5$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{-}4 + {}^{-}5$

$= {}^{-}(4 + 5)$

Work out the sum in the brackets.

$^{-}(4 + 5)$

$= {}^{-}(9)$

Remove the brackets.

$^{-}(9)$

$= {}^{-}9$

Example 2:

$^{-}7 - {}^{-}3$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$^{-}7 - {}^{-}3$

$= {}^{-}7 + {}^{+}3$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$^{-}7 + {}^{+}3$

$= {}^{-}(7 - 3)$

Work out the difference in the brackets.

$^{-}(7 - 3)$

$= {}^{-}(4)$

Remove the brackets.

$^{-}$ **(4)**

= $^{-}$**4**

Let's do Example 3 together:

$^{+}$**9 -** $^{+}$**4**

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

Compare your answer with:

$^{+}$**9 -** $^{+}$**4**

= $^{+}$**9 +** $^{-}$**4**

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

Compare your answer with:

$^{+}$**9 +** $^{-}$**4**

= $^{+}$**(9 – 4)**

Work out the difference in the brackets**.**

Compare your answer with:

$^{+}$**(9 – 4)**

= $^{+}$**(5)**

Remove the brackets.

Compare your answer with:

$^{+}$**(5)**

= $^{+}$**5**

Activity 6

<u>Activity 6</u>

Work out the subtractions below:

(a) $^{+}6 - {}^{-}8$

(b) $^{-}6 - {}^{+}8$

(c) $^{-}\frac{2}{3} - {}^{-}8$

(d) $^{+}6 - {}^{+}8\frac{1}{3}$

(e) $^{+}6.3 - {}^{-}8.9$

Compare your answers with those at the end of the subunit. Continue on if you scored at least 80%. If not, review *subtracting a rational number from another rational number*.

Note it!

Remember:

When subtracting a rational number from another rational number:

1. Change subtraction to addition of the opposite of the rational number which is to be subtracted.
2. Use the rules for adding any two rational numbers.

Solutions to ACTIVITY 6:

(a) $^{+}6 - {}^{-}8$

$^{+}6 - {}^{-}8$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$^{+}6 - {}^{-}8$

$= {}^{+}6 + {}^{+}8$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{+}6 + {}^{+}8$

$= {}^{+}\mathbf{(6 + 8)}$

Work out the sum in the brackets.

$^{+}\mathbf{(6 + 8)}$

$= {}^{+}\mathbf{(14)}$

Remove the brackets.

$^{+}\mathbf{(14)}$

$= {}^{+}\mathbf{14}$

(b) $^{-}6 - {}^{+}8$

$^{-}6 - {}^{+}8$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$^{-}6 - {}^{+}8$

$= {}^{-}6 + {}^{-}8$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{-}6 + {}^{-}8$

$= {}^{-}\,(\mathbf{6 + 8})$

Work out the sum in the brackets.

$^{-}\,(\mathbf{6 + 8})$

$= {}^{-}\,(\mathbf{14})$

Remove the brackets.

$^{-}\,(\mathbf{14})$

$= {}^{-}\mathbf{14}$

(c) $\frac{^{-}2}{3} - {}^{-}8$

$\frac{^{-}2}{3} - {}^{-}8$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$\frac{^{-}2}{3} - {}^{-}8$

$= \frac{^{-}2}{3} + {}^{+}8$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$\frac{^{-}2}{3} + {}^{+}8$

$= {}^{+}\left(8 - \frac{2}{3}\right)$

Work out the difference in the brackets.

$^{+}\left(8-\frac{2}{3}\right)$

$= {}^{+}\left(\frac{22}{3}\right)$

Remove the brackets.

$^{+}\left(\frac{22}{3}\right)$

$= \frac{^{+}22}{3}$

(d) $^{+}6 - {}^{+}8\frac{1}{3}$

$^{+}6 - {}^{+}8\frac{1}{3}$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$^{+}6 - {}^{+}8\frac{1}{3}$

$= {}^{+}6 + {}^{-}8\frac{1}{3}$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$^{+}6 + {}^{-}8\frac{1}{3}$

$= {}^{-}\left(8\frac{1}{3} - 6\right)$

Work out the difference in the brackets.

$^{-}\left(8\frac{1}{3} - 6\right)$

$= {}^{-}\left(2\frac{1}{3}\right)$

Remove the brackets.

$^{-}\left(2\frac{1}{3}\right)$

$= {}^{-}2\frac{1}{3}$

(e) $^{+}6.3 - {}^{-}8.9$

$^{+}6.3 - {}^{-}8.9$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$^{+}6.3 - {}^{-}8.9$

$= {}^{+}6.3 + {}^{+}8.9$

Write the addition of their absolute values in brackets and their sign outside the brackets.

${}^{+}6.3 + {}^{+}8.9$

$= {}^{+}\mathbf{(6.3 + 8.9)}$

Work out the sum in the brackets**.**

${}^{+}\mathbf{(6.3 + 8.9)}$

$= {}^{+}\mathbf{(15.2)}$

Remove the brackets.

${}^{+}\mathbf{(15.2)}$

$= {}^{+}\mathbf{15.2}$

Lesson 4 Multiplying Rational Numbers

Introduction

By the end of this subunit, you should be able to

- multiply two rational numbers with different signs.
- multiply two rational numbers with the same sign.

This subunit is about 8 pages in length.

Multiplication of a rational number by a rational number.

The product, which is the result you get after multiplication, is always positive for any two positive rational numbers. The product of any two negative rational numbers is also positive. But the product of a positive rational number and negative rational number is always negative.

Multiplying any two rational numbers with different signs:

- Multiply their absolute values in brackets with a negative sign outside the brackets;
- Work out the product in the brackets;
- Remove the brackets.

Example 1:

${}^{-}\mathbf{3} \times {}^{+}\mathbf{2}$

Multiply the absolute values in brackets with a negative sign outside the brackets.

$^{-}3 \times {}^{+}2$

$= {}^{-}(3 \times 2)$

Work out the product in the brackets.

$^{-}(3 \times 2)$

$= {}^{-}(6)$

Remove the brackets.

$^{-}(6)$

$= {}^{-}6$

Let us do Example 2 together:

$^{-}2.5 \times {}^{+}2$

Multiply the absolute values in brackets with a negative sign outside the brackets.

____

____

Compare your answer with:

$^{-}2.5 \times {}^{+}2$

$= {}^{-}(2.5 \times 2)$

Work out the product in the brackets.

____

____

Compare your answer with:

$^{-}(2.5 \times 2)$

$= {}^{-}(5)$

Remove the brackets.

____

____

Compare your answer with:

$^{-}(5)$

$= {}^{-}5$

Activity 7

Work out the multiplications below:

Activity 7

(a) $^{+}6 \times {}^{-}8$

(b) $^{-}6 \times {}^{+}8$

(c) $^{-}\frac{2}{3} \times {}^{+}8$

(d) $^{-}6 \times {}^{+}8\frac{1}{3}$

(e) $^{+}6.3 \times {}^{-}8.9$

Compare your answers with those at the end of the subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review *multiplying any two rational numbers with different signs*.

Multiplying any two rational numbers with the same sign:

- Multiply the absolute values;
- Work out the product.

Example 1:

$^{-}2.5 \times {}^{-}2.5$

Multiply the absolute values.

$^{-}2.5 \times {}^{-}2.5$

$= 2.5 \times 2.5$

Work out the product.

2.5×2.5

$= 6.25$

Let us do Example 2 together:

$^{+}\frac{1}{4} \times {}^{+}\frac{3}{4}$

Multiply the absolute values.

____

____

Compare your answer with:

$^{+}\frac{1}{4} \times {}^{+}\frac{3}{4}$

$= \frac{1}{4} \times \frac{3}{4}$

Work out the product.

____

Compare your answer with:

$$\frac{1}{4} \times \frac{3}{4}$$

$$= \frac{3}{16}$$

Activity 8

Work out the multiplications below:

Activity 8

(a) ${}^{-}6 \times {}^{-}8$

(b) ${}^{+}6 \times {}^{+}8$

(c) ${}^{-}\frac{2}{3} \times {}^{-}8$

(d) ${}^{+}6 \times {}^{+}8\frac{1}{3}$

(e) $^{-}6.3\times{}^{-}8.9$

Check your performance against the given solutions at the end of the subunit. Continue if you are satisfied with your score. If not, review *multiplying any two rational numbers with the same sign*.

Note it!

Remember:

When multiplying any two rational numbers with different signs:

1. Multiply the absolute values in brackets with a negative sign outside the brackets;
2. Work out the product in the brackets;
3. Remove the brackets.

When multiplying any two rational numbers with the same sign:

1. Multiply the absolute values;
2. Work out the product.

Solutions to the subunit activities

Solutions to ACTIVITY 7:

(a) $^{+}6\times{}^{-}8$

$^{+}6\times{}^{-}8$

Multiply the absolute values in brackets with a negative sign outside the brackets.

$^{+}6\times{}^{-}8$

$= {}^{-}(\mathbf{6} \times \mathbf{8})$

Work out the product in the brackets.

$^{-}(\mathbf{6} \times \mathbf{8})$

$= {}^{-}(48)$

Remove the brackets.

$^{-}(48)$

$= {}^{-}48$

(b) $^{-}6 \times {}^{+}8$

$^{-}6 \times {}^{+}8$

Multiply the absolute values in brackets with a negative sign outside the brackets.

$^{-}6 \times {}^{+}8$

$= {}^{-}(\mathbf{6} \times \mathbf{8})$

Work out the product in the brackets.

$^{-}(\mathbf{6} \times \mathbf{8})$

$= {}^{-}(48)$

Remove the brackets.

$^{-}(48)$

$= {}^{-}48$

(c) $^{-}\frac{2}{3} \times {}^{+}8$

$^{-}\frac{2}{3} \times {}^{+}8$

Multiply the absolute values in brackets with a negative sign outside the brackets.

$^{-}\frac{2}{3} \times {}^{+}8$

$= {}^{-}\left(\frac{2}{3} \times 8\right)$

Work out the product in the brackets.

$^{-}\left(\frac{2}{3} \times 8\right)$

$= {}^{-}\left(\frac{16}{3}\right)$

Remove the brackets.

${}^{-}\left(\frac{16}{3}\right)$

$= -\frac{16}{3}$

(d) ${}^{-}6 \times {}^{+}8\frac{1}{3}$

${}^{-}6 \times {}^{+}8\frac{1}{3}$

Multiply their absolute values in brackets with a negative sign outside the brackets.

${}^{-}6 \times {}^{+}8\frac{1}{3}$

$= {}^{-}\left(6 \times 8\frac{1}{3}\right)$

Work out the product in the brackets.

${}^{-}\left(6 \times 8\frac{1}{3}\right)$

$= {}^{-}(50)$

Remove the brackets.

${}^{-}(50)$

$= -50$

(e) ${}^{+}6.3 \times {}^{-}8.9$

${}^{+}6.3 \times {}^{-}8.9$

Multiply their absolute values in brackets with a negative sign outside the brackets.

${}^{+}6.3 \times {}^{-}8.9$

$= {}^{-}(\mathbf{6.3} \times \mathbf{8.9})$

Work out the product in the brackets.

${}^{-}(\mathbf{6.3} \times \mathbf{8.9})$

$= {}^{-}(56.07)$

Remove the brackets.

${}^{-}(56.07)$

$= {}^{-}56.07$

Solutions to ACTIVITY 8:

(a) ${}^{-}6 \times {}^{-}8$

${}^{-}6 \times {}^{-}8$

Multiply the absolute values.

${}^{-}6 \times {}^{-}8$

$= 6 \times 8$

Work out the product.

6×8

$= 48$

(b) ${}^{+}6 \times {}^{+}8$

${}^{+}6 \times {}^{+}8$

Multiply the absolute values.

${}^{+}6 \times {}^{+}8$

$= 6 \times 8$

Work out the product.

6×8

$= 48$

(c) ${}^{-}\frac{2}{3} \times {}^{-}8$

${}^{-}\frac{2}{3} \times {}^{-}8$

Multiply the absolute values.

${}^{-}\frac{2}{3} \times {}^{-}8$

$= \frac{2}{3} \times 8$

Work out the product.

$\frac{2}{3} \times 8$

$= \frac{16}{3}$

(d) $^{+}6 \times {}^{+}8\frac{1}{3}$

$^{+}6 \times {}^{+}8\frac{1}{3}$

Multiply their absolute values.

$^{+}6 \times {}^{+}8\frac{1}{3}$

$= 6 \times 8\frac{1}{3}$

Work out the product.

$6 \times 8\frac{1}{3}$

$= 50$

(e) $^{-}6.3 \times {}^{-}8.9$

$^{-}6.3 \times {}^{-}8.9$

Multiply their absolute values.

$^{-}6.3 \times {}^{-}8.9$

$= 6.3 \times 8.9$

Work out the product.

6.3×8.9

$= 56.07$

Lesson 5 Dividing Rational Numbers

Introduction

By the end of this subunit, you should be able to:

- divide a rational number by a rational number, with a different sign.
- divide a rational number by a rational number, with the same sign.

This subunit is about 9 pages in length.

Dividing a rational number by a rational number.

As division by a number is equivalent to multiplication by the reciprocal of the number, you can use the steps in '**multiplying rational numbers**'.

The quotient, which is the result you get after division, is always positive for any two positive rational numbers. The quotient of any two negative rational numbers is also positive. But the quotient of a positive rational number and negative rational number is always negative.

Dividing any two rational numbers with different signs:

- Divide the absolute values in brackets with a negative sign outside the brackets;
- Work out the quotient in the brackets;
- Remove the brackets.

Example 1:

$^{-}4 \div {}^{+}8$

Divide the absolute values in brackets with a negative sign outside the brackets.

$^{-}4 \div {}^{+}8$

$= {}^{-}(4 \div 8)$

Work out the quotient in the brackets.

$^{-}(4 \div 8)$

$= {}^{-}(\frac{1}{2})$

Remove the brackets.

$^{-}(\frac{1}{2})$

$= {}^{-}\frac{1}{2}$ **or -0.5**

Let us do Example 2 together:

$^{-}2.5 \div {}^{+}2$

Divide the absolute values in brackets with a negative sign outside the brackets.

____

____

Compare your answer with:

$^{-}2.5 \div {}^{+}2$

$= {}^{-}(2.5 \div 2)$

Work out the quotient in the brackets.

Compare your answer with:

$^{-}(\mathbf{2.5} \div \mathbf{2})$

$= {}^{-}(\mathbf{1.25})$

Remove the brackets.

Compare your answer with:

$^{-}(\mathbf{1.25})$

$= {}^{-}\mathbf{1.25}$

<u>Activity 9</u>

Activity 9

Work out the divisions below:

(a) $^{+}6 \div {}^{-}8$

(b) $^{-}6 \div {}^{+}8$

(c) $\frac{^{-}2}{3} \div {}^{+}8$

(d) $^{-}6 \div {}^{+}8\frac{1}{3}$

(e) $^{+}6.3 \div {}^{-}7$

Compare your answers with those at the end of the subunit. Continue on if you scored at least 80%. If not, review *dividing any two rational numbers with different signs*.

Dividing any two rational numbers with the same sign:

- Divide the absolute values;
- Work out the quotient.

Examples:

$^{-}\frac{1}{2} \div {}^{-}\frac{1}{4}$

Divide the absolute values.

$^{-}\frac{1}{2} \div {}^{-}\frac{1}{4}$

$= \frac{1}{2} \div \frac{1}{4}$

$= \frac{1}{2} \times \frac{4}{1}$ **(Remember that dividing by a number is the same as multiplying by its reciprocal)**

Work out the quotient.

$\frac{1}{2} \times \frac{4}{1}$

= 2

Let us do Example 2 together:

$^{+}3.4 \div {}^{+}17$

Divide the absolute values.

Compare your answer with:

$3.4 \div {}^{+}17$

$= 3.4 \div 17$

Work out the quotient.

Compare your answer with:

$3.4 \div 17$

$= 0.2$

<u>Activity 10</u>

Work out the divisions below:

Activity 10

(a) $^{-}6 \div {}^{-}8$

(b) $^{+}6 \div {}^{+}8$

(c) $^{-}\frac{2}{3} \div {}^{-}8$

(d) $^{+}6 \div {}^{+}8\frac{1}{3}$

(e) $^{-}6.3 \div {}^{-}9$

Compare your answers with those at the end of the subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review *dividing any two rational numbers with the same sign.*

Note it!

Remember:

When dividing any two rational numbers with different signs:

1. Divide the absolute values in brackets with a negative sign outside the brackets;
2. Work out the quotient in the brackets;

3. Remove the brackets.

When dividing any two rational numbers with the same sign:

1. Divide the absolute values;
2. Work out the quotient.

Solutions to the subunit activities

Solutions to ACTIVITY 9:

(a) $^{-}6 \div {}^{+}8$

$^{-}6 \div {}^{+}8$

Divide the absolute values in brackets with a negative sign outside the brackets.

$^{-}6 \div {}^{+}8$

$= {}^{-}(6 \div 8)$

Work out the quotient in the brackets.

$^{-}(6 \div 8)$

$= {}^{-}(0.75)$

Remove the brackets.

$^{-}(0.75)$

$= -0.75$

(b) $^{+}6 \div {}^{-}8$

$^{+}6 \div {}^{-}8$

Divide the absolute values in brackets with a negative sign outside the brackets.

$^{+}6 \div {}^{-}8$

$= {}^{-}(6 \div 8)$

Work out the quotient in the brackets.

$^{-}(6 \div 8)$

$= {}^{-}(0.75)$

Remove the brackets.

$$^{-}(0.75)$$

$$= -0.75$$

(c) $^{-}\frac{2}{3} \div {}^{+}8$

$^{-}\frac{2}{3} \div {}^{+}8$

Divide the absolute values in brackets with a negative sign outside the brackets.

$$^{-}\frac{2}{3} \div {}^{+}8$$

$$= {}^{-}\left(\frac{2}{3} \div 8\right)$$

Work out the quotient in the brackets.

$$^{-}\left(\frac{2}{3} \div 8\right)$$

$$= {}^{-}\left(\frac{2}{3} \times \frac{1}{8}\right)$$

$$= {}^{-}\left(\frac{1}{12}\right)$$

Remove the brackets.

$$^{-}\left(\frac{1}{12}\right)$$

$$= -\frac{1}{12}$$

(d) $^{-}6 \div {}^{+}8\frac{1}{3}$

$^{-}6 \div {}^{+}8\frac{1}{3}$

Divide the absolute values in brackets with a negative sign outside the brackets.

$$^{-}6 \div {}^{+}8\frac{1}{3}$$

$= {}^{-}\left(6 \div 8\tfrac{1}{3}\right)$

Work out the quotient in the brackets.

${}^{-}\left(6 \div 8\tfrac{1}{3}\right)$

$= {}^{-}\left(\frac{6}{1} \times \frac{3}{25}\right)$

$= {}^{-}\left(\frac{18}{25}\right)$

Remove the brackets.

${}^{-}\left(\frac{18}{25}\right)$

$= -\frac{18}{25}$

(e) ${}^{+}6.3 \div {}^{-}7$

${}^{+}6.3 \div {}^{-}7$

Divide the absolute values in brackets with a negative sign outside the brackets.

${}^{+}6.3 \div {}^{-}7$

$= {}^{-}(6.3 \div 7)$

Work out the quotient in the brackets.

${}^{-}(6.3 \div 7)$

$= {}^{-}(0.9)$

Remove the brackets.

${}^{-}(0.9)$

$= -0.9$

Solutions to ACTIVITY 10:

(a) ${}^{-}6 \div {}^{-}8$

${}^{-}6 \div {}^{-}8$

Divide the absolute values.

${}^{-}6 \div {}^{-}8$

$= \ 6 \div 8$

Work out the quotient.

$6 \div 8$

$= 0.75$

(b) $^{+}6 \div {}^{+}8$

$^{+}6 \div {}^{+}8$

Divide the absolute values.

$^{+}6 \div {}^{+}8$

$= \ 6 \div 8$

Work out the quotient.

$6 \div 8$

$= 0.75$

(c) $^{-}\frac{2}{3} \div {}^{-}8$

$^{-}\frac{2}{3} \div {}^{-}8$

Divide the absolute values.

$^{-}\frac{2}{3} \div {}^{-}8$

$= \ \frac{2}{3} \div 8$

Work out the quotient.

$\frac{2}{3} \div 8$

$= \frac{2}{3} \times \frac{1}{8}$

$= \frac{1}{12}$

(d) $^{+}6 \div {}^{+}8\frac{1}{3}$

$^{+}6 \div {}^{+}8\frac{1}{3}$

Divide the absolute values.

$^{+}6 \div {}^{+}8\frac{1}{3}$

$= \ 6 \div 8\frac{1}{3}$

Work out the quotient.

$6 \div 8\frac{1}{3}$

$= \frac{6}{1} \times \frac{3}{25}$

$= \frac{18}{25}$

(e) $^{-}6.3 \div {}^{-}9$

$^{-}6.3 \div {}^{-}9$

Divide the absolute values.

$^{-}6.3 \div {}^{-}9$

$= \ 6.3 \div 9$

Work out the quotient.

$6.3 \div 9$

$= \ 0.7$

Lesson 6 Demonstrating an Understanding of Classes of Numbers Like Odd, Even, Polygonal (triangular, rectangular or composite and square), Cube and Prime Numbers in a Variety of Situations

Introduction

By the end of this subunit, you should be able to:

- list odd numbers from natural numbers.
- list even numbers from natural numbers.
- list prime numbers from natural numbers.
- list polygonal numbers from natural numbers.
- list cube numbers from natural numbers.

This subunit is about 4 pages in length.

Classes of numbers in natural numbers.

What is an **odd** number?

Activity 11

Here is set of natural numbers:

$\{ \ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots \ \}$

Activity 11

List the set of first five numbers which leave a remainder when divided by 2 from the set of whole numbers.

Compare your answers with those at the end of this subunit.

Therefore, an **odd** number is a natural number which leaves a remainder of 1when divided by 2.

What is an **even** number?

Activity 12

Here is set of natural numbers:

$\{ \ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots \ \}$

Activity 12

List the set of first five numbers which do not leave a remainder when divided by 2 from the set of whole numbers.

Compare your answers with those at the end of this subunit.

Therefore, an **even** number is a whole natural number which leaves a remainder of 0 when divided by 2.

What is a **prime** number?

Activity 13

Here is set of natural numbers:

$\{ \ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots \ \}$

Activity 13

List the set of first five numbers that have only *two factors*.

____

Compare your answers with those at the end of this subunit.

Therefore, a **prime** number is a natural number which has only *two factors, 1 and itself.*

What is a **cube** number?

Activity 14

Here is set of natural numbers:

$\{ \ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \ldots \ \}$

Activity 14

List the set of first five numbers, which can be written as a power, with 3 as the index.

____

Compare your answers with those at the end of this subunit.

Therefore, a **cube** number is a natural number which can be written as a power, with 3 as the index. Cube numbers are also called cubic numbers.

Rectangular or composite numbers:

Composite (rectangular) numbers are natural numbers which have more than two factors.

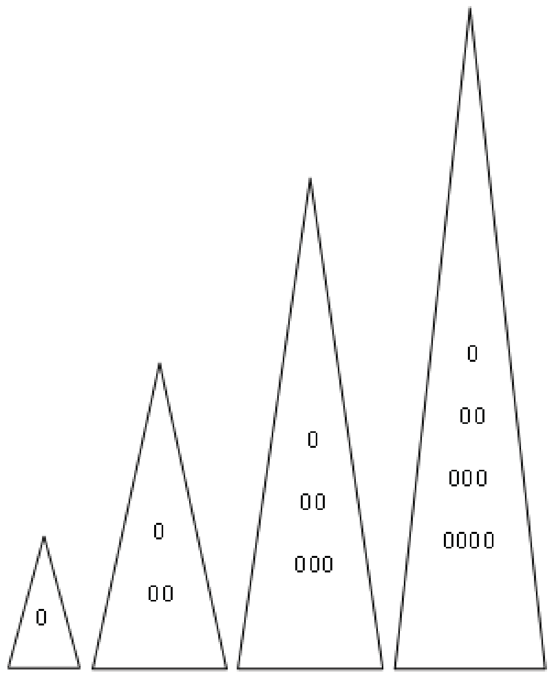
For example, 8 and 9 are composite numbers because 8 has 1,2,4,8 as its factors and 9 has 1,3,9 as its factors.

Polygonal Numbers

These are sequential numbers which form second-order arithmetic sequences, in which the second term of the sequence shows the number of sides in the *polygonal structures* that can be produced from these sequences.

Here are some examples :

1. **Triangular** numbers:



Activity 14

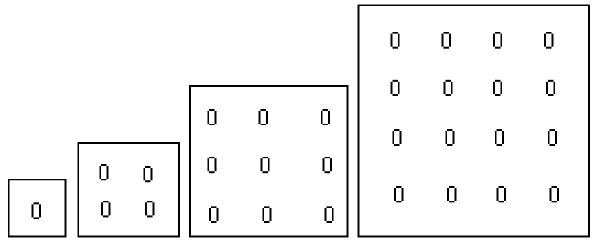
List the set of the number of zeros in the polygons above.

____

____

Compare your answers with those at the end of this subunit.

2. **Square** numbers:



Activity 15

List the set of the number of zeros in the polygons above.

___

Compare your answers with those at the end of this subunit.

Note it!

Remember:

Classes of numbers like odd, even, polygonal (triangular, rectangular or composite and square), cube and prime numbers are obtained from real numbers which are natural numbers.

Solutions to the subunit activities

Solutions to Activity 10

$\{ 1, 3, 5, 7, 9 \}$

This set is the set of first five odd numbers.

Solutions to Activity 11

$\{ 2, 4, 6, 8, 10 \}$

This set is the set of first five even numbers.

Solutions to Activity 12

$\{ 2, 3, 5, 7, 11 \}$

This set is the set of first five prime numbers.

Solutions to Activity 13

$\{ 1, 8, 27, 64, 125 \}$

(Since $1=1^3$, $8=2^3$, $27=3^3$, $64=4^3$, $125=5^3$)

This set is the set of first five cube numbers.

Solutions to Activity 14

$\{1,3,6,10\}$

This set is the set of the first four triangular numbers.

The second term, which is 3, is equal to the number of sides in the polygons; hence the triangles can be drawn.

Solutions to Activity 15

$\{1,4,9,16\}$

This set is the set of the first four square numbers.

The second term, which is 4, is equal to the number of sides in the polygons; hence the squares can be drawn.

Lesson 7 Solving Practical Situations Involving Directed Numbers

Introduction

By the end of this subunit, you should be able to:

- apply knowledge of directed numbers in practical situations.

This subunit is about 5 pages in length.

Directed numbers in problem solving.

Some practical situations, such as calculating temperature changes, distance travelled and calculations of resultant force, involve the use of directed numbers:

<u>When solving problems on practical situations involving directed numbers:</u>

- Change the problem to its mathematical expression;
- Work out the mathematical expression.

Example 1:

One night the temperature dropped from -2 ^{0}C to -10 ^{0}C.

What was the temperature difference?

Change the problem to its mathematical expression.

-2 ^{0}C - -10 ^{0}C

Work out the mathematical expression.

-2 ^{0}C - -10 ^{0}C

= +**8 ^{0}C**

Let us do Example 2 together:

The temperature at 0930 is -6 ^{0}C and the temperature at 1730 is 20 ^{0}C.

Assuming that the temperature rises at a steady rate, find the average temperature between 0930 and 1730.

How do you find the average when a quantity changes steadily?

___

Compare your answer with:

(Initial value + final value) ÷**2**

Change the problem to its mathematical expression.

Compare your answer with:

$(\mathbf{-6}^{0}\mathbf{C} + \mathbf{20}^{0}\mathbf{C}) \div \mathbf{2}$

Work out the mathematical expression.

Compare your answer with:

$(\mathbf{-6}^{0}\mathbf{C} + \mathbf{20}^{0}\mathbf{C}) \div \mathbf{2}$

$= +\mathbf{7}^{0}\mathbf{C}$

Activity 16

Activity 16

Work out the following:

(i) **Selekane always walks to school and back home, after school. The school is 5km from her village. One day, she goes to school and on her way back she rests for 20minutes when she is halfway home.**

(a) What is the total distance she has walked?

(b) How far is Selekane from home then?

(c) If she is 1.8 km from home, how far is she from the school?

(ii) Setŝabelo and Tŝoarelo play by pushing perpendicularly to the faces of a box on a frictionless table.

(a) If **Setŝabelo** pushes with 50 **newtons** and **Tŝoarelo** with 35 **newtons** on the same face, what is the resultant force, in **newtons**, on the box ?

(b) If **Setŝabelo** pushes with 50 **newtons** on one face and **Tŝoarelo** with 35 **newtons** on the opposite face, what is the resultant force, in **newtons**, on the box?

Compare your answers with those at the end of the subunit. Continue on if you scored at least 80%. If not, review *solving practical situations involving directed numbers.*

Note it!

Remember:

When solving problems on practical situations involving directed numbers:

- Change the problem to its mathematical expression;
- Work out the mathematical expression.

You have now completed the last subunit of this unit on types of numbers. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Solutions to ACTIVITY 16:

(i) Selekane always walks to school and back home, after school. The school is 5km from her village. One day, she goes to school and on her way back she rests for 20minutes when she is halfway home.

(a) What is the total distance she has walked**?**

Change the problem to its mathematical expression.

$^{+}$**5km + $^{+}$2.5km**

Work out the mathematical expression.

$^{+}$**5km + $^{+}$2.5km**

= +**7.5km**

(b) How far is Selekane from home then?

Change the problem to its mathematical expression.

$^{+}$**5km - $^{+}$2.5km**

Work out the mathematical expression.

$^{+}$**5km - $^{+}$2.5km**

= +**2.5km**

(c) If she is 1.8 km from home, how far is she from the school?

Change the problem to its mathematical expression.

$^{+}$**5km - $^{+}$1.8km**

Work out the mathematical expression.

$^{+}$**5km - $^{+}$1.8km**

= +**3.2km**

(ii) Setŝabelo and Tŝoarelo play by pushing perpendicularly to the faces of a box on a frictionless table.

(a) If **Setŝabelo** pushes with 50**newtons** and **Tŝoarelo** with 35**newtons** on the same face, what is the resultant force, in **newtons**, on the box**?**

Change the problem to its mathematical expression.

$^{+}$**50newtons** + $^{+}$**35newtons**

Work out the mathematical expression.

$^{+}$**50newtons** + $^{+}$**35newtons**

= +**85newtons**

(b) If **Setŝabelo** pushes with 50 **newtons** on one face and **Tŝoarelo** with 35**newtons** on the opposite face, what is the resultant force, in **newtons**, on the box?

Change the problem to its mathematical expression.

$^{+}$**50newtons -** $^{+}$**35newtons**

Work out the mathematical expression.

$^{+}$**50newtons -** $^{+}$**35newtons**

= +**15newtons**

Unit Summary

Summary

In this unit you learned about all types of numbers on a real number line:

natural (counting) numbers	1, 2, 3 ,4, 5, . . .
whole numbers	zero and all natural (counting) numbers
integers	whole numbers and their opposites
rational numbers	integers and all numbers that can be written as ratios of two integers, **x/y**, with **y** not equal to zero
irrational numbers	real numbers that are *not rational such as* $\sqrt{3}$, $\sqrt[3]{11}$, and - $\sqrt{7}$
ALL ABOVE NUMBERS TOGETHER FORM REAL NUMBERS	

When adding any two rational numbers with the same sign:

1. Write the addition of their absolute values in brackets and their sign outside the brackets;
2. Work out the sum in the brackets;
3. Remove the brackets.

When adding any two rational numbers with different signs:

1. Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets;
2. Work out the difference in the brackets;
3. Remove the brackets.

When subtracting a rational number from another rational number:

1. Change subtraction to addition of the opposite of the rational number which is to be subtracted.
2. Use the rules for adding any two rational numbers which have been studied.

When multiplying any two rational numbers with different signs:

1. Multiply their absolute values in brackets with a negative

sign outside the brackets;

2. Work out the product in the brackets;
3. Remove the brackets.

When multiplying any two rational numbers with the same sign:

1. Multiply their absolute values;
2. Work out the product.

When dividing any two rational numbers with different signs:

1. Divide their absolute values in brackets with a negative sign outside the brackets;
2. Work out the quotient in the brackets;
3. Remove the brackets.

When dividing any two rational numbers with the same sign:

1. Divide their absolute values;
2. Work out the quotient.

An **odd** number is a whole number which leaves a remainder of 1when divided by 2.

An **even** number is a whole number which leaves a remainder of 0 when divided by 2.

A **prime** number is a whole number which has only *two factors, 1 and itself*.

A **cube** number is a whole number which can be written as a power, with 3 as the index.

A **composite (rectangular)** number is a positive integer which has more than two factors.

Polygonal numbers are sequential numbers which form second-order arithmetic sequences, in which the second term of the sequence shows the number of sides in the *polygonal structures* that can be produced from these sequences.

Directed numbers are positive and negative real numbers.

You have completed the material for this unit on types of numbers. You should now spend some time reviewing the content. Once you feel that you can successfully write an exam that covers each of the learning outcomes, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

When you work on this assignment, please observe the time allocated and show your work for each answer.

TOTAL MARKS: 60

TIME: 60 minutes

Work on the following questions:

1. **Complete the table below by filling in YES or NO where appropriate:**

Number	**Counting number**	**Whole number**	**Integer**	**Rational number**
$3\frac{1}{7}$				
$\sqrt{2}$				
2^3				

(9 marks)

2. **Complete the table below by filling in YES or NO where appropriate:**

Number	**Prime number**	**Composite number**	**Odd number**
1			
2			

3			
4			
5			
6			
7			
8			
9			
10			

(30 marks)

3. Work out:

(a) $^{-}6+^{-}3$ (2 marks)

(b) $^{+}7+^{-}8$ (2 marks)

(c) $^{+}5-^{-}8$ (2 marks)

(d) $^{+}1-^{+}9$ (2 marks)

(e) $^{-}6 \times ^{-}3$ (2 marks)

(f) $^{+}7 \times ^{-}8$ (2 marks)

(g) $^{-}6 \div ^{-}3$ (2 marks)

(h) $^{-}6.6 \div ^{+}3$ (2 marks)

4. One winter night in Lesotho, the temperature changed from $7\,^{0}C$ to $-13\,^{0}C$.

What was the temperature difference? (2 marks)

5. Here is a sequence of polygonal numbers:

1, 6, 15, 28, 45, ...

(a) Write down the second term of the sequence. (2 marks)

(b) Which polygonal numbers are in this sequence. (1 mark)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Solutions to the ASSIGNMENT:

1. **Complete the table below by filling in YES or NO where appropriate:**

Number	**Counting number**	**Whole number**	**Integer**	**Rational number**
$3\frac{1}{7}$	**NO**	**NO**	**NO**	**YES**
$\sqrt{2}$	**NO**	**NO**	**NO**	**NO**
2^3	**YES**	**YES**	**YES**	**YES**

2. **Complete the table below by filling in YES or NO where appropriate:**

Number	**Prime number**	**Composite number**	**Odd number**
1	**NO**	**NO**	**YES**
2	**YES**	**NO**	**NO**
3	**YES**	**NO**	**YES**
4	**NO**	**YES**	**NO**
5	**YES**	**NO**	**YES**

6	**NO**	**YES**	**NO**
7	**YES**	**NO**	**YES**
8	**NO**	**YES**	**NO**
9	**NO**	**YES**	**YES**
10	**NO**	**YES**	**NO**

3.Work out:

(a) $^{-}6+^{-}3$

$^{-}\mathbf{6} + {}^{-}\mathbf{3}$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{-}\mathbf{6} + {}^{-}\mathbf{3}$

$= {}^{-}\mathbf{(6+3)}$

Work out the sum in the brackets **.**

$^{-}\mathbf{(6+3)}$

$= {}^{-}\mathbf{(9)}$

Remove the brackets.

$^{-}\mathbf{(9)}$

$= {}^{-}\mathbf{9}$

(b) $^{+}7+^{-}8$

$^{+}7+^{-}8$

Write the difference of their absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$^{+}7+^{-}8$

$= {}^{-}(8-7)$

Work out the difference in the brackets **.**

$^{-}(8-7)$

$= {}^{-}(1)$

Remove the brackets.

$^{-}(1)$

$= {}^{-}1$

(c) $^{+}5-{}^{-}8$

$^{+}5-{}^{-}8$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$^{+}5-{}^{-}8$

$= {}^{+}5+{}^{+}8$

Write the addition of the absolute values in brackets and their sign outside the brackets.

$^{+}5+{}^{+}8$

$= {}^{+}\mathbf{(5+8)}$

Work out the sum in the brackets.

$^{+}\mathbf{(5+8)}$

$= {}^{+}\mathbf{(13)}$

Remove the brackets.

$^{+}\mathbf{(13)}$

$= {}^{+}\mathbf{13}$

(d) $^{+}1-{}^{+}9$

$^{+}1-{}^{+}9$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$^{+}1-{}^{+}9$

$= {}^{+}1+{}^{-}9$

Write the difference of the absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$^{+}1+{}^{-}9$

$= {}^{-}(9-1)$

Work out the difference in the brackets .

$^{-}(9-1)$

$= \, ^{-}(8)$

Remove the brackets.

$^{-}(8)$

$= \, ^{-}8$

(e) $^{-}6 \times ^{-}3$

$^{-}6 \times ^{-}3$

Multiply the absolute values.

$^{-}6 \times ^{-}3$

$= 6 \times 3$

Work out the product.

6×3

$= 18$

(f) $^{+}7 \times ^{-}8$

$^{+}7 \times ^{-}8$

Multiply the absolute values in brackets with a negative sign outside the brackets.

$^{+}7 \times ^{-}8$

$= \, ^{-}(\mathbf{7} \times \mathbf{8})$

Work out the product in the brackets.

$^{-}(\mathbf{7} \times \mathbf{8})$

$= \, ^{-}(56)$

Remove the brackets.

$^{-}(56)$

$= \, ^{-}56$

(g) $^{-}6 \div ^{-}3$

$^{-}6 \div ^{-}3$

Divide the absolute values.

$^{-}6 \div {}^{-}3$

$= \ 6 \div 3$

Work out the quotient.

$6 \div 3$

$= 2$

(h) $^{-}6.6 \div {}^{+}3$

$^{-}6.6 \div {}^{+}3$

Divide the absolute values in brackets with a negative sign outside the brackets.

$^{-}6.6 \div {}^{+}3$

$= \ {}^{-}(6.6 \div 3)$

Work out the quotient in the brackets.

$^{-}(6.6 \div 3)$

$= \ {}^{-}\left(\frac{6.6}{3}\right)$

$= {}^{-}(2.2)$

Remove the brackets.

$^{-}(2.2)$

$= {}^{-}2.2$

4. One winter night in Lesotho, the temperature changed from 7^{0}C to -13^{0}C.

What was the temperature difference?

Change the problem to its mathematical expression.

7^{0}C - -13^{0}C

Work out the mathematical expression.

7^{0}C - -13^{0}C

= +**20^{0}C**

5. Here is a sequence of polygonal numbers:

1, 6, 15, 28, 45, ...

(a) Write down the second term of the sequence.

The second term of the sequence is 6.

(b) Which polygonal numbers are in this sequence.

Hexagonal numbers, since 6 which is the second term, shows that the *polygonal structures* that can be produced from this sequence have six sides.

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Attempt all the questions. Show your work for each answer.

TOTAL MARKS: 30

TIME: 35 minutes

1. Look at the real numbers written below:

$$\left\{2, \sqrt[3]{64}, \sqrt{20}, 16^{\frac{1}{2}}, 18^{\frac{1}{2}}, 19\right\}$$

a) Which of the numbers are natural numbers? (3 marks)

b) Which of the numbers are rational numbers? (3 marks)

c) Which of the numbers are whole numbers? (3 marks)

d) Which of the numbers are prime numbers? (3 marks)

e) Which of the numbers are composite numbers? (3 marks)

2. Work out:

(a) $^{-}11 + ^{-}13$ (2 marks)

(b) $^{+}9.5\times{}^{-}4$ (2 marks)

____

____

(c) $^{+}12\div{}^{-}2$ (2 marks)

____

____

(d) $^{+}10-{}^{+}19$ (2 marks)

____

____

(e) $^{-}4\times{}^{-}9$ (2 marks)

____

____

(f) After buying a new fridge, Selekane noticed that the temperature inside the freezer compartment was $20\,^{0}$C. She wanted to put in her groceries when the temperature inside it had gone down $30\,^{0}$C. Find the temperature at which she put in her groceries. (2 marks)

____

____

3. Here is a sequence of polygonal numbers:

1, 5, 12, 22, 45, ...

(a) Write down the second term of the sequence. (2 marks)

____

____

(b) Which polygonal numbers are in this sequence. (1 mark)

____

____

Check your performance against the given solutions; and if you scored 80% or more, then go on to the next unit; otherwise review *the section(s) on which unsatisfactory performance occurred.*

SOLUTIONS TO ASSESSMENT:

1. Look at the real numbers written below:

$$\left\{2, \sqrt[3]{64}, \sqrt{20}, 18^{\frac{1}{2}}, 19, 25^{\frac{1}{2}}\right\}$$

a) Which of the numbers are natural numbers?

Since $\sqrt[3]{64} = 4$ and $25^{\frac{1}{2}} = 5$,then natural numbers are:

$$\left\{2, \sqrt[3]{64}, 19, 25^{\frac{1}{2}}\right\}$$

b) Which of the numbers are rational numbers?

Since $2 = \frac{2}{1}, \sqrt[3]{64} = 4 = \frac{4}{1}, 19 = \frac{19}{1}$ and $25^{\frac{1}{2}} = 5 = \frac{5}{1}$,then natural numbers are:

$$\left\{2, \sqrt[3]{64}, 19, 25^{\frac{1}{2}}\right\}$$

c) Which of the numbers are whole numbers?

Since $\sqrt[3]{64} = 4$ and $25^{\frac{1}{2}} = 5$,then natural numbers are:

$$\left\{2, \sqrt[3]{64}, 19, 25^{\frac{1}{2}}\right\}$$

d) Which of the numbers are prime numbers?

Natural numbers which have only *two factors, 1 and themselves are*:

$$\{2, 19\}$$

e) Which of the numbers are composite numbers?

$\sqrt[3]{64} = 4$ is the only composite number, as it has more than two factors; namely 1, 2 and 4.

2. Work out:

(a) $^{-}11+^{-}13$

$^{-}\mathbf{11} + {}^{-}\mathbf{13}$

Write the addition of their absolute values in brackets and their sign outside the brackets.

$^{-}\mathbf{11} + {}^{-}\mathbf{13}$

$= {}^{-}\mathbf{(11+13)}$

Work out the sum in the brackets**.**

$^{-}\mathbf{(11+13)}$

$= {}^{-}\mathbf{(\ 24\ )}$

Remove the brackets.

$^{-}\mathbf{(24)}$

$= {}^{-}\mathbf{24}$

(b) $^{+}9.5\times^{-}4$

$^{+}9.5\times^{-}4$

Multiply the absolute values in brackets with a negative sign outside the brackets.

$^{+}9.5\times^{-}4$

$= {}^{-}\mathbf{(\ 9.5\ \times\ 4\ )}$

Work out the product in the brackets.

$^{-}\mathbf{(\ 9.5\ \times\ 4\ )}$

$= {}^{-}(38)$

Remove the brackets.

$^{-}(38)$

$= {}^{-}38$

(c) $^{+}12\div^{-}2$

$^{+}12\div^{-}2$

Divide the absolute values in brackets with a negative sign outside the brackets.

$^{+}12\div^{-}2$

$= {}^{-}(12\div 2)$

Work out the quotient in the brackets.

$^{-}(12 \div 2)$

$= {}^{-}(6)$

Remove the brackets.

$^{-}(6)$

$= -6$

(d) $^{+}10 - {}^{+}19$

$^{+}10 - {}^{+}19$

Change subtraction to addition of the opposite of the rational number which is to be subtracted.

$^{+}10 - {}^{+}19$

$= {}^{+}10 + {}^{-}19$

Write the difference of the absolute values in brackets and the sign of the number with the bigger absolute value outside the brackets.

$^{+}10 + {}^{-}19$

$= {}^{-}(19 - 10)$

Work out the difference in the brackets.

$^{-}(19 - 10)$

$= {}^{-}(9)$

Remove the brackets.

$^{-}(9)$

$= {}^{-}9$

(e) $^{-}4 \times {}^{-}9$

$^{-}4 \times {}^{-}9$

Multiply the absolute values.

$^{-}4 \times {}^{-}9$

$= 4 \times 9$

Work out the product.

4×9

$= 36$

f. After buying a new fridge, Selekane noticed that the temperature inside the freezer compartment was 20^{0}C. She wanted to put in her groceries when the temperature inside it had gone down 30^{0}C. Find the temperature at which she put in her groceries.

Change the problem to its mathematical expression.

$\mathbf{20^{0}C - 30^{0}C}$

Work out the mathematical expression.

$\mathbf{20^{0}C - 30^{0}C}$

$= \mathbf{-10^{0}C}$

The temperature is $\mathbf{-10^{0}C}$**.**

3. Here is a sequence of polygonal numbers:

   1, 5, 12, 22, 45, ...

   (a) Write down the second term of the sequence.

The second term of the sequence is 5.

(b) Which polygonal numbers are in this sequence?

Pentagonal numbers, since 5 which is the second term, shows that the *polygonal structures* that can be produced from this sequence have five sides.

Unit Contents

Unit 3

Unit 3

Limits of Accuracy

In life we make many measurements. We take measurements of length, weight, height, volume and many others. But when taking these measurements we use instruments and our human eyes which are not perfect, this means that there can be small mistakes here and there. To cater for these mistakes we apply what are known as the 'limits of accuracy' to measurements in the real world. This means that the exact measurement falls within the range defined by these limits.

This unit consists of 26 pages and is about 1% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote approximately 10 hours to work on this unit, 6 hours for formal study and 4 hours for self-study and completing assessments/assignments.

The previous unit dealt with types of numbers. You learned about whole numbers, rational numbers, irrational numbers and all the other types. In this unit we are going to look at how accurately we can measure or report on things in the world around us. Thus, in this unit we will be applying the limits of accuracy to those numbers you learned about in unit 2.

This Unit is Comprised of Three Lessons:

Lesson 1 Revising What You Already Know
Lesson 2 Limits of Accuracy
Lesson 3 Solving Simple Problems Involving Limits of Accuracy

Upon completion of this unit you will be able to:

Outcomes

- *give* appropriate upper and lower bounds for data given to a specified accuracy (e.g. measured lengths)
- *obtain* appropriate upper and lower bounds to solutions of simple problems (e.g. the calculation of the perimeter or the area of a rectangle) given data to a specified accuracy

Terminology

Limits of Accuracy :	Boundaries put on a measurement to keep errors and uncertainties within specific limits.
Decimal Numbers:	In this context a decimal number is a number that consists of a zero, a decimal point and one or more numbers after the decimal point.
Mixed numbers:	Numbers that consist of any whole number other than zero, a decimal point and one or more numbers after the decimal point.
Significant figures:	Digits of a number that carry meaning contributing to its precision.
Lower bound:	An element of a set which is lesser than or equal to every element of that set.
Upper bound:	An element of a set which is greater than or equal to every element of that set.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations

- Online Courses

Lesson 1 Revising What You Already Know

At the end of this sub-unit you should be able to:

- Round any number to the specified degree of accuracy.
- Count the number of significant figures in whole numbers, decimals and mixed numbers.
- Give rough estimations of numerical expressions.

This sub-unit consists of approximately 6 pages.

This unit mainly deals with limits of accuracy. However, before introducing any new material, we will just remind ourselves of some important concepts that we will need to prepare for the discussion on limits of accuracy.

In secondary school you learned how to round off numbers to the given degree of accuracy and how to estimate calculations. You rounded off numbers to the desired number of significant figures or decimal places. You also learned how to count the number of significant figures in whole numbers, mixed numbers and decimal numbers.

Now work on Activity 1 and see how much you can still remember.

Activity

Activity 3.1

1. Round off the following numbers to the nearest ten:

a) 1238

b) 578

c) 688.90

2. Round of the following numbers to the nearest hundredth:

a) 0.1578

b) 3.498

c) 423.024

3. Round off each of the following numbers to the nearest whole number:

a) 23.8

b) 17.4

c) 99.9

d) 7 888.8

4. Round off each of the following numbers to the number of decimal places indicated in the bracket:

a) 45.901 (1 dp)

b) 0.0123 (2 dp)

c) 0.44991 (3 dp)

5. Give the number of significant figures in each of the following numbers:

a) 231

b) 23.1

c) 5 501

d) 0.0023

e) 6 090

f) 23.01

g) 0.56

h) 9 900 000

i) 36.9

j) 0.00020

k) 9 871

l) 76.0

m) 0.004500

6. Round off the following numbers to the number of significant figures given in the brackets:

a) 321 (2 sf)

b) 6 302 (1 sf)

c) 45 509 (3 sf)

d) 1.214 (3 sf)

e) 67.010 (4 sf)

f) 5.00001 (3 sf)

g) 0.0122 (2 sf)

h) 0.9910 (3 sf)

i) 0.0100 (2 sf)

7. Make a rough estimation of the numerical values of each of the following expressions:

a) 2.64×2178

b) $8.974 + 32.23 - 15.87$

c) 0.5701×45

d) $\frac{986 \times 0.0891}{99}$

e) $\frac{2.76 \times 0.0421}{194}$

f) 8.12^2

8)

A rectangle has the following dimensions, length 17.2m and width 6.2m, estimate the area of the rectangle.

Compare your answers with those given at the end of the sub-unit. I hope you were able to get all the answers correct. If you were unable to answer all of the questions correctly, revisit your secondary school textbook or check the following notes and get some help.

Tip

As you were working out the questions of activity 1 you should have put in mind the following points:

- When counting the number of significant figures in whole numbers, you count from the left to the last non-zero digit.
- When counting the number of significant figures in mixed numbers, count all numbers.
- When counting the number of significant figures in decimals, count to the right starting with the first non zero-digit.
- When calculating the estimate of an expression we round off each number in the expression correct to one significant figure, then work out the value of the resulting expression.
- By convention the final answer in estimation is always approximated to 1 significant figure.

Model Answers

Activity 3.1

1. a) 1238 to the nearest ten is 1240.

   b) 578 to the nearest ten is 580.

   c) 688.90 to the nearest ten 690.

2. a) 0.1578 to the nearest hundredth is 0.16.

   b) 3.498 to the nearest hundredth is 3.50.

   c) 423.024 to the nearest hundredth is 423.02.

3. a) 23.8 to the nearest whole number is 24.

   b) 17.4 to the nearest whole number 17.

   c) 99.9 to nearest whole number is 100.

   d) 7888.8 to the nearest whole number is 7889.

4. a) 45.901 to 1 dp is 45.9.

   b) 0.0123 to 2 dp is 0.01.

   c) 0.44991 to 3dp is 0.450.

5. a) 231 has 3 sf.

   b) 23.1 has 3 sf.

c) 5501 has 4 sf.

d) 0.0023 has 2 sf.

e) 6090 has 3 sf.

f) 23.01 has 4 sf.

g) 0.56 has 2 sf.

h) 9900000 has 2 sf.

i) 36.9 has 3 sf.

j) 0.00020 has 2 sf.

k) 9871 has 4 sf.

l) 76.0 has 3 sf.

m) 0.004500 has 4 sf.

6. a) 321 to 2 sf is 320.

b) 6302 to 1 sf is 6000.

c) 45509 to 3 sf is 45500.

d) 1.214 to 3 sf is 1.21. Note that we do not write the last zero because a zero after a number and a decimal point is significant.

e) 67.010 to 4 sf is 67.01.

f) 5.00001 to 3 sf is 5.00.

g) 0.0122 to 2 sf is 0.012. We count to the right starting with the first non-zero digit.

h) 0.9910 to 3 sf is 0.991.

i) 0.0100 to 2 sf is 0.010.

7. Making rough estimations we round each number in the expression correct to 1 sf.

a) $2.64 \times 2178 \approx 3 \times 2000 = 6000$.

b) $8.974 + 32.23 - 15.87 \approx 9 + 30 - 20 = 19 \approx 20$.

c) $0.5701 \times 45 \approx 0.6 \times 50 = 30$.

d) $\frac{986 \times 0.0891}{99} \approx \frac{1000 \times 0.09}{100} = 0.9$

e) $\frac{2.76 \times 0.0421}{194} \approx \frac{3 \times 0.04}{200} = 0.0006$

f) $8.12^2 \approx 8^2 = 64$.

8. Area = length × width

$= 17.2\text{m} \times 6.2\text{m}$

$\approx 20\text{m} \times 6\text{m}$

$= 120m^2 \approx 100m^2$, we leave the answer to an estimated calculation in 1 sf.

Now that you remember how to deal with estimation and significant figures you can carry on with the limits of accuracy.

Lesson 2 Limits of Accuracy

At the end of this sub-unit you should be able to:

- *give* appropriate upper and lower bounds (limits of accuracy) for data given to a specified accuracy.

This sub-unit consists of approximately 3 pages.

I hope you still have the introduction in mind that the limits of accuracy are applied to any measurement so that the little human mistakes that can happen when taking measurements are taken care of.

Consider the following situation:

In the laboratory Puleng takes a measurement of the volume of water in a measuring cylinder. To her eye, the volume looks to be 80 ml. She asks her lab partner, Teboho, to check the measurement, and he determines that the cylinder contains 81 ml. of water. Which one of them is right?

The answer is that Puleng and Teboho are both right. Because of the limitations of the human eyes and slight differences in positioning the cylinder when taking the measurement, it is not possible to determine how much water is in it with a greater degree of accuracy.

It would be wrong to suggest that their answer is more accurate than is actually possible under the circumstances, so to eliminate any confusion we apply the bounds or limits of accuracy to the measurement. To Puleng it seems the volume is 80ml, but the limits of accuracy indicate that this measurement could still be anywhere in the range between 75ml and 85ml. These are all numbers which when rounded to 1 significant figure (as 80 has one significant figure) will give 80. So the limits of accuracy of the exact volume are given in the form of an inequality as:

$75\text{ml} \leq \text{exact volume} < 85\text{ml}.$

The inequality sign on the left side shows equality because when rounding all the numbers between 75 and 80 with 75 included, to one significant figure we get 80. On the other side the inequality sign on the right hand side shows no equality because 85 when rounded to one significant figure gives 90 instead of 80. However, when all the numbers between 80 and 85 are rounded to one significant figure, the result is 80. So this is how we apply the limits of accuracy.

75ml is called the **lower bound** of the limits of accuracy and 85 is said to be the **upper bound** of the limits of accuracy.

Consider the following examples.

Example 1:

Give the limits of accuracy of the following numbers:

a) 55

b) 17.3

Solution:

a) The limits of accuracy of 55 should include all numbers which when rounded to two significant figures will give 55 as it has two significant figures.

So the limits are:

$54.5 \leq$ exact number < 55.5

b) In the same way as in a) above the limits of accuracy of 17.3 should include all numbers which when rounded to one decimal place will give 17.3 as it has one decimal place.

So the limits are:

$17.25 \leq$ exact number < 17.35

Now work on the following activity on the limits of accuracy.

Activity

Activity 3.2

The following quantities are given to the stated accuracy. Write down the limits of accuracy of the measurements in the form of an inequality.

1. 10 cm to the nearest cm.

2. 3 mℓ to the nearest m ℓ.

3. 15 km to the nearest km.

4. 50 kg to the nearest kg.

5. 13m to the nearest m.

6. 39.763km to the nearest m.

7. 0.003ℓ to the nearest m ℓ.

8. 15.62 A to the nearest 0.01A.

9. 7.1Ω to the nearest 0.1 Ω.

10. 0.93m to the nearest cm.

When you are done answering all the questions, compare your answers with those given at the end of the sub-unit on solving problems involving limits of accuracy.

Lesson 3 Solving Simple Problems Involving Limits of Accuracy

By now you should know how to apply the limits of accuracy to a measurement. In this section you will learn how to solve problems that involve limits of accuracy.

At the end of this sub-unit you should be able to:

- *obtain* appropriate upper and lower bounds to solutions of simple problems given data to a specified accuracy.

This sub-unit consists of ? pages.

Consider the following examples:

Example 1:

A car took roughly 3 hours (with the measurement taken to the nearest hour) to travel 360km (measured to the nearest ten kilometres). What are the lower and the upper bounds of the average speed to the nearest km/h?

Solution

First we know that the average speed is given by total distance over total time, i.e.

$$S = \frac{total\ dis\tan ce}{total\ time} = \frac{d}{t}$$

To get the lower and the upper bounds of the average speed we first have to know the lower and the upper bounds of both distance and time.

The limits of accuracy of time should include all numbers of which when rounded to the nearest hour give an answer of 3 hours.

So the limits of accuracy of time are: $2.5h \leq t < 3.5h$

In the same way, the limits of accuracy of distance should include all numbers of which when rounded to the nearest ten should give 360km.

So the limits of accuracy of distance are as follows:

$355km \leq d < 365km$

Now that we know the limits of accuracy of distance and time we can find the upper and lower bounds of the average speed.

Just take a minute and think of how you would find the lower bound of the average.

____

____

____

Compare your answer with the following.

The lower bound of the average speed are given by the shortest or smallest time divided by the longest or largest time. That is, the

$$\text{lower bound of average speed} = \frac{shortest\ dis\tan ce}{longest\ time}$$

This makes sense because the lower bound is the smallest of the range, so if the distance is already the shortest and it is divided by the largest value (time), then the output should be the smallest that could be found.

$$\text{So the lower bound} = \frac{shortest\ dis\tan ce}{longest\ time} = \frac{355km}{3.5h} = 101km/h$$

Now can you find the upper bound? Just give it a try.

____

____

____

Compare your answer with the following:

The upper bound of the average speed is given by the longest distance divided by the shortest time. That is, the

$$\text{upper bound of average speed} = \frac{\textit{longest distance}}{\textit{shortest time}}$$

This also makes perfect sense because the upper bound is the largest of the range, so if the distance is the longest and it is divided by the shortest time, then the output should be the largest possible outcome.

$$\text{Now the upper bound} = \frac{\textit{longest distance}}{\textit{shortest time}} = \frac{365km}{2.5h} = 146km/h$$

Example 2

A triangle has base 5 cm and height 4 cm to the nearest cm. Find to the nearest 0.01cm², the limits of accuracy of the area of the triangle.

Solution

The limits of accuracy of the base are:

$4.5\text{cm} \le \text{base} < 5.5\text{cm}$

The limits of accuracy of the height are:

$3.5\text{cm} \le \text{height} < 4.5\text{cm}$

Area= base × height

Now the lower bound of the area is given by:

Lower bound of Area= smallest base × smallest height = 4.5cm × 3.5cm= 15.75cm²

And the upper bound of area is given by:

Upper bound Area= largest base × largest height = 5.5cm × 4.5cm = 24.75cm²

So the limits of accuracy of area to the nearest 0.01cm² are:

$15.75\text{cm}^2 \le \text{area of triangle} < 24.75\text{cm}^2$.

Note again that the lower bound has the symbol of equality because both the smallest height and the smallest base had the symbol of equality, and that the upper bound of area has no symbol of equality because the largest height and the largest base both had no symbol of equality.

Now work on activity 3.3 to apply your new knowledge to solving problems involving limits of accuracy.

Activity

Activity 3.3

1. A cuboid has the following dimensions: length 9m width 7m and height 10m all measured to the nearest meter. Copy and complete the following statements, with

length, width and height to the nearest m, area to the nearest m² and volume to the nearest m³:

a) ______ $\le$ length < ______

b) ______ $\le$ width < ______

c) ______ $\le$ height < ______

d) ______ $\le$ base area < ______

e) ______ $\le$ volume < ______

2. A car travels a distance of 120km to the nearest km in 3 hours to the nearest hour. It then changes speed and travels further at 80km/h to the nearest km/h in 2 hours to the nearest hour.

a) Find the least and the greatest possible average speed of the car in the first part of the journey.

b) Find the greatest possible distance travelled in the second part of the journey.

c) Find the upper bound of the average speed of the entire journey.

3. A car starts from rest and attains a maximum speed of 20m/s to the nearest m/s after 10s to the nearest s. Find the least possible acceleration of the car.

Hint: Acceleration a is given by

$$a = \frac{final\ speed - initial\ speed}{time}$$

4. If $2 \le x \le 5$, $-4 \le y \le -1$ and $5 \le z$ 8, Calculate the lower and the upper bounds of :

a) x + y

b) y – x

c) $\frac{xy}{z}$

5. The radius of a circle is measured as 11mm correct to the nearest millimetre.

i) Write down the least possible value of the radius.

ii) Taking π to be $\frac{22}{7}$, calculate the least possible value of the circumference.

When you have finished answering all the questions compare your answers with those given at the end of the sub-unit.

Model Answers

Activity 3.2

1. 9.5cm ≤ true length < 10.5cm.
2. 2.5mℓ ≤ true volume < 3.5mℓ
3. 14.5km ≤ true distance < 15.5km.
4. 49.5kg ≤ true distance < 50.5kg.
5. 12.5m ≤ true distance < 13.5m.

6) 39.7625km ≤ true distance < 39.7635km or 39762.5m ≤ true distance < 39 763.5m, because 1000m = 1km.

7. 0.0025ℓ ≤ true volume < 0.0035ℓ or 2.5mℓ ≤ true volume < 3.5mℓ.
8. 15.615A ≤ true current < 15.625A.

9. $7.05\Omega \leq$ true resistance $< 7.15\Omega$.

10. $0.925m \leq$ true distance $< 0.935m$ or $92.5cm \leq$ true distance $< 93.5cm$.

Activity 3.3

1. a) $8.5m \leq$ length $< 9.5m$.

   b) $6.5m \leq$ width $< 6.5m$

   c) $9.5m \leq$ height $< 10.5m$.

   d)

$$lower\ bound = smallest\ length \times smallest\ width = 8.5m \times 6.5m = 55.3m^2$$

$$upper\ bound = l\arg est\ lenght \times l\arg est\ width = 7.5m \times 10.5m = 78.8m^2$$

Therefore, the limits of accuracy of the base area are: $55.3m^2 \leq$ base area $< 78.8m^2$.

   e)

$$lower\ bound = smallest\ area \times smallest\ height = 55.3m^2 \times 9.5m = 525.4m^3$$

$$upper\ bound = l\arg est\ area \times l\arg est\ height = 78.8m^2 \times 10.5m = 827.4m^3$$

Therefore, the limits of accuracy of volume are: $525.4m^3 \leq$ volume $< 827.4m^3$.

2. a) The limits of accuracy of distance and time and time are:

   $119.5km \leq$ true distance $< 120.5km$

   $2.5h \leq$ true time $< 3.5h$

So the least possible average speed S

$$= \frac{shortest\ dis\tan ce}{longest\ time} = \frac{119.5km}{3.5h} = 32.1km/h$$

The greatest possible average speed S =

$$\frac{longest\ dis\tan ce}{shortest\ time} = \frac{120.5km}{2.5h} = 48.2km/h$$

   b) Limits of accuracy of speed and time are:

   $79.5km/h \leq$ actual speed $< 80.5km/h$

   $1.5h \leq$ actual time $< 2.5h$.

The greatest possible distance = greatest speed × longest time= 80.5km/h ×2.5h=2.1.3km.

c) The upper bound of the average speed of the entire journey =

$$\frac{upper\ bound\ of\ \ first\ speed + upper\ bound\ of\ second\ speed}{2}$$

$$= \frac{48.2km/h + 80.5km/h}{2}$$

$$= 64.4km/h$$

3.

The limits of accuracy of the final velocity are: 19.5m/s ≤ v < 20.5m/s.

The limits of accuracy of time are: 9.5s ≤ true time < 10.5s.

So the least possible acceleration is

$$a = \frac{smallest\ velocity}{largest\ time} = \frac{19.5m/s}{10.5s} = 1.9m/s$$

4.

a) The lower bound of x + y is given by the lower bound of x + lower bound of y.

So the lower bound of x + y= 2+ -4 = -2.

The upper bound of x + y is given by the upper bound of x + the upper bound of y.

So the upper bound of x + y = 5 – 1 =4.

b) The lower bound of y - x is given by the lower bound of y - upper bound of x.

So the lower bound of y - x =-4 -5 = -9.

The upper bound of y -x is given by the upper bound of y - the lower bound of x.

So the upper bound of x - y = -1 -2 =-3.

c) The lower bound of $\frac{xy}{z}$ is given by

$$\frac{lower\ bound\ of\ \ x \times lower\ bound\ of\ \ y}{upper\ bound\ of\ \ z} = \frac{2 \times -4}{8} = \frac{-8}{8} = -1.$$

The upper bound of $\frac{xy}{z}$ is given by:

$$\frac{upper\ bound\ of\ \ x \times upper\ bound\ of\ \ y}{lower\ bound\ of\ \ z} = \frac{5 \times -1}{5} = \frac{-5}{5} = -1.$$

5.

i) The least possible value of the radius is 10.5 mm.

ii) $c = \pi d = 2\pi r$.

So the least possible value of the circumference is

$$c = 2 \times \frac{22}{7} \times 10.5mm$$
$$= 66mm$$

Unit Summary

Summary

In this unit you learned that:

- When counting the number of significant figures in whole numbers, you count from the left to the last non-zero digit.
- When counting the number of significant figures in mixed numbers, count all numbers
- When counting the number of significant figures in decimals, count to the right starting with the first non zero-digit.
- When calculating the estimate of an expression we round each number in the expression correct to one significant figure, then work out the value of the resulting expression.
- By convention the final answer in estimation is always approximated to 1 significant figure.
- When taking measurements or approximating numbers, the limits of accuracy which consist of the lower and the upper bound are applied to the measurement to ensure accuracy.
- The limits of accuracy are written as: $a \leq x < b$, where a is the lower bound and b the upper bound.
- The lower limit has an inclusive inequality and the upper limit has a strict inequality because the upper limit does not round down to the true measurement. Instead it rounds up to a different number altogether.

The next unit after this is on algebraic manipulations.

Assignment

You are advised to spend 40 minutes on this assignment. It carries 50 marks in all and the marks for each question are shown in parentheses.

Show all the necessary workings.

The use of calculators is not permitted.

1. Round the following numbers to the nearest 1000:

a) 68 769
(2 marks)

b) 792 347
(1 mark)

c) 999 899
(1 mark)

d) 88 500
(1 mark)

2. Round of the following numbers to the nearest 100:

a) 78 531
(1 mark)

b) 8 079
(1 mark)

c) 3 011
(1 mark)

d) 720
(1 mark)

e) 2 993
(1 mark)

3. Round off the following numbers to the number of decimal places indicated in the brackets:

a) 3.12 (1 dp)
(1 mark)

b) 2.56 (1 dp)
(1 mark)

c) 3.98 (1 dp)
(1 mark)

d) 1.003 (1 dp)
(1 mark)

e) 0.013 (2 dp)
(1 mark)

f) 9.999 (2 dp)
(1 mark)

4. How many significant figures does each of the following numbers show?

a) 7 538
(1 mark)

b) 20 450
(1 mark)

c) 0.00971
(1 mark)

d) 0.005100
(1 mark)

e) 56.2310
(1 mark)

f) 2 341.0
(1 mark)

5. Write the following numbers to the number of significant figures shown in the brackets:

a) 48 599 (1 sf)
(1 mark)

b) 7 899 (3 sf)
(1 mark)

c) 990 (1 sf)
(1 mark)

d) 483.7 (1 sf)
(1 mark)

e) 2.5728 (3 sf)
(1mark)

f) 14.952 (3 sf)
(1 mark)

g) 0.085 (1 sf)
(1 mark)

h) 0.0019 (1 sf)
(1mark)

i) 0.002300 (2 sf)
(1 mark)

6. 1 mile is 1760 yards. Estimate the number of yards in 11.5 miles.
(3 marks)

7. Estimate the answers to the following:

a) $\frac{5.3 \times 11.2}{2.1}$
(2 marks)

b) $\frac{9.8^2}{4.7^2}$
(2 marks)

c) $\frac{18.8 \times (7.1)^2}{(3.1)^2 \times (4.9)^2}$
(2 marks)

8. Each of the following numbers is correct to one decimal place. State the limits of accuracy in the form of inequalities as $a \leq x < b$, where a is the lower bound and b the upper bound.

a) 3.8
(1 mark)

b) 15.6
(1 mark)

c) 1.0
(1 mark)

d) -0.2
(1 mark)

9. The capacity of a swimming pool is given as $740m^3$ correct to two significant figures.

a) Calculate the lower and the upper bounds of the pool's capacity and using x cubic metres for the capacity, express the range of values in which x must lie as an inequality.
(2 mark)

10. A farmer measures the dimensions of his rectangular field to the nearest 10m. The length is recorded as 570m and the width is recorded as 340m.

a) Calculate the lower and the upper bounds of the length.
(2 marks)

b) Using W metres for the width, express the range of values in which W must lie as an inequality.
(2 marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Model Answers to the Assignment

1.

a) 68769 to the nearest 1000 is 69 000.

b) 792 347 to the nearest 1000 is 792 000.

c) 999 899 to the nearest 1 000 is 1 000 000.

d) 88 500 to the nearest 1000 is 89 000.

2.

a) 78 531 to the nearest 100 is 78 500.

b) 8 079 to the nearest 100 is 8100.

c) 3 011 to the nearest 100 is 3000.

d) 720 to the nearest 100 is 700.

e) 2 993 to the nearest 100 is 3000.

3.

a) 3.12 to 1 dp = 3.1.

b) 2.56 to 1 dp = 2.6.

c) 3.98 to 1dp= 4.0.

d) 1.003 to 1 dp=1.0.

e) 0.013 to 2 dp = 0.01.

f) 9.999 to 2 dp = 10.00.

4.

a) 7538 has 4sf

b) 20450 has 4 sf.

c) 0.00971 has 3 sf.

d) 0.005100 has 4 sf.

e) 56.2310 has 6 sf.

f) 2341.0 has 5 sf.

5.

a) 48 599 to 1 sf =50 000.

b) 7 899 to 3 sf = 7900.

c) 990 to 1 sf = 1 000.

d) 483.7 to 1 sf = 500.

e) 2.5728 to 3 sf = 2.57.

f) 14.952 to 3 sf = 15.0.

g) 0.085 to 1 sf = 0.09.

h) 0.0019 to 1sf = 0.002.

i) 0.002300 to 2 sf = 0.0023.

6.

1 mile is approximately 2 000 yards, the number of yards is approximately 10 miles.

So in 11.5 miles there is approximately

$$\frac{2000\ yards \times 10\ miles}{1\ mile} = 20\ 000.$$

7.

a) $\frac{5.3 \times 11.2}{2.1} \approx \frac{5 \times 10}{2} = 25.$

b) $\frac{9.8^2}{4.7^2} \approx \frac{10^2}{5^2} = \frac{100}{25} = 4.$

c) $\frac{18.8 \times 7.2^2}{3.1^2 \times 4.9^2} \approx \frac{20 \times 7^2}{3^2 \times 5^2} = \frac{20 \times 49}{9 \times 25} = 4.35 \approx 4.$

8.

a) $3.75 \leq x < 3.85$

b) $15.55 \leq x < 15.65$

c) $0.95 \leq x < 1.05$

d) $-0.25 < x \leq -0.15$

9.

The limits of accuracy of the pool are: $735m^3 \leq 740\ m^3 < 745m^3$.

10.

a) Upper bound of length is 575m.

And the lower bound of length is 565m.

b) $335m \leq W < 345m$.

Assessment

You are advised to spend 30 minutes on this assignment. It carries 35 marks all in all and the marks for each question are shown in parentheses.

Show all the necessary workings.

The use of calculators is **not** permitted.

1. Round off the following numbers to the degree of accuracy shown in brackets:

a) 3 621 (nearest 100)
(1 mark)

b) 8 976 (nearest 10)
(1 mark)

c) 56 862 (nearest 1000)
(1 mark)

d) 986 589 (nearest 10 000)
(1 mark)

2. Round off the following numbers to the number of decimal places shown in the brackets:

a) 4.61 (1 dp)
(1 mark)

b) 8.583 (1 dp)
(1 mark)

c) 0.00277 (3 dp)
(1 mark)

d) 9.953 (1 dp)
(1 mark)

3. Round off the following numbers to the number of significant figures shown in the brackets:

a) 0.7765 (1 sf)
(1 mark)

b) 834.97 (2 sf)
(1 mark)

c) 687 453 (3 sf)
(1 mark)

d) 42.6 (1 sf)
(1 mark)

4. Make rough estimations of the numerical values of the following expressions:

a) 0.0321×1846
(2 marks)

b) $\dfrac{34.49 \times 0.700}{1.83}$
(3 marks)

5. A cuboid's dimensions are given as 3.973m by 2.4 m by 3.16 m. Estimate its volume, giving your answer to an appropriate degree of accuracy. (3 marks)

6. A school measures the dimensions of its rectangular playing field. The length was recorded as 350 m to the nearest 10 metre and the width as 200 m to the nearest 100m. Express the ranges in which the length and width lie using inequalities.
(3 marks)

7. The mass of sack of vegetables is given as 7.8kg correct to 1 dp. Write the limits of accuracy of the mass.
(2 marks)

8. The following numbers are expressed correct to three significant figures. Present the limits of accuracy of each number using inequalities:

a) 254
(1 mark)

b) 50.5
(1 mark)

c) 1.00
(1 mark)

9.

A basketball stadium has 13492 seats.

During a season a basketball team played 26 matches and every seat was sold for each match. At each match a seat cost $18.80.

By writing each value correct to 1 significant figure, estimate the total amount of money paid to watch these matches during the season.
(3 marks)

10. The distance travelled by an object was 230m, correct to the nearest 10m. The time taken was 7 seconds, correct to the nearest second.

i) Complete the following statements
(2 mark)

______m≤ distance< ________m

_______s≤ time < ________s

ii) What was the least possible average speed for the whole journey?
(2 marks)

____

____

____

____

____

____

____

Send your answers and work to your tutor for marking.

Unit Contents

Unit 4

Unit 4

Algebraic Manipulation

Introduction

When letters and numbers are used together, the mathematics is called algebra. We have already occasionally used letters instead of numbers. In this unit, we are going to do more work on algebra. We are going to work with both linear and non linear expressions.

This unit consists of 54 pages. This is approximately 2% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 20 hours to work on this unit, 15 hours for formal study and 5 hours for self-study and completing assessments/assignments.

When reading the following learning outcomes, think about them as a guide to what you should focus on while studying this unit.

This Unit is Comprised of Three Lessons:

Lesson 1 Multiplying Algebraic Expressions
Lesson 2 Multiplication of Other Algebraic Expressions
Lesson 3 Factorisation

Outcomes

Upon completion of this unit you will be able to:

- *multiply* a monomial by a polynomial
- *expand* products of algebraic expressions
- *extract* common factors using brackets
- *factorise* where possible expressions of the form $ax + ay$; $ax + bx + kay + kby$; $a^2x^2 - b^2y^2$; $a^2 + 2ab + b^2$; $ax^2 + bx + c$

Terminology

Term: Either a single number, called a constant, or variable, or the product of several numbers and/or variables.

Expression: An expression is formed when terms are combined by either addition or subtraction.

Linear expression: An expression that has the highest power on the variable as 1.

Non linear expression: An expression that has the highest power on the variable of power greater than 1 or less than 1, but never 1.

Factor: Factors of numbers are numbers that divide into another exactly.

Factorisation: Writing a number or an expression as a product of its factors.

Monomial: An algebraic expression with one term.

Binomial: An algebraic expression with two terms.

Trinomial: An algebraic expression with three terms.

Polynomial: An algebraic expression that has many terms ("poly" means "many")

of the form $c_nx^n + c\ v_{n-1}x^{n-1} + ...c_1x + c_0$ with

where n is a positive integer and $c_0, c_1,. ....c_n$ are real numbers with $c_n \neq 0$

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Multiplying Algebraic Expressions

By the end of this subunit, you should be able to

- multiply a monomial by a polynomial.
- expand products of algebraic expressions.

Multiplying a Monomial by a Polynomial.

We will first consider examples of multiplying a monomial by a polynomial. To multiply a monomial by a polynomial, we multiply each term of the polynomial by the monomial. This is a direct application of the distributive property.

Reflection

The distributive property says
If a, b, and c are real numbers, then

$a(b + c) = a \times b + a \times c$

and

$(b + c)a = b \times a + c \times a$

a and $(b + c)$ are factors. The factor without brackets is said to be distributed as a factor of each term within the brackets.

Examples

$$\begin{aligned} 3(4 + 5) &= 3(4) + 3(5) \\ &= 12 + 15 \\ &= 27 \end{aligned}$$

$$\begin{aligned} (8 + 2)\,6 &= 8(6) + 2(6) \\ &= 48 + 12 \\ &= 60 \end{aligned}$$

The distributive property is normally challenged with the BEDMAS rule.
The BEDMAS rule states that the standard order of operations is such that
We start with the operation inside the **B**rackets.
next **E**xponents and **R**oots
followed by **M**ultiplication and **D**ivision (from left to right if consecutive)
and lastly **A**ddition and **S**ubtraction (from left to right if consecutive)

Going back to our examples

$3(4 + 5) = 3(9)$
$= 27$

$(8 + 2)\,6 = (10)6$
$= 60$

One would then ask, why bother with the distributive property when BEDMAS still gives the same result. Let us look at the next example:

$x(y + 27)$

y and 27 are not like terms. This says we cannot add them together. That therefore says working with the addition in the brackets will only work if like terms are involved. The distributive property works for all situations.

$$\begin{aligned} x(y + 27) &= (x \times y) + (x \times 27) \\ &= xy + x27 \\ &= xy + 27x \end{aligned}$$

Can the distributive property be extended to the sums involving any other number of terms?

____

Compare your answer to the following:

Yes!

The distributive property is the key tool in multiplying expressions.

When multiplying algebraic expressions:

- multiply numerical coefficients together, then.
- list all the variables that occur in the terms being multiplied and write them in alphabetical order; since it makes it easier to read when the problems become more involved.
- remember the laws of indices; add the exponents of like variables.

Let us consider these examples.

Example 1

a (3x + 4 + 6y)

We have a monomial *a* , and our polynomial 3x + 4 + 6y

Multiplying them out, using the distributive property:

a (3x + 4 + 6y) = a(3x) + a(4) + a(6y)

The factor, *a*, without brackets is said to be distributed as a factor of each term within the brackets.

= a3x + a4 + a6y

As a matter of convention, the figures are written before the letters:

= 3ax + 4a + 6ay

Example 2

a (x + y - z) = ax + ay - az

Example 3

-3 (ab + d + 5ab)

The factor, -*3*, without brackets is distributed as a factor of each of the terms in (ab + d + 5ab).

-3 (ab + d + 5ab) = -3 × ab + -3 × d + -3 × 5ab

Note it!

Adding a negative number is equivalent to subtracting a positive number of the same value.

$$= -3ab + -\ 3d + -15ab$$

-3ab and -15ab are like terms. We collect like terms.

$$= -3ab + -\ 15ab + -3d$$
$$= -18ab + -3d$$
$$= -18ab - 3d$$

Example 4

$3x\ (x + y - z + ax^2)$

The factor, 3x, is distributed as a factor of each of the terms in $(x + y - z + ax^2)$

$3x\ (x + y - z + ax^2) = 3x\ (x + y - z + ax^2)$

$$= [(3x \times x) + (3x \times y) - (3x \times z) + (3x \times ax^2)]$$

$3x \times x$ and $3x \times ax^2$ can be simplified with the help of the laws of indices. Remember that when no power is explicitly stated, the power is assumed to be 1.

$$= [(3x^1 \times x^1) + (3x \times y) - (3x \times z) + (3.\ x^1 \times a \times x^1 \times x^1)]$$

$$= [(3x^{1+1}) + (3x\times y) - (3x\times z) + (3\times x^1\ a\times x^{1+1})]$$

$$= [(3x^2) + (3x \times y) - (3x \times z) + (3x^{1+1+1}\ a)]$$

$$= [(3x^2) + (3x \times y) - (3x \times z) + (3a \times x^3)]$$

$$= 3x^2 + 3xy - 3x\,z + 3ax^3$$

We have a term that has power 2 and another that has power 3. We normally write the terms in descending order.

$$= 3ax^3 + 3x^2 + 3xy - 3x\,z$$

Example 5

$2\{3a + 5(b + c)\}$

We have a monomial *2*, and our polynomial is $\{3a + 5(b + c)\}$

This polynomial has two sets of brackets. It is best, as a rule, to begin with the innermost brackets and work outwards.

$$\begin{aligned} 2\{3a + 5(b + c)\} &= 2\{3a + (5 \times b + 5 \times c)\} \\ &= 2\{3a + 5b + 5c\} \\ &= 2 \times 3a + 2 \times 5b + 2 \times 5c \\ &= 6a + 10b + 10c \end{aligned}$$

Example 6

$x^2(x^4 + 5a) = x^6 + 5ax^2$

Activity 1

Work out:

1. $4y(6x - 8)$

2. $2a(-a - b)$

3. $-5z(x - 9y)$

4. $-(b - 4)$

5. $a(a^2 - 3a + 2)$

6. $4b^2(b^2 + 2b - 3)$

7. $-5y[5c + y (y^2 - 2s^4 + 11)]$

8. $xyz[(-3 (xyz^3 + x^3yz^2 + x^2y^3z)]$

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Lesson 2 Multiplication of Other Algebraic Expressions

Next, we will consider the multiplication of other algebraic expressions in two brackets.
If you need to multiply two brackets together, we multiply each term in the first pair of brackets by every term in the second pair of brackets.
This multiplication of one polynomial by another is called **expansion.**

Multiplying out the factors of an expression so as to get rid of the brackets is called **expanding** the brackets.

Example 1

Work out $(x + 3)(x + 5)$

This is best done by the "FOIL" method. This is what it says:

	$(x + 3)(x + 5)$		
F	Multiply out the First terms	$(x + 3)(x + 5)$	$x \times x = x^2$

O	Then multiply out the Outer terms	$(x + 3)(x + 5)$	$x \times 5 = 5x$
I	Next multiply the Inner terms	$(x + 3)(x + 5)$	$3 \times x = 3x$
L	Finish off by multiplying out the Last terms	$(x + 3)(x + 5)$	$3 \times 5 = 15$

Then add them all up

$$(x + 3)(x + 5) = x^2 + 5x + 3x + 15$$

We collect like terms

$$(x + 3)(x + 5) = x^2 + (5x + 3x) + 15$$
$$= x^2 + 8x + 15$$

Example 2

In this example, one of the first terms looks different from what we have had so far. It has a constant and a variable. That will not change anything. The rules are not changed.

$$(6x - 5)(x + 7) = 6x \times x + (6x \times 7 + -5 \times x) + -5 \times 7$$
$$= 6x^2 + (42x + -5x) + -35$$

We collect like terms

$$= 6x^2 + 37x - 35$$

Example 3

$$(3x + 1)(3x + 2) = 3x \times 3x + (3x \times 2 + 1 \times 3x) + 1 \times 2$$
$$= 9x^2 + (6x + 3x) + 2$$

We collect like terms

$$= 9x^2 + 9x + 2$$

There are some special products that we are going to look at. They are:

1. **Square of a sum : $(a + b)^2$**

$(a + b)^2 = (a + b)(a + b)$

Using the "FOIL" method;

$$= [(a \times a) + (a \times b) + (b \times a) + (b \times b)]$$
$$= a^2 + ab + ba + b^2$$

Collecting like terms, ab and ba

$$= a^2 + 2ab + b^2$$

Example 4

Expand, $(x + 5)(x + 5)$

$(x + 5)(x + 5) = (x + 5)^2$

$$= [(x \times x) + (x \times 5) + (5 \times x) + (5 \times 5)]$$
$$= x^2 + x5 + 5x + 5^2$$

Collecting like terms, x5 + 5x

$$= x^2 + 10x + 25$$

Actually we did not have to go through all these steps, we could have straight away fitted the result like this:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(x + 5)^2 = x^2 + 2x5 + 5^2$$

$$= x^2 + 10x + 25$$

Square of a difference: $(a - b)^2$

$(a - b)^2 = (a - b)(a - b)$

Using the "FOIL" method;

$= [(a \times a) + (a \times -b) + (-b \times a) + (-b \times -b)]$
$= a^2 – ab - ab + b^2$
$= a^2 - 2ab + b^2$

Example 5

Expand, (x - 5) (x - 5)

$(x - 5)(x - 5) = (x - 5)^2$

$= [(x \times x) + (x \times -5) + (-5 \times x) + (-5 \times -5)]$
$= x^2 + x \times -5 + -5 \times x + -5^2$

$= x^2 - 10x + 25$

Again we did not have to go through all these steps; we could have straight away fitted the result in this:

$$(a - b)^2 = a^2 - 2ab + b^2$$

$(x - 5)(x - 5) = (x - 5)^2 = x^2 - 2x5 + -5^2$
$= x^2 - 10x + 25$

2. **Difference of two squares : (a + b) (a - b)**

(a + b) (a - b)

Using the "FOIL" method;

$= [(a\times a) + (a\times -b) + (b\times a) + (b\times -b)]$
$= a^2 – ab + ab - b^2$
$= a^2 - 2ab + b^2$
$= a^2 - b^2$

Example 6

Expand, (x - 5) (x + 5)

$(x - 5)(x + 5) = [(x \times x) + (x \times 5) + (-5 \times x) + (-5 \times 5)]$
$= x^2 + x5 + -5x + -5^2$

Collecting like terms, x5 + -5x

$= x^2 + -25$

$= x^2 - 25$

OR

$(x - 5)(x + 5) = x^2 - 5^2$
$= x^2 – 25$

The examples that we have had so far have had letters and constants. We can have situations where we are using letters throughout. That will not change anything. The rules are not changed.

Example 7

Expand, $(x + y)(x + y)$

This is square of a sum

$(x + y)^2 = x^2 + 2xy + y^2$

Example 8

Expand, $(x - y)(x - y)$

This is square of a difference

$(x - y)^2 = x^2 - 2xy + y^2$

Example 9

Expand, $(f - g)(f + g)$

This is the difference of two squares

$(f - g)(f + g) = f^2 - g^2$

That will not change anything. The rules are not changed.

These special products can also be used to provide an easy way of calculating squares of numbers, when calculators are not used.

Example 10

13^2

This we can workout with the help of the square of a sum
We can write 13 as 10 + 3

$13^2 = (10 + 3)^2$

This is a square of a sum: $(a + b)^2 = a^2 + 2ab + b^2$

$13^2 = (10 + 3)^2$

$= [(10 \times 10) + (10 \times 3) + (3 \times 10) + (3 \times 3)]$

$= 100 + 30 + 30 + 9$

$= 100 + 60 + 9$

$= 169$

Example 11

99^2

This we can workout with the help of the square of a difference:
$(a - b)^2 = (a - b)(a - b)$

$99^2 = (100 - 1)^2$
$= [(100 \times 100) - (100 \times 1) - (1 \times 100) + (1 \times 1)]$

$= 100^2 - 2(100)(1) + 1^2$
$= 10\,000 - 200 + 1$

$= 9800 + 1$
$= 9801$

Example 12

$(79 + 21)(79 - 21)$

$= 79^2 - 21^2$
$= 6\,241 - 441$
$= 5\,800$

Example 13

6.25^2

$6.25^2 = (6 + 0.25)^2$

$= [(6\times6) + (6\times0.25) + (0.25\times6) + (0.25\times0.25)]$
$= 36 + 1.5 + 1.5 + 0.0625$
$= 36 + 3 + 0.0625$
$= 39.0625$

Activity 2

Calculate:

1) 102^2

2) 10.5^2

Check your performance against the given solutions at the end of this subunit. Continue if you are satisfied with your ability to answer the questions. If not, review the above content and try the activity again.

Activity 3

Expand the following:

1. $(x + 3)(x + 2)$

2. $(x - 3)(x - 2)$

3. $(x - 5)(x - 4)$

4. $(2x + 3y)(x + 2y)$

5. $(3x + 2)(2x + 4)$

6. $(x - y)(2x - 3y)$

7. $(-x + 2)(2x - 1)$

8. $(-2x - 8)(x + 7)$

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Activity 4

Expand the following:

1. $(x + y)^2$

2. $(2x + 3)^2$

3. $(x - 4y)^2$

4. $(2a - y)^2$

5. $(5x - 2y)^2$

Compare your answers with those at the end of this subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content and work through the activity again.

Key Points to Remember

The key points to remember in this subunit on multiplication of algebraic expressions are:

- numerical coefficients are multiplied together,
- all the variables that occur in the terms being multiplied are listed and written in alphabetical order; since it makes it easier to read when the problems become more involved

- when we multiply algebraic expressions, we need to remember the laws of indices. Add the exponents of like variables

- use the "FOIL" method to multiply two brackets together

- there are some special products; they are:

Square of a sum : $(a + b)^2 = (a + b)(a + b)$

Square of a difference : $(a - b)^2 = (a - b)(a - b)$

Difference of two squares : $(a + b)(a - b) = a^2 - b^2$

Answers to Activity 1

1. $4y(6x - 8) = 4y \times 6x - 4y \times 8$
   $= 24xy - 32y$

2. $2a(-a - b) = 2a \times -a - 2a \times b$
   $= 2\text{-}a^2 - 2ab$
   $= -2a^2 - 2ab$

3. $-5z(x - 9y) = -5z \times x - -5z \times 9y$
   $= -5xz - -45y\,z$
   $= -5xz + 45y\,z$

4. $-(b - 4) = -1(b - 4)$
   $= -1 \times b - -1 \times 4$
   $= -1b + 1 \times 4$
   $= -b + 4$

5. $a(a^2 - 3a + 2) = a \times a^2 - a \times 3a + a \times 2$
$= a^3 - 3a^2 + 2a$

6. $4b^2(b^2 + 2b - 3) = 4b^2 \times b^2 + 4b^2 \times 2b - 4b^2 \times 3$

$= 4b^4 + 8b^3 - 12b^2$

7. $-5y[5c + y(y^2 - 2s^4 + 11)] = 5y[5c + (y^3 - 2s^4 y + 11y)]$
$= 5y[5c + y^3 - 2s^4 y + 11y)]$
$= 25cy + 5y^4 - 10s^4 y^2 + 55y^2)]$

8. $xyz[(-3(xyz^3 + x^3yz^2 + x^2y^3z)] = xyz[(-3xyz^3 - 3x^3yz^2 - 3x^2y^3z)]$
$= -3x^2y^2z^4 - 3x^4y^2z^3 - 3x^3y^4z^2$

Answers to Activity 2

102^2

$102^2 = (100 + 2)^2$

$= [(100 \times 100) + (100 \times 2) + (2 \times 100) + (2 \times 2)]$

$= 10\,000 + 200 + 200 + 4$
$= 10\,000 + 400 + 4$
$= 10\,404$

10.5^2

$10.5^2 = (10 + 0.5)^2$

$= [(10 \times 10) + (10 \times 0.5) + (0.5 \times 10) + (0.5 \times 0.5)]$

$= 100 + 5 + 5 + 0.25$
$= 100 + 10 + 0.25$

$= 110.25$

Answers to Activity 3

1. $(x + 3)(x + 2) = x^2 + (3x + 2x) + 6$
   $= x^2 + 5x + 6$

2. $(x - 3)(x - 2) = x^2 + (-3x + -2x) + -6$
   $= x^2 + -5x + -6$
   $= x^2 - 5x – 6$

3. $(x - 5)(x - 4) = x^2 + (-4x + -5x) + -20$
   $= x^2 + -9x + -20$
   $= x^2 - 9x - 20$

4. $(2x + 3y)(x + 2y) = 2x^2 + (4xy + 3xy) + 6y^2$
   $= 2x^2 + 7xy + 6y^2$

5. $(3x + 2)(2x + 4) = 6x^2 + (12x + 4x) + 8$
   $= 6x^2 + 16x + 8$

6. $(x - y)(2x – 3y)\,) = 2x^2 - 3xy – 2xy + 3y^2$
   $= x^2 - 9x - 20$

7. $(-x + 2)(2x – 1)\,) = -2x^2 + x + 4x - 2$

   $= -2x^2 + 5x - 2$

8. $(-2x - 8)(x + 7)\,) = -2x^{2-}-14x - 8x - 56$

   $= -2x^2 - 22x - 56$

Answers to Activity 4

1. $(x + y)^2 = (x + y)\ (x + y)$
   $= x^2 + 2xy + y^2$

2. $(2x + 3)^2 = (2x + 3)\ (2x + 3)$
   $= 4x^2 + 2(2x.3) + 3^2$
   $= 4x^2 +12x + 9$

3. $(x - 4y)^2 = (x - 4y)\ (x - 4y)$
   $= x^2 - 2\ (x.4y) + (4y)^2$
   $= x^2 - 2\ (x.4y) + 16y^2$
   $= a^2 - 8xy + 16y^2$

4. $(2a - y)^2 = (2a - y)\ (2a - y)$
   $= 4a^2 - 2\ (2a.y) + (y)^2$
   $= 4a^2 - 4ay + y^2$

5. $(5x - 2y)^2 = (5x - 2y)\ (5x -2\ y)$
   $= 25x^2 - 2\ (5x\times2y) + (2y)^2$
   $= 4a^2 - 20xy +4y^2$

Lesson 3 Factorisation

By the end of this subunit, you should be able to:

- extract common factors using brackets.
- factorise where possible expressions of the form $ax + ay$; $ax + bx + kay + kby$; $a^2x^2 – b^2y^2$; $a^2 + 2ab + b^2$; $ax^2 + bx + c$

Factorisation is the reverse of the work we did in the previous subunit. Factorisation simply means finding the factors (the things that divide evenly into) of a number or expression.

In your Junior Certificate Course you did work on factorisation of whole numbers and some expressions. You factorised expressions by taking out a common factor.

In this unit, you will continue to work on factorisation of expressions by taking out common factors. You will also learn to find the factors of expressions of different types using other methods.

Reflection

Factorise the following:

As a reminder, to factorise a number means to write it as a product of its factors.

(a) 2
(b) 5
(c) 15
(d) 12

(a) Factors of 2 are 1 and 2
1 divides into 2 exactly 2 times
2 divides exactly once into 2

(b) Factors of 5 are 1 and 5
1 divides into 5 exactly 5 times
5 divides exactly once into 5

(c) Factors of 15 are 1, 3, 5 and 15
1 divides into 15 exactly 15 times

3 divides into 15 exactly 5 times

(d) Factors of 12 are 1, 2, 3 4, 6 and 12
1 divides into 12 exactly 12 times
2 divides into 12 exactly 6 times
3 divides into 12 exactly 4 times

Factorising by Taking Out a Common Factor

A simple rule in factorising any expression is first to look for common factors and to "take out" these common factors.

Example 1

3(4) + 3(5)

The expression 3(4) + 3(5) has two terms, 3(4) and 3(5). Each of the terms has a common factor of 3. This is extracted using brackets giving:

$3(4) + 3(5) = 3(4 + 5)$

Example 2

$4x + 4y + 4z$

This expression has three terms. All three have a common factor, 4. Taking it out: $4x + 4y + 4z = 4(x + y + z)$

4 and $x + y + z$ are factors of $4x + 4y + 4z$

Example 3

Extract common factors in $xy + 27x$

$xy + 27x$ has two terms, and each has x in them; x is a common factor of xy and $27x$

$xy + 27x = x(y) + x(27)$
$= x(y + 27)$

Example 4

$7x^2 + 21x + 14xy$

This expression has three terms. All three have two common factors, 7 and x.

It is advisable that we write the variables with powers without the powers.

$7x^2 + 21x + 14xy = 7\times x\times x + 21x + 14xy$

$= 7(x\times x + 3x + 2xy)$

Then taking out x:

$= 7x(x + 3 + 2y)$

Note it!

If an expression has two or more common factors, you can "take them out" in one step.

$7x^2 + 21x + 14xy$

Taking out 7 and x:

$7x^2 + 21x + 14xy = 7x(x + 3 + 2y)$

Example 5

Extract common factors in $ax^2 – a^2x$

As needed, you can write the variables with powers without them.

$ax^2 - a^2x = a \times x \times x - a \times a \times x$

Each of the two terms has two common factors, *a* and *x*

$= ax(x - a)$

Note it!

An important thing to remember is that when you have factorised, and you then multiply these factors (brackets) together, you must end up with what you had originally.

Let us multiply the factors that we got in the examples above and check what we get.

1. $3(4 + 5) = 3(4) + 3(5)$
2. $4(x + y + z) = 4x + 4y + 4z$
3. $x(y + 27) = xy + 27x$
4. $7x(x + 3 + 2y) = 7x^2 + 21x + 14xy$
5. $a \times x(x - a) = a \times x \times x - a \times a \times x$

   $= ax^2 - a^2x$

Activity 1

Factorise the following, completely:

1. $3x - 6$

2. $6x + 4$

3. $6y^2 - 4y$

4. $10xy + 5xz$

5. $3x^2y - 9x^3y^3$

6. $8a^3bc - 12ab^3cd + 4b^4c^2d^2$

After completing the questions, compare your answers to the correct answers at the end of this subunit. Take the time needed to understand each answer before continuing.

"Grouping" and Taking Out a Common Factor

Sometimes only a few terms in an expression have a factor in common.
For example $ax + ay + bx + by$

It is only **ax** and **ay** that have a common factor **a**
It is only **ax** and **bx** that have a common factor **x**
It is only **ay** and **by** that have a common factor **y**
It is only **bx** and **by** that have a common factor **b**

We use the method called **"grouping" and taking out a common factor**. Group the terms by identifying patterns of common factors, then take out the common factor in each group.

Example 1

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \end{aligned}$$

$(x + y)$ is a common factor of the two terms $a(x + y) + b(x + y)$.

We take out this common factor:

Taking out $(x + y)$ from $a(x + y)$ leaves a,
and taking out $(x + y)$ from $b(x + y)$ leaves b

$$a(x + y) + b(x + y) = (x + y)(a + b)$$

Even if we were to group the terms differently from what we have above, the result should be the same.

$$\begin{aligned} ax + ay + bx + by &= (ax + bx) + (ay + by) \\ &= x(a + b) + y(a + b) \end{aligned}$$

$(a + b)$ is a common factor of the two terms $x(a + b) + y(a + b)$

Taking out $(a + b)$ from $x(a + b)$ leaves x,
and taking out $(a + b)$ from $y(a + b)$ leaves y

$$x(a + b) + y(a + b) = (a + b)(x + y)$$

$(a + b)$ and $(x + y)$ are factors of $ax + ay + bx + by$

Activity 2

Factorise the following:

1. $ac + ad + 2bc + 2bd$

2. $km - kn + lm - ln$

3. $3xy + 21x - 2y - 14$

4. $3x^2 + 6x - 4x - 8$

5. $a^3 - 3a^2 + 6a - 12$

6. $x^3 - x^2 + x + 1$

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Factoring Quadratic Expressions: Familiar Products

In a previous topic, we used the "FOIL" method to expand the following:

$(a + b)\,(a + b) = a^2 + 2ab + b^2$
$(a - b)\,(a - b) = a^2 - 2ab + b^2$
$(a + b)\,(a - b) = a^2 - b^2$
These may now be familiar patterns.

We want to factorise them that is do "FOIL" in reverse.

Example 1

Factorise $x^2 + 6x + 9$
You may have noticed that it matches this pattern:
$a^2 + 2ab + b^2$
Let us then write $x^2 + 6x + 9$ to match this pattern.
$x^2 + 6x + 9 = x^2 + 2.x.3 + 3^2$
$= (x + 3)\,(x + 3)$
$= (x + 3)^2$

Example 2

Factorise $p^2 – q^2$
This pattern is recognizable. It is the difference of two squares.
$p^2 – q^2 = (p + q)\,(p - q)$

Remember that if you did the reverse process of expansion, the +pq and –pq terms cancel each other out.

Example 3

Factorise $16x^2 - 25y^2$
This is the difference of two squares.
$16x^2 - 25y^2 = 4^2 \times x^2 - 5^2 \times y^2$
Using one of the laws of indices: $x^n \times y^n = (xy)^n$
$= (4\times x)^2 - (5\times y^2)$
$= (4x)^2 - (5y^2)$
$= (4x + 5y)\ (4x - 5y)$

Example 4

Factorise $45x^2 - 20y^2$
We are tempted to say this is the difference of two squares, but 45 and 20 are not perfect squares. They only become perfect squares when their common factor is taken out.
$45x^2 - 20y^2 = 5(9x^2 - 4y^2)$
$= 5[(3^2\times x^2) - (2^2\times y^2)]$
$= 5[(3x)^2 - (2y)^2]$
$= 5[(3x + 2y)\ (3x - 2y)]$

Activity 3

Factorise the following:

1. $x^2 - 49$

2. $2x^2 - 50$

3. $x^2 - 16\ x^2$

4. $4a^2 - 64b^2$

5. $1 - 100y^2$

6. $x^3 - 16x$

7. $9a^2 - 16x^2$

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Factoring Quadratic Expressions: Trial and Inspection

Sometimes we cannot recognise any familiar pattern in a quadratic expression. For example, $x^2 + 9x + 18$

The expressions we are going to work with have this pattern: $ax^2 + bx + c$

We factorise with the help of the following steps:

The expression will still have two factors, (x + □) (x + □)
the first term in each bracket is x to give x^2
we look for numbers that go into the boxes. These are factors of **c**
these factors of **c** should add up to **b**

Example 1

$x^2 + 9x + 18$

c = 18

Factors of 18 are 1 and 18, 2 and 9, 3 and 6, -1 and -18, -2 and -9, -3 and -6.

b = 9 1 and 18 do not add up to 9
-1 and -18 do not add up to 9
2 and 9 do not add up to 9
-2 and -9 do not add up to 9
-3 and -6 do not add up to 9
3 and 6 add up to 9

$x^2 + 9x + 18 = (x + 3)(x + 6)$

Example 2

$x^2 + 7x + 12$

c = 12

Factors of 12 are 1 and 12, 2 and 6, 3 and 4, -1 and -12, -2 and -6, -3 and -4.

b = 7 1 and 12 do not add up to 7
-1 and -12 do not add up to 7
2 and 6 do not add up to 7
-2 and -6 do not add up to 7
-3 and -4 do not add up to 7
3 and 4 add up to 7;

$x^2 + 7x + 12 = (x + 3)(x + 4)$

Example 3

$x^2 - 6x + 8$

The product of the two numbers required is positive (+8) and their sum is negative (-6); so both numbers must be negative.

Factors of 8 are 1 and 8, 2 and 4, -1 and -8, - 2 and - 4

Since both numbers must be negative, we can ignore 1 and 8, 2 and 4.

b = -6
-1 and -8 do not add up to -6
-2 and- 4 add up to -6

$x^2 - 6x + 8 = (x + -2)(x + -4)$ which is normally written $(x - 2)(x - 4)$

Example 4

$x^2 – x - 6$

The product of the two numbers required is negative (-6). The negative product indicates that one of the numbers is negative and the other is positive, and their sum is negative (-1).

Factors of -6 are -1 and 6, - 2 and 3, 1 and -6, 2 and - 3

We are aware that because of the negative product, one of the numbers is negative and the other is positive. We therefore test the pairs with one being positive, the other negative.

b = -1	1 and -6 do not add up to -1
	-1 and 6 do not add up to -1
	-2 and 3 do not add up to -1
	2 and -3 add up to -1

$$x^2 – x - 6 = (x + 2)(x + -3)$$
$$= (x + 2)(x - 3)$$

Example 5

$2x^2 – x - 2$

The term $2x^2$ has factors 2x and x.

The factors are still (2x + □) (x + □)

The product of the two missing terms is -2.

Factors of -2 are:
-2 and 1,
1 and -2,
2 and -1, and
-1 and 2.

Using -2 and 1, we find

$2x^2 - x - 2 = (2x + -2)(x + 1)$
$= (2x - 2)(x + 1)$

We expand the brackets to check the answer

$(2x - 2)(x + 1) = 2x \times x + 2x \times 1 + -2 \times x + -2 \times 1$

$= 2x^2 + 2x + -2x + -2$
$= 2x^2 + 2x - 2x - 2$

$= 2x^2 + 0 - 2$

This is **not** the correct factorisation.

Still using -2 and 1, but having 1 in the first bracket and having -2 in the second bracket, we find

$2x^2 - x - 2 = (2x + 1)(x + -2)$
$= (2x + 1)(x - 2)$

We expand the brackets to check the answer.

$(2x + 1)(x - 2) = 2x \times x + 2x \times -2 + 1 \times x + 1 \times -2$

$= 2x^2 + -4x + 1x + -2$
$= 2x^2 + -3x + -2$
$= 2x^2 + -3x - 2$

Still, this is **not** the correct factorization.

Using -1 and 2, we find

$2x^2 - x - 2 = (2x + -1)(x + 2)$
$= (2x - 1)(x + 2)$

We expand the brackets to check the answer.

$(2x - 1)(x + 2) = 2x \times x + 2x \times 2 + -1 \times x + -1 \times 2$

$= 2x^2 + 4x + -1x + -2$
$= 2x^2 + 2x - 1x - 2$

$= 2x^2 + x - 2$

This is **not** the correct factorization.

Still using-1 and 2, but having 2 in the first bracket and having -1 in the second bracket, we find

$$2x^2 - x - 2 = (2x + 2)(x + -1)$$
$$= (2x + 2)(x - 1)$$

We expand the brackets to check the answer

$$(2x + 2)(x - 1) = 2x.x + 2x\times-1 + 2\times x + 2\times-1$$

$$= 2x^2 + -x + 2x + -2$$
$$= 2x^2 - x + 2x - 2$$
$$= 2x^2 + x - 2$$

Not the required factorization, yet again! And we have run out of options. This says not every expression can be factorised.

Note it!

Not every expression can be factorised. Later in the course, we will use the quadratic formula to find the solutions to these expressions.

Activity 4

Factorise, where possible, the following:

1. $x^2 - 5x + 6$

2. $x^2 - 5x - 6$

3. $x^2 + 5x + 6$

4. $x^2 + 2x - 15$

5. $x^2 + 12x + 36$

6. $x^2 – 2x - 1$

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Factoring Quadratic Expressions: Trial and Inspection (continued)

In all our examples so far, the coefficient of x^2 has been 1. We can have situations where the coefficient of x^2 is any other number.

Example 1

$2x^2 + 15x + 7$

The term $2x^2$ has factors 2x and x.

The factors are (2x + □) (x + □)

The product of the two missing terms is 7.

Factors of 7 are
7 and 1or
1 and 7.

We are immediately tempted to say this will not work as 1 + 7 does not add up to 15, they add up to 8, which is far less than 15.

We are, however, quick to remember that 1 and 7 are not the only ones that are used to build the middle term. In the FOIL method Multiplication of the outer terms and the inner terms give the middle term

Trying out the possibilities, let's substitute 1 and 7 in (2x + □) (x + □)

$2x^2 + 15x + 7 = (2x + 1)(x + 7)$

We expand the brackets to check the answer

$(2x + 1)(x + 7) = 2x\times x + 2x\times 7 + 1\times x + 1\times 7$

$= 2x^2 + 14x + x + 7$
$= 2x^2 + 15x + 7$

This is the correct factorisation.

We expand the brackets to check the other combination (2x + 7) (x + 1), that indeed it is an incorrect factorisation.

$(2x + 7)(x + 1) = 2x\times x + 2x\times 1 + 7\times x + 7\times 1$

$= 2x^2 + 2x + 7x + 7$
$= 2x^2 + 9x + 7$

Indeed this is not the correct factorisation. The middle term is 9x, and not 15x.

Example 2

$6x^2 + 5x - 4$

The term $6x^2$ has factors 6x and x.
2x and 3x

The term 4 has factors 4 and 1.
2 and 2
Remember one on these pairs should add to 5

Your guess is as good as mine, as to how much work is going to be involved!

We can reduce the amount of work involved by "temporarily eliminating" some of the possible combinations with the following:

The use of 6x and x, factors of $6x^2$, is more likely to give us a middle term that is greater than 5x, which is the term under consideration. We will therefore start using 3x and 2x. If this does not work then we can go back and try 6x and 1x.

The product of the two numbers required is negative (-4). The negative product indicates that one of the numbers is negative and the other is positive. Their sum is positive (+5). We therefore test the pairs with one number being positive, the other negative.
The possible combinations of the two numbers are:

-4 and 1
1 and -4
4 and -1
-1 and 4
2 and -2
-2 and 2

We start by using -4 and 1. Since this is essentially trial and error, you can start with any pair.

$6x^2 + 5x – 4 = (3x - 4) (2x + 1)$

We expand the brackets to check the answer

$= 3x\times2x + 3x\times1 + -4\times2 x +- 4\times1$
$= 6x^2 + 3x + -8x + -4$
$= 6x^2 + 3x - 8x - 4$
$= 6x^2- 5x - 4$

This is **not** the correct factorisation.

Similarly, (3x + 1) (2x – 4) does not work.

Next we try 4 and -1

$6x^2 + 5x - 4 = (3x + 4)(2x - 1)$

We expand the brackets to check the answer

$$= 3x\times2x + 3x\times-1 + 4\times2x + 4\times-1$$
$$= 6x^2 + -3x + 8x + -4$$
$$= 6x^2 - 3x + 8x - 4$$
$$= 6x^2 + 5x - 4$$

This is the correct factorisation.

Activity 5

Factorise, where possible, the following:

1. $2x^2 - x - 3$

2. $2x^2 - 3x - 5$

3. $3x^2 - 13x - 10$

4. $6p^2 - 18p + 12$

5. $9x^2 - 12x + 4$

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on factorisation are:

When given an expression to factorise, try the various methods of factorisation in succession

take out any common factors
look for familiar patterns
use the method of trial and inspection

when you have factorised, and you then multiply these factors (brackets) together, you must end up with what you had originally

not every expression can be factorised

You have now completed the last subunit of this unit on algebraic manipulation. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Answers to Activity 1

1. $3x - 6 = 3(x - 2)$
2. $6x + 4 = 2(3x + 2)$
3. $6y^2 - 4y = 2y(3y - 2)$
4. $10xy + 5xz = 5x(2y + z)$
5. $3x^2y - 9x^3y^3 = 3x^2y\ (1 - 3xy^2)$
6. $8a^3bc - 12ab^3cd + 4b^4c^2d^2 = 4c(2a^3b - 3ab^3d + 4b^4cd^2)$

Answers to Activity 2

1. $ac + ad + 2bc + 2bd = (ac + ad) + (2bc + 2bd)$
   $= a(c + d) + 2b(c + d)$
   $= (c + d)\ (a + 2b)$

2. $km - kn + lm - ln = (km - kn) + (lm - ln)$
   $= k(m - n) + l(m - n)$
   $= (k + l)\ (m - n)$

3. $3xy + 21x - 2y - 14 = (3xy + 21x) - (2y + 14)$
   $= 3x(y + 7) - 2(y + 7)$
   $= (3x - 2)\ (y + 7)$

4. $3x^2 + 6x - 4x - 8 = (3x^2 + 6x) - (4x + 8)$
   $= 3x(x + 2) - 4(x + 2)$
   $= (3x - 4)\ (x + 2)$

5. $a^3 - 3a^2 + 6a - 12 = (a^3 - 3a^2) + (6a - 12)$
   $= a^2\ (a - 3) + 6(a - 2)$
   $= (a^2 + 6)\ (a - 2)$

6. $x^3 - x^2 + x - 1 = (x^3 - x^2) + (x - 1)$
   $= x^2\ (x - 1) + 1(x - 1)$
   $= (x^2 + 1)\ (x - 1)$

Answers to Activity 3

1. $x^2 – 49 = (x – 7) (x + 7)$

2. $2x^2 – 50 = 2(x^2 – 25)$
   $= 2[(x – 5) (x + 5)]$

3. $x^2 – 16 x^2 = (x – 4x) (x + 4x)]$

4. $4a^2 – 64b^2 = (2a – 8b) (2a + 8b)$

5. $1 – 100y^2 = (1 – 10y) (1 + 10y)$

6. $x^3 – 16x = x(x^2 – 16)$
   $= x[(x – 4) (x + 4)]$

7. $9a^2 – 16x^2 = (3a – 4x) (3a + 4x)$

Answers to Activity 4

1. $x^2 – 5x + 6 = (x – 3) (x - 2)$
2. $x^2 – 5x – 6 = (x – 6) (x + 1)$
3. $x^2 + 5x + 6 = (x + 3) (x + 2)$
4. $x^2 + 2x - 15 = (x - 3) (x + 5)$
5. $x^2 + 12x + 36 = (x + 6) (x + 6) = (x + 6)^2$
6. $x^2 – 2x - 1$ cannot be factorised

Answers to Activity 5

1. $2x^2 – x – 3 = (2x + 1) (x - 3)$
2. $2x^2 – 3x – 5 = (2x - 5) (x + 1)$
3. $3x^2 – 13x – 10 = (3x + 2) (x - 5)$
4. $6p^2 – 18p + 12 = (3p - 3) (2p - 4)$
5. $9x^2 – 12x + 4 = (3x - 2) (3x - 2) = (3x - 2)^2$

Unit Summary

Summary

In this unit you learned that:

- distributive property is the key tool in multiplying expressions.
- expansion is the reverse of factorization.
- factorisation is the reverse of multiplying out, or expansion.
- factorisation can be done.

  (a) by taking out a common factor

  (b) by grouping

  (c) with patterns: square of a sum, square of a difference and the difference of two squares

  (d)with trial and inspection method

- some expressions cannot be factorised.

You have completed the material for this unit on algebraic manipulation. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

1. **Answer All Questions.**
2. **Show all the necessary working.**

Total marks = 50 **Time**: 1hour

1. **Work out**

(a) $x(x - 7)$ [1]

(b) $9xy(p + q + r)$ [2]

(c) $(4s + t)6s$ [2]

(d) $-2a(3 - 5a)$ [2]

(e) $(x + 8)(x + 5)$ [2]

(f) $(2a - 3b)(a + 4b)$ [2]

(g) $(x - 5)^2$ [2]

(h) $(5h + k)^2$ [2]

(i) $(3x + 4)(x - 2)$ [2]

Factorise the following:

(a) $9x + 3y$ [1]

(b) $12cm + 16dm$ [2]

(c) $2pq - 6q^2$ [3]

(d) $x + y - ax - ay$ [3]

(e) $ax^2 - a y^2 + bx^2 - by^2$ [3]

(f) ah + bh + ch + ap + bp + cp [3]

(g) $3x^2 - 6x + x - 2$ [3]

(h) $x^3 - 3 x^2 + x - 3$ [3]

(i) $x^2 - 2x - 3$ [3]

(j) $20x^2 - 17x - 3$ [3]

(k) $6x^2 - 5x + 1$ [3]

(l) $2x^2 - 19x + 35$ [3]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

(a) $x(x - 7) = x^2 - 7$
(b) $9xy(p + q + r) = 9xyp + 9xy\ q + 9xyr$
(c) $(4s + t)6s = 24s^2 + 6ts$
(d) $-2a(3 - 5a) = -6a + 10a^2$
(e) $(x + 8)\ (x + 5) = x^2 + 8x + 5x + 40$

$= x^2 + 13x + 40$

(f) $(2a - 3b)\ (a + 4b) = 2a\times a + 2a\times 4b - 3b\times a - 3b\times 4b$

$= 2a^2 + 8ab - 3ba - 12b^2$

$= 2a^2 + 8ab - 3ba - 12b^2$

$= 2a^2 + 5ab - 12b^2$

(g) $(x - 5)^2 = (x - 5)(x - 5)$
$= x^2 - 5x - 5x + 25$
$= x^2 - 10x + 25$

(h) $(5h + k)^2 = (5h + k)(5h + k)$
$= 25h^2 + 5hk + 5kh + k^2$
$= 25h^2 + 10hk + k^2$

(i) $(3x + 4)(x - 2) = 3x^2 - 6x + 4x - 8$
$= 3x^2 - 2x - 8$

2.

(a) $9x + 3y = 3(3x + y)$

(b) $12cm + 16dm = 4m(3c + 4d)$

(c) $2pq - 6q^2 = 2q(p - 3q)$

(d) $x + y - ax - ay = x - ax + y - ay$
$= (x - ax) + (y - ay)$
$= x(1 - a) + y(1 - a)$
$= (1 - a)(x + y)$

(e) $ax^2 - a y^2 + bx^2 - by^2 = ax^2 + bx^2 - ay^2 - by^2$
$= (ax^2 + bx^2) - (a y^2 + by^2)$
$= x^2(a + b) - y^2 (a + b)$
$= (x^2 - y^2)(a + b)$

(f) $ah + bh + ch + ap + bp + cp$
$= (ah + bh + ch) + (ap + bp + cp)$
$= h(a + b + c) + p(a + b + c)$
$=(h + p)(a + b + c)$

(g) $3x^2 - 6x + x - 2 = (3x^2 - 6x) + (x - 2)$
$= 3x(x - 2) + (x - 2)$

$= (3x + 1)(x - 2)$

(h) $x^3 - 3x^2 + x - 3 = (x^3 - 3x^2) + (x - 3)$

$= x^2(x - 3) + 1(x - 3)$

$= (x^2 + 1)(x - 3)$

(i) $x^2 - 2x - 3 = (x + 1)(x - 3)$

(j) $20x^2 - 17x - 3 = (20x + 3)(x - 1)$

(k) $6x^2 - 5x + 1 = (2x - 1)(3x - 1)$

(l) $2x^2 - 19x + 35 = (2x - 5)(x - 7)$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

1. **Answer All Questions.**
2. **Show all the necessary working.**

Total marks = 42 **Time**: 45 minutes

1. Simplify

(a) $2(x + y - z)$ [1]

(b) $a(x^2 + x - 3)$ [2]

(c) $\frac{3}{4}(4x - 12y - 8y)$ [3]

(d) $x(a + b) - y(b - c)$ [3]

(e) $(b - c)(x + y)$ [3]

(f) $(k + 1)(2k + 1)$ [3]

(g)$(3a + 2)^2$ [3]

(h) $2x - 7)^2$ [3]

2. Factorise the following:

(a) $3x – 6$ [1]

(b) $a(x – 2) + b(x – 2)$ [2]

(c) $3x^2 - 12$ [3]

(d) $(a - b)^2 - c^2$ [3]

(e) $9 - 36x^2$ [3]

(f) $x^2 - 15x + 50$ [3]

(g) $x^2 + 4x + 4$ [3]

(h) $9x^2 - 4$ [3]

Answers

1.Simplifying

(a) $2(x + y - z) = 2x + 2y - 2z$
(b) $a(x^2 + x - 3) = ax^2 + ax - 3a$
(c) ¾ $(4x - 12y - 8y) = 3x - 9y - 6y$
(d) $x(a + b) - y(b - c) = ax + bx - by + cy$
(e) $(b - c)(x + y) = bx + by - cx - cy$
(f) $(k + 1)(2k + 1) = 2k^2 + 3k + 1$
(g) $(3a + 2)^2 = 9a^2 + 12a + 4$
(h) $(2x - 7)^2 = 4x^2 - 28x + 49$

2. Factorising

(a) $a(x - 2) + b(x - 2) = (a + b)(x - 2)$
(b) $3x^2 - 12 = 3(x^2 - 4)$
(c) $3x - 6 = 3(x - 2)$
(d) $(a - b)^2 - c^2 = [(a - b) - (c)][(a + b) + (c)]$
(e) $9 - 36x^2 = (3 - 6x)(3 + 6x)$
(f) $x^2 - 15x + 50 = (x - 5)(x - 10)$
(g) $x^2 + 4x + 4 = (x + 2)^2$
(h) $9x^2 - 4 = (3x - 2)(3x + 2)$

Unit Contents

Unit 5

Unit 5

Linear Equations

Introduction

Linear equations are part of our everyday life. Consider the following situation; you have 5 books in your bag, someone comes and puts x more book in your bag so that the total is now nine. How many books have been put in your bag? 4 because 5+4=9. Now your equation is 5+ x=9. Once you use an equal sign or once you say equals to, you already have your equation.

In this unit we are coming to work with linear equations. You will soon know what kind of equations are linear equations.

This unit consists of 106 pages. This unit is about 5% of the whole course, so plan your time accordingly. As reference, you will need to devote 35 hours to work on this unit, 25 hours for formal study and 10 hours for self-study and completing assessments/assignments.

In unit 4 you worked with algebraic manipulations. Algebraic manipulations are a direct introduction to linear equations. In this unit you will need almost all the information you gathered in unit 4. In mathematics information and skills builds upon itself.

This Unit is Comprised of Five Lessons:

Lesson 1 Problem of an Equation
Lesson 2 Solving Linear Equations
Lesson 3 Equations with Numerical Denominators
Lesson 4 Equations with Algebraic Denominators
Lesson 5 Simultaneous Linear Equations

Outcomes

Upon completion of this unit you will be able to:

- *define* the term 'linear equation' and explain how it differs from other types of equations;
- *give* examples of where linear equations can be used in contexts drawn from everyday life;
- *solve* linear equations involving fractions.
- *solve* simultaneous equations with two unknowns by the elimination method.
- *solve* simultaneous equations with two unknowns by substitution method.
- *solve* simultaneous equations with two unknowns by matrix method.
- *solve* simultaneous equations by graphing.

Terminology

Linear equation: An algebraic equation in which each term is either a constant or a product of a constant and a variable and the unknown variable has the highest power as one.

Simultaneous equations: A set of algebraic equations which will be satisfied by one solution set at the same time.

Elimination method: An algebraic method of solving linear equations where a variable with equal coefficients is removed from the equation by addition, or subtraction of the equations. When the coefficients are not equal the two equations are multiplied by appropriate factors to get equations (multiples) which have equal coefficients in one of the variables, then elimination is carried as usual where coefficients are equal.

Substitution method: An algebraic method of solving linear equations where one variable is expressed in terms of the other in one of the equations, then this expression is substituted for the variable in the other equation, the remaining variable is solved.

Matrix method: An algebraic method of solving linear equations where the equations are written in matrix form and rules of matrices operations are applied to find the solution to the original equations.

Numerical denominators: A bottom number in a fraction which is a counting number.

Algebraic denominators: a bottom number in a fraction which is an algebraic expression.

LCM Method: An algebraic and arithmetic method where two (or more) fractions are added or subtracted by first changing them to their equivalent fractions with the same denominator (this denominator is the lowest common multiple).

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
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Lesson 1 Problem of an Equation

Mathematics in general is about describing relationships and patterns. For you to be able describe a relationship or a pattern you should develop an eye to recognise the structure in situations you are faced with. How does the structure of the situation help? The recognition of the structure of the situation will help you to form a generalised statement of the situation. Once you are able to form a generalised statement you can use mathematics to solve the problem you are faced with.

How can you use equations in everyday life situations? The answer will be provided as you read further.

Upon completion of this sub-unit you should be able to:

- *define* the term 'linear equation' and explain how it differs from other types of equations;
- *give* examples of where linear equations can be used in contexts drawn from everyday life;

In Unit 4 you learnt about terms and expressions. There are times when two expressions are linked with an equal sign. The expression with an equal sign is called an **equation**. In an equation, at least one unknown (variable) must be present.

For example,

$4b + 6 = 26$,

$n - 2 = 8$,

$x + \frac{1}{4} = 1\frac{1}{2}$,

$x + 2y = 3$,

$1.8x - 0.4y = 10.6$

are all equations. But,

$3 + 4 = 7$,

$6 + 6 = 6 + 6$,

$4a + 3b$

are **not** equations. The first two have no unknowns (variable(s)), the third has **no equal sign**, it is an expression.

For an equation to be a **linear equation**, it has to have terms of the power 1only.

For example,

$3x + 7 = 12$ is a linear equation in one unknown (x),

$\frac{3}{2a} = 5$ is a linear equation in one unknown (a),

$2x + 3y = 6$ is a linear equation in two unknowns (x *and* y).

But $x^2 + 3x - 4 = 0$ is **not** a linear equation as it contains x^2 (x to the second power).

Linear equations can be used almost every where in real life where the situation can be represented (modelled) by a relationship of some kind. The following examples will highlight this. These are just some of the possibilities not all possibilities where linear equations can be used.

Example 1

A packet of sweets costs M p and a packet of chocolate costs M2.50 more. The total cost is M10.50. Calculate the cost of each packet.

Solutions to Example 1

A packet of sweets costs M p and a packet of chocolate costs M2.50 more. The total cost is M10.50. Calculate the cost of each packet.

Let the cost of packet of sweets be represented by c.

So, you can form an equation:

Total cost = cost of packet of sweets + cost of packet of chocolate

$$M10.50 = Mp + (Mp + M2.50)$$

$$10.50 = p + (p + 2.50)$$

$$2p + 2.50 = 10.50$$

$$2p + 2.50 - 2.50 = 10.50 - 2.50$$

$$2p = 8.00$$

$$\frac{2p}{2} = \frac{8.00}{2}$$

$$p = 4.00$$

The cost of packet of sweets = M4.00

The cost of packet of chocolate = Mp + M2.50
= M4.00 + M2.50
= M6.50

Example 2

The length of a rectangular door mat is $(2 + x)$ cm and the width is 59 cm. If the area is 6018 cm^2. Find the length.

Solution to Example 2

The equation which connects the area of the rectangle to the size of its sides is

$$\text{Area of rectangle} = lw$$

$$6018\ cm^2 = (2 + x)cm \times 59\ cm$$

$$6018 = 59(2 + x)$$

$$6018 - 118 = 118 - 118 + 59x$$

$$5900 = 59x$$

$$\frac{5900}{59} = \frac{59x}{59}$$

$$100 = x$$

The length of the mat = (2 + x) cm

= (2 + 100) cm

= 102 cm

Example 3

A tourist is using a vehicle to tour the country. After three days he found that the total distance travelled is 495 km. He travelled $1\frac{1}{2}$ times farther on the second day and twice as far as the first day on the third day. How far did he travel on each day?

Solution to Example 3

Let d be the number of kilometres travelled on the first day.

So, you can form an equation

$$\text{total distance travelled} = \text{day 1 distance} + \text{day 2 distance} + \text{day 3 distance}$$

$$495\ km = d\ km + 1\tfrac{1}{2}d\ km + 2d\ km$$

$$495\ km = 4\tfrac{1}{2}\ d\ km$$ (collect like terms)

$$\frac{495\ km}{4\frac{1}{2}} = \frac{4\frac{1}{2}d\ km}{4\frac{1}{2}}$$ $\left(495 \times \frac{2}{9} = \frac{990}{9} = 110\right)$

$$110\ km = d\ km$$

For day 1 distance travelled = 110 km

For day 2 distance travelled = $1\frac{1}{2}d$ km

$= 1\frac{1}{2} \times 110$ km $\quad (\frac{3}{2} \times 110 = \frac{330}{2} = 165)$

$= 165$ km

For day 3 distance travelled = $2d$ km

$= 2 \times 110$ km

$= 220$ km

It is good to look and get guidance from worked examples, but the old Chinese saying says 'I hear I forget, I see I remember, I do I understand' meaning that getting the chance to do yourself will be more helpful than just seeing how to do. Therefore, turn the word problems in activity 5.1 into linear equations then solve them to be able to answer the questions asked.

Activity 5.1

Activity

Solve the following by first forming an equation from the given problem.

a) 2 times a certain number increased by 6 is 8. What is the number?

b) Palesa's mass is 10kg more than her brother's mass. The sum of their masses is 110kg. Find the mass of each.

c) When a number is divided by 2, the answer is the same as when 15 is subtracted from 3 times the number. What is the number?

d) 4 straws of length x cm and 3 straws of length 2x cm have to be cut from a 60 cm length of straw. What is the length of each type of straw?

Check your answers against the model answers provided below, if you are confident in the process of turning word problems into appropriate linear equations to help answering the associated questions then move to the next sub-unit of solving linear equations.

Model answers to activity 5.1

1. $\text{number} \rightarrow 2 \text{ times number} \rightarrow$
$2 \text{ times number and increase by } 6 \rightarrow$
$2 \text{ times number and increase by } 6 \text{ answer is } 8$

Let the number be represented by *n*.

$n \rightarrow 2n \rightarrow 2n + 6 \rightarrow 2n + 6 = 8$

To find the number you solve for *n*.

$2n + 6 = 8$

$2n + 6 - 6 = 8 - 6$ (subtract 6 both sides)

$2n = 2$

$\frac{2n}{2} = \frac{2}{2}$ (divide by 2 both sides)

$n = 1$

The number is 1.

2. $\text{Palesa's mass} + \text{her brother's mass} = 110 \text{ kg}$

Let the brother's mass be represented by *m*.

$m + (m + 10) = 110$

To find the mass of each, first find the value of *m*.

$$m + (m + 10) = 110 \qquad \textit{(multiply brackets)}$$

$$2m + 10 = 110$$

$$2m + 10 - 10 = 110 - 10 \qquad \text{(subtract 10 both sides)}$$

$$2m = 100$$

$$\frac{2m}{2} = \frac{100}{2} \qquad \text{(divide by 2 both sides)}$$

$$m = 50$$

So, Palesa's mass = 50 kg + 10 kg

= 60 kg

Brother's mass = 50 kg

3. *number → divide number by 2 → divide number by 2 equals 3 times number subtract 15*

Let number be represented by *n*.

$n \rightarrow \frac{n}{2} \rightarrow \frac{n}{2} = 3n - 15$

To find the number, find the value of *n*.

$$\frac{n}{2} = 3n - 15$$

$$2\left(\frac{n}{2}\right) = 2(3n - 15) \qquad \textit{(multiply by 2 both sides)}$$

$$n = 6n - 30$$

$$n - 6n = 6n - 6n - 30 \qquad \textit{(subtract 6n both sides)}$$

$$-5n = -30$$

$$\frac{-5n}{-5} = \frac{-30}{-5} \qquad \textit{(divide by 5 both sides)}$$

$$n = 6$$

The number is 6.

4. $4(x \text{ cm}) + 3(2x \text{ cm}) = 60 \text{ cm}$

$4x + 6x = 60$

To find the length of each type, first find value of *x*.

$$4x + 6x = 60 \quad \textit{(collect like terms)}$$

$$10x = 60$$

$$\frac{10x}{10} = \frac{60}{10} \quad \textit{(divide by 10 both sides)}$$

$$x = 6$$

4 straws are of 6 cm in length,

3 straws are of 12 cm in length ($2 \times 6\ cm = 12\ cm$)

From the activity, generally the process of solving real life problems using equations is:

1. Decide on the unknown quantity and represent it by a letter. Remember to state the units being used where possible.
2. Form an equation which generalises the facts given in the problem about the unknown quantity,
3. Solve the equation then answer the associated question(s) by using the solution of the equation.

Now, you are at the end of the sub-unit problem to an equation, move to the next sub-unit of solving linear equations which looks at some of the methods used to solve similar type of equations.

Lesson 2 Solving Linear Equations

The skill of solving the linear equation is very important because it is used across a number of fields. For instance, it can be used in problems occurring in business, industry and science. Consider the following situation.

In Lesotho, like in many developing countries, the government is establishing infrastructure for investors to come and invest in industries with the purpose of creating jobs for the citizens, especially textile industries. However, the investors agree to come and establish the industries (factories) with the main purpose of making profit in the business.

Let us say that it will require two million Maloti (M) to equip the factory and to start up the production line for a certain type of jeans (casual trousers). The recurrent cost (for labour, materials and overheads) of producing one pair of jeans is thirty Maloti. The owners and managers of the business will then decide the price at which the jeans are sold to suppliers so that a profit is made.

Suppose each pair of jeans is sold to a wholesale supplier for seventy Maloti. The owners and managers of the business need to know how many pairs of jeans will have to be sold to recover both the initial cost of getting the factory running and the recurrent cost of producing each pair of jeans? In business this is referred to as the "breakeven point" – the stage where income covers all expenses. At the breakeven point, the business is not yet making any profit, but neither is it losing money. What happens is that you recover the expenses but do not make extra money (profit).

This is the kind of situation in life where you can do your business calculations (homework) even before you run a losing battle in your business.

Let us now look at how you can use your knowledge and skills with linear equations to find the information necessary to make informed decisions.

Let *j* represent the number of jeans sold.

Remember, in order to break even, the total cost of producing jeans must be equal to the income from sales of the jeans.

The cost of producing jeans $= 2\ 000\ 000 + 30j$

Cash from sales of j jeans $= 70j$

At the break-even point you have:

Income from sales of *j* jeans = the cost of producing jeans

$70j = 2\ 000\ 000 + 30j$ (subtract 30*j* both sides)

$70j - 30j = 2\ 000\ 000$

$40j = 2\ 000\ 000$ (divide by 40 both sides)

$\frac{40j}{40} = \frac{2\ 000\ 000}{40}$

$j = 50\ 000$

- In order to break even, 50 000 pairs of jeans have to be sold.
- This also means that the business will make a profit on every pair of jeans that it can sell above the breakeven point of 50 000.

Notice that the equation used here is a linear equation (the highest power of the variable *j* is 1). This is the kind of equation that can be used to calculate the information needed for small businesses and income-generating activities, such as selling fruits and vegetables.

First let us use the four basic operations to solve one-step equations. Basically, one-step equations are solved simply by doing the inverse operation to both sides of the equation.

A. Addition

Example

Solve the following equations

a) $a - 5 = -12$

b) $d - 3.5 = 19$

Solution

a) $a - 5 = -12$

$\underline{\quad +5 = \quad +5}$

$a \quad = \quad -7$

b) $d - 3.5 = 19$

$\underline{+3.5 = +3.5}$

$d \quad = 21.5$

B. Subtraction

Example

Solve the following equations

a) $y + 2.1 = 23$

b) $m + 8 = -25$

Solution

a) $y + 2.1 = 23$

$\underline{\quad -2.1 = -2.1}$

$Y \quad = 20.9$

b) $m + 8 = -25$

$\underline{\quad -8 = -8}$

$m \quad = -33$

C. Multiplication

Example

Solve the following equations

a) $\frac{p}{-8} = -3$

b) $\frac{2}{3}(a) = 4$

Solution

a) $\frac{p}{-8} = -3$

multiply both sides by -8

$\frac{-8}{1} \times \frac{p}{-8} = -3 \times -8$

Simplify both sides

$p = 24$

b) $\frac{2}{3}(a) = 4$

multiply both sides by $\frac{3}{2}$

$\frac{3}{2} \times \frac{2}{3}(a) = 4 \times \frac{3}{2}$

$a = 6$

D.Division

Example

Solve the following equation.

$5x = -31$

Solution

$5x = -31$

Divide both sides by 5.

$\frac{5x}{5} = \frac{-31}{5}$

Simplify both sides.

$x = -6\frac{1}{5}$

The steps above can be combined to solve longer equations. Do not be afraid of words longer equations. They only mean the equations which require more than one operation to arrive at the solution. The working is still similar to what you have seen above. However, to help you along, examples below will take you slowly showing step by step working.

Example

If the task says solve the following equations, then it requires you to find the value of the variable that will make the statement true. The worked examples below will show you how to combine the operations when solving an equation.

a) $5 - 3x = 26$

Solution

$5 - 3x = 26$

Subtract 5 from both sides of the equation.

$5 - 5 - 3x = 26 - 5$

Simplify both sides.

$-3x = 21$

Divide both sides by -3

$\frac{-3x}{-3} = \frac{21}{-3}$

Simplify both sides

$x = -7$

Note that the same equation could also be done by choosing to start with division

$$5 - 3x = 26$$

$\frac{(5-3x)}{3} = \frac{26}{3}$ (divide both side by 3)

$\frac{5}{3} - \frac{3x}{3} = \frac{26}{3}$ (simplify division where possible)

$\frac{5}{3} - x = \frac{26}{3}$ (subtract $\frac{5}{3}$ both sides)

$\frac{5}{3} - \frac{5}{3} - x = \frac{26}{3} - \frac{5}{3}$ (simplify the subtraction)

$-x = \frac{21}{3}$ (multiply by -1 both sides)

$x = \frac{-21}{3}$ (simplify the division)

$$x = -7$$

In the first approach subtraction was done first followed by division, in the second approach division was done first followed by subtraction then multiplication. Which approach look easier to follow to reach to the solution?

b) 7 (b – 1) = 21

Solution

7 (b – 1) = 21

Divide both sides by 7

$$\frac{7(b-1)}{7} = \frac{21}{7}$$

Simplify both sides.

b – 1 = 3

Add 1 to both sides of the equation.

b – 1+1 = 3 + 1

Simplify both sides.

b = 4

Even here the equation could have be worked as follows:

$7(b - 1) = 21$ (multiply the bracket)

$7b - 7 = 21$

$7b - 7 + 7 = 21 + 7$ (add 7 both sides)

$7b = 28$

$\frac{7b}{7} = \frac{28}{7}$ (divide by 7 both sides)

$b = 4$

Is there a significant difference between the two approaches? Any noticeable difficulty or more working?

c) 5n + 2 = 2n +17

Solution

5n + 2 = 2n + 17

Subtract 2n from both sides

$5n + 2 - 2n = 2n + 17 - 2n$

Simplify both sides.

$3n + 2 = 17$

Subtract 2 from both sides

$3n + 2 - 2 = 17 - 2$

Simplify both sides.

$3n = 15$

Divide both sides by 3

$$\frac{3n}{3} = \frac{15}{3}$$

Simplify both sides.

$n = 5$

d) $1 - 3(p+1) = p - (2p - 1)$

Solution

$1 - 3(p+1) = p - (2p - 1)$

Remove brackets

$1 - 3p - 3 = p - 2p + 1$

Simplify both sides.

$-3p - 2 = -p + 1$

Add 2 to both sides

$-3p - 2 + 2 = -p + 1 + 2$

Simplify both sides.

$-3p = -p + 3$

Add p to both sides

$-3p + p = -p + 3 + p$

Simplify both sides.

$-2p = 3$

Divide both sides by -2

$$\frac{-2p}{-2} = \frac{3}{-2}$$

Simplify both sides.

$$p = \frac{-3}{2}$$

You will find that many times it is easiest to solve by starting with addition or subtraction before multiplication or division.

Now, can you use the same skills to work out the equations in activity 5.2.

Activity

Activity 5.2

Solve the following equations

a) $7p - 8 = 6$

b) $4(3a + 2) = 20$

c) $8s = 28 + s$

d) $6 - 5r = 19r$

e) $18u - 6 = 2u + 12$

f) $6t + 2 = 2t + 8$

g) $3a - 5 = 6 - 2(3 - a)$

I hope up to now you have continued to use the skills in the activity, you can check your work against the model answers provided below. Remember that if your work looks different it does not mean your work is wrong.

Answers to Activity 5.2

a) $7p - 8 = 6$

7p – 8 + 8 = 6 + 8

$7p = 14$

7p ÷ 7 = 14 ÷ 7

p = 2

b) $4(3a + 2) = 20$

12a + 8 = 20

12a + 8 – 8 = 20 – 8

12a ÷ 12 = 12 ÷ 12

a = 1

c) $8s = 28 + s$

8s – s = 28 + s – s

7s ÷ 7 = 28 ÷ 7

s = 4

d) $6 - 5r = 19r$

6 – 5r + 5r = 19r + 5r

6 ÷ 24 = 24r ÷ 24

$\frac{1}{4} = r$

e) 18u – 6 + 6 = 2u + 12 + 6

18u – 2u = 2u – 2u + 18

16u ÷ 16 = 18 ÷ 16

f) 6t – 2t + 2 = 2t – 2t + 8

4t + 2 – 2 = 8 – 2

4t ÷ 4 = 6 ÷ 4

$u = \frac{9}{8}$ $\qquad\qquad$ $t = \frac{3}{2}$

g) $3a - 5 = 6 - 2(3 - a)$

$3a - 5 = 6 - 6 + 2a$

$3a - 2a - 5 = 2a - 2a$

$a - 5 + 5 = 0 + 5$

$a = 5$

You have done a lot up to here, now let us move to the equations with numerical denominators and use the same skills again.

Lesson 3 Equations with Numerical Denominators

Most of the work you have been doing in solving equations has been to balance the equation by doing the same operation on both sides. You will continue to use balancing, but the equations are going to change slightly.

You have seen one situation in business where a problem gives an equation. The fact is, there are many situations which can give equations in life. Consider a wealthy man who has made a will (a written document instructing how the wealth is to be given to chosen beneficiaries and certified by lawyer) where the beneficiaries are the child to be born and his wife (the wife was expecting a child at the time the will was written); the child is to receive two-thirds of the wealth if it is a boy and the wife one-third of the wealth. But, if it is a girl, is to receive one-third and the wife two-thirds. When the wife gives birth, there were twins a boy and a girl. How should the wealth be divided to satisfy the will?

At the end of this sub-unit you should be able to:

- define the term 'numerical denominators',
- identify equations with numerical denominators,
- Solve linear equations involving numerical denominators.

There are six pages in this sub-unit.

In unit 2 you learnt about Types of numbers. Can you recall which type of numbers have denominators? If you need help to remember review work done in unit 2.

In the will problem above, if the newly born child would be a boy he would be entitled to two-thirds of the wealth $\left(\frac{2}{3}\right)$. The number indicates that the wealth is divided into three equal parts by the bottom number. Do you still remember the name of this bottom number? The parts the boy will get are indicated by the top number. What is the name given to this top number?.

The three at the bottom is the numerical denominator because three is a counting number.

For example, $\frac{2}{3}$, $\frac{1}{6}$ are fractions with numerical denominators because the bottom numbers 3 and 6 are counting numbers.

Example 1

$$\frac{2r}{3} = -4$$

Multiply both sides by 3

$$\frac{2r}{3} \times 3 = -4 \times 3$$

Simplify both sides.

$$2r = -12$$

Divide both sides by 2

$$\frac{2r}{2} = \frac{-12}{2}$$

Simplify both sides.

$$r = -6$$

Example 2

$$\frac{3x-1}{4} = 5$$

Multiply both sides by 4

$$\frac{3x-1}{4} \times 4 = 5 \times 4$$

Simplify both sides.

$$3x - 1 = 20$$

Add 1 to both sides

$$3x - 1 + 1 = 20 + 1$$

Simplify both sides.

$$3x = 21$$

Divide both sides by 3

$$\frac{3x}{3} = \frac{21}{3}$$

Simplify both sides.

$x = 7$

Example 3

$\frac{y+5}{3} = 17$

Multiply both sides by 3

$\frac{y+5}{3} \times 3 = 17 \times 3$

Simplify both sides.

$y + 5 = 51$

Subtract 5 from both sides

$y + 5 - 5 = 51 - 5$

Simplify both sides.

$y = 46$

Example 4

$\frac{3(a-1)}{5} = 6$

Multiply both sides by 5

$\frac{3(a-1)}{5} \times 5 = 6 \times 5$

Simplify both sides

$3a - 3 = 30$

Add 3 to both sides

$3a - 3 + 3 = 30 + 3$

Simplify both sides

$3a = 33$

Divide both sides by 3

$$\frac{3a}{3} = \frac{33}{3}$$

$a = 11$

Example 5

$\frac{2p}{3} + 1 = \frac{7p}{9}$

Multiply each term in the equation by 6 (the LCM of the denominators)

$$\frac{2p}{3} \times 6 + 1 \times 6 = \frac{7p}{2} \times 6$$

Simplify both sides

$$4p + 6 = 21p$$

Subtract 4p from both sides of the equation

$$4p + 6 - 4p = 21p - 4p$$

Simplify both sides

$$6 = 17p$$

Divide by 17 on both sides

$$\frac{6}{17} = \frac{17p}{17}$$

Simplify both sides

$$\frac{6}{17} = p \quad \left(\text{or } p = \frac{6}{17}\right)$$

Activity

Activity 5.3

Solve the following equations. Show all of the work you do to arrive at an answer.

a) $\frac{s+8}{3} = 5$

b) $\frac{x-2}{4} = \frac{x-3}{3}$

c) $\frac{2d}{7} + \frac{2}{3} = 1$

d) $\frac{r}{8}+3=7$

e) $\frac{(b-1)}{5}-\frac{(2-b)}{6}=\frac{1}{3}$

f) $\frac{x+2}{4}=1+\frac{x-2}{2}$

g) $\frac{y-1}{2}-\frac{1}{3}=\frac{2y+1}{6}$

Model Answers

Activity 5.3

a) $\frac{c+8}{3}\times 3 = 5\times 3$ (multiply both sides by 3)

$c+8-8 = 15-8$ (subtract 8 both sides)

$c = 7$

b) $\frac{x-2}{4}\times 20 = \frac{x-3}{5}\times 20$ (multiply both sides by 20; LCM of denominators)

$5x-4x-10 = 4x-4x-12$ (subtract 4x both sides)

$x-10+10 = -12+10$ (add 10 both sides)

$x = -2$

c) $\frac{3d}{7}\times 21+\frac{2}{3}\times 21 = 1\times 21$ (multiply both sides by 21; LCM of denominators)

$9d+14-14 = 21-14$ (subtract 14 both sides)

$\frac{9d}{9} = \frac{7}{9}$ (divide by 9 both sides)

$d = \frac{7}{9}$

d) $\frac{e}{5}\times 5+3\times 5 = 7\times 5$ (multiply by 5 both sides)

$e+15-15 = 35-15$ (subtract 15 both sides)

$e = 20$

e) $\frac{(b-1)}{5}\times 30-\frac{(2-b)}{6}\times 30 = \frac{1}{3}\times 30$ (multiply by 30 both sides)

$6b-6-10+5b = 10$ (collect like terms)

$11b-16+16 = 10+16$ (add 16 both sides)

$\frac{11b}{11} = \frac{26}{11}$ (divide by 11 both sides)

$b = 1\frac{10}{11}$

f) $\frac{x+2}{4}\times 4 = 1\times 4+\frac{x-2}{2}\times 4$ (multiply by 4 both sides)

$$x - x + 2 = 4 + 2x - x - 4 \quad \text{(subtract x both sides)}$$

$$2 = x$$

g) $\frac{y-1}{2} \times 6 - \frac{1}{3} \times 6 = \frac{2y+1}{6} \times 6$ (multiply by 6 both sides)

$$3y - 2y - 3 - 2 = 2y - 2y + 1$$ (collect then subtract 2y both sides)

$$y - 5 + 5 = 1 + 5$$ (add 5 both sides)

$$y = 6$$

The important thing to remember is:

Get rid of the denominator by multiplying all the terms of the equation by the lowest common multiple of the denominators.

You have now completed the sub-unit on how to solve linear equations with numerical denominators using the LCM (Lowest Common Multiple of denominators) method. Do a quick review of the content then continue on to solving linear equations with algebraic denominators.

Lesson 4 Equations with Algebraic Denominators

The rules of operating the fraction equations with algebraic denominators are the same as for numerical fractions. Do you still remember the meaning of algebraic denominator? Just like in numerical fractions where we have the bottom number as a counting number or an integer; algebraic fractions have the bottom number as an algebraic expression.

At the end of this sub-unit you be able to:

- Solve linear equations involving algebraic denominators.

There are six pages on this sub-unit.

Linear equations occur quite often in our everyday life, they sometimes involve algebraic denominators not numerical denominators. Similar to equations with numerical denominators these algebraic denominators have to be cleared before one can solve the equations with ease. How are they cleared?

Let us start by looking at worked examples to illustrate the working process in solving linear equations with algebraic denominators.

Example 1

$\frac{4}{y+3} + \frac{9}{2} = 4$

Multiply each term by the LCM of the denominators which is 2(y+3)

$\frac{4}{y+3} \times 2(y+3) + \frac{9}{2} \times 2(y+3) = 4 \times 2(y+3)$

Simplify both sides by multiplying out the brackets

$8 + 9y + 27 = 8y + 24$

Simplify the left side of the equation

$9y + 35 = 8y + 24$

Subtract 8y from both sides.

$9y - 8y + 35 = 8y - 8y + 24$

Simplify both sides

$y + 35 = 24$

Subtract 35 from both sides of the equation

$y + 35 - 35 = 24 - 35$

Simplify both sides

$y = -11$

Example 2

$$\frac{5}{p-2} \quad \frac{1}{p-3} = \frac{2}{p-2}$$

Multiply each fraction by (p - 2)(p – 3)the LCM of the denominators

$$\frac{5(p-2)(p-3)}{p-2} - \frac{1(p-2)(p-3)}{p-3} = \frac{2(p-2)(p-3)}{p-2}$$

Simplify both sides

$$5(p-3) - (p-2) = 2(p-3)$$

Remove the brackets

$$5p - 15 - p + 2 = 2p - 6$$

Simplify left side of the equation

$$4p - 13 = 2p - 6$$

Subtract 2p from both sides of the equation

$$4p - 2p - 13 = 2p - 2p - 6$$

Simplify both sides

$$2p - 13 = -6$$

Add 13 to both sides

$4p - 13 + 13 = -6 + 13$

Simplify both sides

$$4p = 7$$

Divide both sides by 4

$\frac{4p}{4} = \frac{7}{4}$

Simplify both sides

$p = 1\frac{3}{4}$

Example 3

$$\frac{3}{x-3} = \frac{x}{x-3} - \frac{3}{2}$$

Multiply both sides by LCM of the denominators which is $2(x - 3)$

$\frac{3}{x-3} \times 2(x-3) = \frac{x}{x-3} \times 2(x-3) - \frac{3}{2} \times 2(x-3)$

Simplify both sides

$$6 = 2x - 3x + 9$$

Subtract 9 to both sides of the equation.

$$6 - 9 = -x + 9 - 9$$

Simplify both sides

$$-3 = -x$$

Divide by -1 on both sides

$$\frac{-3}{-1} = \frac{-x}{-1}$$

Simplify both sides

$$3 = x \quad (\text{or } x = 3\ )$$

Example 4

$$\frac{1}{a} - \frac{1}{2a} = 3$$

Multiply both sides by 2a (LCM of the denominators)

$$\frac{1 \times 2a}{a} - \frac{1 \times 2a}{2a} = 3 \times 2a$$

Simplify both sides

$$2 - 1 = 6a$$

Simplify left sides

$$1 = 6a$$

Divide both sides by 6

$$\frac{1}{6} = \frac{6a}{6}$$

Simplify both sides

$$\frac{1}{6} = a \quad \text{or} \quad a = \frac{1}{6}$$

Now it's time for you to try solving some problems on your own. Good luck working on the following equations!

Activity

Activity 5.4

Solve the following equations. Show all of the work you do to arrive at an answer.

a) $\frac{2}{f}+3=\frac{4}{f}$

b) $5-\frac{1}{2h}=4$

c) $\frac{7}{b+1}=\frac{3}{2}$

d) $\frac{7}{9}=\frac{3a-8}{a+1}$

e) $\frac{1}{x-1}=\frac{1}{2x+2}$

Model Answers

Activity 5.4

a) $\frac{2}{f} + 3 = \frac{4}{f}$ (multiply all terms by f)

$2 - 2 + 3f = 4 - 2$ (subtract 2 on both sides)

$\frac{3f}{3} = \frac{2}{3}$ (divide by 3 on both sides)

$f = \frac{2}{3}$

b) $3 - \frac{1}{2h} = 4$ (multiply all terms by 2h)

$6h - 6h - 1 = 8h - 6h$ (subtract 6h on both sides)

$-1 = 2h$ (divide by 2 on both sides)

$-\frac{1}{2} = h$ (or $h = \frac{-1}{2}$)

c) $\frac{7}{b+1} = \frac{3}{2}$ (multiply all terms by 2(b+1))

$14 = 3(b + 1)$ (multiply the bracket)

$14 - 3 = 3b + 3 - 3$ (subtract 3 on both sides)

$\frac{11}{3} = \frac{3b}{3}$ (divide by 3 on both sides)

$3\frac{2}{3} = b$

d) $\frac{7}{9} = \frac{3a-5}{a+1}$ (multiply all terms by 9(a+1) L.C.M)

$7(a + 1) = (3a - 5)9$ (multiply the brackets)

$7a + 7 = 27a - 45$ (subtract 7a on both sides)

$7a - 7a + 7 = 27a - 7a - 45$ (add 45 on both sides)

$\frac{52}{20} = \frac{20a}{20}$ (divide by 20 on both sides)

$a = 2\frac{3}{5}$

e) $\frac{1}{x-1} = \frac{1}{2x+5}$ (multiply all terms by $(x-1)(2x+5)$)

$2x + 5 = x - 1$ (subtract x on both sides)

$2x - x + 5 = x - x - 1$ (subtract 5 on both sides)

$x + 5 - 5 = -1 - 5$

$x = -6$

The important thing to remember is:

remove the denominator(s) by multiplying all the terms of the equation by the algebraic lowest common multiple of the denominators.

You have now completed the sub-unit on the method of solving linear equations involving algebraic denominators. Do a quick review of the content and then continue on to the next sub-unit, which presents a method for solving simultaneous linear equations with two unknowns

Lesson 5 Simultaneous Linear Equations

You have considered simple linear equations having only one unknown quantity (variable). However, there are some situations which will require more than one variable, and their solution will also require you to find the value of more than one variable.

In this sub-unit, we will only look at linear equations with two unknown quantities.

At the end of this sub-unit you should be able to:

- *solve* simultaneous equations with two unknowns by the elimination method.
- *solve* simultaneous equations with two unknowns by substitution method.
- *Solve* simultaneous equations with two unknowns by matrix method.
- *solve* simultaneous equations by graphing.

Generally, to find the solution of linear equations with two unknowns you will use similar working as when there is one unknown. However, the only important difference is that you need to form two equations from the information of the situation.

Equations with two unknowns (variables) (see Figure 5.1) are called simultaneous linear equations. Solving simultaneous linear equations requires you to find the values of the two unknowns at the same time, meaning the solution you get will be true for both equations. Some of the methods used to solve for the unknown quantities in simultaneous equations are elimination, substitution, matrix and graphing. We will look at the elimination method first, substitution second, matrix third, and graph method fourth.

A. The Elimination Method

To solve this type of equation, you should first eliminate one of the two unknowns. This means you manipulate the equation so that you can get rid of (or eliminate) one of the two unknowns. This leaves you with only one unknown variable, and you already know how to find the value of a single variable. Then, you continue working to find the value of the second unknown variable using the value of the first.

At the end of this sub-unit you should be able to:

- Solve simultaneous equations with two unknowns by the elimination method.

There are nine (9) pages on this sub-sub-unit on solving simultaneous linear equations in two unknowns by elimination.

The method of elimination works the same way as you were solving for one unknown, you have solved the equations when you have found the values of the two unknowns. The found values must make both equations true statements.

The key step is to make the coefficients (Figure 5.1) of one of the unknowns equal but different in signs in the two equations.

This is done by adding or subtracting one equation from the other. Addition and subtraction needed here was reviewed at the beginning of this unit.

However, this will not always be straightforward. In some cases you may have to multiply or divide one or both of the two equations by a number before you can add or subtract one from the other. This means you have to look at the equations, think, then decide what to do before you start manipulating them. In other words, plan what you are going to do before you do it, then follow your plan.

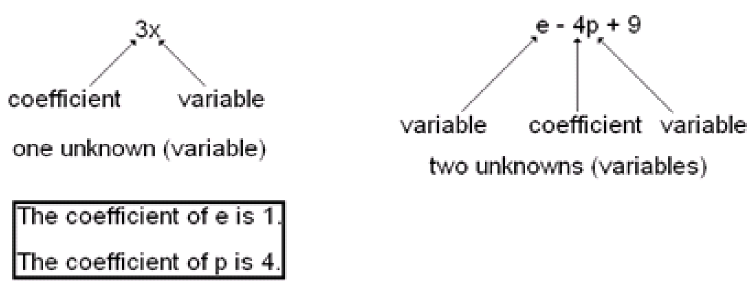


Figure 5.1 : coefficient and variable.

Example 1

Given the following equations, find the values of x and y.

$2x + y = 8$ (Equation 1)

$3x - 2y = 5$ (Equation 2)

Solution

The first step is to eliminate one of variables in the two equations. So you choose which one before you eliminate. How do you choose?

Note that there are two variables so the choice is really between *x and y only*. For now, , let us do both of them and to see how much work is needed for each choice.

To eliminate x in the two equations:

You look at the coefficient of *x in both equations.* In the first equation, the coefficient is 2, while in the second equation it is 3. Obviously these coefficients are not numerically equal. Since they are not already equal, you look for the LCM of 2 and 3. The LCM of 2 and 3 is 6. Now the task is to make 6 the coefficient of x in both equations.

To achieve this in the first equation, multiply the 2 which is already there by 3. That will give you 6. Note that you do not want to change the equations, so you must multiply all the terms in the first equation by 3. This gives you the following:

$6x + 3y = 24$ (Equation 3)

To achieve the same in the second equation, you need to multiply the 3 that is already there by 2 in order to make the coefficient of x equal to 6. Remember that, once you multiply one term in an equation by a number, you must multiply all the other terms by the same number. The result is the equation below:

$6x - 4y = 10$ (Equation 4)

Now, you have equal coefficients of x. The next step is to eliminate the x. Here again you have to make a choice to add the two equations or to subtract one from the other. If you add (Equation 3) and (Equation 4) together you get:

$(6x + 3y) + (6x - 4y) = 24 + 10$

$(6x + 6x) + (3y - 4y) = 34$

$12x - y = 34$ (Equation 5)

Have you eliminated the *x?* NO, there are still two variables. This means that adding the equations here was the wrong choice, because it will not give what you want, which is to eliminate one of the two unknown variables.

Now let us try subtracting one of the two equations from the other. If you subtract (Equation 4) from (Equation 3) you get:

$(6x + 3y) - (6x - 4y) = 24 - 10$

$(6x - 6x) + (3y + 4y) = 24$

$7y = 14$ (Equation 6)

Have you eliminated the *x?* Definitely YES, there is no *x* in Equation 6. You could also have subtracted Equation 3 from Equation 4. What would the result of that operation be?

$-7y = -14$

When we reverse the signs on both sides, this becomes:

$7y = 14$

Now you can continue to find the value of *y*. Remember you worked equations of this type at the beginning of the unit where you were solving one-step equations by using the inverse of the operation in the equation. Here in Equation 6, *y* is multiplied by 7. The inverse of multiplication is division. Therefore you choose to divide both sides by 7 to get:

$y = 2$ (Equation 7)

Next, you use the value of *y* to find the value of *x*. You will achieve that by using one of the starting equations, either Equation 1 or Equation 2. Using Equation 1 you get:

$2x + y = 8$

Substituting 2 for *y*, we get:

$2x + 2 = 8$

$2x = 6$

$x = 3$

Find the value of *x* using Equation 2.

Do you get the same value or is your answer different?

The solution is $x = 3$ and $y = 2$

It is a good practice to check your solution. How? *Take the values of x and y and substitute into any of the starting equations to see if you get a true statement!*

That is, in Equation 1 you get

$$2(3) + (2) = 8$$

$$6 + 2 = 8$$

$$8 = 8$$

The statement $8 = 8$ is true, so the solution is correct.

Check using the second equation at the start of this example (Equation 2).

Note that, the choice might have been to eliminate *y* not *x*. If you chose that option, your work would be as follows:

To eliminate *y* in the two equations, multiply all terms in the first equation by 2 (so that the coefficient of *y* will be equal in both equations, luckily they are also opposite)

$4x + 2y = 16$ (Equation 1)

$3x - 2y = 5$ (Equation 2)

Add the two equations (so that the sum of *y* coefficients is zero when you add; it cancels out):

$7x = 21$ (Equation 3)

Now you can realise that *y* has been eliminated – this is why this approach is called the elimination method.

Solve the above equation by dividing both sides by 7 (a one-step equation) to get:

$$x = 3$$

Substitute 3 for *x* in Equation 2

$$3(3) - 2y = 5$$

Solve the above equation for *y*

$$9 - 2y = 5$$

$$-2y = -4$$

$y = 2$

Therefore, $x = 3$ and $y = 2$

Check the solution obtained

First equation: Left hand side (LHS): $2(3) + 2 = 6 + 2 = 8$

Right hand side (RHS):8

Second equation: Left hand side: $3(3) - 2(2) = 9 - 4 = 5$

Right hand side: 5

LHS = RHS because $8 = 8$ and $5 = 5$ are true statements.

Therefore,

$x = 3$ and $y = 2$ is a solution.

The explanation above is rather long-winded. A lot of the steps shown above are not usually written down when solving equations. Instead, it is usually done in our heads as mental work. However, I have shown it here to clarify the whole process of solving linear equations.

When you are asked to solve simultaneous equations, you need not show how you can work using all the choices available. Instead, you just use one of the options. This means that different people can solve the same problem using different options. There is nothing wrong with this; as long as the process is correct; the result will be the same. However some choices may take a lot of working before you get to the solution.

For the next examples you will not see all the steps written down, but I hope you will remember the choices you have to make at each step and that there are other ways you could have gone about solving the equation.

Example 2

We will continue and look at another set of equations the steps are much similar but some details and options will be left for you to try for yourself.

$3x + 2y = 6$ (1)

$x - 3y = 2$ (2)

Solution

Multiply the first equation by 3 and the second equation by 2 (the choice is to eliminate y)

$9x + 6y = 18$

$2x - 6y = 4$

Add the two equations

$11x = 22$

Solve the above equation for x

$x = 2$

Substitute 2 for x in equation (2)

$2 + 3y = 2$

Solve the equation for y

$3y = 0$

$y = 0$

Therefore, $y = 0$ and $x = 2$

Example 3

Solve the following equations in two unknowns.

$y - 4x = -10$ (1)

$2y + 4x = 16$ (2)

Solution

Add the two equations (the signs of the coefficients of x are different and coefficients are equal)

$3y = 6$

Solve the above equation for y

$y = 2$

Put 2 for y in the first equation

$2 - 4x = -10$

Solve the equation for x

$-4x = -12$

$x = 3$

Therefore, $y = 2$ and $x = 3$

Example 4

$3x - 2y = 17$ (1)

$7x - 2y = 45$ (2)

Solution

The two equations have equal coefficients of y, but if you attempt to add them, will they cancel each other? No, but if you use what you learnt in unit 2 about directed numbers you subtract equation 1 from equation 2, as this will get rid of the y.

$$7x - 2y = 45$$

$$(3y \quad 2y = 17)$$

Subtract the two equations, the signs of coefficients of y are different and the coefficients are equal. The result is

$4x = 28$

Solve the above equation

$x = 7$

Put 7 for x in the second equation

$7(7) - 2y = 45$

Solve the equation above

$49 - 2y = 45$

$-2y = -4$

$y = 2$

Therefore, $x = 7$ and $y = 2$ is a solution.

Note, here in example 4 you had to multiply first equation by -1 before adding the two equations. Note, you would obtain same result if you subtract the first equation from the second equation instead of multiplying by -1 .

Now, look at example 5.

Example 5

$x + 3y = 26$ (1)

$x - 5y = -6$ (2)

Solution

Subtract the second equation from the first equation

$8x = 32$

Solve the equation above

$x = 4$

Put 4 for x in the second equation

$1(4) - 5y = -6$

Solve the equation above

$4 - 5y = -6$

$-5y = -10$

$y = 2$

Therefore, $y = 2$ and $x = 4$ is a solution

Now, move to example 6, i hope you are now getting the idea.

Example 6

$6x - 4y = 9$ (1)

$4x - 2y = 7$ (2)

Choosing to eliminate x , multiply equation (1) by 4 and multiply equation (2) by -6 to get

$24x - 16y = 36$ (3)

$-24x + 12y = -42$ (4)

Add the two equations to get

$-4y = -6$ (5)

Divide both sides by -4 to get

$\frac{-4y}{-4} = \frac{-6}{-4}$

$y = \frac{3}{2}$

$y = 1\frac{1}{2}$

Substitute $y = \frac{3}{2}$ into equation (2) to get value for *x*.

$4x - 2\left(\frac{3}{2}\right) = 7$

$4x - 3 = 7$

$4x = 10$

$x = \frac{5}{2}$

$= 2\frac{1}{2}$

Solution is $x = 2\frac{1}{5}$ and $y = 1\frac{1}{5}$.

Example 7

$7x + 2y = 22$ (1)

$2x + 3y = -1$ (2)

Choosing to eliminate y, multiply equation (1) by 2 and equation (2) by 7 to get

$14x + 4y = 44$ (3)

$14x + 21y = -7$ (4)

Now, to make *y* disappear, subtract equation (4) from equation (3) to get

$-17y = 51$ (5) (divide by -17 both sides)

$y = -3$

Substitute $y = -3$ into equation (2) to get

$2x + 3(-3) = -1$ (simplify brackets)

$2x - 9 = -1$ (add 9 both sides)

$2x = 8$ (divide by 2 both sides)

$x = 4$

Solution is $x = 4$ and $y = -3$.

Example 8

$2x + 3y - 5 = 0$ (1)

$3x - 4y - 7 = 0$ (2)

Choosing to eliminate *y,* multiply equation (1) by 4 and equation (2) by 3 to get

$8x + 12y - 20 = 0$ (3)

$9x - 12y - 21 = 0$ (4)

To get rid of *y*, *add* equation (3) and (4) to get

$17x - 41 = 0$

$17x = 41$

$x = \frac{41}{17}$

Substitute $x = \frac{41}{17}$ into equation (1) to get

$2\left(\frac{41}{17}\right) + 3y - 5 = 0$ (multiply brackets)

$\frac{82}{17} + 3y - 5 = 0$ (collect like terms; $\left(\frac{82}{17} - \frac{85}{17}\right)$)

$3y - \frac{3}{17} = 0$ (add $\frac{3}{17}$ both sides)

$3y = \frac{3}{17}$ (divide both sides by 3)

$y = \frac{1}{17}$

Solution is $x = \frac{41}{17}$ *and* $y = \frac{1}{17}$.

Activity

Activity 5.5

Solve the following equations with two unknown variables by using the elimination method. Please show all the steps you used to arrive at your answer.

a) $x + y = 1$

$x + 3y = 9$

b) $4x + 5y = 33$

$3x + 2y = 16$

c) $2a + b = 8$

$-a + b = -1$

d) $c + 3d = 15$

$3c + 2d = 38$

e) $r + 5s + 9 = 0$

$3s - r + 15 = 0$

f) $7u + 2v = 49$

$2u + 2v = 24$

Answers

a) $2y = 8$ subtract equation 1 from equation 2, then divide both sides by 2

$y = 4$

Using equation 1, substitute $y = 4$ to find value of x .

$x + 4 = 1$

$x = -3$

$y = 4$ and $x = -3$ is a solution

b) $12x + 15y = 99$ multiply equation 1 by 3 and equation 2 by 4

$12x + 8y = 64$

Subtract equation 2 from equation 1

$7y = 35$

$y = 5$

Using equation 2, substitute $y = 5$ to find value of.

$3x + 2(5) = 16$

$3x + 10 = 16$

$3x + 10 - 10 = 16 - 10$

$3x = 6$

$x = 2$

$x = 2$ and $y = 5$ is a solution.

c) $3a = 9$ subtract equation 2 from equation 1

$a = 3$

Using equation 2, substitute $a = 3$ to find b.

$-(3) + b = -1$

$-3 + 3 + b = -1 + 3$

$b = 2$

$a = 3 \text{ and } b = 2$ is a solution.

d) $3c + 9d = 45$ multiply equation 1 by 3

$3c + 2d = 38$

Subtract equation 2 from equation 1 to eliminate . Then find .

$7d = 7$ divide both sides by 7

$d = 1$

Using equation 1, substitute $d = 1$ to find c.

$c + 3(1) = 15$

$c + 3 = 15$

$c + 3 - 3 = 15 - 3$

$c = 12$

$d = 1 \text{ and } c = 12$ is a solution.

e) Add equation 1 and equation 2 to eliminate. Then, find.

$6s + 24 = 0$ divide both sides by 6

$s + 4 = 0$

$s = -4$

$r = 3 \text{ and } s = -4$ is a solution

f) Subtract equation 2 from equation 1 to eliminate. Then, find u.

$5u = 25$ divide both sides by 5

$u = 5$

Using equation 2, substitute $u = 5$ to find.

$$2(5) + 2v = 24$$

$$10 - 10 + 2v = 24 - 10$$ subtract 10 from both sides

$$2v = 14$$ divide both sides by 10

$$v = 7$$

$u = 5\ and\ v = 7$ is a solution.

You have now done the first of the four methods. It has been a lot of work to get this far. Congratulations! You are now left with three methods to learn. Have a good time on this learning journey. The next method is substitution.

B. The Substitution Method

The substitution method is very similar to the elimination approach you learned about in the last sub-unit. The idea here is still to eliminate one unknown variable, but the difference is in how we eliminate it. With the substitution method, we make either unknown the subject of one of the two equations, and then substitute the expression equal to the unknown which has been made the subject into the other equation.

Remember, when you are checking the solution of an equation, you want to find whether the expression on the left-hand side (LHS) is equal to the expression on the right-hand side (RHS). If LHS = RHS, then it is a true statement. This means that the expressions on the LHS and on the RHS have equal values, as that is the definition of an equation. You can use this knowledge to replace one number with the other; we say you substitute one for the other. That is, you can replace a number on the LHS by a number on the RHS (or RHS by LHS).

In the substitution method, you start to replace one unknown by the equal expression to eliminate it. Then, find the value of the remaining unknown by using the same steps you were using in elimination method when there is one unknown left in the equation.

At the end of this sub-sub-unit you should be able to:

- Solve simultaneous equations with two unknowns by substitution method.

There are nine pages on this sub-sub-unit.

The following examples will help you understand how this method works.

Example 1

$3y - 2x = 11$ (1)

$y + 2x = 9$ (2)

make y the subject of the equation in the second equation.

$y + 2x - 2x = 9 - 2x$ (subtract $2x$ from both sides)

$y = 9 - 2x$

Notice that LHS = y, and RHS = $9 - 2x$. These sides are equal, so you can replace y by $9 - 2x$ in equation 1 $3y - 2x = 11$. Thus,

Substitute $9 - 2x$ for y in the first equation.

$3y - 2x = 11$

$3(9 - 2x) - 2x = 11$

Solve the new equation for x, using the skills you were using in elimination method from here onwards. Notice that y is no longer in the equation, you have x only.

$3(9 - 2x) - 2x = 11$ (multiply brackets)

$27 - 6x - 2x = 11$ (collect like terms)

$27 - 8x = 11$ (subtract 27 both sides (or add $8x$ both sides))

$-8x = -16$

$x = 2$

Put 2 for x in the second equation

$y + 2x = 9$

$y + 2(2) = 9$ (substitute 2 for x and multiply out)

$y + 4 = 9$ (subtract 4 both sides)

$y = 5$

Therefore, $x = 2$ and $y = 5$ is a solution.

Check: substitute $x = 2$ and $y = 5$ into first and second equations. If these answers are correct both equations will be true!

Check: $3y - 2x = 11$ (1)

$3(5) - 2(2) = 11$

$$15 - 4 = 11$$

$$11 = 11$$

LHS = RHS

Check: $y + 2x = 9$ (2)

$$5 + 2(2) = 9$$

$$5 + 4 = 9$$

$$9 = 9$$

LHS = RHS

Example 2

$x + 3y = 10$ (1)

$2x + 5y = 16$ (2)

Choosing to make *x,* the subject in equation (1) to get

$x + 3y = 10$ (subtract 3*y* both sides)

$x = 10 - 3y$ (3)

Note that LHS = x and RHS = $10 - 3y$. The sides are equal, so substitute *x* by $10 - 3y$ in equation (2) to get

$2x + 5y = 16$

$2(10 - 3y) + 5y = 16$ (4)

Solve for y . Equation (4) has one unknown! You work as you have done when solving equations with one unkown.

$2(10 - 3y) + 5y = 16$ (simplify brackets)

$20 - 6y + 5y = 16$ (collect like terms)

$20 - y = 16$ (subtract 20 both sides)

$-y = -4$ (divide by -1 both sides)

$y = 4$

Use $y = 4$ in equation (1) to find value of x .

$x + 3y = 10$

$x + 3(4) = 10$ (simplify brackets)

$x + 12 = 10$ (subtract 12 both sides)

$x = -2$

Solution is $x = -2$ and $y = 4$.

Example 3

$3x - 2y = 7$ (1)

$x + 2y = 5$ (2)

Make x the subject in equation (2)

$x + 2y - 2y = 5 - 2y$ (subtract $2y$ on both sides)

$x = 5 - 2y$ (3)

Substitute for x in equation (1) using equation (3) to get

$3x - 2y = 7$

$3(5 - 2y) - 2y = 7$ (4)

Solve for y in equation (4)

$15 - 6y - 2y = 7$ (multiply brackets)

$15 - 8y = 7$ (collect like terms)

$15 - 15 - 8y = 7 - 15$ (subtract 15 both sides)

$-y = -8$

$\frac{-y}{-1} = \frac{-8}{-1}$ (divide by -1 both sides)

$y = 8$

Substitute the value of $y = 8$ into equation (2) to find the value of x.

$x + 2y = 5$

$x + 2(8) = 5$ (substitute y = 8)

$x + 16 = 5$

$x + 16 - 16 = 5 - 16$ (subtract 16 both sides)

$x = -11$

$x = -11$ and $y = 8$ is a solution.

Check the solution by using any of the two equations (1) or (2).

Example 3

$5y = x$ (1)

$x - 3y = 6$ (2)

Put 5y for x in the second equation

$5y - 3y = 6$

$8y = 6$

$y = \frac{3}{4}$

Substitute $\frac{3}{4}$ for y in equation (1).

$5y = x$

$5\left(\frac{3}{4}\right) = x$

$\frac{15}{4} = x$

$x = 3\frac{3}{4}$

Therefore, $x = 3\frac{3}{4}$ and $y = \frac{3}{4}$

Example 4

$y = 5x - 17$ (1)

$y = 3x - 9$ (2)

Since $y = 5x - 17$ and $y = 3x - 9$, then $5x - 17 = 3x - 9$.

Each right side is equal to y

Solve for x

$5x - 17 = 3x - 9$.

$2x = 8$

$x = 4$

Now substitute 4 for x in the first equation.

$y = 5x - 17$

$y = 5(4) - 17$

$y = 20 - 17$

$y = 3$

Therefore, $x = 4$ and $y = 3$

Example 5

$-3x + 2y = 21$ (1)

$-4x + 6y = 38$ (2)

Choosing to make y the subject in equation (1) to get

$-3x + 2y = 21$

$2y = 21 + 3x$ (3)

$y = \frac{21+3x}{2}$ (4)

Here, with careful look in equation (2) you could use equation (3) (try on yourself) but here we will use (4) to find value of x .

$-4x + 6\left(\frac{21+3x}{2}\right) = 38$

$-4x + 3(21 + 3x) = 38$ (simplify brackets)

$-4x + 63 + 9x = 38$ (collect like terms)

$5x + 63 = 38$ (subtract 63 both sides)

$5x = 25$ (divide by 5 both sides)

$x = 5$

Use $x = 5$ in equation (2) to find value of y.

$-4x + 6y = 38$

$-4(5) + 6y = 38$ (simplify brackets)

$-20 + 6y = 38$ (add 20 both sides)

$6y = 58$ (divide by 6 both sides)

$y = \frac{58}{6}$

$= 9\frac{2}{3}$

Solution is $x = 5$ and $y = 9\frac{2}{3}$

Example 6

$9p - 7q = 33$ (1)

$7p - 9q = 15$ (2)

Choosing to make *p* subject in equation (2) to get

$7p - 9q = 15$

$7p = 15 + 9q$

$p = \frac{15+9q}{7}$ (3)

Substituting (3) into equation (1) to find value of *q*.

$9p - 7q = 33$ (1)

$9\left(\frac{15+9q}{7}\right) - 7q = 33$ (substitute *p* and multiply brackets)

$\frac{135+81q}{7} - 7q = 33$ (multiply by 7 all terms)

$135 + 81q - 49q = 231$ (collect like terms)

$32q = 231 - 135$

$32q = 96$ (divide both sides by 32)

$q = 3$

Now, use equation (1) and $q = 3$ to find the value of *p*.

$9p - 7q = 33$

$9p - 7(3) = 33$ (substitute *q*)

$9p - 21 = 33$ (add 21 both sides)

$9p = 33 + 21$ (simplify)

$9p = 54$ (divide both sides by 9)

$p = 6$

Solution is $p = 6$ and $q = 3$.

Activity

Activity 5.6

Now solve the following equations by substitution

a) $y = 3x - 5$

$3 - 5x = y$

b) $2x - 3y = 11$

$y = 4x - 7$

c) $2x + 11 = y$

$5x - 4y = -32$

d) $x - 2y = 14$

$x + 3y = 9$

e) $2y + 3x = 4$

$y + x = 1$

f) $3y + 2x - 17 = 0$

$2y + 5x - 15 = 0$

Model Answers

Activity 5.6

a) In both equations y *is the subject. Therefore any equation can be used. Substitute* $y = 3x - 5$ *into equation 2.*

$$3 - 5x = 3x - 5$$

$3 - 5x + 5x = 3x + 5x - 5$ *(add 5x on both sides)*

$3 + 5 = 8x - 5 + 5$ *(add 5 on both sides)*

$\frac{8}{8} = \frac{8x}{8}$ *(divide both sides by 8)*

$1 = x$

Using equation 1, substitute $x = 1$*, to find y*

$y = 3(1) - 5$ *(multiply bracket)*

$y = 3 - 5$ *(subtract)*

$y = -2$

$x = 1 \text{ and } y = -2$ is a solution.

b) *In equation 2, y is already a subject. Substitute* $y = 4x - 7$ *into equation 1.*

$2x - 3(4x - 7) = 11$ *(multiply bracket)*

$2x - 12x + 21 = 11$ *(collect like terms)*

$-10x + 21 - 21 = 11 - 21$ *(subtract 21 on both sides)*

$\frac{-10x}{-10} = \frac{-10}{-10}$ *(divide by -10 on both sides)*

$x = 1$

Substitute $x = 1$ *into equation 2, to find y*

$y = 4(1) - 7$ *(multiply bracket)*

$y = 4 - 7$ *(subtract)*

$y = -3$

$x = 1 \text{ and } y = -3$ is a solution.

c) *In equation 1, y is already a subject. Substitute* $2x + 11 = y$ *into equation 2.*

$5x - 4(2x + 11) = -32$ *(multiply bracket)*

$5x - 8x - 44 = -32$ *(collect like terms)*

$-3x - 44 + 44 = -32 + 44$ *(add 44 on both sides)*

$\frac{-3x}{-3} = \frac{12}{-3}$ *(divide by -3 on both sides)*

$x = -4$

Substitute $x = -4$ *into equation 1, to find y*

$2(-4) + 11 = y$ *(multiply bracket)*

$-8 + 11 = y$ *(add)*

$3 = y$

$x = -4$ and $y = 3$ is a solution.

d) Remember any of the two unknowns can be made the subject. In both equations it is easier to make x the subject. Making x the subject in equation 1; $= 14 - 2y$. Then substitute for x in equation 2.

$(14 + 2y) + 3y = 9$ *(remove bracket)*

$14 + 2y + 3y = 9$ *(collect like terms)*

$14 - 14 + 5y = 9 - 14$ *(subtract 14 on both sides)*

$\frac{5y}{5} = \frac{-5}{5}$ *(divide by 5 on both sides)*

$y = -1$

Substitute $y = -1$ into equation 1, to find x

$x - 2(-1) = 14$ *(multiply bracket)*

$x + 2 - 2 = 14 - 2$ *(subtract 2 on both sides)*

$x = 12$

$x = 12$ *and* $y = -1$ is a solution.

e) Make y the subject in equation 2. Substitute $y = 1 - x$ into equation 1, to find x.

$2(1 - x) + 3x = 4$ *(multiply bracket)*

$2 - 2x + 3x = 4$ *(collect like terms)*

$2 - 2 + x = 4 - 2$ *(subtract 2 on both sides)*

$x = 2$

Substitute $x = 2$ into equation 2, to find y

$y + 2 = 1$

$y + 2 - 2 = 1 - 2$ *(subtract 2 on both sides)*

$y = -1$

$x = 2$ *and* $y = -1$ *is a solution.*

f) Make y the subject in equation 2

$2y + 5x - 15 + 15 = 0 + 15$ *(add 15 on both sides)*

$2y + 5x - 5x = 15 - 5x$ *(subtract 5x on both sides)*

$\frac{2y}{2} = \frac{15-5x}{2}$ *(divide both sides by 2)*

$y = \frac{15-5x}{2}$

Substitute for y in equation 1

$3\left(\frac{15-5x}{2}\right) + 2x - 17 = 0$ *(multiply bracket)*

$\frac{45-15x}{2} + 2x - 17 = 0$ *(multiply all terms by 2 to remove fraction)*

$45 - 15x + 4x - 34 = 0$ *(collect like terms)*

$45 - 34 - 15x + 4x = 0$

$11 - 11 - 11x = 0 - 11$ *(subtract 11 on both sides)*

$\frac{-11x}{-11} = \frac{-11}{-11}$ *(divide by -11 on both sides)*

$x = 1$

To find y, substitute $x = 1$ *into equation 2.*

$2y + 5(1) - 15 = 0$ *(multiply bracket)*

$2y + 5 - 15 = 0$

$2y - 10 = 0$ *(collect like terms)*

$2y - 10 + 10 = 0 + 10$ *(add 10 on both sides)*

$\frac{2y}{2} = \frac{10}{2}$ *(divide by 2 both sides)*

$y = 5$

$x = 1$ and $y = 5$ is a solution.

You have now completed the sub-unit which presented the substitution method of solving linear equations. Go to the beginning of the sub-unit and do a quick review of the content before moving on. Once you are confident that you have mastered the substitution method, you can turn to the next sub-unit which looks at how you can solve linear equations using matrices.

C. The Matrix Method

In this section you are going to learn how to solve linear equations with two unknown variables using the matrix method. Unit 7 is on matrices in case you need to look more on matices.

Simultaneous equations sometimes arise from matrix multiplication for example,

If $\begin{pmatrix} 2 & -3 \\ 6 & 14 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 11 \\ 10 \end{pmatrix}$ then multiplying the left hand side gives

$$\begin{pmatrix} 2a - 3b \\ 6a + 14b \end{pmatrix} = \begin{pmatrix} 11 \\ 10 \end{pmatrix}$$

Therefore $2a - 3b = 11$ (equal matrices))

$6a + 14b = 10$

At the end of this sub-sub-unit on solving linear equations in two unknowns by matrix method you should be able to:

- *write* equations in matrix form,
- *find* the determinant of a matrix,
- *find* the inverse of a matrix,
- *multiply* matrices,
- *solve* simultaneous equations with two unknowns by matrix method.

There are seventeen (17) pages on this sub-sub-unit on solving linear equations in two unknowns by matrix method.

The following examples will help you learn how to solve simultaneous equations by matrix method.

Example 1

$3x + 2y = 4$

$x + y = 1$

It helps to number the equations, so they are numbered below.

$3x + 2y = 4$ (1)

$x + y = 1$ (2)

Step 1

Rewrite the equations in matrix form

$$\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Step 2

Find the determinant of the 2 by 2 matrix $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ by

- multiplying the elements in the leading diagonal then subtracting the product of the elements on the other diagonal

  That is,

$$\det \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = (3 \times 1) - (2 \times 1)$$

(Same as) $$\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = (3 \times 1) - (2 \times 1)$$

$$= 3 - 2$$

$$= 1$$

Step 3

Find the inverse of the 2 by 2 matrix $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

First find the new matrix by re-arranging the positions of the elements of the 2 by 2 matrix $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ as follows

a) $\begin{pmatrix} 3 & \\ & 1 \end{pmatrix}$ exchange positions of these elements (leading diagonal)

b) $\begin{pmatrix} & 2 \\ 1 & \end{pmatrix}$ negative of these elements (the other diagonal)

The new matrix is $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

Then, to get the inverse

the inverse $= \frac{1}{det}(new\ matrix)$

(reciprocal of the determinant multiplied by the new matrix)

the inverse is $\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ (det = 1)

Step 4

Multiply both sides of the equations as written in **step 1** by the inverse of the original 2by2 matrix on the **left hand side**

$$\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Then,

LHS = inverse matrix multiplied by original matrix equals a unit matrix

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$,

RHS = $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\times 4 + -2\times 1 \\ -1\times 4 + 3\times 1 \end{pmatrix} = \begin{pmatrix} 4-2 \\ -4+3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Since, LHS = RHS

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (property of multiplication by unit matrix)

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (equal matrices, corresponding elements are equal)

Therefore, $x = 2$ and $y = 1$ is a solution.

Example 2

$y = 3x - 5$ (1)

$y = -5x + 3$ (2)

Re-arrange the equations so that they can be re-written in matrix form.

$y - 3x = -5$ (subtracted $3x$ on both sides)

$y + 5x = 3$ (added $5x$ on both sides)

Re-arrange the positions of x and y.

$-3x + y = -5$

$5x + y = 3$

Now that the equations are re-arranged, you can write them in matrix form.

Step 1

Rewrite the equations in matrix form

$$\begin{pmatrix} -3 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

Step 2

Find the determinant of $\begin{pmatrix} -3 & 1 \\ 5 & 1 \end{pmatrix}$

$$det\begin{pmatrix} -3 & 1 \\ 5 & 1 \end{pmatrix} = (-3\times 1) - (1\times 5)$$

$$= -3 - 5$$

$$= -8$$

Then, the new matrix $\begin{pmatrix} 1 & -1 \\ -5 & -3 \end{pmatrix}$

after re-arranging the elements in the leading diagonal and changing the signs of the elements in the other diagonal.

That is,

a) $\begin{pmatrix} -3 & \\ & 1 \end{pmatrix}$ exchange positions of these elements

b) $\begin{pmatrix} & 1 \\ 5 & \end{pmatrix}$ negative of these elements

to get

the new matrix as $\begin{pmatrix} 1 & -1 \\ -5 & -3 \end{pmatrix}$

Find the inverse of the 2 by 2 matrix

$$\textbf{the inverse} = \frac{1}{det}(new\ matrix)$$

$$= \frac{1}{-8}\begin{pmatrix} 1 & -1 \\ -5 & -3 \end{pmatrix}$$

Step 3

Multiply both sides of the equations as written in **step 1** by the inverse of the original 2 by 2 matrix on the **left hand side**

$$\frac{1}{-8}\begin{pmatrix} 1 & -1 \\ -5 & -3 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-8}\begin{pmatrix} 1 & -1 \\ -5 & -3 \end{pmatrix}\begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

LHS $= \frac{1}{-8}\begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -8\frac{1}{-8}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ (factor out -8 to get unit matrix)

RHS $= \frac{1}{-8}\begin{pmatrix} -5-3 \\ 25-9 \end{pmatrix} = \frac{1}{-8}\begin{pmatrix} -8 \\ 16 \end{pmatrix} = -8\frac{1}{-8}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (factor out -8)

Since LHS = RHS

$$-8\frac{1}{-8}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -8\frac{1}{-8}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$ (property of multiplication by unit matrix)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Therefore, $x = 1$ and $y = -2$ is a solution.

Example 3

$2x + 11 = y$ (1)

$5x - 4y = -32$ (2)

Equation 1 requires some re-arranging so that it is written in the form similar to equation 2.

$2x + 11 - 11 = y - 11$ (subtract 11 on both sides)

$2x - y = y - y - 11$ (subtract y on both sides)

$2x - y = -11$

Now, the equations are

$2x - y = -11$ (equation 1 re-arranged)

$5x - 4y = -32$ (same as equation 2 above)

In this form, they can be written in matrix form.

Step 1

Rewrite the equations in matrix form

$$\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -11 \\ -32 \end{pmatrix}$$

Step 2

Find the determinant of the 2 by 2 matrix $\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}$

$$\det\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix} = (2 \times -4) - (-1 \times 5)$$

$$= -8 + 5$$

$$= -3$$

Then, the new matrix

a) $\begin{pmatrix} 2 & \\ & -4 \end{pmatrix}$ exchange positions of these elements

b) $\begin{pmatrix} & -1 \\ 5 & \end{pmatrix}$ negative of these elements

to get

the new matrix as $\begin{pmatrix} -4 & 1 \\ -5 & 2 \end{pmatrix}$

Find the inverse of the 2 by 2 matrix

the inverse = $\frac{1}{det}(new\ matrix)$

$= \frac{1}{-3}\begin{pmatrix} -4 & 1 \\ -5 & 2 \end{pmatrix}$

Step 3

Multiply both sides of the equations as written in **step 1** by the inverse of the original 2 by 2 matrix on the **left** hand **side**

$$\frac{1}{-3}\begin{pmatrix} -4 & 1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-3}\begin{pmatrix} -4 & 1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} -11 \\ -32 \end{pmatrix}$$

LHS = $\frac{1}{-3}\begin{pmatrix} -8+5 & 4-4 \\ -10-10 & 5-8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-3}\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ = -3$\frac{1}{-3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ (factor out -3 to get unit matrix)

RHS = $\frac{1}{-3}\begin{pmatrix} -4 & 1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} -11 \\ -32 \end{pmatrix} = \frac{1}{-3}\begin{pmatrix} 44-32 \\ 55-64 \end{pmatrix} = \frac{1}{-3}\begin{pmatrix} 12 \\ -9 \end{pmatrix}$ = -3$\frac{1}{-3}\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ (factor out -3)

Since LHS = RHS

$$-3\frac{1}{-3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\frac{1}{-3}\begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ (property of multiplication by unit matrix)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

Therefore, $x = -4$ and $y = 3$ is a solution.

Activity

Activity 5.7

Now use the skills you learned in the above examples to solve the following equations.

Use a matrix method to solve the following simultaneous equations.

a) $5x + 7y = 3$

$3x + 5y = 4$

b) $4x + 2y = 14$

$3x - y = 3$

c) $3y - 4 = x$

$2x + 6y = 24$

d) $y = 3x - 3$

$3x - y = 3$

e) $y = 4x - 3$

$y = 2x + 3$

f) $y = 3x$

$2x + 3y = 22$

Model Answers

Activity 5.7

a) $5x + 7y = 3$

$3x + 5y = 4$

Step 1

Write equations in matrix form

$\begin{pmatrix} 5 & 7 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Step 2

Find the determinant of the 2 by 2 matrix $\begin{pmatrix} 5 & 7 \\ 3 & 5 \end{pmatrix}$

$$\det\begin{pmatrix} 5 & 7 \\ 3 & 5 \end{pmatrix} = (5 \times 5) - (7 \times 3)$$

$$= 25 - 21$$

$$= 4$$

Then, the new matrix

$\begin{pmatrix} 5 & -7 \\ -3 & 5 \end{pmatrix}$

Find inverse matrix of the 2 by 2 matrix

the inverse = $\frac{1}{\det}(new\ matrix)$

$= \frac{1}{4}\begin{pmatrix} 5 & -7 \\ -3 & 5 \end{pmatrix}$

Step 3

Multiply equations in matrix form (**step 1**) by inverse matrix (**step 2**) on the left hand side on both sides.

$$\frac{1}{4}\begin{pmatrix} 5 & -7 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5 & -7 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \text{(row by column)}$$

$$\frac{1}{4}\begin{pmatrix} 25-21 & 35-35 \\ -15+15 & -21+25 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 15-28 \\ -9+20 \end{pmatrix} \quad \text{(simplify)}$$

$$\frac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -13 \\ 11 \end{pmatrix} \quad \text{(factor out 4 on left side)}$$

$$4\frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -13 \\ 11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -13 \\ 11 \end{pmatrix} \quad \text{(unit matrix multiplication)}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -13 \\ 11 \end{pmatrix}$$

Therefore, $x = -\frac{13}{4}$ and $y = \frac{11}{4}$ is a solution.

b) $4x + 2y = 14$

$3x - y = 3$

Step 1

Write equations in matrix form

$$\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 3 \end{pmatrix}$$

Step 2

Find the determinant of the matrix $\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}$

$$\det\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} = (4 \times -1) - (2 \times 3)$$

$$= -4 - 6$$

$$= -10$$

Then, the new matrix

$$\begin{pmatrix} -1 & -2 \\ -3 & 4 \end{pmatrix}$$

Find inverse matrix

the inverse = $\frac{1}{det}(new\ matrix)$

$$= \frac{1}{-10}\begin{pmatrix} -1 & -2 \\ -3 & 4 \end{pmatrix}$$

Step 3

Multiply equations in matrix form (**step 1**) by inverse matrix (**step 2**) on the left hand side on both sides.

$$\frac{1}{-10}\begin{pmatrix} -1 & -2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -1 & -2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 14 \\ 3 \end{pmatrix}$$ (row by column)

$$\frac{1}{-10}\begin{pmatrix} -4-6 & -2+2 \\ -12+12 & -6-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -14-6 \\ -42+12 \end{pmatrix}$$ (simplify)

$$\frac{1}{-10}\begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-10}\begin{pmatrix} -20 \\ -30 \end{pmatrix}$$ (factor out -10 on both sides)

$$-10\frac{1}{-10}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -10\frac{1}{-10}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$ (unit matrix multiplication)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Therefore, $x = 2$ and $y = 3$ is a solution.

c) $5y - 4 = x$

$2x + 6y = 24$

Step 1

Write equations in matrix form

$$\begin{pmatrix} 1 & -5 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 24 \end{pmatrix}$$

Step 2

Find the determinant of the 2 by 2 matrix $\begin{pmatrix} 1 & -5 \\ 2 & 6 \end{pmatrix}$

$$det\begin{pmatrix} 1 & -5 \\ 2 & 6 \end{pmatrix} = (1 \times 6) - (-5 \times 2)$$

$$= 6 + 10$$

$$= 16$$

Then, the new matrix

$$\begin{pmatrix} 6 & 5 \\ -2 & 1 \end{pmatrix}$$

Find inverse matrix

the inverse = $\frac{1}{det}(new\ matrix)$

$$= \frac{1}{16}\begin{pmatrix} 6 & 5 \\ -2 & 1 \end{pmatrix}$$

Step 3

Multiply equations in matrix form (**step 1**) by inverse matrix (**step 2**) on the left hand side on both sides.

$$\frac{1}{16}\begin{pmatrix} 6 & 5 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -5 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 6 & 5 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 24 \end{pmatrix}$$ (row by column)

$$\frac{1}{16}\begin{pmatrix} 6+10 & -30+30 \\ -2+2 & 10+6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{16}\begin{pmatrix} -24+120 \\ 8+24 \end{pmatrix}$$ (simplify)

$$\frac{1}{16}\begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 96 \\ 32 \end{pmatrix}$$ (factor out 16 on both sides)

$$16\frac{1}{16}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 16\frac{1}{16}\begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$ (unit matrix multiplication)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Therefore, $x = 6$ and $y = 2$ is a solution.

d) $y = 3x - 3$

$3x - y = 3$

Step 1

Write equations in matrix form

$$\begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

Step 2

Find the determinant of the 2 by 2 matrix $\begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix}$

$$det\begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix} = (-3 \times 1) - (1 \times -1)$$

$$= -3 + 1$$

$$= -2$$

Then, the new matrix

$\begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix}$

Find inverse matrix

$$\textbf{the inverse} = \frac{1}{det}(new\ matrix)$$

$$= \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix}$$

Step 3

Multiply equations in matrix form (**step 1**) by inverse matrix (**step 2**) on the left hand side on both sides.

$$\frac{1}{-2}\begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \end{pmatrix}$$ (row by column)

$$\frac{1}{-2}\begin{pmatrix} -3+1 & 1-1 \\ -3+3 & 1-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 3-5 \\ -3-15 \end{pmatrix}$$ (simplify)

$$\frac{1}{-2}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -2 \\ -18 \end{pmatrix}$$ (factor out -2 on both sides)

$$-2\frac{1}{-2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\frac{1}{-2}\begin{pmatrix} 1 \\ 9 \end{pmatrix}$$ (divide by -2 on both sides)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$$ (unit matrix multiplication)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$$

Therefore, $x = 1$ and $y = 9$ is a solution.

e) $y = 4x - 2$

$y = 2x + 5$

Step 1

Write equations in matrix form

$$\begin{pmatrix}-4 & 1\\ -2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-3\\ 5\end{pmatrix}$$

Step 2

Find the determinant of the 2 by 2 matrix $\begin{pmatrix}-4 & 1\\ -2 & 1\end{pmatrix}$

$$det\begin{pmatrix}-4 & 1\\ -2 & 1\end{pmatrix}=(-4\times 1)-(1\times -2)$$

$$=-4+2$$

$$=-2$$

The new matrix

$$\begin{pmatrix}1 & -1\\ 2 & -4\end{pmatrix}$$

Find inverse matrix

the inverse = $\frac{1}{det}(new\ matrix)$

$$=\frac{1}{-2}\begin{pmatrix}1 & -1\\ 1 & -3\end{pmatrix}$$

Step 3

Multiply equations in matrix form (**step 1**) by inverse matrix (**step 2**) on the left hand side on both sides.

$$\frac{1}{-2}\begin{pmatrix}1 & -1\\ 2 & -4\end{pmatrix}\begin{pmatrix}-4 & 1\\ -2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{-2}\begin{pmatrix}1 & -1\\ 2 & -4\end{pmatrix}\begin{pmatrix}-3\\ 5\end{pmatrix}$$ (row by column)

$$\frac{1}{-2}\begin{pmatrix}-4+2 & 1-1\\ -8+8 & 2-4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{-2}\begin{pmatrix}-3-5\\ -6-20\end{pmatrix}$$ (simplify)

$$\frac{1}{-2}\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{-2}\begin{pmatrix}-8\\ -26\end{pmatrix}$$ (factor out -2 on both sides)

$$-2\frac{1}{-2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-2\frac{1}{-2}\begin{pmatrix}4\\ 13\end{pmatrix}$$ (divide by -2 on both sides)

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}4\\ 13\end{pmatrix}$$ (unit matrix multiplication)

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}4\\ 13\end{pmatrix}$$

Therefore, $x=4$ and $y=13$ is a solution.

f) $y = 3x$

$2x + 3y = 22$

Step 1

Write equations in matrix form

$$\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 22 \end{pmatrix}$$

Step 2

Find the determinant of the 2 by 2 matrix $\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}$

$$det\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix} = (3 \times 3) - (-1 \times 2)$$

$$= 9 + 2$$

$$= 11$$

Then, the new matrix

$$\begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix}$$

Find inverse matrix

the inverse = $\frac{1}{det}(new\ matrix)$

$$= \frac{1}{11}\begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix}$$

Step 3

Multiply equations in matrix form (**step 1**) by inverse matrix (**step 2**) on the left hand side on both sides.

$$\frac{1}{11}\begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 3 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 22 \end{pmatrix} \quad \text{(row by column)}$$

$$\frac{1}{11}\begin{pmatrix} 9+2 & -3+3 \\ -6+6 & 2+9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 0+22 \\ 0+66 \end{pmatrix} \quad \text{(simplify)}$$

$$\frac{1}{11}\begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 22 \\ 66 \end{pmatrix} \quad \text{(factor out 11 on both sides)}$$

$$11\frac{1}{11}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 11\frac{1}{11}\begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad \text{divide by 11 on both sides)}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \quad \text{(unit matrix multiplication)}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

Therefore, $x = 2$ and $y = 6$ is a solution.

.

You have now completed the sub-unit which looked at solving linear equations using the matrix method. Congratulations, we have almost finished with this topic. Before attempting the next sub-unit, do a quick review of the content of this sub-unit

D. The Graph Method

In mathematics, objects are represented by points. If you can look at an apple fruit falling off from a tree to the ground and try to represent its path from the height of the branch to the ground after every second of its fall you will appreciate this statement. What path do you get?

In this section we are going to learn how to solve equations in two unknown by graph method.

At the end of this sub-sub-unit on solving simultaneous equations by graph method you should be able to:

- Solve simultaneous equations with two unknowns by graph method.

There are five pages in this sub-sub-unit.

This is the last of the methods we will look at on this module. This method of graph is considered the most inaccurate of the four, just approximate answers are obtained. This is largely caused by the reading skill of the graph, the scale used to draw the graph. Also the method takes much time to reach to the solution as compared to the other algebraic methods considered. How does the method work?

Basically you need to:

1. find the coordinates, this is achieved by making a table of values for the equations to be plotted.
2. choose a scale to use on the two axes (x and y).
3. plot the points using table of values.
4. join the points.
5. read the coordinates of the point of intersection of the two graphs.
6. the solution is the x and y value of the point of intersection.

Learning Guidance (e.g. worked examples)

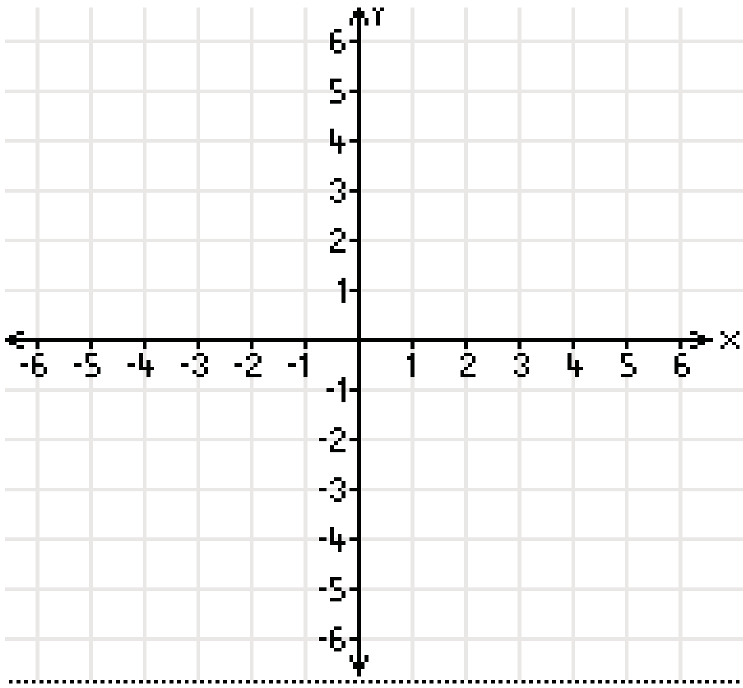
Examples

1.

a) Draw the graphs for the following equations

$3x + 5y = 15$

$3x - 5y = 15$



b) Mark the point of intersection.

c) Write down the coordinates of the point of intersection.____________

d) Write down the value for x and y at the point of intersection. x =_____and y =________.

Solution

-

a)

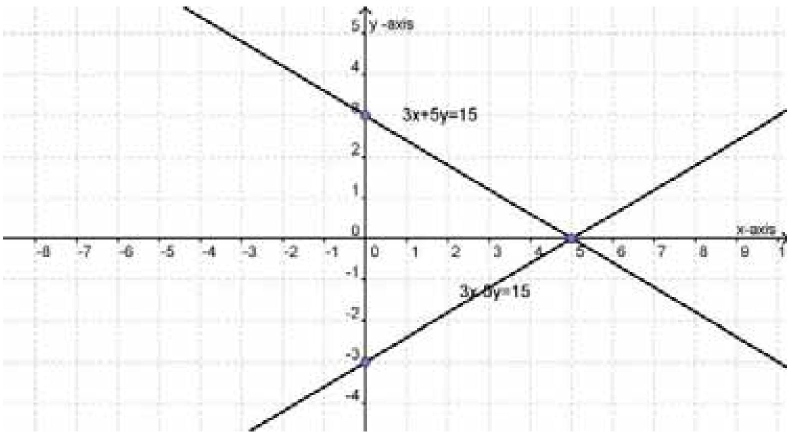
Table of values for $3x + 5y = 15$

X	0	3	5
Y	3	1.2	0

Table of values for $3x - 5y = 15$

X	0	3	5
Y	- 3	- 1.2	0

b)



c)

look at b) above

d)

the coordinates of point of intersection (5,0)

e)

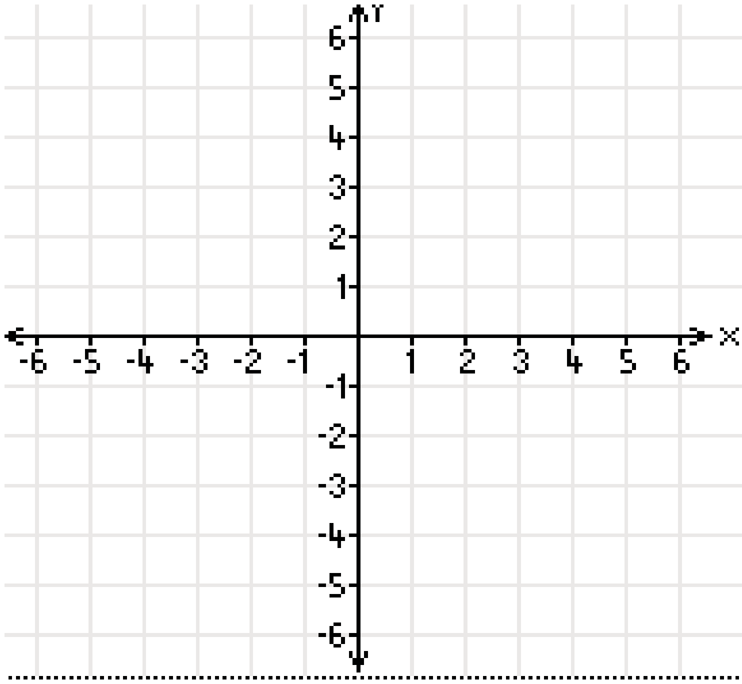
Solution is $x = 5$ and $y = 0$

2.

b) Solve the following simultaneous equations by graph method

$3x + 2y = 12$

$3x - 2y = 0$



b) Find the table of values, at least three points are needed (the x- intercept, y-intercept and any other point).

c) Using the coordinates from the table of values plot and draw the lines using the provided grid.

d) Read the coordinates of the point of intersection

e) The coordinates of point of intersection give the solution that satisfy both equations.

Now compare your answers with the ones given below.

Solution

-

a)

$3x + 2y = 12$

If x = 0: $2y = 12 \Rightarrow y = 6$ (y – intercept)

If y = 0; $3x = 12 \Rightarrow x = 4$ (x – intercept)

If x = 2; $6 + 2y = 12 \Rightarrow 2y = 6 \Rightarrow y = 3$

Table of values

X	0	2	4
Y	6	3	0

$3x - 2y = 0$

If x = 0; $-2y = 0 \Rightarrow y = 0$

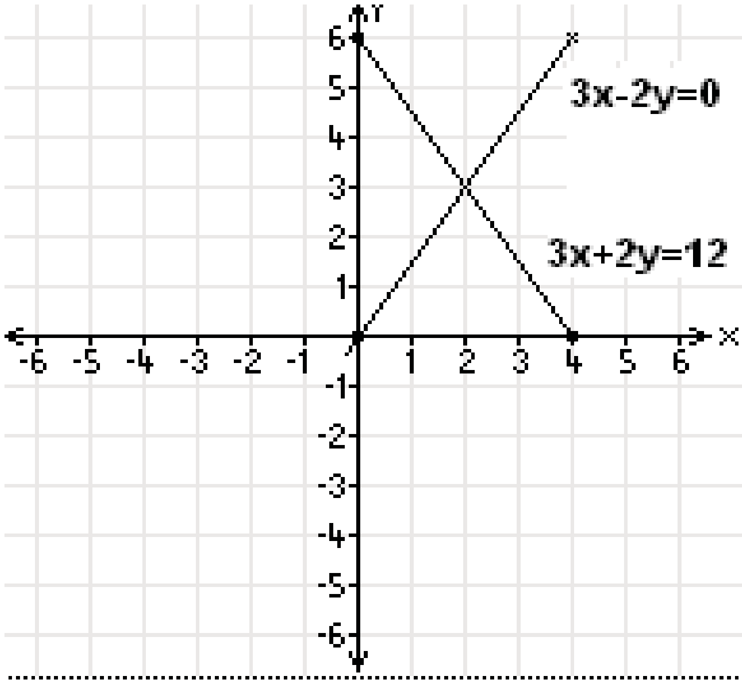
If x = 2; $6 - 2y = 0 \Rightarrow -2y = -6 \Rightarrow y = 3$

If x = 4; $12 - 2y = 0 \Rightarrow -2y = -12 \Rightarrow y = 6$

Table of values

X	0	2	4
Y	0	3	6

b)



c)

look at b) above

d)

Coordinates of point of intersection (2,3)

e)

solution is $x = 2$ and $y = 3$

Now you should have realised that at the point of intersection the x and the y coordinate values on one line are the same as the x and y coordinate values on the other line.

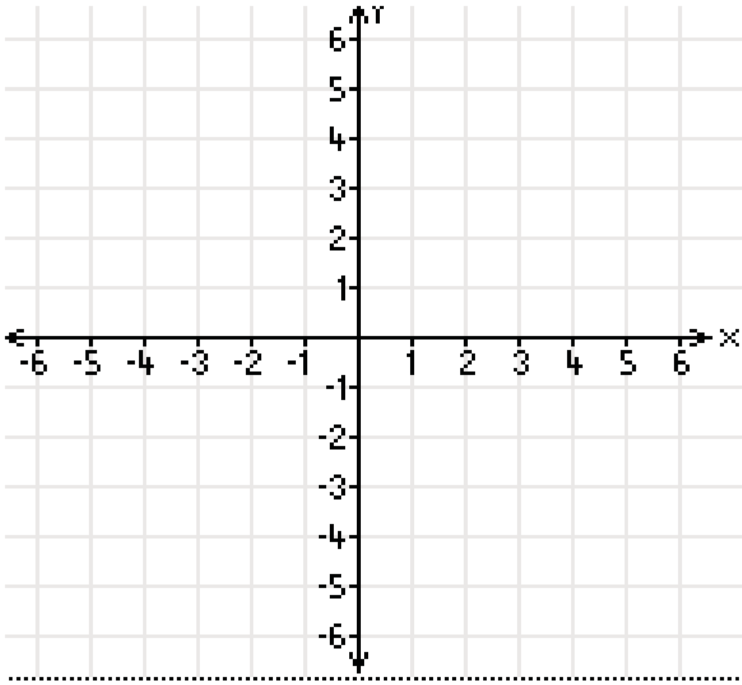
Activity

Activity 5.8

You can now use the skills you learned from the examples above to solve the following equations.

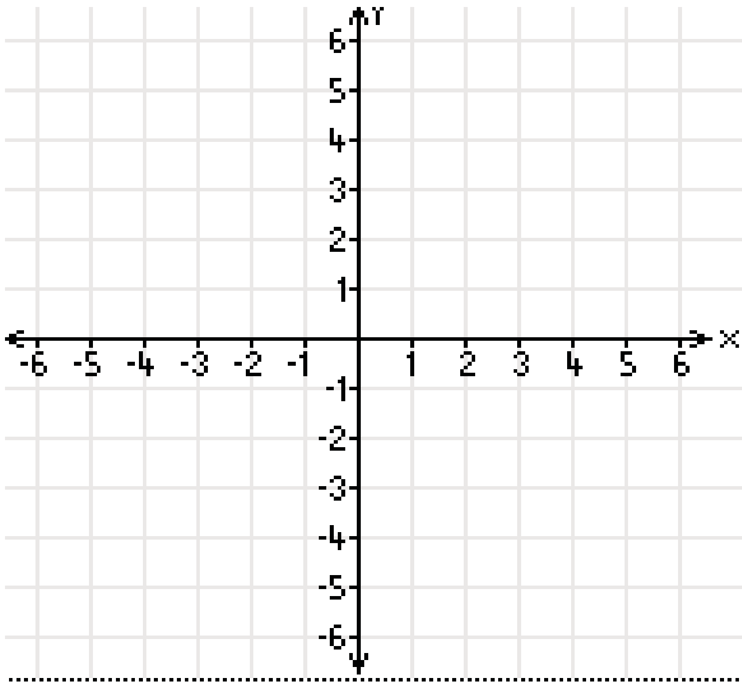
1. Solve the following equations:

a) On the same diagram, draw the graphs of the equations $y = 2x + 1$ and $x + 2y = 2$ for values of x from -3 to 3.



b) Use you graph to solve the simultaneous $y = 2x + 1$ and $x + 2y = 2$.

2.a) Draw the graphs of the equations $y = 3x - 3$ and $y = x + 5$ on the same diagram, for values of x from -2 to 3

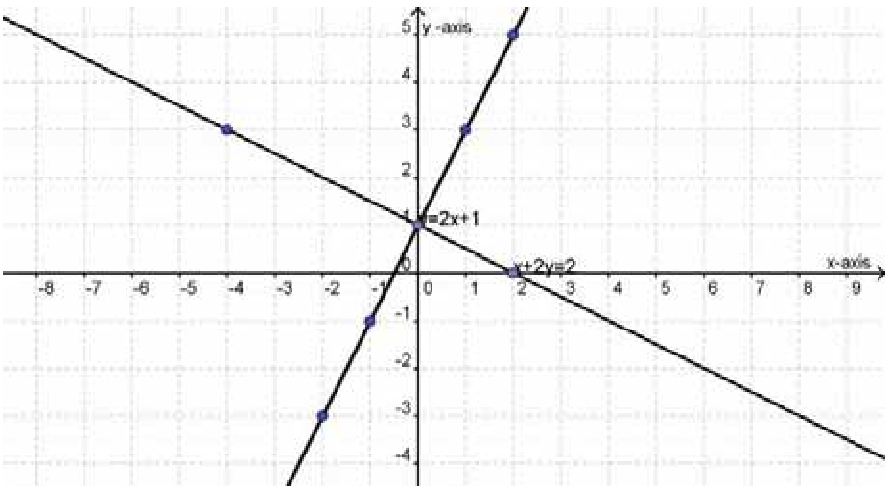


b) Use you graph to solve the simultaneous $y = 3x - 3$ and $y = x + 5$.

Check whether you have correctly drawn your graphs below. It has been a long way up to here so once again congratulations for the effort you are applying.

Answers

1.a)



b)

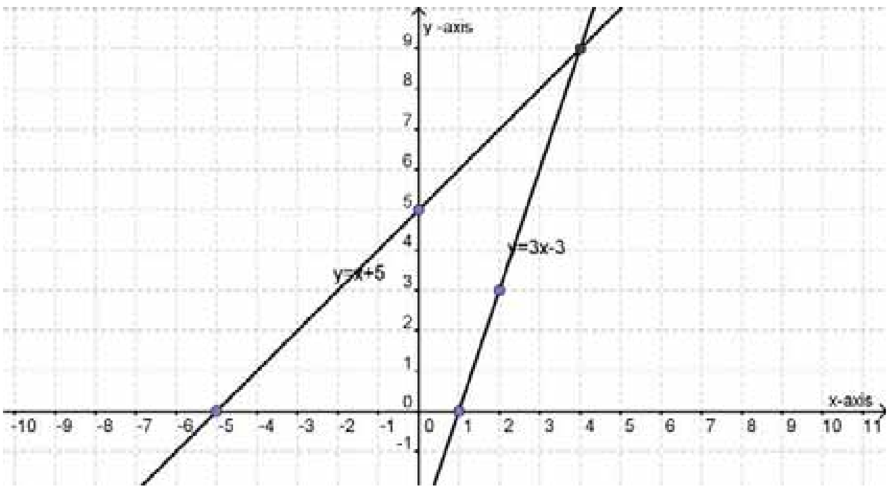
look at a) above

c)

look at a) above

d)

2.a)



You have now completed the method of solving linear equations by graph, do a quick review of the content of this method of solving linear equations and then continue.

You have now completed the last sub-unit on Linear Equations. Do a quick review of the entire content of this unit and then continue onto the unit summary.

Unit Summary

Summary

In this unit you learned that

Solving linear equations can be achieved by using a number of methods.

Whatever method you are using remember to do the same thing (operation) on both sides of the equation.

- One step equations (where there is one operation) are solved by any of the following depending on the operation in the equation:
  - Addition
  - Subtraction
  - Multiplication
  - Division

  If the equation shows that addition has been done, to solve you will use subtraction. If the equation shows subtraction has been done you will solve by using addition. Similarly, if the equation shows multiplication has been done, you will solve by using division. Whereas, if the equation shows division you will use multiplication to solve for the variable. The pairs of operations are inverse of each other.

- More than one operation in an equation means you will need a combination of operations to solve the equation. However, you have to look which operation is in the equation to choose the appropriate inverse operation to use in solving for the variable.
- To solve linear equations with numerical denominators use the Lowest Common Multiple (LCM) of the denominators to clear the fraction(s), then solve by using the inverse operations of the operations in the equation.
- To solve linear equations with algebraic denominators use the Lowest Common Multiple (LCM) of the algebraic denominators to clear the fraction(s) as in numerical denominators, then solve the resulting equation (with no fractions) using the inverse operations of the operations in the equation.
- To solve linear equations with two unknowns; the four methods dealt with here were
  - Elimination
  - Substitution
  - Matrix
  - Graph

- Elimination method

  One variable is removed from the equation by addition, or subtraction, when the variable has equal coefficients. However, when the variable coefficients are not equal you need to multiply the equations by suitable factors to result in equal coefficients for the two equations then use either addition or subtraction to eliminate the variable. Once one variable is removed (eliminated) you solve for the remaining variable. Then, use the value of the variable you found in any of the original equations to find the value of the second variable.

- Substitution method

  This method uses the same idea of eliminating one variable, but the variable is eliminated by expressing one variable in terms of the other variable (make one variable a subject in one of the equation and substitute into the other equation). Once, you substitute one variable only one variable will remain in the resulting equation. Solve for the remaining variable. Then, use the found value of the variable to find the other variable by using one of the original equations.

- Matrix method

  The main idea being used in this method is the fact that the multiplication of a matrix by its inverse matrix result in an identity matrix.

  The steps of the method are:

  Step 1

  Write the equations in a matrix form

  Step 2

  Find the inverse matrix, if it exist (the determinant must not be zero; if it is zero there is no specific solution)

  Step 3

  Multiply the equation in matrix form (in step 1) by inverse on the left hand side and solve for the unknown variables by comparing corresponding elements after working out each of the sides of the equation. Note that the left hand side will be equal to identity matrix multiplied by the variables matrix which leaves variables matrix by property of identity multiplication, so after simplifying the right hand side you then compare corresponding elements on both sides to get the solution.

- Graph method

In this method the solution is given by the coordinates of the intersection of the two graphs.

Steps of the method:

Step 1

Find the table of values (coordinates of points to plot) at least two points are needed to draw a straight line, but use three,

Step 2

Choose the scale for both axes based on the values in you table of values. Then, plot the points from the table of values. Join the plotted points to form a straight line,

Step 3

Read the coordinates of the intersection of the two lines. The solution is the x value and the y value of this coordinates.

I hope you found the material presented to you useful and enjoyable to work through, but above all that it helped you to prepare for examinations and life. I hope your knowledge in solving linear equations is strengthened and your confidence in solving linear equations has increased.

In particular, you have gained confidence in solving:

1. linear equations involving both numerical and algebraic fractions.
2. simultaneous equations with two unknowns using elimination method.
3. simultaneous equations with two unknowns using substitution method.
4. simultaneous equations with two unknowns using matrix method
5. simultaneous equations with two unknowns using graph method.

You have completed the material for this unit on Linear Equations. You should spend some time reviewing the content in detail. Once you are confident that you can successfully write an examination on the concepts, try assignment 5. Check your answers with those provided and clarify any misunderstandings.

Assignment

Assignment

Assignment 5

You should be able to complete this assignment on Linear Equations in 90 minutes.

[Total marks: 50]

1. Solve for t

$2t + 3 = 15$ (3)

2. Solve for x

$7 + 5x = 32$ (3)

3. Solve for w

$\frac{3}{w} = \frac{3}{16}$ (3)

4. Solve for v

$4v = 24$ (2)

5. Solve for *x*

$4x + 2x = 3 - 6(2x - 3)$ (4)

6. Solve for *p*

$1 - 3(p + 1) = p - (2p -)$ (4)

7. Solve for *b*

$7(b - 1) = 21$ (4)

8. Solve for *a*

$3a - 5 = 6 - 2(3 - a)$ (4)

9. Solve for *r*

$\frac{2x}{3} = -4$ (3)

10. Solve for *x*

$\frac{x - [illegible]}{[illegible]} = \frac{x - [illegible]}{6}$ (4)

11. Solve for *z*

$\frac{z + 8}{5} = 9$ (3)

12. Solve the following simultaneous equations by the elimination method.

$9x - 3y = 32$

$4x - 3y = 27$ (3)

13. Solve the following simultaneous equations by the substitution method.

$x + 2y = 8$

$y = \frac{1}{3}x + 9$ (3)

14. Solve the following simultaneous equations by the matrix method.

$$\frac{1}{2}x + y = 5$$

$$x - 2y = 4 \qquad (4)$$

15. Solve the following simultaneous equations by the graph method.

$$y = x + 5$$

$$y = \frac{-1}{3}x - 1 \qquad (4)$$

Answers to Assignment 5

1.

$$2t + 3 = 15$$

$$2t + 3 - 3 = 15 - 3$$

$$2t = 12$$

$$\frac{2t}{2} = \frac{12}{2}$$

$$t = 6$$

2.

$$7 + 5x = 32$$

$$7 - 7 + 5x = 32 - 7$$

$$5x = 25$$

$$\frac{5x}{5} = \frac{25}{5}$$

$$x = 5$$

3.

$$\frac{8}{w} = \frac{8}{16}$$

$$w \times \frac{8}{w} = \frac{8}{16} \times w$$

$$\frac{16}{8} \times 8 = \frac{8}{16} w \times \frac{16}{8}$$

$$16 = w$$

4.

$$4v = 24$$

$$\frac{4v}{4} = \frac{24}{4}$$

$$v = 6$$

5.

$$4x + 2x = 6 - 6(2x - 5)$$

$$4x + 2x = 6 - 12x + 30$$

$$4x + 2x + 12x = 6 - 12x + 12x + 30$$

$$\frac{18x}{18} = \frac{36}{18}$$

$$x = 2$$

6.

$$1 - 3(p + 1) = p - (2p - 1)$$

$$1 - 3p - 3 = p - 2p + 1$$

$$1 - 3p + 3p - 3 = p - 2p + 3p + 1$$

$$1 - 3 = 2p + 1$$

$$-2 - 1 = 2p + 1 - 1$$

$$-3 = 2p$$

$$\frac{-3}{2} = \frac{2p}{2}$$

$$\frac{-3}{2} = p$$

7.

$$7(b - 1) = 21$$

$$7b - 7 = 21$$

$$7b - 7 + 7 = 21 + 7$$

$$\frac{7b}{7} = \frac{28}{7}$$

$$b = 4$$

8.

$$3a - 5 = 6 - 2(3 - a)$$

$$3a - 5 = 6 - 6 + 2a$$

$$3a - 2a - 5 = 6 - 6 + 2a - 2a$$

$$a - 5 = 0$$

$$a - 5 + 5 = 0 + 5$$

$$a = 5$$

9.

$$\frac{2r}{3} = -4$$

$$3 \times \frac{2r}{3} = -4 \times 3$$

$$2r = -12$$

$$\frac{2r}{2} = \frac{-12}{2}$$

$$r = -6$$

10.

$$\frac{x-2}{4} = \frac{x-3}{6}$$

$$12 \times \frac{x-2}{4} = \frac{x-3}{6} \times 12$$

$$3(x-2) = 2(x-3)$$

$$3x - 6 = 2x - 6$$

$$3x - 2x - 6 = 2x - 2x - 6$$

$$x - 6 = -6$$

$$x - 6 + 6 = -6 + 6$$

$$x = 0$$

11.

$$\frac{z+8}{3} = 5$$

$$3 \times \frac{z+8}{3} = 5 \times 3$$

$$z + 8 = 15$$

$$z + 8 - 8 = 15 - 8$$

$$z = 7$$

12. Elimination Method

$9x - 5y = 52$ (1)

$4x - 3y = 27$ (2)

Multiply equation 1 by 3 and equation 2 by 5 to get

$27x - 15y = 156$ (3)

$20x - 15y = 135$ (4)

Subtract equation 4 from equation 3 to get

$27x - 15y = 156$

$\underline{20x - 15y = 135}$

$7x \quad = 21$ (y has been eliminated)

Now, solve for x

$7x = 21$

$\frac{7x}{7} = \frac{21}{7}$

$x = 3$

Then, use $x = 3$ in any of the original equation to find value of y, here we will use equation 1

$9(3) - 5y = 52$

$27 - 5y = 52$

$27 - 27 - 5y = 52 - 27$

$-5y = 25$

$\frac{-5y}{-5} = \frac{25}{-5}$

$y = 5$

The solution is $x = 3$ and $= 5$.

13. Substitution Method

$x + 2y = 8$ (1)

$y = \frac{4}{3}x + 9$ (2)

Substitute y in equation 1 by the expression equal to y from equation 2 to get

$x + 2\left(\frac{1}{3}x + 9\right) = 8$ (note that y has been eliminated)

Solve for *x*

$$x + 2\left(\frac{1}{3}x + 9\right) = 8$$

$$x + \frac{2}{3}x + 18 = 8$$

$$\frac{5}{3}x + 18 - 18 = 8 - 18$$

$$\frac{5}{3}x = -10$$

$$\frac{\frac{5}{3}x}{\frac{5}{3}} = \frac{-10}{\frac{5}{3}}$$

$$x = \frac{-30}{5}$$

$$x = -6$$

Now, put $x = -6$ into equation 1 to find value of *y*

$$-6 + 2y = 8$$

$$-6 + 6 + 2y = 8 + 6$$

$$2y = 14$$

$$\frac{2y}{2} = \frac{14}{2}$$

$$y = 7$$

Solution is $x = -6$ and $y = 7$.

14. Matrix Method

$$y - x = 3$$

$$y + \frac{1}{3}x = -1$$

Note that the equations need to be re-arranged.

$$-x + y = 3$$

$\frac{1}{3}x + y = -1$

Now, re-write in matrix form

$$\begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Find the determinant of matrix $\begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 1 \end{pmatrix}$

$$= (-1 \times 1) - (1 \times \tfrac{1}{3})$$

$$= -1 - \tfrac{1}{3}$$

$$= -1\tfrac{1}{3}$$

$$= \tfrac{-4}{3}$$

Find a new matrix by re-arranging elements of matrix $\begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 1 \end{pmatrix}$ by interchange positions of -1 and 1 diagonal and change the sign of 1 and $\frac{1}{3}$ diagonal

The new matrix is $\begin{pmatrix} 1 & -1 \\ \frac{-1}{3} & -1 \end{pmatrix}$

Find the inverse

$\frac{1}{det}(new\ matrix)$ (det is short for determinant)

$$\frac{1}{\frac{-4}{3}}\begin{pmatrix} 1 & -1 \\ \frac{-1}{3} & -1 \end{pmatrix} = \frac{-3}{4}\begin{pmatrix} 1 & -1 \\ \frac{-1}{3} & -1 \end{pmatrix}$$

Then, multiply the matrix form of the equations by the inverse to get

$$\frac{-3}{4}\begin{pmatrix} 1 & -1 \\ \frac{-1}{3} & -1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ \frac{1}{3} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-3}{4}\begin{pmatrix} 1 & -1 \\ \frac{-1}{3} & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Now, let us work out LHS and RHS separately

$$LHS = \frac{-3}{4}\begin{pmatrix}1 & -1\\ \frac{-1}{3} & -1\end{pmatrix}\begin{pmatrix}-1 & 1\\ \frac{1}{3} & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$= \frac{-3}{4}\begin{pmatrix}1\times -1 + 1\times\frac{1}{3} & 1\times 1 + 1\times 1\\ \frac{-1}{3}\times -1 + -1\times\frac{1}{3} & \frac{-1}{3}\times 1 + -1\times 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$= \frac{-3}{4}\begin{pmatrix}-1\frac{1}{3} & 0\\ 0 & -1\frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$= \frac{-3}{4}\begin{pmatrix}\frac{-4}{3} & 0\\ 0 & \frac{-4}{3}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$= \frac{-3}{4}\left(\frac{-4}{3}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$ (factor out $\frac{-4}{3}$)

$$= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$RHS = \frac{-3}{4}\begin{pmatrix}1 & -1\\ \frac{-1}{3} & -1\end{pmatrix}\begin{pmatrix}3\\ -1\end{pmatrix}$$

$$= \frac{-3}{4}\begin{pmatrix}1\times 3 + 1\times -1\\ \frac{-1}{3}\times 3 + -1\times -1\end{pmatrix}$$

$$= \frac{-3}{4}\begin{pmatrix}\frac{3}{2}\\ \frac{2}{3}\end{pmatrix}$$ (multiply out to simplify)

$$= \begin{pmatrix}\frac{-9}{8}\\ \frac{-1}{8}\end{pmatrix}$$

Now, we have seen above that LHS = RHS

So,

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac{-9}{8}\\ \frac{-1}{8}\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\frac{-9}{8}\\ \frac{-1}{8}\end{pmatrix}$$ (property of unit matrix)

By comparing corresponding elements on both sides we have

$x = \frac{-9}{2}$ and $y = \frac{-1}{8}$

Solution is $x = \frac{-9}{2}$, $y = \frac{-1}{8}$

Assessment

Assessment

You should be able to complete this assessment on Linear Equations in 60 minutes.

[Total marks: 30]

1. Solve the equations

   a) $2y = 8$ (2)

   b) $3p + 4 = 3 - 2(p - 5)$ (2)

2. Solve the equations

   a) 5 – 2(3x – 1) = 2x + 1 (2)

b) $\frac{2}{2t} = \frac{3}{4}$

(2)

3. Solve the simultaneous equations by the elimination method.

$5x - 2y = 16,$

$2x - 3y = 13.$

(3)

4. Solve the equations.

a) $3p + 6 = 7p - 10$

(3)

b) $\frac{3x - 2}{2} = \frac{x}{3}$ (4)

5. Solve the simultaneous equations by the substitution method.

$2x - 3y = 13$

$3x + y = 3$ (4)

6. Solve the following simultaneous equations by the matrix method

$2x - y = 16$

$3x + 2y = 17$ (4)

7. Solve the following simultaneous equations by the graph method.

$$3x + 2y = 12$$
$$3x - 2y = 0$$ (4)

Answers to Assessment 5

1.

a)

$$2y = 8$$
$$\frac{2y}{2} = \frac{8}{2}$$
$$y = 4$$

b)

$$5p + 4 = 8 - 2(p - 3)$$
$$5p + 4 = 8 - 2p + 6$$

$$3p + 2p + 4 = 8 - 2p + 2p + 6$$
$$5p + 4 - 4 = 14 - 4$$
$$5p = 10$$
$$p = 2$$

2.

a)

$$5 - 2(3x - 1) = 2x + 1$$
$$5 - 6x + 1 = 2x + 1$$
$$6 - 6x = 2x + 1$$
$$6 - 6x + 6x = 2x + 6x + 1$$
$$6 - 1 = 8x + 1 - 1$$
$$5 = 8x$$
$$\frac{5}{8} = x$$

b)

$$\frac{2}{5t} = \frac{3}{4}$$
$$20t \times \frac{2}{5t} = \frac{3}{4} \times 20t$$
$$8 = 15t$$
$$\frac{8}{15} = \frac{15t}{15}$$
$$\frac{8}{15} = t$$

3.

Elimination Method

$5x - 2y = 16$ (1)

$2x - 3y = 13$ (2)

Multiply equation 1 by 3 and equation 2 by 2 to get

$15x - 6y = 48$ (3)

$4x - 6y = 26$ (4)

Now subtract equation 4 from equation 3 to get

$11x = 22$ (y has been eliminated)

Solve for x

$$\frac{11x}{11} = \frac{22}{11}$$

$$x = 2$$

Then, substitute $x = 2$ into equation 2

$$2(2) - 3y = 13$$

$$4 - 3y = 13$$

$$4 - 4 - 3y = 13 - 4$$

$$-3y = 9$$

$$\frac{-3y}{-3} = \frac{9}{-3}$$

$$y = -3$$

Solution is $x = 2$ and $y = -3$.

4.

a)

$$3p + 6 = 7p - 10$$

$3p - 3p + 6 = 7p - 3p - 10$ (subtract $3p$ both sides)

$6 = 4p - 10$ (subtract 10 both sides)

$16 = 4p$ (divide by 4 both sides)

$$4 = p$$

b)

$$\frac{2x-2}{5} = \frac{x}{3}$$

$$15 \times \frac{2x-2}{5} = \frac{x}{3} \times 15$$

$$3(2x - 2) = 5x$$

$$6x - 6 = 5x$$

$$6x - 5x - 6 = 5x - 5x$$

$$x - 6 = 0$$

$$x - 6 + 6 = 0 + 6$$

$$x = 6$$

5. Substitution Method

$2x - 3y = 13$ (1)
$3x + y = 5$ (2)

Notice that, in equation 2 can be re-arranged so that it becomes

$y = 5 - 3x$ (make y the subject)

Substitute this expression for y into equation 1 to get

$2x - 3(3 - 3x) = 13$

Solve for x

$2x - 9 + 9x = 13$ (remove brackets)

$11x - 9 + 9 = 13 + 9$ (collect like terms)

$11x = 22$ (add 9 both sides)

$\frac{11x}{11} = \frac{22}{11}$ (divide by 11 both sides)

$x = 2$

Now, use $x = 2$ in equation 2 to find value of y

$3(2) + y = 5$

$6 + y = 5$

$y = -3$

Solution is $x = 2$, $y = -3$

6. Matrix Method

$2x - y = 16$ (1)
$3x + 2y = 17$ (2)

Re-write in matrix form

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ 17 \end{pmatrix}$$

Find the determinant

$$(2 \times 2) - (-1 \times 3) = 4 + 3$$
$$= 7$$

Find inverse

$$\frac{1}{det}(new\ matrix)$$

(Remember new matrix is found by re-arranging the elements of the original matrix)

So, inverse is

$$\frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$$

Then multiply the equations in matrix form by the inverse both sides to get

$$\frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 16 \\ 17 \end{pmatrix}$$

$$LHS = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 4+3 & -2+2 \\ -6+6 & 3+4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{7}(7)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ (factor out 7)

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$RHS = \frac{1}{7}\begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 16 \\ 17 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 32 + 17 \\ -48 + 34 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} 49 \\ -14 \end{pmatrix}$$

$$= \frac{1}{7}(7)\begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

Remember that these are two equal sides of an equation, so

LHS = RHS

Meaning that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \qquad \text{(property of unit matrix)}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$$

Comparing corresponding elements on both sides gives

$x = 7 \text{ and } y = -2$

Solution is $x = 7$, $y = -2$

Unit Contents

Unit 6

Unit 6

Changing The Subject of the Formula

Introduction

Formulas are written so that a single variable, the "unknown", is on the left hand side of the equation. The rest of the variables, the "known" variables, are written on the right hand side of the equation.

The unknown is the **subject of the formula.**

We **change the subject of the formula** by choosing and carrying out a series of operations on both sides of the equation so that the new subject is left standing alone on the left hand side, and all the other variables and constants are collected on the right hand side.

This unit consists of 25 pages. This is approximately 1% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 15 hours to work on this unit, 10 hours for formal study and 5 hours for self-study and completing assessments/assignments.

When reading the following learning outcomes, think about them as a guide to what you should focus on while studying this unit.

This Unit is Comprised of Four Lessons:

Lesson 1 Defining and Isolating The Subject of the Formula
Lesson 2 Changing The Subject of the Formula Using Factorisation
Lesson 3 Changing the Subject of the Formula with Fractions
Lesson 4 Changing The Subject of the Formula With Powers and Roots

Upon completion of this unit you will be able to:

Outcomes

- *change* the subject of the formula involving roots, powers, fractions and factorisation to a specified variable.

Terminology

Formula: An equation expressing the relation between two or more variables a formula cannot be solved, until the variables are replaced with their values.

Roots: A number that, when multiplied by itself a certain number of times, equals the number that you have:
2 is the fourth root of 16.

Powers: A convenient way of writing multiplication of a number by itself. They are sometimes called indices or exponents.

Factor: Factors of numbers are numbers that divide into another exactly.

factorisation: Writing a number or an expression as a product of its factors.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Defining and Isolating The Subject of the Formula

Introduction

By the end of this subunit, you should be able to

- define the subject of the formula

Defining and Isolating the Subject of the Formula

In the formula for the area of a triangle,

A = ½b × h

the variable on the left, A, is called **the subject of the formula**. This is the variable not known.

The rest of the variables, which are the “known” variables are written on the right hand side of the equation. This is the **rule** that tells us how to calculate the subject, the area, if we are given the base and the height.

If the area and the base are already known, and we need to calculate the height, then it would make more sense to have a formula of the form

h =.......

This says we are making **h** the subject of the formula. This is the variable not known. We will have to isolate **h**, step by step.

A and **b**, which are the known variables are going to be written on the right hand side of the equation.

We are **changing the subject of the formula**. We do this by choosing and carrying out a series of operations on both sides of the equation so that the new subject is left standing alone on the left hand side, and all the other variables and constants, if there are any, are collected on the right hand side.

We can use the balancing principle of equations to transform it to an equivalent formula in which the unknown variable is the subject.

When we change the subject of the formula, we apply the same rules as we have applied to solve normal equations:

Whatever you do to the left hand side, you must do to the right hand side.

These are the things that we can do:

- add the same variable or constant to both sides
- subtract the same variable or constant from both sides
- multiply both sides by the same variable or constant
- divide both sides by the same variable or constant
- square both sides
- take the square root of both sides

Reflection

Reflection

When we change the subject of the formula, we apply the same rules as we have applied to solving normal equations.

Let us remind ourselves of how we solve equations.

Solve for x in this equation, $2x + 4 = 12$

Solution

To solve the equation $2x + 4 = 12$, we have to find the value of x, that will make the statement true. To do this, we need to isolate a single x on the left and end with the number on the right.

Remember **whatever you do to the left hand side, you must do to the right hand side.**

$$\begin{array}{rcl} 2x + 4 &=& 12 \\ \underline{\quad -4} && \underline{-4} \\ 2x &=& 8 \end{array}$$ (subtracting 4 from both sides)

$$\frac{2x}{2} = \frac{8}{2}$$ (dividing both sides by 2)

$$x = 4$$

Example 1

If $P = R + ST$, make S the new subject.

This means we have to isolate S, step by step, so that we end with S = ….

$P = R + ST$

Subtracting R from both sides

$$\begin{array}{rcl} P &=& R + ST \\ \underline{\quad -R} && \underline{-R} \\ P - R &=& ST \end{array}$$

Now divide by T. Here we make an assumption that $T \neq 0$, to avoid division by 0.

$$\frac{P-R}{T} = \frac{ST}{T}$$

$$\underline{P - R} = S$$

T

As a matter of convention, the subject of the formula is written on the left hand side.

$$S = \frac{P - R}{T}$$

Example 2

If $I = Prt$, make r the new subject.

We have to isolate r , step by step . This says P and t have to be removed from the right hand side.

We do not need to do the division in two separate steps, we can just do it in one step.

We divide both sides by P and t. Again here we make an assumption that $Pt \neq 0$, to avoid division by 0.

$$\frac{I}{Pt} = \frac{Prt}{Pt},$$

$$\frac{I}{Pt} = r$$

$$r = \frac{I}{Pt}$$

Activity 1

Transform these formulae to make the letter in brackets the new subject

1. $A = 2\pi rh$ (h)

2. $p = a + bt$, (t)

3. $y = 2x + 1$ (x)

4. $p = -3q + 6$ (q)

5. $ax + b = 0$ (x)

6. $x = 2a + d(n-1)$ (n)

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on defining and isolating the subject of the formula are:

- It is a matter of convention that formulas are written such that the subject of the formula is on the left hand side.

- The subject of the formula can be changed by doing to the right hand side whatever you do to the left hand side.

In the next sub unit, we are going to look at changing the subject of the formula using factorisation

Answers:

1. $A = 2\pi rh$

$$\frac{A}{2\pi r} = \frac{2\pi rh}{2\pi r}$$

$$\frac{A}{2\pi r} = h$$

2. p = a + bt

-a -a

p - a= bt

$$\frac{p}{b} - \frac{a}{b} = \frac{bt}{b}$$

$$\frac{p}{b} - \frac{a}{b} = t$$

$$\frac{p-a}{b} = t$$

$$t = \frac{p-a}{b}$$

3. y = 2x + 1

y = 2x + 1

-1 -1

y – 1 = 2x

$$\frac{y-1}{2} = x$$

$$x = \frac{y-1}{2}$$

4. $p = -3q + 6$
   $p - 6 = -3q$

   $\frac{p-6}{-3} = q$

5. $ax + b = 0$

   $ax = -b$

   $x = \frac{-b}{a}$

6. $x = 2a + d(n-1)$

7.
   $x = 2a + d(n-1)$
   $\underline{-2a \quad -2a}$
   $x - 2a = +d(n-1)$

   $\frac{x-2a}{d} = +\frac{d(n-1)}{d}$

   $\frac{x-2a}{d} = +n-1$

   $\frac{x-2a}{d} + 1 = +n$
   $n = \frac{x-2a}{d} + 1$

Lesson 2 Changing The Subject of the Formula Using Factorisation

By the end of this subunit, you should be able to

- change the subject of the formula using factorisation

Changing the Subject of the Formula With the New Subject in Two Terms

Factorisation becomes useful when changing the subject of the formula with the new subject in two terms.

Example 1

If $S = P + Prt$, make P the new subject.

We have P in two terms. It is a common factor in these two terms. We isolate it with the help of **factorisation.**

$S = P + Prt$

Factorising the right hand side;

$S = P(1 + rt)$

We now divide both sides by $1 + rt$. Again, we make an assumption that $1 + rt \neq 0$, to avoid division by 0.

$$\frac{S}{1 + rt} = \frac{P(1 + rt)}{1 + rt}$$

$$\frac{S}{1 + rt} = P$$

$$P = \frac{S}{1 + rt}$$

Example 2

If $2s - 3 = st + 6$, make s the new subject.

We have two terms that contain s, the new subject.
Let us put them on one side

$2s - st = 6 + 3$

$2s - st = 9$

s is now a common factor in the two terms on the LHS. We isolate it with the help of **factorisation.**

$s(2 - t) = 9$

Factorising the right hand side;

We now divide both sides by $2 - t$.
Again, we make an assumption that $2 - t \neq 0$, to avoid division by 0.

$$\frac{s(2-t)}{2-t} = \frac{9}{2-t}$$

$$s = \frac{9}{2-t}$$

Activity 2

Transform these formulae to make the letter in brackets the new subject

1. p = at + bt, (t)
2. v = au + bu (u)
3. m = pq – pr (p)

Compare your answers with those at the end of this subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content and work through the activity again.

Key Points to Remember

The key point to remember in this subunit on changing the subject of the formula using factorisation are:

- If the two terms containing the new subject are on different sides of the equation, put them on one side. Continue with factorisation.

Answers:

1. p = at + bt

   p = t(a + b)

   $$\frac{p}{a+b} = t$$

   $$t = \frac{p}{a+b}$$

2. v = au + bu

   v = u(a + b)

   $$\frac{v}{a+b} = u$$

   $$u = \frac{v}{a+b}$$

3. m = pq – pr

$$m = p(q - r)$$

$$\frac{m}{q-r} = p$$

$$p = \frac{m}{q-r}$$

Lesson 3 Changing the Subject of the Formula with Fractions

Introduction

By the end of this subunit, you should be able to

- change the subject of a formula with fractions

Changing the Subject of the Formula with Fractions

When in a formula there are fractions, clear the fractions by multiplying throughout by the Least Common Denominator (LCM).

Example 1

If $r = \frac{2mI}{B(n+1)}$ make m the new subject

First remove the fraction. Do this by multiplying both sides of the equation by the LCM. The LCM of B(n + 1) *and* 1 is B(n + 1)

$$B(n+1) \times r = \frac{2mI}{B(n+1)} \times B(n+1)$$

$$B(n+1) \times r = 2mI$$

We divide both sides by $2I$. $(2I \neq 0)$

$$\underline{B(n+1)} \times \underline{r} = \underline{2mI}$$

$2I \qquad\qquad 2I$

$\frac{B(n+1) \times r}{2I} = m$

$m = \frac{B(n+1) \times r}{2I}$

$m = \frac{Br(n+1)}{2I}$

Example 2

Change the formula to give the new subject

$E = \frac{W}{W+x}$ to W

$(W + x) \times E = \frac{W}{W+x} \times (W + x)$	Each of the terms is multiplied by the LCM of 1 and W + x which is W + x
$(W + x) \times E = W$	
$(W + x) \times E = W$	Removing the brackets on the LHS
$WE + Ex = W$	
$WE + Ex = W$	Put "W" terms on the LHS
$WE - W = -Ex$	
$W(E - 1) = -Ex$	Factorising the LHS
$\frac{W(E-1)}{E-1} = \frac{-Ex}{E-1}$ $(E - 1 \neq 0)$	Divide both sides by E - 1
$W = \frac{-Ex}{E-1}$	

Activity 3

Transform these formulae to make the letter in brackets the new subject

1. $y = \frac{8 - 5x}{x}$ (x)

2. $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$ (u)

3. $y = \frac{3x - 1}{2x}$ (x)

4. $T = \frac{2t + a}{t + b}$ (t)

5. $G = 5r - \frac{r - s}{g}$ (r)

Check your performance against the given solutions at the end of this subunit. Continue if you are satisfied with your ability to answer the questions. If not, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on changing the subject of the formula with fractions are:

- if the equation contains fractions, we simplify the equation by clearing the fractions first

Answers:

1. $y = \frac{8 - 5x}{x}$

$xy = 8 - 5x$

$xy + 5x = 8$

$x(y + 5) = 8$

$x = \frac{8}{y + 5}$

2. $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$

$$\text{fuv}.\ \frac{1}{f} = \text{fuv}\ .\ \frac{1}{u} + \text{fuv}\ .\ \frac{1}{v}$$

uv = fv + fu
uv - fu = fv

u(v – f) = fv

$u = \frac{fv}{(v - f)}$

3. $y = \frac{3x - 1}{2x}$

$$2x\ .\ y = 2x\ .\ \frac{3x-1}{2x}$$

2xy = 3x - 1
2xy – 3x = - 1
x(2y – 3) = - 1
$x = \frac{-1}{2y - 3}$

4. $T = \frac{2t + a}{t + b}$

T(t + b) = 2t + a
Tt + Tb = 2t + a
Tt – 2t = a - Tb
t(T – 2) = a - Tb
$t = \frac{a - Tb}{T - 2}$

5. $G = 5r - \frac{r - s}{g}$

$$G + \frac{r-s}{g} = 5r$$

Gg + r – s = 5r

$r - 5r = s - Gg$

$r(1 - 5) = s - Gg$

$r(-4) = s - Gg$

$-4r = s - Gg$

$$r = \frac{s - Gg}{-4}$$

Lesson 4 Changing The Subject of the Formula With Powers and Roots

Introduction

By the end of this subunit, you should be able to

- change the subject of a formula with powers and roots

Changing the Subject of the Formula with Powers and Roots

When in a formula there are powers, we clear powers with corresponding roots.

We clear power 2 by taking the square root of both sides.

We clear power 3 by taking the cube root of both sides, and so on.

When in a formula there are roots, we clear roots with corresponding powers.

We clear square root by raising both sides to power 2.

We clear cube root by raising both sides to power 3, and so on.

Example 1

If $A = \pi r^2$, make *r* the new subject.

We start off by dividing both sides by π.

$$\frac{A}{\pi} = \frac{\pi r^2}{\pi}$$

$$\frac{A}{\pi} = r^2$$

In this case we will clear power 2 by taking the square root of both sides.

$$\frac{A}{\pi} = r^2$$

$$\sqrt{\frac{A}{\pi}} = \sqrt{r^2}$$

$$\sqrt{\frac{A}{\pi}} = r$$

$$r = \sqrt{\frac{A}{\pi}}$$

Example 2

If $y = 3\sqrt{h}$, make h the new subject.

$y = 3\sqrt{h}$

divide both sides by 3

$$\frac{y}{3} = \frac{3\sqrt{h}}{3}$$

$$\frac{y}{3} = \sqrt{h}$$

Squaring both sides

$$\left(\frac{y}{3}\right)^2 = \left(\sqrt{h}\right)^2$$

$$\left(\frac{y}{3}\right)^2 = h$$

$$\left(\frac{y^2}{3^2}\right) = h$$

$$\frac{y^2}{9} = h$$

$$h = \frac{y^2}{9}$$

Activity 4

Transform these formulae to make the letter in brackets the new subject

1. $N = \sqrt{c}$ (c)
2. $x^2 + y^2 = z^2$ (y)
3. $\sqrt{x-3} = A$ (x)
4. $E = kmv^2$ (v)
5. $v^2 = u^2 + 2as$ (u)

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on changing the subject of the formula with powers and roots are:

- When in a formula there are powers, we clear powers with corresponding roots.
- When in a formula there are roots, we clear roots with corresponding powers.

You have now completed work on this unit on algebraic manipulation. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Answers

1. $N = \sqrt{c}$

$\sqrt{N} = c$

$c = \sqrt{N}$

2. $x^2 + y^2 = z^2$

$y^2 = z^2 - x^2$

$\sqrt{y^2} = \sqrt{z^2} - \sqrt{x^2}$

$y = z - x$

3. $\sqrt{x-3} = A$

$\left(\sqrt{x-3}\right)^2 = A^2$

$x - 3 = A^2$

$x = A^2 + 3$

4. $E = kmv^2$

$\frac{E}{km} = \frac{kmv^2}{km}$

$\frac{E}{km} = v^2$

$\sqrt{\frac{E}{km}} = v$

$v = \sqrt{\frac{E}{km}}$

5. $v^2 = u^2 + 2as$

$v^2 - 2as = u^2$

$\sqrt{v^2 - 2as} = u$

$u = v^2 - 2as$

Unit Summary

Summary

In this unit you learned that when changing the subject of the formula:

- you choose and carry out a series of operations on both sides of the equation so that the new subject is left standing alone on the left hand side, and all the other variables and constants, if there are any, are collected on the right hand side;
- using factorisation, if the two terms containing the new subject are on different sides of the equation, put them on one side, then factorise;
- clear the fractions by multiplying throughout by the Least Common Denominator (LCM);
- when, there are powers, we clear powers with corresponding roots;
- when there are roots, we clear roots with corresponding powers;

You have completed the material for this unit on algebraic manipulation. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

1. **Answer All Questions.**
2. **Show all the necessary working.**

Total marks = 12 **Time**: 15 minutes

Transform these formulae to make the letter in brackets the new subject.

1. p = 8q – 1 (q) [2]

2. mx – my = z (m) [2]

3. $T = \dfrac{x}{x + y}$ (x) [3]

4. $m = 5\sqrt{t}$ (t) [2]

5. $R = 2t^2 + \dfrac{1}{u}$ (t) [3]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1. $p = 8q - 1$

$p + 1 = 8q$

$\frac{p+1}{8} = q$

$q = \frac{p+1}{8}$

2. $mx - my = z$

$m(x - y) = z$

$m = \frac{z}{x-y}$

3. $T = \frac{x}{x+y}$

$T(x + y) = x$

$Tx + Ty = x$

$Tx - x = -Ty$

$x(T - 1) = -Ty$

$x = \frac{-Ty}{T-1}$

4. $m = 5\sqrt{t}$

$\frac{m}{5} = \sqrt{t}$

$\left(\frac{m}{5}\right)^2 = t$

$\frac{m^2}{25} = t$

$$t = \frac{m^2}{25}$$

5. $$R = 2t^2 + \frac{1}{u}$$

$$Ru = 2ut^2 + 1$$

$$Ru - 1 = 2ut^2$$

$$\frac{Ru - 1}{2u} = t^2$$

$$\sqrt{\frac{Ru - 1}{2u}} = t$$

$$t = \sqrt{\frac{Ru - 1}{2u}}$$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

1. **Answer All Questions.**
2. **Show all the necessary working.**

Total marks = 14 **Time**: 20 minutes

Transform these formulae to make the letter in brackets the new subject

1. P = am + b (m) [2]
2. y = ⅔x - ¼ (x) [3]
3. A = $\pi r^2 + 2\pi rh$ (h) [3]
4. S = $\dfrac{\sqrt{b^2 - 4a^2c}}{2a}$ (a) [3]
5. g = $\dfrac{2p - q}{5}$ (q) [3]

Answers

1. m = $\dfrac{P - b}{a}$

2. x = $\dfrac{12y + 3}{8}$

3. h = $\dfrac{A - \pi r}{2\pi r}$

4. a = $\pm\sqrt{\dfrac{b^2}{4s^2 + 4c}}$

5. q = 5g – 2p

Unit Contents

Unit 7

Unit 7

Matrices

Introduction

Matrices can be used as a store of information.

Liteboho is a driver in a soft drink company. His work is to deliver some cases of one-litre bottles of various soft drinks to customers. Before he goes home he prepares orders for the next day. The orders can be represented as follows:

	Fanta	Coca-cola	Sprite	Stoney
Matau	1	2	0	1
Shoaepane	2	4	1	5
Liako	0	3	2	3

a) How many rows are there? ______________________

b) How many columns are there? ___________________

c) Shoaepane's drinks are in row____________________

d) Liako does not order Fanta. We know this because of what is written in row____________ and column__________________

e) How many cases should Liteboho deliver altogether? ______________

Compare your answers with those given below.

Model Answers

a) 3 rows b) 4columns c) row 2

d) row 3, column 1 e) 24 cases

I hope you got all the answers correct. This idea will be covered in detail below

If Liteboho can remember the headings, the information in the table above can be written as follows:

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 5 \\ 0 & 3 & 2 & 3 \end{pmatrix}$$

We have represented the information in **matrix form**.

In the Junior Secondary Certificate Mathematics course you learned addition, subtraction and multiplication of matrices. We will review those skills in this unit. You are also going to learn how to find the determinant and the inverse of a matrix.

This unit consists of 44 pages. This is 2% of the whole course, so plan your time accordingly. As reference, you will need to devote 20 hours to work on this unit, 15 hours for formal study and 5 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Five Lessons:

Lesson 1 Addition and Subtraction of Matrices
Lesson 2 Multiplication of Matrices
Lesson 3 The Determinant of a Matrix
Lesson 4 Identity Matrix
Lesson 5 Inverse of a Matrix

Upon completion of this unit you will be able to:

Outcomes

- *add* matrices of any order;
- *subtract* matrices of any order;
- *multiply* matrices of any order;
- *find* the determinant of 2 by 2 matrices;
- *find* the inverse of 2 by 2 matrices.

Terminology

Null or Zero matrix: A matrix whose entries are all zeros.

Zero matrix **O** = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Determinant of a matrix: The product of the numbers in the leading diagonal minus the product of the numbers in the other diagonal.

Order of a matrix: The number of rows followed by the number of columns. If matrix A = $\begin{pmatrix} 1 & 5 & 1 \\ 3 & 4 & 2 \end{pmatrix}$, then the order of a matrix is 2×3.

Unit or Identity matrix: A 2×2 matrix that does not change the entries of the matrix it is multiplying.

Identity matrix **I** = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Scalar: A number multiplying each element of the matrix.

Lesson 1 Addition and Subtraction of Matrices

In Arithmetic, there are rules for addition, and subtraction. Can these rules be applied to matrices?

In unit 5 on Linear Equations you learnt about solving linear equations using the matrix method. Do you still remember the order of a matrix? In this Unit you will use the order of a matrix to determine whether the rules of addition and subtraction can be applied.

In this section you are going to remind yourself about addition and subtraction of matrices.

At the end of this sub-unit you should be able to:

- *add* matrices of any order.
- *subtract* matrices of any order.

There are six pages in this sub-unit.

Activity

Activity 7.1

In this activity you are going to learn how to add matrices.

The tables below shows the amounts of bread, sugar, and milk used by Rono and Kati families over a period of two weeks:

.

1st Week	Rono	Kati
Bread (loaves)	7	9

Sugar (kg)	4	5
Milk (litres)	12	8

2nd Week	Rono	Kati
Bread (loaves)	6	7
Sugar (kg)	5	3
Milk (litres)	10	6

a) Represent the above information in the matrix form. Let A be the matrix for the information for the 1st week, and B be the matrix for the information for the 2nd week.

A = B =

b) How many kilograms of sugar were used by the Rono family in two weeks? _________________

c) How many litres of milk were used by the Rono family in two weeks?

____

d) The Kati family used________________ litres of milk in two weeks.

e) How many loaves of bread were used by the Kati family in two weeks?

____

f) Based on your answers for questions b) through e), how much bread, sugar and milk were used by the two families in two weeks? Represent your answer in matrix form.

Compare your answers with those given below.

Model Answers

Activity 7.1

a)

$A = \begin{pmatrix} 7 & 9 \\ 4 & 5 \\ 12 & 8 \end{pmatrix}$ $B = \begin{pmatrix} 6 & 7 \\ 5 & 3 \\ 10 & 6 \end{pmatrix}$

b) 9kg

c) 22 litres

d) 14 litres

e) 14 loaves

f) $\begin{pmatrix} 13 & 16 \\ 9 & 8 \\ 22 & 14 \end{pmatrix}$

In the above activity, you added values for the same items in the two tables. You also added corresponding entries in the two matrices to get entries for the matrix required in (f). It is important to note that matrices can only be added if they are of the same order. That is both matrices have the same number of rows and columns.

Activity

Activity 7.2

In this activity you are going to learn about the subtraction of matrices.

Consider the following case.

‘Malebohang owns a book store, which stocks some books with come in two types – those with hard covers and those with soft covers. She wants to find out which type her customers prefer for three different titles: books A, B and C. The following table shows the sales of the three books over a period of two months.

Month 1

	Book A	Book B	Book C
Hard cover type	13	7	6
Soft cover type	28	19	42

Month 2

	Book A	Book B	Book C
Hard cover type	10	5	2
Soft cover type	34	19	55

a) How many copies of Book A, soft cover type, were bought during the first month? ___________

b) How many copies of Book C, hard cover type, were sold during the second month? ______

c) Represent the information in the two tables in matrices.

d) What is the order of each matrix?

____

e) What is the total number of copies of each type of book sold over the two months? Use matrices to work out your answer.

f) Find the difference in the total sales of the three Books over the two-month period. Support your answer with calculations using matrices.

g) Which cover type is preferred by the store's customers? Support your answer with some explanation:

____

Now compare your answers with those given below.

Answers

a) 13

b) 2

c) Month 1: $\begin{pmatrix} 13 & 7 & 2 \\ 28 & 19 & 42 \end{pmatrix}$

Month 2: $\begin{pmatrix} 10 & 5 & 6 \\ 34 & 19 & 55 \end{pmatrix}$

d) 2 by 3

e) $\begin{pmatrix} 23 & 12 & 8 \\ 62 & 38 & 97 \end{pmatrix}$

f) $\begin{pmatrix} -3 & -2 & 4 \\ 6 & 0 & 13 \end{pmatrix}$

g) The soft cover type is favoured by customers. More copies of the soft cover type were sold for each book in this study.

Did you get all the answers correct? If so well done! If not then you should go through the activity again.

Note it!

When adding or subtracting matrices, we operate on corresponding entries.

Matrices can only be added or subtracted if they are of the same order, that is they have the same number of rows and columns.

Activity

Activity 7.3

Work out the following;

1. a) $\begin{pmatrix} 6 \\ 1 \\ 8 \end{pmatrix} - \begin{pmatrix} 8 \\ 0 \\ 5 \end{pmatrix}$

b). $\begin{pmatrix} 8 & 3 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 1 & 7 \end{pmatrix}$

c) $\begin{pmatrix} -5 & 1 & 13 \\ 2 & -4 & -7 \end{pmatrix} - \begin{pmatrix} 4 & 5 & 11 \\ 2 & -9 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 7 \\ 3 & 1 & 2 & 4 \end{pmatrix} + \begin{pmatrix} -5 & 2 & 4 & -7 \\ 3 & -1 & 5 & 0 \\ 2 & 6 & -3 & -8 \end{pmatrix}$

2.Let $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 5 & 6 \end{pmatrix}$ $B = \begin{pmatrix} 5 & -2 & 3 \\ 7 & 0 & 13 \\ 12 & 1 & 4 \end{pmatrix}$ $C = \begin{pmatrix} 2 & 1 \\ 0 & 5 \\ 3 & 4 \end{pmatrix}$

$D = \begin{pmatrix} 7 & 3 & 6 \\ 1 & 2 & 0 \end{pmatrix}$ $and\ E = \begin{pmatrix} 8 & 0 & -1 \\ -6 & 3 & 0 \\ 1 & 2 & 2 \end{pmatrix}$

Work out the following

a) A + D

b) B – E

c) A + B

d) D – C

Model Answers

Activity 7.3

1. a) $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} 10 & 7 \\ 2 & -2 \end{pmatrix}$

c) $\begin{pmatrix} -9 & -4 & 2 \\ 0 & 5 & -8 \end{pmatrix}$

d) $\begin{pmatrix} -3 & 5 & 8 & -2 \\ 4 & 1 & 10 & 7 \\ 5 & 7 & -1 & -4 \end{pmatrix}$

2a) $A + D = \begin{pmatrix} 8 & 1 & 9 \\ -3 & 5 & 6 \end{pmatrix}$

b) $B - E = \begin{pmatrix} -8 & -2 & 4 \\ 13 & -3 & 13 \\ 11 & -1 & 2 \end{pmatrix}$

c) Addition cannot be done because the matrices do not have the same order.

d) Subtraction cannot be done because the matrix D and matrix C do not have the same order.

The key points to remember on addition and subtraction of matrices are:

Addition,

1. two matrices can be added only when they have the same number of rows and the same number of columns (same order). Can you add if the number of rows are equal and the number of columns are not equal?

2. to add two matrices, you add the elements in corresponding positions in each matrix.

Subtraction,

1. two matrices can be subtracted only when they have the same number of rows and the same columns (same order). Can you subtract if the number of rows are not equal but the number of columns are equal?
2. to subtract two matrices, you subtract the elements in corresponding positions in each matrix.

You have now completed the sub-unit on addition and subtraction of matrices. Go back to the beginning of the sub-unit and briefly review the content. When you think you have mastered the topic, move on to the next sub-unit.

Lesson 2 Multiplication of Matrices

You have just learned that addition and subtraction of matrices is possible only when the number of rows and columns of the two matrices are the same. Do you think this also applies to multiplication of matrices?

Do you remember the steps of solving simultaneous equations by matrix method? This you learned in unit 5 Linear equations;

Step 1

Write equations in matrix form

Step 2

Find the determinant

Find the inverse

Step 3

Multiply equations matrix form by inverse, then simplify both sides to a stage where you can compare corresponding elements on both sides of the equations for the values of the two unknowns.

Do you think these are related to multiplication of matrices? How?

In this sub-unit you are going to learn how to multiply a matrix by a scalar and another matrix.

At the end of this sub-unit you should be able to:

- define the term 'scalar' and explain how they are used;
- multiply a matrix by a scalar;
- *multiply* matrices of any order.

There are nine pages in this sub-unit.

Multiplication of a Matrix by a Scalar

Activity

Activity 7.4

Consider the following case

Mr. Nku has two children in primary school. One is in Class 4 and the other is in Class 6.

The following table shows the stationary required at school for their respective classes each semester (session).

	Class 4	Class 6
Pens	2	3
Pencils	2	1
Crayons	3	2
Erasers	1	2

a) Represent this information in a matrix form.

b) If Mr. Nku buys all the stationary for the whole year at the beginning of the year, how many items will he buy for each child?

Now compare your answers with the ones below.

Model Answers

Activity 7.4

a) $\begin{pmatrix} 2 & 3 \\ 2 & 1 \\ 3 & 2 \\ 1 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 4 & 6 \\ 4 & 2 \\ 6 & 4 \\ 2 & 4 \end{pmatrix}$

Did you get them all correct? If so, good! If not you should go through the activity again.

But, look how matrix in b) is obtained from matrix in a); each element of matrix in a) was multiplied by 2 (a year has two semesters (or sessions)) to get the corresponding element in b).

The number 2 here is a multiplier. It multiplies all the elements of a matrix. The 2 is called a scalar.

Note it

Now you should have found that:

- When multiplying a matrix by a scalar, we multiply **all entries** of the matrix by the **scalar.**

Activity 7.5

Activity

Now work out the following

a) $4\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

b) $2\begin{pmatrix} 6 & 7 \\ -3 & -5 \\ 1 & 2 \end{pmatrix}$

c) $3\begin{pmatrix} 2 & 5 & -1 \\ 2 & 0 & 1 \end{pmatrix}$

d) $\frac{1}{2}\begin{pmatrix} 10 & 2 & 8 \end{pmatrix}$

Model Answers

Activity 7.5

a) $\begin{pmatrix} 12 \\ 20 \\ 4 \end{pmatrix}$

b) $\begin{pmatrix} 12 & 14 \\ -6 & -10 \\ 2 & 4 \end{pmatrix}$

c) $\begin{pmatrix} 6 & 15 & -3 \\ 6 & 0 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 5 & 1 & 4 \end{pmatrix}$

When multiplying a matrix by a scalar, all elements of the matrix are multiplied by the same number.

Multiplying One Matrix by Another

Activity

Activity 7.6

In this activity you are going to learn how to multiply a matrix by another matrix.

Lintle has a catering business. She serves two types of meat in every meal. These are beef and chicken.

Beef cost M45 per kg.

Chicken cost M20 per kg

One day she bought 5kg of beef and 7kg of chicken for her business.

On another occasion she bought 6kg of beef and 2kg of chicken.

a) Complete the following tables representing the above information.

	Beef	Chicken
Occasion 1		
Occasion 2		

	Price
Beef	
Chicken	

b) How much money did Lintle spend on occasion 1?

____

c) How much money did Lintle spend on the second occasion?

____

d) Represent the information in the tables above in matrix form.

We can use matrix multiplication to work out how much money Lintle spends on beef and chicken.

You can check your answers for the above activity as you work through the following example.

Example 1

We will use the information in the above activity.

We can represent the above data in two matrices.

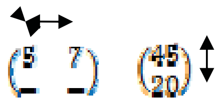
$\begin{pmatrix} 5 & 7 \\ 6 & 2 \end{pmatrix}$ and $\begin{pmatrix} 45 \\ 20 \end{pmatrix}$

Money spent on beef on occasion 1 = 5 × M45 = M225

Money spent on chicken on occasion 1 = 7 × M20 = M140

Total money spent on occasion 1 for meat = M (225 + 140) = M365

This means that the first row of the 2by2 matrix is multiplied by the 2by1 matrix as follows



Money spent on beef on occasion 2 = 6 × M45 = M270

Money spent on chicken on occasion 2 = 2 × M20 = M40

Total money spent on meat on occasion 2 = M (270 + 40) = M310

We can show the money spent as occasion 1 M365

occasion 2 M310

or as a matrix $\begin{pmatrix} 365 \\ 310 \end{pmatrix}$

We can represent the calculation as a multiplication of matrices

$$\begin{pmatrix} 5 & 7 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 45 \\ 20 \end{pmatrix} = \begin{pmatrix} 365 \\ 310 \end{pmatrix}$$

Using this example we can formulate the rules for multiplying matrices.

Note it!

Rules for multiplying matrices

- The number of columns of the first matrix must be equal to the number of rows in the second matrix.
- Multiply the elements of a row of the first matrix by the elements of a column of the second matrix and add the products.
- The order of the product matrix is the number of rows of the first matrix by the number of the columns in the second matrix. For example, if the order of the second matrix is ***3by2*** and the order of the second matrix is ***2by4*** then the order of the product will be ***3by4***. That is ***3by2*** and ***2by4*** gives ***3by4.***

Now consider the following examples.

Example 2

Find the product of the following

$$\begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

How does the order of the matrix help one to check the answer of the matrix multiplication?

Write down the instruction for multiplying a matrix by another matrix to explain how the multiplication is carried out.

Compare your answer with the sample answers given below. I hope you are doing well up to now! If not go through the solution and compare it with the two questions above. If you need more help go over the example slowly and carefully again. Then read 'Note it!' above.

Solution

Order of the first matrix =1by2

Order of the second matrix = 2by1

Order of the product matrix = 1by1

Multiply the first element of the row in the first matrix by the element of the column in the second matrix: $2 \times 4 = 8$

Multiply the second element of the row in the first matrix by the second element of the column in the second matrix: $5 \times 3 = 15$ and add them up, $8 + 15 = 23$

$$\begin{pmatrix} 2 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} =: \begin{pmatrix} 2 \times 4 + 5 \times 3 \end{pmatrix} = (23)$$

Example 3

Calculate $\begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -4 & 3 \\ 6 & 2 \end{pmatrix}$

Solution

The orders of the matrices are 2by2 and 2by2. Therefore the order of the answer matrix is also 2by2.

$$\begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} -4 & 3 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 \times -4 + 5 \times 6 & 1 \times 3 + 5 \times 2 \\ 3 \times -4 + 2 \times 6 & 3 \times 3 + 2 \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 26 & 7 \\ 0 & 13 \end{pmatrix}$$

Example 4

Calculate $\begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & -2 \\ 3 & 1 & 0 \end{pmatrix}$

Solution

The orders of the matrices are 2by2 and 2by3. Therefore the order of the product matrix is 2by3

$$\begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & -2 \\ 3 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \times 4 + 5 \times 3 & 4 \times 2 + 5 \times 1 & 4 \times -2 + 5 \times 0 \\ 3 \times 4 + 1 \times 3 & 3 \times 2 + 1 \times 1 & 3 \times -2 + 1 \times 0 \end{pmatrix}$$

$$= \begin{pmatrix} 16 + 15 & 8 + 5 & -8 + 0 \\ 12 + 3 & 6 + 1 & -6 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} 31 & 13 & -8 \\ 15 & 7 & -6 \end{pmatrix}$$

Example 5

Calculate $\begin{pmatrix} -2 & 0 \\ 1 & -5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$

Solution

The order of the resulting matrix is 3by2.

$$\begin{pmatrix} -2 & 0 \\ 1 & -5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -2\times 3+0\times 2 & -2\times 0+0\times 1 \\ 1\times 3\pm 5\times 2 & 1\times 0\pm 5\times 1 \\ 3\times 3+4\times 2 & 3\times 0+4\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 \\ -6 & -5 \\ 17 & 4 \end{pmatrix}$$

Identity Matrix for Multiplication

In previous units you learned about the zero matrix as an identity matrix of addition. The identity matrix does not change other matrices it is added to.

Let us now find out the identity matrix for multiplication.

Activity

Activity 7.7

1. Work out

a) $\begin{pmatrix} 3 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$

b) $\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$

c) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \\ 7 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$

2. a) What do you notice about your answers to question 1a), b) and c) above?

b) Write down the answer to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In your own words can you say what happens when a matrix is multiplied by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Now compare your answers with the ones given below

Answers to activity 7.7

1. a) $\begin{pmatrix} 3 & -4 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \\ 7 & 9 \end{pmatrix}$

2. a) The matrix that is multiplied by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has not changed in each case. For example

$$\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$$

b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Note it!

In conclusion:

- Multiplying any matrix by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not change the matrix.
- Therefore, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called an **identity matrix** for multiplication.

You have now completed the sub-unit on multiplication of matrices. Go back to the beginning of the sub-unit and do a quick review of the content of this sub-unit before going on to the next sub-unit on the determinant of a matrix.

Lesson 3 The Determinant of a Matrix

You know what a matrix is. What about a determinant? The determinant is a number which is a result of the product of the elements in the leading diagonal of a matrix minus the product of the elements in the other diagonal of the same matrix.

In the previous sub-unit, you have seen that there are some matrices which will multiply together to give an identity matrix. How can one know which matrices will do this multiplication and which ones will not? This can be determined without trying the actual matrix multiplication by finding the determinant of the matrix. If the determinant is a positive or negative number then there will be a matrix which will multiply to give the identity matrix. If the determinant is zero, there is no matrix to multiply with to give the identity matrix.

At the end of this sub-unit you should be able to:

- *define* a term 'determinant',
- *find* the determinant of 2 by 2 matrices.

There are three pages in this sub-unit.

For the 2 by 2 matrix B = $\begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$,

the numbers $(2 \times 4) - (5 \times 1) = 3$.

The number 3 is called the **determinant** of the matrix.

In matrix B above the numbers 2 and 4 lie on the **leading diagonal** of the matrix, while the numbers 5 and 1 lie on the **other diagonal** (figure 7.1 below). Therefore the determinant of a matrix is found by subtracting the product of the numbers in the other diagonal from the product of the numbers in the leading diagonal.

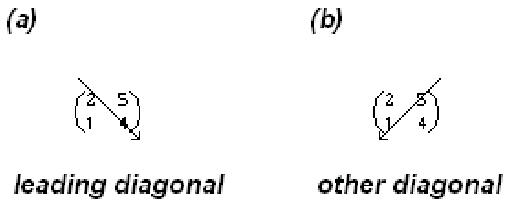


Figure 7.1

Example

Look at the matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}$$

To find a determinant,

- multiply elements in the leading diagonal; in this case 1 and 5 are in the leading diagonal so

$$(1)(5) = 5$$

- multiply elements in the other diagonal; in this case 2 and 2 are in the other diagonal so

$$(2)(1) = 2$$

- then, subtract the product of the other diagonal from the product of the leading diagonal, that is,

$$5 - 2 = 3$$

- the determinant of matrix $\begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}$ is 3.

Note that a determinant of a matrix is a number.

Example 1

Look how the determinants of the following matrices are found.

a) $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

product of leading diagonal elements – product of other diagonal elements

$(1)(5) - (2)(2) = 5 - 4 = 1$

b) $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$

product of leading diagonal elements – product of other diagonal elements

$(3)(5) - (2)(4) = 15 - 8 = 7$

You can now compare your answers with those given below. For each answer remember that you subtract the product of the elements in the other diagonal from the product of the elements in the leading diagonal.

You should realise that determinant can be a positive or a negative number. Some matrices have a determinant equal to zero.

A positive or negative determinant means the matrix will have an inverse matrix. However, a zero determinant means the matrix has no inverse matrix. Why? The answer will be found when finding the inverse.

Therefore, a determinant can be used to determine whether you can find the inverse or you can not.

Activity

Activity 7.8

Now answer the following question.

1. $\begin{pmatrix} -5 & -2 \\ 6 & 5 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix}$

4. $\begin{pmatrix} 6 & -9 \\ -2 & 3 \end{pmatrix}$

5. The determinant of the matrix $\begin{pmatrix} b & b \\ 5 & 2 \end{pmatrix} = 9$.

   Find b

Note it

For the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the determinant of M = ad−bc

The determinant is denoted by det (M) or $|M|$

Thus det (M) = $|ad - bc|$

Solutions to Activity 7.8

Remember that to find a determinant

product of leading diagonal elements – product of other diagonal elements

1. $(-3 \times 5) - (-2 \times 6) = -3$
2. $(3 \times 3) - (4 \times 4) = -5$
3. $(2 \times 8) - (4 \times 4) = 0$
4. $(6 \times 3) - (-9 \times -2) = 0$
5. $2b - 5b = 9$ (collect like terms)

   $b(2 - 5) = 9$

   $b(-3) = 9$ (divide both sides by -3)

   $b = \frac{9}{-3}$

   $b = -3$

You have now completed the sub-unit on the determinant of a matrix, do a quick review of the content of this sub-unit on determinant of a matrix and then continue on to inverse of a matrix. However, try to remember how you find the determinant because determinant will be used in the next sub-unit.

Lesson 4 Identity Matrix

In unit 5 Linear equations, you solved many of the equations by doing the same operation on both sides. When doing these operations, the numbers are chosen in such a way that in addition the result is zero, while in multiplication the result is one.

Zero is the identity element in addition while one is identity element in multiplication.

What is the identity element in matrices? This is what you are going to study in this sub unit at the end of which you should be able to:

- *define* identity matrix.
- *identify* a 2 by 2 identity matrix.

There are seven pages in this sub-unit.

Example

In these examples you are going see how some matrix multiplication gives identity matrix and how identity matrix works in multiplication.

1. Find the determinants for the following matrices

a) $\begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$

$$\det\begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix} = (2)(8) - (3)(5) = 16 - 15 = 1$$

b) $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$

$$\det\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = (3)(7) - (4)(5) = 21 - 20 = 1$$

2. Work out

a) $\begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}\begin{pmatrix} 8 & -3 \\ -5 & 2 \end{pmatrix} =$

$$= \begin{pmatrix} 2\times 8 + 3\times -5 & 2\times -3 + 3\times 2 \\ 5\times 8 + 8\times -5 & 5\times -3 + 8\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 16 + -15 & -6 + 6 \\ 40 + -40 & -15 + 16 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b) $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} =$

$$= \begin{pmatrix} 3\times 7 + 4\times -5 & 3\times -4 + 4\times 3 \\ 5\times 7 + 7\times -5 & 5\times -4 + 7\times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 21 + -20 & -12 + 12 \\ 35 - 35 & -20 + 21 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

c) What do you notice about the answers to 2 a) and b)?

The answers are identity matrix.

d) Multiply the answers you got in 2a) and b) by the determinants found in 1a) and b) respectively.

$$1\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$1\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

Identity matrix multiplied by 1 (identity for multiplication) answer is identity matrix.

The matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ is called the identity (or unit) matrix, it is an identity in matrix multiplication just like one is identity in number multiplication.

The identity matrix is the matrix which gives the same answer as the matrix with the one it is being multiplied with.

Now, proceed to the activity to use the skills you have seen in the above examples.

Activity

Activity 7.9

1. Find the determinants for the following matrices.

a) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 13 & 7 \\ 8 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 12 & 2 \\ 10 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 5 & 10 \\ 2 & 4 \end{pmatrix}$

2. Work out the following

a) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} =$

b) $\begin{pmatrix} 13 & 7 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -7 \\ -3 & 13 \end{pmatrix} =$

c) What do you notice about the answers to a) and b)?

Now compare your answers with those that are given below. Did you work all the problems out correctly? If so, good! Otherwise, do the activity carefully. Remember, to find a determinant you subtract the product of the elements on the other diagonal from the product of the elements on the leading diagonal.

To multiply two matrices;

- multiply row 1 by column 1 to get element of row 1, column 1 of the answer matrix
- multiply row 1 by column 2 to get element of row 1 column 2 of the answer matrix
- multiply row 2 by column 1 to get element of row 2 column 1 of the answer matrix
- multiply row 2 column 2 to get element of row 2 column 2 of the answer matrix

Some matrices multiplication can result in identity matrix.

But, what kind of matrices will result in an identity matrix?

Answers to Activity 7.9

1.

a) determinant $= 2 \times 3 - 1 \times 4 = 2$

b) determinant $= 13 \times 2 - 7 \times 3 = 5$

c) determinant $= 12 \times 2 - 10 \times 2 = 4$

d) determinant $= 4 \times 1 - -2 \times -3 = -2$

e) determinant $= 5 \times 4 - 2 \times 10 = 0$

2. a) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 2 \times 3 + 1 \times -4 & 2 \times -1 + 1 \times 2 \\ 4 \times 3 + 3 \times -4 & 4 \times -1 + 3 \times 2 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

b) $\begin{pmatrix} 13 & 7 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -7 \\ -3 & 13 \end{pmatrix} = \begin{pmatrix} 13 \times 2 + 7 \times -3 & 13 \times -7 + 7 \times 13 \\ 3 \times 2 + 2 \times -3 & 3 \times -7 + 2 \times 13 \end{pmatrix}$

$$= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

c) The answer for a) gives $2 \times \mathbf{I} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and for b) the answer gives $5 \times \mathbf{I} = 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In the first activity matrix multiplication resulted into an identity matrix. Therefore we say one of the matrices in those pairs is an inverse of the other. For example

$$\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, the matrix $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$ is an inverse of the matrix $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$.

In the second activity matrix multiplication resulted into $2 \times \mathbf{I}$ and $5 \times \mathbf{I}$ respectively.

If we divide the product matrices by the determinants 2 and 5 respectively we will get an identity matrix.

For example

$$\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The key points to remember are:

1. the identity matrix is the matrix that will give the same matrix as the one it is multiplied with.
2. the identity matrix for matrices multiplication is the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
3. identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ results when a matrix is multiplied by its inverse matrix.

You can now proceed to the next sub-unit on inverse of a matrix as they are very much related.

Lesson 5 Inverse of a Matrix

You have seen that there are matrices multiply other matrices to give a unit matrix. Which matrices will do this and which will not do this?

Are these matrices and inverses related? How are they related? By the way what is an inverse matrix, and how do you get this inverse matrix?

Before attempting to find answers to these questions, let us start by recollecting the previous knowledge which will help us to answer these questions. You have used inverse for addition and inverse for multiplication in arithmetic. A number and its inverse gives an identity. What is the identity for addition? What is the identity for multiplication?

You already have seen that an identity does not change other elements. Also, from the previous unit the matrix and other matrix give the identity matrix, what is the name of this other matrix?

In this sub unit you are going to learn about the inverse of a matrix. At the end of this sub-unit you should be able to:

- *define* an inverse of a matrix.
- *find* the inverse of 2 by 2 matrices.

An inverse matrix is a matrix that will multiply with another matrix to give an identity matrix. How do you find the inverse matrix? Look at the following example.

Example

Now the following example will help you to learn how to find an inverse of a 2by2 matrix. Study each step as you will apply the steps in the following exercise.

To find the inverse of the following matrix

A= $\begin{pmatrix} 3 & 5 \\ -5 & -7 \end{pmatrix}$

Solution

Find the determinant of a matrix

det $A = -21 + 25 \qquad ((3 \times -7) - (5 \times -5))$

$= 4$

Exchange the elements in the leading diagonal

$\begin{pmatrix} -7 & \\ & 3 \end{pmatrix}$

Change the signs of elements in the other diagonal

$\begin{pmatrix} -7 & -5 \\ 5 & 3 \end{pmatrix}$

Divide each entry by the determinant or multiply each entry by $\frac{1}{det}$

$A^{-1} = \frac{1}{4}\begin{pmatrix} -7 & -5 \\ 5 & 3 \end{pmatrix}$

Note that A^{-1} *is another way of writing the inverse of matrix A.*

Activity

Activity 7.10

1. Find the inverses of the following matrices.

a) $\begin{pmatrix} 4 & 11 \\ 1 & 3 \end{pmatrix}$

b) $\begin{pmatrix} -2 & -4 \\ 5 & 8 \end{pmatrix}$

c) $\begin{pmatrix} -4 & 2 \\ -5 & 3 \end{pmatrix}$

d) $\begin{pmatrix} -2 & -3 \\ -4 & -9 \end{pmatrix}$

2. a) The matrix M satisfies the equation

$3M + 4\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} = M.$

Find M, expressing it in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Solutions to Activity 7.10

Remember to find an inverse

1. Find the determinant.
2. Form a matrix from the matrix you are looking for its inverse by interchanging the positions of the elements in the leading diagonal and changing the signs of the elements in the other diagonal.
3. Multiply by the reciprocal of the determinant (or divide by determinant).

1.a) $det = (4 \times 3) - (11 \times 1) = 12 - 11 = 1$

Inverse = $\begin{pmatrix} 3 & -11 \\ -1 & 4 \end{pmatrix}$

b) $det = (-2 \times 8) - (-4 \times 5) = -16 + 20 = 4$

inverse = $\frac{1}{4}\begin{pmatrix} 8 & 4 \\ -5 & -2 \end{pmatrix}$

c) $det = (-4 \times 3) - (2 \times -5) = -12 + 10 = -2$

inverse = $-\frac{1}{2}\begin{pmatrix} 3 & -2 \\ 4 & -4 \end{pmatrix}$

d) $det = (-2 \times -9) - (-3 \times -4) = 18 - 12 = 6$

inverse = $\frac{1}{6}\begin{pmatrix} -9 & 3 \\ 4 & -2 \end{pmatrix}$

2.a) $M = \begin{pmatrix} -4 & 2 \\ -6 & 0 \end{pmatrix}$

You have now completed the sub-unit on inverse of a matrix.

The key points to remember on this sub-unit are:

1. to find a determinant of a matrix:
   a. subtract the product of the elements in the other diagonal from the elements in the leading diagonal
2. steps in finding an inverse of a 2 by 2 matrix are:
   a. form a new matrix from the matrix you are looking for its inverse by:
      i. interchanging the positions of the elements in the leading diagonal and change the sign of the elements in the other diagonal.
   b. Multiply the reciprocal of the determinant by the new

matrix you formed in 2. (a) above.

3. do a quick review of the content of this sub-unit on inverse of a matrix and then continue. Remember; find the determinant before you find the inverse all the time.

You have now completed the last part of this unit on Matrices. Do a quick review of the entire content of this unit and then continue onto the unit summary.

Unit Summary

Summary

In this unit you learned that

- When adding or subtracting matrices, we operate on corresponding entries.
- When multiplying a matrix by a scalar, we multiply **all entries** of the matrix by the **scalar.**
- **Rules for multiplying a matrix by another matrix are:**
    1. The number of columns of the first matrix must be equal to the number of rows of the second matrix.
    2. Multiply the elements of a row of the first matrix by the elements of a column of the second matrix and add the products.
    3. The order of the product matrix is the number of rows of the first matrix by the number of the columns in the second matrix. For example, if the order of the first matrix is ***3by2*** and the order of the second matrix is ***2by4***, then the order of the product will be ***3by4***. That is ***3by2*** and ***2by4*** gives ***3by4.***
    4. When multiplying any matrix by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, it does not change the matrix. Therefore the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called an **identity matrix** for multiplication.

- **a determinant**
    1. A determinant is a number found by subtracting a

product of the elements in the other diagonal from the product of the elements in the leading diagonal of a matrix. (See 2 below)

2. For the matrix M = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the determinant of M = $|ad - bc|$

   The determinant can also be written as det (M) or $|M|$

   Thus det (M) = $|ad \quad bc|$

3. If a determinant is positive or negative number then the matrix will have another matrix which will multiply it to give an identity matrix.

4. If the determinant is zero there will be no matrix to multiply to give an identity matrix.

- **an inverse**

  1. an inverse of a matrix is a matrix which will multiply a matrix to give an identity matrix,

  2. To find an inverse matrix

     i. Find a determinant, if it is zero no inverse, (stop, there is no matrix), if not zero continue to ii.

     ii. form a new matrix from the matrix you are looking for its inverse by

        1. interchanging the positions of the elements in the leading diagonal and change the sign of the elements in the other diagonal.

     iii. Multiply the reciprocal of the determinant by the new matrix you formed in 2. (i) above.

  3. The inverse of matrix M is denoted by

     $$M^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{for } ad - bc \neq 0$$

     where (ad – bc) is a determinant of the matrix.

You have completed the material for this unit on matrices. You should spend some time reviewing the content in detail.

Once you are confident that you can successfully write an examination on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings.

Your last step is to complete the assessment 7. Once you have completed the assessment 7, proceed to the next unit, which explores how mathematics can be used for commercial purposes.

Assignment

Assignment

You should be able to complete this assignment on Matrices in 60 minutes.

The total marks available for this assignment is 34, and the marks allotted to each question or part of a question are indicated in (parentheses).

1. The table shows the results of matches for 4 football teams.

	Win	Draw	Lose
Likhopo	3	2	1
Matlama	2	1	3
Lioli	2	3	1
Bantu		2	3

a) Write the information in matrix form. (3)

What is the order of the matrix? (1)

2. If $\begin{pmatrix} 2a & 7 \\ 5 & 3b \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 5 & 12 \end{pmatrix}$

Find the value of a and b. (4)

3. Given that $X = \begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} -2 & -4 \\ -1 & 3 \end{pmatrix}$

Find: a) X+Y b) 2XZ c) Z−Y d) YZ

(8)

4. Given that $\begin{pmatrix} a & 3 \\ 2 & b \end{pmatrix} + \begin{pmatrix} 5 & c \\ d & 6 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 5 \end{pmatrix}$

Find the value of a, b, c and d (2)

5. The table for the results of matches for 4 netball teams is as follows:

	Win	Draw	Lose
Matlama	3	2	1
Arsenal	2	1	3
LMPS	1	2	3
LDF	2	3	1

The points given are Winning 3
Draw 1
Lose 0

a) Represent the information in matrix form. (3)

b) Find the total points for each of the four teams and put them in the form of a matrix. (1)

6. If $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 4 & 3 \end{pmatrix}, C = \begin{pmatrix} -1 & -2 \\ 3 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 3 \\ -2 & 6 \end{pmatrix}$.

a) Find the determinants of A, B, C and D (4)

b) Write down the inverses of A, B, C and D. (8)

Model Answers to Assignment

1.

a) $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$ b) 4×3

2.

Compare corresponding elements on the LHS and on the RHS of the equation.

$2a = 6$

$a = \frac{6}{2}$ (divide by 2 both sides)

$a = 3$

Compare corresponding elements on the LHS and on the RHS of the equation.

$3b = 12$

$b = \frac{12}{3}$ (divide by 3 both sides)

$b = 4$

3.

a) $X + Y = \begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}$

$\begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}$

b) $2XZ = \begin{pmatrix} -16 & 8 \\ 2 & -26 \end{pmatrix}$

$2\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -2 & -4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} -2 & -4 \\ 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 4 \times -2 + 8 \times -1 & 4 \times -4 + 8 \times 3 \\ 2 \times -2 + -6 \times -1 & 2 \times -4 + -6 \times 3 \end{pmatrix}$

$= \begin{pmatrix} -16 & 8 \\ 2 & -26 \end{pmatrix}$

c) $Z - Y = \begin{pmatrix} -2 & -4 \\ -1 & 3 \end{pmatrix}$

$\begin{pmatrix} -2 & -4 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ -1 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

4.

$a = 1 \quad b = -3 \quad c = -1 \quad d = 0$

Compare corresponding elements on the LHS and on the RHS of equation.

$a + 5 = 6$

$a = 6 - 5$

$a = 1$

$b + 6 = 3$

$b = 3 - 6$

$b = -3$

$3 + c = 2$

$c = 2 - 3$

$c = -1$

$2 + d = 2$

$d = 2 - 2$

$d = 0$

5.

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 11 \\ 7 \\ 5 \\ 9 \end{pmatrix}$$

Matlama= 11points,

Arsenal = 7 points,

LMPS = 5 points,

LDF = 9 points

6.

a) $detA = 5,\ detB = -3,\ detC = 5,\ detD = 0$

$detA = (2 \times 4) - (3 \times 1) = 8 - 3 = 5$

$detB = (-1 \times 3) - (0 \times 4) = -3 - 0 = -3$

$detC = (-1 \times 1) - (-2 \times 3) = -1 + 6 = 5$

$detD = (-1 \times 6) - (3 \times -2) = -6 + 6 = 0$

b) $A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}$

$B^{-1} = \frac{-1}{3}\begin{pmatrix} 3 & 0 \\ -4 & -1 \end{pmatrix}$

$C^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix}$

There is no inverse for matrix D

Assessment

Assessment

You should be able to complete this assessment on Matrices in 90 minutes.

A total of 20 marks have been allotted for this assessment. The marks available for each question or part of a question are shown in (parentheses).

1. Given that

$$A = \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -\frac{4}{5} & -\frac{3}{7} \end{pmatrix}$$

Find

a. $A + B$ (2)

b. $A - B$ (2)

2. Given that $D = \begin{pmatrix} 4 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix}$, $E = \begin{pmatrix} 3 & -4 & -1 \\ 0 & 6 & 2 \end{pmatrix}$,

$F = \begin{pmatrix} 2 & 1 \end{pmatrix}$, $G = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}$

Find

a. $2D - E$ (2)

b. FD (1)

c. DG (1)

d. $4D$ (1)

e. EH (2)

3. Given that $L = \begin{pmatrix} 3 & -4 \\ 1 & 6 \end{pmatrix}$, $M = \begin{pmatrix} 5 & -2 \\ 7 & 4 \end{pmatrix}$

Find

a. The value of determinant of the matrix L

(2)

b. The value of determinant of the matrix M
(2)

4. Given that $N = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

Find the inverse of the matrix N
(4)

Model Answers to Assessment

1.

a. $A + B = \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} + \begin{pmatrix} \frac{1}{5} & \frac{5}{8} \\ \frac{-1}{8} & \frac{-3}{8} \end{pmatrix}$

$= \begin{pmatrix} 3 + \frac{1}{5} & -1 + \frac{5}{8} \\ -4 + \frac{-1}{8} & 7 + \frac{-3}{8} \end{pmatrix}$

$= \begin{pmatrix} 3\frac{1}{5} & \frac{-3}{8} \\ -4\frac{1}{8} & 6\frac{5}{8} \end{pmatrix}$

b. $A - B = \begin{pmatrix} 3 & -1 \\ -4 & 7 \end{pmatrix} - \begin{pmatrix} \frac{1}{5} & \frac{5}{8} \\ \frac{-1}{8} & \frac{-3}{8} \end{pmatrix}$

$= \begin{pmatrix} 3 - \frac{1}{5} & -1 - \frac{5}{8} \\ -4 - \frac{-1}{8} & 7 - \frac{-3}{8} \end{pmatrix}$

$= \begin{pmatrix} 2\frac{4}{5} & -1\frac{5}{8} \\ -3\frac{7}{8} & 7\frac{3}{8} \end{pmatrix}$

2.

a. $2D - E = 2\begin{pmatrix} 4 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix} - \begin{pmatrix} 5 & -4 & -1 \\ 0 & 6 & 2 \end{pmatrix}$

$= \begin{pmatrix} 8 & -6 & 0 \\ 0 & 12 & -4 \end{pmatrix} - \begin{pmatrix} 5 & -4 & -1 \\ 0 & 6 & 2 \end{pmatrix}$

$= \begin{pmatrix} 3 & -2 & 1 \\ 0 & 6 & -6 \end{pmatrix}$

b. $FD = \begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 4 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix}$

$= \begin{pmatrix} 2 \times 4 + 1 \times 0 & 2 \times -3 + 1 \times 6 & 2 \times 0 + 1 \times -2 \end{pmatrix}$

$= \begin{pmatrix} 8 & 0 & -2 \end{pmatrix}$

c. $DG = \begin{pmatrix} 4 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 4 \times -2 + 3 \times 5 + 0 \times 1 \\ 0 \times -2 + 6 \times 5 + -2 \times 1 \end{pmatrix}$

$= \begin{pmatrix} -8 + 15 + 0 \\ 0 + 30 - 2 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ 28 \end{pmatrix}$

d. $4D = 4\begin{pmatrix} 4 & -3 & 0 \\ 0 & 6 & -2 \end{pmatrix}$

$= \begin{pmatrix} 8 & -12 & 0 \\ 0 & 24 & -8 \end{pmatrix}$

3.

a. $\det L = (3 \times 6) - (-4 \times 1)$\\

$= 18 + 4$

$= 22$

b. $\det M = (3 \times 4) - (-2 \times 7)$

$= 12 + 14$

$= 26$

4.

Step 1

Find the determinant of matrix N

$\det N = (1 \times 4) - (2 \times 3)$

$= 4 - 6$

$= -2$

Step 2

Form a new matrix from matrix N by:

- Interchanging the positions of 1 and 4 on the leading diagonal and changing the signs of 2 and 3 on the other diagonal, in this case the signs are both positive, so they change to negative for both 2 and 3 to become -2 and -3.

Step 3

Multiply the new matrix by $\frac{1}{det}$ to get the inverse.

$$\textit{Inverse of matrix } N = \frac{1}{det}\begin{pmatrix} 4 & 2 \\ -3 & 1 \end{pmatrix}$$

$$\textit{Inverse of matrix } N = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & \frac{1}{-2} \end{pmatrix}$$

Unit Contents

Unit 8

Unit 8

Commercial Mathematics

Introduction

Welcome to another interesting unit in this course. This unit consists of 70 pages. This is approximately 3% of the whole course. Plan your time so that you can complete the whole course on schedule. . As reference, you will need to devote 25 hours to work on this unit, 15 hours for formal study and 10 hours for self-study and completing assessments/assignments.

This unit on commercial mathematics is one of many that helps learners appreciate the study of mathematics. It touches on business concepts and everyday life. We are going to deal with time, money, interest, discount, profit and loss, tax, and budgeting.

All these topics are extensive, so we are not going to cover them in depth in this course. If you wish to learn about some of them in greater depth, you can try a related course, such as grade 12 accounting.

When working with money, you cannot help but be careful with place value that is where to place the decimal point. For example, if you work in a shop and are careless in writing up the price of an item that sells for M 500, a customer may buy it for only M50. It is very important that you revisit what you have already learned about place value, which is covered in unit 3 of this course.

When reading the following learning outcomes, think about them as a guide to what you should focus on while studying this unit.

This Unit is Comprised of Eight Lessons:

Lesson 1 Converting Times Between the 12-hour and the 24-hour Clock
Lesson 2 Currency Conversion
Lesson 3 Simple Interest
Lesson 4 Compound Interest
Lesson 5 Discount
Lesson 6 Profit and Loss
Lesson 7 Tax
Lesson 8 Budgeting

Upon completion of this unit you will be able to:

Outcomes

- *convert* times between the 12-hour and 24-hour clock;
- *solve* problems involving money and *convert* from one currency to another;
- *solve* problems on personal and household finance involving earnings, simple interest, compound interest, discount, profit and loss, tax and budgeting using data;
- *solve* practical problems using information from charts and tables.

Terminology

Currency:	The particular type of money in use in a country. The currency in Lesotho is the Maloti.
Earnings:	Money that you get by working, regardless of whether you run your own business, have a salaried job or work in the informal sector.
Interest:	An amount paid to borrow money or an amount earned for lending money.
Principal:	Money lent or borrowed.
Term of a loan:	The period the borrower takes to pay back a loan.
Maturity value:	The money that results after addition of interest in a given period is.
Discount:	An amount that reduces a bill or a price of an item.
Profit:	Money gained by a business.
Loss:	A failure to make a profit; or the amount by which the cost of an article is greater than the selling price.
Tax:	A sum of money paid in accordance with the law to the government according to income, property, goods bought, etc.
Budget:	A plan of how to spend money
Per annum:	A calculation that occurs each year

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Converting Times Between the 12-hour and the 24-hour Clock

By the end of this subunit, you should be able to:

- explain the differences between the 12-hour and the 24-hour clock;
- *convert* times between the 12-hour and 24-hour clock.

This subunit is about 4 pages in length.

It is about time that we talk about time!

Most people agree that time is money and they try doing things on time. If you do not respect time you will definitely fall behind with your work.

Do you remember the last time you had an appointment with a friend and he or she was late? You nearly lost your mind!

We can talk about time under the 12-hour clock and the 24-hour clock. These are not separate instruments for measuring time, but different ways of representing the hours in a day. Some digital clocks and watches use the 24-hour system, but a traditional clock/watch has only twelve numerals arranged in a circle.

Under the 12-hour clock, the day is split into two periods of 12 hours.

The first period runs from midnight to noon that is midday. These hours are Ante Meridian (Latin for "before the sun passes through its highest point", which occurs at noon). In their short form, we write this as A.M. or a.m. or am.

The second period runs from noon to midnight. These are Post Meridian (Latin for "after noon") hours, written in short as P.M. or p.m. or pm.

Example 1

Jane has an appointment with her dentist at 8 in the morning. In the 12-hour clock, this is written 8:00 a.m.

Example 2

Jane goes to bed at 8 at night.

In the 12-hour clock, this is written 8:00 p.m.

With the 24-hour clock system, the day is not split. Instead, it runs from midnight to midnight. The day starts at 00:00, which is midnight and ends at 23:59, a minute before midnight.

When we get to midday, that is 12:00 noon, we continue counting. 1:00 in the afternoon becomes 13:00; 2:00 in the afternoon is shown as 14:00, and so forth till we get to midnight, when the clock resets to 00:00.

Example 3

Jane has an appointment with her dentist at 8 in the morning. In the 24-hour clock, this is written 08:00 or 08H00 or 08h00

Example 4

Jane goes to bed at 8 at night.

In the 24-hour clock, this is written 20:00 or 20H00 or 20h00

Both systems are used in the world, but the 12-hour clock is still the one that dominates. The 24-hour system of representing time is widely used in the military and by airlines, where the possibility of confusion between a.m. and p.m. could have disastrous consequences.

Activity

Activity 8.1

Complete the table given below

12-Hour Clock	*24-Hour Clock*
12 midnight (the day begins)	00:00
	01:00
	02:00
3:00 a.m	
4:00 a.m	
5:00 a.m	
	06:00
	07:00
	08:00
	09:00
	10:00
11:00 a.m	
12:00 noon	
1:00 p.m	
2:00 p.m	
3:00 p.m	
4:00 p.m	
	17:00
	18:00

	19:00
	20:00
9:00 p.m	
10:00 p.m	
11:00 p.m	
12 midnight (the day ends)	24:00, when the clock resets to 00:00

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on converting times between the 12 hour and 24 – hour clock are:

- Under the 12- hour clock, the day is split into two periods of 12 hours each.
- A.M. or a.m or am is used to indicate the period between midnight and midday or the morning hours
- P.M. or p.m or pm. is used to indicate after midday till midnight, which are the afternoon and evening hours
- Under the 24-hour clock, the day is not split, it runs from midnight to just before midnight, which is the start of the following day.

Model Answers

Activity 8.1

12 – Hour Clock	24 – Hour Clock
12 midnight (day begins)	00:00
1:00 a.m	01:00
2:00 a.m	02:00

3:00 a.m	**03:00**
4:00 a.m	**04:00**
5:00 a.m	**05:00**
6:00 a.m	06:00
7:00 a.m	07:00
8:00 a.m	08:00
9:00 a.m	09:00
10:00 a.m	10:00
11:00 a.m	**11:00**
12:00 noon	**12:00**
1:00 p.m	**13:00**
2:00 p.m	**14:00**
3:00 p.m	**15:00**
4:00 p.m	**16:00**
5:00 p.m	17:00
6:00 p.m	18:00
7:00 p.m	19:00
8:00 p.m	20:00
9:00 p.m	**21:00**
10:00 p.m	**22:00**
11:00 p.m	**23:00**
12 midnight (day ends)	24:00, but the new day begins immediately afterwards at 00:00

Lesson 2 Currency Conversion

By the end of this subunit, you should be able to

- solve problems involving money and convert from one currency to another

This subunit is about 3 pages in length.

A group of tourists from England is in Lesotho for a visit. The money used in England is called the pound sterling, commonly referred to simply as "the pound" or "the British pound". The pound is not used in Lesotho, where the Maloti is the national currency.

What do these tourists need to do?

____

____

Yes, they will need to change their pounds into Maloti.

Everyday millions of people travel into countries other than their own. They travel for various reasons; for business or just for pleasure. They need money. Different countries use money with different names. We say they use different **currencies**.

One therefore needs the currency of the country being visited. This allows one to pay for everything without a problem.

The table below gives the currencies of some of the different countries of the world.

Country	Currency	Symbol
Kingdom of Lesotho	Maloti	M
Republic of South Africa	Rands	R
Republic of Botswana	Pula	P
Republic of Namibia	Namibian Dollar	N$
Seychelles	Seychelles Rupee	SCR, SR or SRe

Canada	Canadian Dollar	C$

These currencies have both coins and notes, that is paper money.

The value of one currency as compared with another is called the **exchange rate**.

Sometimes the exchange rate between two currencies is fixed. For example, the Maloti can be exchanged with the South African Rand at a fixed rate of Maloti 1.00 to ZARand 1.00. In this case, because the currencies are equal to one another, we say they are at parity.

More frequently, the exchange rate between two currencies varies from day to day. There are different exchange rates for buying and selling currency. You can “buy” or “sell” foreign currency at a bank or *bureau de change*. The rates change on a daily basis.

If for example you have Maloti and you want American dollars, you will be assisted with the buying exchange rate. When you want your Maloti back, the changer will use the selling exchange rate.

It is very important that you change just enough money to cover the amount you expect to spend in the other country. The reason for this is that every time you change money into another currency, you lose some of it. Those helping you at the bank or *bureau de change* take a commission on the transaction to cover their costs and make a small profit.

Example

Mrs Forrester lives in USA and has decided to visit South Africa. She has US$1 000 which she wants to change to South African Rand. The current buying exchange rate is

US$ 1.00 = ZAR 7.05

The agent charges commission at the rate of 1% per transaction.

How much will she receive in Rands?

Solution

First, we need to subtract the commission the agent charges on the transaction.

$$\frac{1}{100} \times \text{US\$1 000} = \text{US\$10}$$

This means Mrs Forrester has US$990 to exchange for Rand.

Let x equal the amount of money (in South African Rand) that Mrs Forrester will receive for the dollars. Then:

X = Amount in US$ × the exchange rate (Rand/Dollars)

US$ 1.00 = ZAR 7.05

$$\frac{US\$990 \times ZAR7.05}{US\$1}$$

The units (US$) on both the top and the bottom of the division line cancel each other out, leaving the answer in South African Rand.

X = ZAR 6 979.50

Activity

Activity 8.2

1. The currency conversion rates between three countries are shown.

US $	South African Rand	Euro
1	6.99	0.73

Convert

(a) US$60 to Euros

(b) ZAR 575 to US$

(c) ZAR 2 690 to Euros

2. Siphiwe travels from Zambia to Namibia.

   He needs to have 1 500 Namibian dollars (NAD).

   (a) How many Zambian Kwacha (ZMK) does he need to change?

   (b) He is left with NAD 300 which he changes to US dollars. How many US dollars will he get?

   ZMK 1.00 = NAD 0.0014

   USD 1.00 = NAD 7.15

Compare your answers with those at the end of this subunit. Be sure that you understand each answer before continuing. If you *have any misunderstandings, review this content and work through the activity again.*

Key Points to Remember

The key point to remember in this subunit on solving problems involving money and converting from one currency to another are:

- Exchange rates can vary from day to day
- every time you change money into another currency, you lose some of it as commission

Model Answers

Activity 8.2

1.

(a) US$ 60 to Euros

US$ 1.00 = €0.73
US$ 60.00 = X

$$\frac{\text{US\$60} \times \text{€}0.73}{\text{US\$1}} = X$$

€43.80= X

(b) ZAR575 to US$

1US$ = R6.99
= R575

$$\frac{\text{R575} \times \text{1US\$}}{\text{R6.99}} = X$$

$82.26= X

(c) R2 690 to Euros

R6.99 = € 0.73

R1 = X $\frac{R1 \times € 0.73}{R6.99}$ = 0.10

R1 = € 0.10

R2 690 = X

$\frac{R2\ 690 \times € 0.10}{R1}$ = X

€ 269 = X

2.

1ZMK = 0.0014 NAD

X = 1 500NAD

$\frac{1\ 500NAD \times 1\ ZMK}{0.0014\ NAD}$ = X

1 071 428.57ZMK = X

1USD = 7.15NAD

= 300 NAD

$\frac{300NAD \times 1\ US\$}{7.15\ NAD}$ = X

41.96 NAD = X

Lesson 3 Simple Interest

In life we sometimes run short of cash and have to borrow money from the bank, companies or people etc. When money is lent, the lender charges the borrower an extra fee for the privilege of using his/her money. The money lent is called the **principal,** and the extra fee, **interest**. The money that results after addition of interest in a given period is called a maturity value. The interest on the money borrowed is paid in intervals after a fixed period of time, such as monthly, half-yearly; annually, etc.

It is a good idea for you to know how much extra you will have to pay back so that you can be prepared. This means that when the borrower asks for their money back with interest, you'll have enough to clear your debt.

Interest is not only charged when you borrow money, it is also paid by the bank when you make a deposit to a savings account. When you deposit your money in a bank, you are essentially loaning it to them.

The money you deposited increases due to the interest that the bank puts into your account at fixed intervals. The interest can be paid daily, monthly, once a year, or at the end of the investment period. One of the most common intervals is once a year, which is usually referred to with the phrase *per annum*. These are Latin words that mean "for one year".

There are several types of interest that can be charged on the money borrowed or deposited. In this subunit we will look at the two types of interest: (a) simple interest and (b) compound interest. We will look at how the two types of interest differ and how to calculate each.

By the end of this subunit, you should be able to:

- discuss the differences between simple and compound interest;
- define the terms: principal, interest and maturity value;
- calculate rate given the principal and time;
- calculate time given the rate and the principal;
- *solve* problems on simple interest

Simple Interest

Simple interest is the most basic type. It is expressed as a percentage of the principal. It is proportional to the principal and to the time for which the money is invested (in the case of a deposit in a bank) or before the money needs to be repaid (in the case of a loan). This means that the more money you borrow or lend, the more the interest. Similarly, the longer the time the money is borrowed, the greater the interest will be.

Suppose you borrow M500.00 from the bank with the condition that the money will be charged a simple interest of 5% per annum. This means that at the end of the first year the total amount you have to repay – the principal plus the interest - will increase by 5%.

Let us work out your actual interest:

Your interest at the end of the first year will be:

$$= \frac{5}{100} \times M500 = M25.00$$

You will owe the bank $M500.00 + M25.00 = M525.00$

Let us say that you borrowed the M500.00 for two years. The simple interest charged for the second year will still be charged on the principal (i.e. M 500, which was the original amount you borrowed). So in two years the interest will be M500.00 or 10%. Every year M25.00 will be added to your loan if the principal amount of M500.00 has not been reduced by repayments.

NOTE: With simple interest, the amount of interest is calculated only on the principal and not on the total amount you owe (principal plus interest).

Normally when you borrow money, you will be expected to pay it back at certain intervals. The 5% will be charged annually on the balance of the principal that is left after taking into account any payments you made.

Calculating Simple Interest

Since we are calculating simple interest, the amount of extra money we pay is the same every year.

If I is the interest, the rate is denoted by R which is a percentage and the loan or money borrowed is the principal P, then:

$$I = P \times R$$

If the time before the loan is repaid (also known as the 'term') is more than one year (e.g. 2 years)

$$I = P \times R \times \text{Number of Years}$$

Two years is the time the money should be charged interest for. If T represents that time/term, our formula for simple interest becomes

$$I = P \times R \times T$$

Remember the rate is a percentage. A percentage is a fraction whose denominator is 100. So this means that the above formula can be written as follows:

$$I = \frac{PRT}{100}$$

$$R = \frac{100I}{PT}$$

$$T = \frac{100I}{PR}$$

We can change the subject of this formula to find either the time (T), the rate (R) or the principal (P)

Example 1

Find the simple interest on M300 in 5 years at 4% per annum

$$\text{I} = \frac{PRT}{100} = \frac{M300.00 \times 4 \times 5}{100} = M60.00$$

Example 2

'Mamukone goes to her husband's cousin and asks to borrow M525 for a period of three years. Her relative tells her the total amount she will have to pay back is M588. If simple interest is being charged, what is the rate of interest?

$$I = A - P = M588.00 - M525.00 = M63.00$$

$$R = \frac{100I}{PT} = \frac{100 \times 63}{525 \times 3} = \frac{4 \times 63}{21 \times 3} = 4\%$$

Example 3

A sum of money invested at 3% per annum simple interest amounts to M280.00 after 4 years. Find the sum invested

The original amount in terms of a percentage is 100%

The 100% increases by 3% after an interest is added.

This means that M280.00 which is the maturity value = 103%

103% = M280.00

100% = ?

$$\frac{100\% \times M280.00}{103\%}$$

$M271.84$

Example 4

Thabo invested an amount of M600.00. The rate of simple interest was 5% per annum, yielding total interest of M180.00. For how long was the money invested?

$$T = \frac{100I}{PR}$$

$$T = \frac{100I}{PR}$$

$$T = \frac{100 \times M180.00}{M600 \times 5} = 6 years$$

Activity 3

Activity 8.3

Answer all questions

1. Lineo wants to invest M680.00 in a bank over 4 years at 5% interest per annum. Calculate the interest the money will accumulate after 4 years Find the simple interest on the principal amount of M680.00 over a term of 4 years at 5% per annum

2. Thabiso borrowed M12 100 from the bank at 3% simple interest per annum. Find the simple interest he will owe the bank after 5 years if he does not pay anything on the loan.

3. If Lineo pays M37.50 each year as simple interest on a loan of M500.00 for 3 years, what is the interest rate (in per cent) per annum on this money?

4. Lerato borrowed M280 for a period of 4years. If the amount she pays each year as simple interest is M39.20, find the annual interest rate (in per cent).

5. Sebeo put M220 in the Post Bank. If the simple interest he earns on this money at 4% per annum is M44, find the term of the investment.

6. A total sum of M720.50 has been invested at the simple interest rate of 4%, yielding interest of M72.05 per annum. For how long has the principal been invested?

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above the solutions in the worked examples above. Note once again, how changing the subject of the formula changes the way you need to go about solving the problem. When you have spent some time reviewing the content of the subunit, try the activity again and see if you can't improve your score.

Key Points to Remember

The key points to remember in this subunit on simple interest are:

- The original sum of money borrowed or lent is called the principal;
- The additional money charged on top of the principal is called interest
- The types of interest dealt with in this unit are (a) simple interest and (b) compound interest
- Simple interest is charged on the original amount borrowed or lent
- The formulae for calculating simple interest $= \frac{PRT}{100}$

  Where:

  P = principal

  R = rate of interest, and

  T = the term of the loan (the time before the money has to be repaid)
- The total amount that must be repaid after the addition of interest to the principal over a given period is called the maturity value of the loan or investment.

Model Answers

Activity 8.3

1.

$$I = \frac{PRT}{100}$$

$$I = \frac{M680.00 \times 5\% \times 4 years}{100} = M136.00$$

2.

$$I = \frac{PRT}{100}$$

$$I = \frac{M121.00 \times 3\% \times 5 years}{100} = M18.15$$

3.

$$R = \frac{100I}{PT}$$

$$R = \frac{100 \times M37.50}{M500.00 \times 3} = \frac{3750}{1500.00} = 2.5\%$$

4.

$$R = \frac{100I}{PT}$$

$$R = \frac{100 \times M39.20}{M280.00 \times 4} = \frac{3920}{1120.00} = 3.5\%$$

5.

$$T = \frac{100I}{PR}$$

$$T = \frac{100 \times M44.00}{M220.00 \times 4} = \frac{4400}{880} = 5 years$$

6.

$$T = \frac{100I}{PR}$$

$$T = \frac{100 \times M72.05}{M720.50 \times 4} = \frac{7205}{2880} = 2.5 years$$

Lesson 4 Compound Interest

Compound interest is a type of interest that is paid on both the principal and also on any interest accumulated from past years. It's often used when someone re-invests any interest they earn along with their original investment. For example, if I got 5% interest on my M500.00 investment, the first year and I re-invested the interest along with the original sum, then in the second year I would get 5% interest on the principal of M500.00 plus the interest of M25.00 that I reinvested. This yields interest of M 26.25 in the second year. Over time, compound interest will add up to much more money than simple interest.

By the end of this subunit, you should be able to:

- differentiate between simple and compound interest,
- calculate compound interest,
- *solve* problems using the formula for compound interest.

This subunit is about 3 pages in length.

To derive the formulae for calculating compound interest, let's look at the following example:

M700.00 is invested at the bank at an interest rate of 4% compounded annually. Over the term of the investment, we can work out the maturity value of the investment for each year using the formula for simple interest. The only difference is that the amount we insert in the formula changes from year to year because we need to include the accumulated interest. The maturity value is the total amount of the investment, including the principal and accumulated interest over the term of the investment.

Can you remember the formula for simple interest from the previous subunit? If not, go back and have a look.

YEAR	*INTEREST*	*MATURITY VALUE*
1st	$\frac{4}{100} \times M700 = M28$	$M700 + M28 = M728$
2nd	$\frac{4}{100} \times M728 = M29.12$	$M728 + M29.12 = M757.12$
3rd	$\frac{4}{100} \times M757.12 = M30.28$	$M757 + M30.28 = M787.28$
4th	$\frac{4}{100} \times M787.28 = M31.49$	$M787.28 + M31.49 = M818.77$

5th	$\frac{4}{100} \times M818.77 = M32.75$	$M818.28 + M32.75 = M851.03$

If the rate of interest is 4% per annum, this means that the maturity value at the end of any particular year is equal to 100% + 4% of the value at the start of the year. So multiplying the money invested by the rate of interest and adding the interest to the principal gives us the same amount as multiplying that principal by $\frac{104}{100}$. The equation looks like this:

$$\frac{4}{100} \times M700 + M700 = \frac{104}{100} \times M700$$

Another way to get the maturity value is to:

multiply the principal by $\frac{104}{100}$

because the M700 represents 100% of the principal. Therefore:

$$\text{Maturity value (A)} = \frac{M700 \times 104\%}{100} = M728.00$$

YEAR	FORMULA FOR MATURITY VALUE		MATURITY VALUE
1st	$M700.00 \times \frac{104}{100} =$	$M700.00 \times \left(\frac{104}{100}\right)^1$	M728.00
2nd	$M700.00 \times \frac{104}{100} \times \frac{104}{100} =$	$M700.00 \times \left(\frac{104}{100}\right)^2$	M757.12
3rd	$M700.00 \times \frac{104}{100} \times \frac{104}{100} \times \frac{104}{100} =$	$M700.00 \times \left(\frac{104}{100}\right)^3$	M787.40
4th	$M700.00 \times \frac{104}{100} \times \frac{104}{100} \times \frac{104}{100} \times \frac{104}{100} =$	$M700.00 \times \left(\frac{104}{100}\right)^4$	M818.90
5th	$M700.00 \times \frac{104}{100} \times \frac{104}{100} \times \frac{104}{100} \times \frac{104}{100} \times \frac{104}{100} =$	$M700.00 \times \left(\frac{104}{100}\right)^5$	M851.66

If you look at the third column in the table above, a pattern begins to emerge. As you know formulae are mathematical representations of patterns. The pattern indicates that the original sum of money invested (the Principal) can be multiplied by the value $\left(\frac{104}{100}\right)^n$ where n is the number of years that interest has been accumulating.

Also note that this multiplying value can be as follows

$$\left(\frac{104}{100}\right)^n = \left(\frac{100+4}{100}\right)^n = \left(\frac{100}{100}+\frac{4}{100}\right)^n = \left(1+\frac{4}{100}\right)^n$$

This shows that the maturity value after n years

$$\text{A} = p\times\left(1+\frac{4}{100}\right)^n = p\times(1+r)^n$$

Where:

A = the maturity value of the investment or loan after n years

p = principal (the sum of money originally invested)

n = the number of years the money is in the bank (if invested) or repaid (if borrowed)

Let us now use the formula to work out some problems on compound interest.

Example 1

Find the compound interest on M200 in 3 years at 4% per annum

$$\frac{4}{100}\times M200 = M8.00$$

$$\frac{4}{100}\times M208 = M8.32$$

$$\frac{4}{100}\times M216.32 = M8.65$$

$$M8.65 + M216.32 = M224.97$$

$$I = M224.97 - M200 = M24.97$$

Or

$A = P\left(1 + \frac{r}{100}\right)^n$

$A = 200\left(1 + \frac{4}{100}\right)^3$

$A = 200 \times (1.04)^3$

$A = 200 \times 1.125$

$A = M225$

$I = M225 - M200 = M25$

Example 2

Find the compound interest onM360 in 4years at 3% per annum

$A = 360\left(1 + \frac{3}{100}\right)^4$

$A = 360(1.03)^4$

$A = 360 \times 1.1255$

$A =$

405.18

$I = M405.18 - M360.00 = M45.18$

Example 3

Find the compound interest on M300 in 3 years at 3% per annum

$A = P\left(1 + \frac{3}{100}\right)^3$

$A = 300 \times (1.03)^3$

$A = M327.82$

$I = M327.82 - M300 = M27.82$

Example 4

A father leaves a legacy of M500,000 to his son. The money invested in the bank at 3% compound interest equals M 562,754.405 after a certain period. For how many years has the money been invested?

$$A = P\left(1+\frac{3}{100}\right)^n$$

$$M562{,}754.405 = 500{,}000(1.03)^n$$

$$\frac{M562{,}754.405}{500{,}0000} = 1.03^n$$

$$1.1255 = 1.03^n$$

$$changingto \log s$$

$$.05135 = .012837 \times n$$

$$n = 4 years$$

Activity

Activity 8.4

Answer all questions

1. Find the compound interest on M370.50 in 3 year at 4% per annum.

2. Elias invests M600 on January 1st at compound interest of 4% per annum. Calculate the total amount he has in the bank at the end of 3 years.

3. ‘Machobane opens a savings account at her local bank. The account yields compound interest at the rate of 4% per annum. At the end of 5 years, the interest due on her account amounts to M120. How much principal did she put in her savings account at the beginning of the period?

4. A savings certificate costing M1.50 is worth M2.05 at the end of two years. What is the annual percentage rate of compound interest on the certificate?

5.The M100 invested in the bank at 4% interest amounts to M219.00 after a certain period. For how long was the money invested in the bank.

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on compound interest are:

- Compound interest is an interest that is paid on both the principal and any interest from past years.
- The formulae for calculating compound interest is

  $$A = P\left(1 + \frac{r}{100}\right)^n$$

  Where,

  A- maturity value,

  P - principal, r-rate of interest and

  n - number of years the money is invested

Model Answers

Activity 8.4

1.

$$I = \frac{PRT}{100}$$

$$I = \frac{M680.00 \times 5\% \times 4\,years}{100} = M136.00$$

2.

$$I = \frac{PRT}{100}$$

$$I = \frac{M121.00 \times 3\% \times 5\,years}{100} = M18.15$$

3.

$$R = \frac{100I}{PT}$$

$$R = \frac{100 \times M37.50}{M500.00 \times 3} = \frac{3750}{1500.00} = 2.5\%$$

4.

$$R = \frac{100I}{PT}$$

$$R = \frac{100 \times M39.20}{M280.00 \times 4} = \frac{3920}{1120.00} = 3.5\%$$

5.

$$T = \frac{100I}{PR}$$

$$T = \frac{100 \times M44.00}{M220.00 \times 4} = \frac{4400}{880} = 5\,years$$

6.

$$T = \frac{100I}{PR}$$

$$T = \frac{100 \times M72.05}{M720.50 \times 4} = \frac{7205}{2880} = 2.5\,years$$

Lesson 5 Discount

By the end of this subunit, you should be able to

- *solve* problems on discount

This subunit is about 3 pages in length.

In the previous subunit we talked about earning and paying interest. The interest earned is ours to spend! We can spend it on clothes, but there are times when you wish you could pay less than the marked price? When you were out shopping last weekend, didn't you see that nice jacket you so badly wanted but you didn't have enough money to buy?

Yes! I thought so!

Sometimes you can bargain or negotiate with the sales person to reduce the asking price. However, in some shops, you don't need to bargain because they are offering a discount.

A discount is a reduction of the selling price to boost sales or to get rid of stock that is no longer needed. When a discount is given, customers save money on specific products or services.

A discount is usually expressed as a percentage.

Example 1

A blanket is sold for M550. A customer is offered 5% discount. How much does the customer actually pay?

Solution

$$\text{Discount} = \frac{5}{100} \times 550$$

= M27.50

Amount paid = Original Price – Discount

= M550 – M27.50

= M522.50

The customer actually pays M522.50

Example 2

1. A car was sold for M28 000. If the discount on the car was 20%, how much was the discount in Maloti?

Solution

Before the discount, the car is said to be at 100 %. After the discount of 20%, the car will be at 100% - 20% = 80%

Let x be the value of the car before the discount. In order to find x, we can use the equation:

$$M28{,}000 = 80\%$$

$$x = 100\%$$

$$x = \frac{M28{,}000 \times 100\%}{80\%}$$

$$x = M35{,}000$$

The original price was M35 000

Discount (D) = Original Price (OP) – Selling Price (SP)

= M35 000 – M28 000

= M7 000

Activity

Activity 8.5

Answer all questions

1. Complete this table

Item	Original Price	Discount	New Price
Dress	£99.50	50%	
Laptop		48%	€3 120
Coffee table	M3 600		M3 240

2. Palesa passed by a store that is having a sale and saw (accepted)a M200 pair of shoes for M150.
   Find the discount amount and the discount percentage.

3. A dress is reduced from M650 to M450.

(a) Find the discount amount and the discount percentage.

(b) If it is reduced by a further 25%, how much can you expect to pay for this dress? (all accepted)

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on discount are:

- discount is a reduction of the selling price.

Model Answers

Activity 8.5

1. Complete this table

Item	Original Price	Discount	New Price
dress	£99.50	50%	**£49.75**
laptop	**M6 500**	48%	€3 120
Coffee table	M3 600	**10%**	M3 240

$$\text{Discount} = \frac{50}{100} \times £99.50$$
$$= £49.75$$

$$\text{Amount paid} = \text{Original Price} - \text{Discount}$$
$$= £99.50 - £49.75$$
$$= £49.75$$

€3 120 - 48%
X - 100%

$$X = \frac{€3\ 120 \times 100\%}{80\%}$$

X = M6 500

Discount = Original Price – Selling Price

M360 = M3 600 – M3 240

$$\frac{M360}{M3\ 600} \times 100 = 10\%$$

2. M200 - M150 = M50 (discount amount)

   discount percentage: $\frac{M50}{M200} \times 100 = 25\%$

3. A dress is reduced from M650 - M455 = M195

   discount percentage: $\frac{M195}{M650} \times 100 = 30\%$

   It is to be reduced further by 25%. How much can one expect to pay for this dress?

   $\frac{25}{100} \times M455 = M113.75$

   M455 - M113.75 = M341.25

Lesson 6 Profit and Loss

By the end of this subunit, you should be able to

- *solve* problems on profit and loss

This subunit is about 4 pages in length.

In the subunits on interest, we talked about how you can calculate the interest you will earn if you invest money. That is one way of making more money, but there are others. Another way is to go into business. Everybody who goes into business aims for a **profit**. Some people will say profit is the measure of how well a business is doing.

In your junior secondary certificate mathematics course you did some work on profit and loss. The simplest definition of profit is the amount a seller receives when the selling price is **more** than the cost price. The cost price is the price at which goods are bought and the selling price is the price for which the goods are sold. This relationship can be expressed in the following equation:

Profit = Selling Price (SP) – Cost Price (CP)

The percentage profit you earn is usually calculated on the cost price, as shown in the following formula:

$$\text{Percentage profit} = \frac{\text{Profit}}{\text{Cost Price}} \times 100$$

Example 1

Tefo buys a bag of oranges for M16. He sells them one by one to passersby on the street. After selling them, he finds that he has collected a total of M20. What profit did he make?

Solution

M16 is the cost price and M20 is the selling price.

Profit = Selling Price (SP) – Cost Price (CP)

= M20 – M16

= M4

$$\text{Percent profit} = \frac{\text{Profit}}{\text{Cost Price}} \times 100$$

$$\text{Percent profit} = \frac{4}{16} \times 100$$

= 25%

Example 2

A man wants to make 15% profit on the sale of his cellphone. If he bought it for M750, how much would he sell it for?

Solution

$$\frac{15}{100} \times M750 = M112.50$$

The cellphone will be sold for M750 + M112.50 = M862.50

Unfortunately, you cannot make a profit all the time; sometimes you have to sell goods at a **loss**. A loss is made if the selling price is **less** than the cost price.

You can use the same formulae as we used for profit to calculate both the total amount of the loss and the percentage loss. Try this with the following example:

Example 3

Sarah bought a used car for $4 000. After a few months, she discovered that maintaining the care was very expensive, so she decided to buy a motor scooter instead. In order to raise money for this, she sold the car to her friend for $2 500. How much money did she lose on the transaction, and what was the percentage loss?

Solution A

If we insert these figures into the formula for Profit, what happens?

Profit = Selling Price (SP) – Cost Price (CP)
Profit = $ 2 500 - $ 4 000
Profit = $ -1 500

In this case, the profit is a negative number, which indicates a loss. By inserting this figure in the formula for percent profit, we get the following:

$$\text{Percent profit} = \frac{Profit}{Cost\ Price} \times 100$$

$$= \frac{\$-1500}{\$4000} \times 100$$

$$= \$ -37.5\%$$

Again, this approach results in a negative percentage, which shows that a loss was made.

An alternative way of calculating a loss is to reverse the two expressions in the formula as shown below:

Loss = Cost Price (CP) – Selling Price (SP)

Since this approach is intended to calculate Loss, the resulting answer is a positive number.

As with the profit percentage, the percentage of a loss is usually calculated on the cost price.

$$\text{Percentage loss} = \frac{Loss}{Cost\,Price} \times 100$$

Solution B
If we adopt this alternative approach to solving the example above, the calculations look like this:

Loss = Cost Price (SP) – Selling Price (CP)
\$1 500 = \$4 000 - \$2 500

$$\text{Percent loss} = \frac{1500}{4000} \times 100$$

$$= 37.5\%$$

Example 4

(a) Lebohang bought a car for M150 000. When she sold it, she made a loss of 21%. How much did Lebohang sell the car for?

(b) A salesperson had made a profit of 20% when he sold the car to Lebohang for M150 000. How much did the salesperson pay for the car?

Solution

(a) The loss is $\frac{21}{100} \times$ M150 000 = M31, 500

M150 000 - M31, 500 = M118, 500

Lebohang sold the car for M118, 500

(b) The salesperson bought the car at what we can call the original price, which is the cost price. Let us call it Y.

Profit = Selling Price – Cost Price

Profit $= M150,000 - Y$

We also know that profit $= \frac{20}{100} \times$ Cost price (Y)

Therefore

$$M150,000 - Y = \frac{20}{100} \times \text{cost price (Y)}$$

Solving for the cost price,

$$M15,000\,000 - 100Y = 20 \times \text{cost price (Y)}$$

$$M15,000\,000 = 20 \times \text{cost price (Y)} + 100Y$$

$$M15,000\,000 = 120Y$$

$$Y = M125\,000$$

OR

We can let the cost price be at 100%.

In order to get the selling price, the percentage profit is added to the 100%.

So the selling price of M150 000 is at 120%

M150 000 - 120%
X - 100%

$$X = \frac{M150\,000 \times 100\%}{120\%}$$

$$X = M125\,000$$

The original price is M125 000

We can check the answer this way:

Profit = selling price – cost price

25 000 = 150 000 – 125 000

$$\text{Percent profit} = \frac{Profit}{Cost\ Price} \times 100$$

$$\text{Percent profit} = \frac{25000}{125000} \times 100$$
$$= 20\%$$

Example 5

Morena pays M40 for a box of pears and M30 for a box of bananas.

When all the bananas and the pears have been sold, he finds that he has made M50 from the pears and M39 from the bananas.

He wants to know which product was more profitable so that he can buy more of one that is more profitable the next day.

Solution

Profit from pears = M50 – M40

= M10

$$\text{Percent profit} = \frac{10}{40} \times 100$$
$$= 25\%$$

Profit from bananas = M39 – M30

= M9

$$\text{Percent profit} = \frac{9}{30} \times 100$$
$$= 30\%$$

Buying more bananas would be a better option for the following day, as he is making 30% profit on sales of bananas. Pears are not as profitable, as his profit margin is only25%.

Activity

Activity 8.6

Answer all Questions

1. An item bought for M30 is sold at M36. Calculate the percentage profit.

2. A shopkeeper buys an article for M800. He wishes to make a profit of 25%. What should his selling price be?

3. A book bought for M75 is sold for M60. What is the percentage loss?

4. At a bookshop a pen is sold for M14 thereby making a profit of 40%. Calculate the cost of a pen.

5. A chemist buys a batch of 200 bottles for M3 000. When they arrive at his shop, he finds that a quarter of the bottles are broken. The rest are sold at M22.00 each.
   Calculate the profit or loss.

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on profit and loss are:

- There is a profit when the selling price is **more** than the cost price.
- Profit = Selling Price (SP) – Cost Price (CP).
- Percentage profit = $\frac{Profit}{Cost\ Price} \times 100$
- There is a loss if the selling price is **less** than the cost price.
- Loss = Cost Price (SP) – Selling Price (CP)
- Percentage loss = $\frac{Loss}{Cost\ Price} \times 100$

Model Answers

Activity 8.6

1.

M30 is the cost price and M36 is the selling price.

Profit = Selling Price (SP) – Cost Price (CP)

= M36 – M30

= M6

$$\text{Percent profit} = \frac{Profit}{Cost\ Price} \times 100$$

$$\text{Percent profit} = \frac{6}{30} \times 100$$

= 20%

2.

The cost price which is M800 is at 100%.

The selling price, X, is at 120%

M800 - 100%
X - 120%

$$X = \frac{M800 \times 120\%}{100\%}$$

X = M960
The original price is M960

3.

Loss = Cost Price (SP) – Selling Price (CP)
M15 = M75 – M60

$$\text{Percent loss} = \frac{15}{75} \times 100$$

= 20%

4.

The selling price of M14 is at 140%

The cost price is at 100%.

M14 - 140%
X - 100%

$$X = \frac{M14 \times 100\%}{140\%}$$

X = M10

The cost price is M10

5.

$\frac{25}{100} \times 200 = 50$ broken bottles

200 – 50 = 150 bottles that are not broken

150 bottles × M22 = M3 300

M3 300 – M3 000 = M 300 profit

Lesson 7 Tax

By the end of this subunit, you should be able to

- Explain the difference between VAT and PAYE
- *solve* problems on tax

This subunit is about 4 pages in length.

Tax is money collected by governments based on individual incomes, the profits from business, the valuation of properties, or the sale of goods and services. Tax is used to support public services. These services include education, health, public roads, bridges, etc.

There are different types of taxes, but in this subunit we will only deal with Income Tax and Value Added Tax.

Income Tax

It is normal for a person to look forward to his or her first pay slip after starting a new job. However, many people are disappointed when they discover that they are getting less than they expected. Generally, they forget about the tax that is deducted from their pay before they receive it - income tax to be specific!

Income Tax, also known as Pay As You Earn (PAYE), is the tax paid by all persons earning an income from salaried employment. Anyone who is employed by another person or company, as well as those who are self-employed, must pay tax on their salaries. PAYE is also paid by people who make their money by other means, for example business men and business women, but only if that income comes from a salary or wages.

Not all of the income is taxed. Income tax is calculated only on your taxable income. The allowances that reduce the amount of money to be taxed vary from country to country. Examples of some of these allowances are children's allowance and disability allowance. Normally, tax payers are allowed to deduct a specified amount from their gross income before the percentage tax is applied to the net income.

Some countries use a system of tax credits instead of deductions. The percentage tax is applied to the gross income, but some of that tax is 'given back' as a tax credit. Taxpayers are allowed to reduce the total amount of tax they owe by subtracting the tax credit.

When the tax has been calculated, it is subtracted from the gross salary to give the net salary, sometimes called **take-home pay.**

Different countries have different policies on how to calculate income tax, but most countries use a progressive income tax system. This means that people who earn higher incomes pay a higher rate of tax compared to their counterparts who earn lower incomes.

Income Tax is normally calculated on a yearly or *per annum* basis.

Given below is the income tax structure in Lesotho:

Where chargeable income per annum is:	*Tax Rate*
• less than M 22 727	No tax is due
• between M 22 727 and M40 368	22% on income in this band
• greater than M 40 368	35% on the excess

In addition, anyone who pays tax is allowed a tax credit of M5 000.

Example 1

Mr Mosito earns M1 600 per month. How much tax does he pay per month if he lives in Lesotho?

Solution

We calculate Mr Mosito's annual income

$$M1{,}600 \times 12 = M14{,}400$$

Because his gross income is below M 22 727 per annum, Mr Mosito does not pay tax.

Example 2

Mrs Mafa earns M8 500 per month. How much tax does she pay per month if she lives in Lesotho?

Solution

We calculate Mrs Mafa's annual income

M8 500 × 12 = M102 000

First, we need to calculate the tax she must pay at the lower rate of 22%

$$\frac{22}{100} \times \text{M}40\ 368 = \text{M}8\ 880.96$$

Any income over M40 368 is referred to as 'excess' and is taxed at the higher rate of 35%.

M102 000 - M40 368 = M61 632

$$\frac{35}{100} \times \text{M}61\ 632 = \text{M}21\ 571.20$$

Total tax = M8 880.96 + M21 571.20

= M30 452.16

If there was no tax credit of M5 000, Mrs Mafa would be liable for a total tax bill of M30 452.16.

Now because of the tax credit, she will pay

M30 452.16 – M5 000 = M25 452.16

Mrs Mafa will pay total tax of M25 452.16 per annum

To calculate how much tax is deducted from Mrs Mafa's salary each month, you divide the total amount of tax by 12 months:

M25 452.16 ÷ 12 = M2 121.01 per month

Example 3

Robert earns M2 500 per month. How much tax does he pay per annum if the tax rate is 27% on his gross income?

Solution

We calculate Robert's annual income

M2 500 × 12 months = M30 000

Annual tax calculated at a rate of 27% is

$\frac{27}{100} \times$ M30 000 = M8 100

OR

We could calculate Robert's monthly tax

Tax calculated at a rate of 27% is

$\frac{27}{100} \times$ M2 500 = M675

M675 × 12 months = M8 100

Example 4

Chippo earns M5 300 per month. She has a child with disability. In her country, this reduces the amount of money that has to be taxed by M2 880. How much tax does she pay per annum if the tax rate is 32%?

Solution

Chippo's annual income = M5 300 × 12 months = M63 600

Chippo is taxed on M63 600 – M2 880 = M60 720

Annual tax calculated at a rate of 32% is

$\frac{32}{100} \times$ M60 720= M19 430.40

Value Added Tax (VAT)

Value Added Tax (VAT) is a tax on goods and services supplied to customers. It is normally already included in the price of the item. It allows one to say "the price you see is the price you pay."

A registered business is charged with the responsibility for collecting VAT when supplies are made to customers. Registered businesses are also responsible for declaring and paying VAT on supplies made. In return, they can claim a credit for any VAT they pay on goods and services bought for their business. So, a factory manufacturing clothing must declare and pay VAT on all the finished garments they supply to customers. However, they can claim a credit for the VAT they pay on supplies of raw materials, machinery and other inputs required to keep the business running.

VAT rates are set by the governments in different countries and can change from time to time. At the time of writing, VAT in Lesotho is 14%on all goods and services with the exception of basic foodstuffs, such as milk and maize meal.

Example 1

You are about to buy goods to the value of £350 excluding VAT.
Find the total amount you are going to pay if VAT is 10%

Solution

The VAT is 10% of £350

$$\frac{10}{100} \times £350 = £35$$

Total amount = £350 + £35 = £385

Example 2

When buying a new lounge suite, Molemo paid M8 050. The price included VAT at 15%.
Calculate the original price of the lounge suite excluding VAT.

Solution

We can let the original price of the lounge suite be at 100%.

So the price of the lounge suite at M8 050 is at 115%

$$M8{,}050 = 115\%$$
$$x = 100\%$$
$$x = \frac{M8{,}050 \times 100\%}{115\%}$$
$$x = M7{,}000$$

The original price of the lounge suite is M7 000, excluding VAT.

Activity

Activity 8.7

Answer All Questions

1. A woman buys a dress which is priced at M650 plus VAT at 15%. How much does she actually pay for the dress?

2. An item is priced at $60 inclusive of VAT at 20%. What is its price exclusive of VAT?

3. Complete this table

Gross Salary Per Year	Income Tax Calculations	Net Salary Per Year
M75 000	15%	
M180 000	M24 000 is tax free and the taxable income is taxed at 50%	
M125 000	Gross salary up to M40 000 taxed at the lower rate of 25% and excess taxed at the upper rate of 35% and the tax credit is M5 000	
M38 400	taxed at 23% Child allowance of M180 per month	

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on tax are:

- taxes are one of the most common ways for governments to raise money needed for public services.
- income tax, also known as Pay As You Earn (PAYE) is the tax charged on the income of individuals working in salaried employment.
- different countries have different policies on how to calculate income tax.
- Value Added Tax (VAT) is a form of tax levied on goods and services supplied inside a country.
- VAT rates differ from country to country.

Model Answers

1. The VAT is 15% of M650

$$\frac{15}{100} \times \text{M}650 = \text{M}97.50$$

Total amount = M650 + M97.50= M747.50

2. The price excluding VAT is at 100%.

   The price including VAT is at 120%.

   $60 - 120%
   X - 100%

   $$X = \frac{\$60 \times 100\%}{120\%}$$

   X = $50

   The price exclusive of VAT is $50

3.

Gross Salary Per Year	Income Tax Calculations	Net Salary Per Year
M75 000	15%	$\frac{15}{100} \times$ **M75 000= M11 250**

		M75 000 - M11 250 = M63 750
M180 000	M24 000 is tax free and the taxable income is taxed at 50%	**M180 000 – M24 000 = M156 000** $\frac{50}{100}$ × **M156 000= M78 000** **M180 000 – M78 000 = M102 000**
M125 000	Gross salary between M0 and M40 000 taxed at the lower rate of 25% and excess taxed at the upper rate of 35% and the tax credit is M5 000	$\frac{25}{100}$ × **M40 000= M10 000** **M125 000 – M40 000 = M85 000** $\frac{35}{100}$ × **M85 000= M29 750** **M10 000 + M29 750 = M39 750** **M39 750 - M5 000 (tax credit) = M34 750** **M125 000 – M34 750(tax paid)= M92 250**
M38 400	taxed at 23% Child allowance of M180 per month	**Child allowance of M180 × 12 months = M2 160** **M38 400 – M2 160(child allowance)= M36 240** $\frac{23}{100}$ × **M36 240=**

		M8 335.20

Lesson 7 Budgeting

By the end of this subunit, you should be able to:

- *solve* problems on budgeting.

This subunit is about 4 pages in length.

A **budget** is a plan that estimates income and expenditure. Individuals, families, organisations, businesses and governments all draw up budget to show how much money they expect to take in and spend, normally over a period of one or more years. A budget helps to analyse exactly where money earned goes and how funds can best be allocated.

In this subunit, we will focus on budgeting for household purposes. For individuals and households, a budget is about control, it is not at all meant to restrict you. It simply helps one to avoid overspending.

The starting point is to put down all your earnings.

Then list all your outgoings, that is all the money you spend to maintain the household and meet the expenses of family members. It will help if you can categorise your expenses. These are some of the categories that you can use and some of the expenses that should be included under each.

1. **Investment and savings**

2. **Home**
   For people who are renting they will have rent here, whilst some may have mortgages or bonds on their houses.

3. **Food**
   This is a must for all! This category includes all the groceries and other foodstuffs that we buy to prepare at home. It also includes money spent on take-away food or eating out.

4. **Debt Payments**
   In this category we may have loans, credit cards, instalments one has to pay for furniture or clothes bought on credit

5. **Utilities**
   We have electricity, water, telephone (landline and cell)

6. **School Expenses**
   Fees, books, stationery, school trips, examination fees, etc. are included under this heading.

7. **Transportation**
   Petrol, vehicle services, car wash, and bus or taxi fares for children when travelling to and from school.

8. **Other Expenses**
   Here you can include all expenses that do not come under any of the other categories.

Example 1

The table below gives Mr and Mrs Matsoso's monthly budget for the month of January, 2011

	Actual Amount	Sub-Totals
Income		
Salary: Mr Matsoso	M 8 900	
Mrs Matsoso	M10350	
Rental Income (from the letting of property they own but do not live in)	M 2 000	
Total		**M21 250**
Expenses		
Investment and savings	M2 000 M 500 M 430	
Food Groceries	M2 500	
Debt Payments Instalment for Furniture Store Installment for Clothes Store	M 2 330 M 980	
Utilities Electricity Telephone : Land line Cell: Mr Matsoso Mrs Matsoso	M 550 M 150 M 600 M 450	
School Expenses School fees	M6 000	

Transportation Petrol Bus fare for Children Car Wash	 M 600 M 270 M 150	
Others Gardener Housemaid	 M 550 M 890	
Total		**M 18 950**
Difference		**M 2 400**

Mr and Mrs Matsoso have a **surplus** of M2 400 because their income is more than their outgoings. If their expenses were more than their income, they would have a **shortage or deficit**.

Example 2

The table below gives Mr and Mrs Matsoso's monthly budget for the month of February, 2011

	Actual Amount	Sub-Totals
Income		
Salary: Mr Matsoso	M 8 900	
Mrs Matsoso	M10 350	
Rental Income	M 2 000	
Total		M21 250
Expenses		
Investment and savings	 M 2 000 M 500 M 430	
Food Groceries	 M 2 500	
Debt Payments Instalment for Furniture Store Installment for Clothes Store	 M 2 330 M 980	
Utilities Electricity Telephone : Land line Cell: Mr Matsoso Mrs Matsoso	 M 550 M 150 M 600 M 450	

School Expenses		
Transportation		
Petrol	M 600	
Bus fare for Children	M 270	
Car Wash	M 150	
Vehicle Service	M 1560	
Others		
Gardener	M 550	
Housemaid	M 890	
House maintenance	M 7 040	
Total		**M 21 550**
Difference		M 300

This time around there is a shortage of M300. Mr and Mrs Matsoso have overspent. This calls for cutting down on certain items. We could suggest that they cut down back on the use of their cell phones and on some items in their grocery list or buy store brand items instead of brand-name labels.

Activity

Activity 8.8

1. Lineo has total earnings amounting to M3 500. Her expenses for this month are M5 320. Does she have a shortage or a surplus?

2. Tefo spends 20% of his earnings on groceries. If he earns M2 500 a month, how much does he spend on groceries?

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on budgeting are:

- A budget is a plan for earning or spending money.

You have now completed work on this unit on commercial mathematics. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Model Answers

Activity 8.8

1. There is a shortage of M1 820 as earnings are less than expenses.
2. $\frac{20}{100} \times$ M2 500 = M500

Unit Summary

Summary

In this unit you learned that:

- When using the 12- hour clock system to express time, the day is split into two periods each of 12 hours.
- A.M. or a.m. or am is used to indicate the morning hours, between midnight and midday.
- P.M. or p.m. or pm is used to indicate the afternoon and evening hours, between midday and midnight.
- With the 24-hour clock system, the day is not split. Instead it runs from midnight to midnight of the following day.
- The term *exchange rate* is the amount of currency in one country that is required to obtain a single unit of currency in another country. The exchange rate can be fixed or it can vary from day to day.
- Every time you change money into another currency, you lose some of its value as commission.
- The original sum of money borrowed or lent is called the principal.
- The amount charged on money that is borrowed or loaned is called interest.
- The types of interest dealt with in this unit are (a) simple interest and (b) compound interest.
- Simple interest is charged on the principal (the original amount borrowed or lent) at the same rate over the term of the loan.
- The formulae for calculating simple interest $= \dfrac{PRT}{100}$

  Where P-principal, R-rate of interest and T- the time the money is lent or borrowed.
- The amount that results after interest is added to the principal over a given period is called the maturity value of the loan.
- Compound interest is interest that is paid on both the principal and any interest accumulated over past years.

- The formulae for calculating compound interest is:

  $$A = P\left(1 + \frac{r}{100}\right)^n$$

  Where,

  B- maturity value,

  P - principal, r-rate of interest and

  n - number of years the money is invested

- A discount is any reduction in the selling price of goods or services.

- There is a profit when the selling price is **more** than the cost price.

- Profit = Selling Price (SP) – Cost Price (CP)

- Percentage profit = $\frac{Profit}{Cost\ Price} \times 100$

- There is a loss if the selling price is **less** than the cost price

- Loss = Cost Price (CP) – Selling Price (SP)

- Percentage loss = $\frac{Loss}{Cost\ Price} \times 100$

- Taxes are one of the most common ways for governments to raise money needed for public services.

- Income Tax, also known as Pay As You Earn (PAYE), is the tax charged on the income of individuals in salaried employment.

- Different countries have different policies on how to calculate income tax.

- Value Added Tax (VAT) is a form of tax levied on goods and services supplied to customers.

- VAT rates differ from country to country and may also vary depending on the type of goods and services sold.

- A budget is a plan for earning or spending money.

You have completed the material for this unit on Commercial Mathematics. You should now spend some time reviewing the content in detail.

Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have.

Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

1. **Answer All Questions.**
2. **Show all of the work you do to arrive at an answer.**

Total marks = 30 **Time**: 30 mins

1. Letlotlo pays 15% of his salary as income tax. His salary is M2, 300 per month.

   (a) How much income tax does he pay in

   (i) a month?

   (ii) a year?

   (b) How much does Letlotlo take home at the end of the month?

   [5]

2. The cost price of a dress is M450. It is then sold for M540.

   (a) Calculate the profit made on the dress.

   (b) Express the profit as a percentage of the cost price.

   [5]

3. A telephone bill including VAT at 14% is M934.80. How much was the bill before VAT was added?

[4]

4. An item is sold for M 635 at a loss of 20%. What was the cost price?

[4]

5. Lintle lives in South Africa. She wants to visit Botswana.

   If P1.00 = R1.04

   How many Pula will she get for R7 000

[4]

6. A man invested M9 500 in a bank.

   (a) Calculate simple interest if the money is invested for 3 years at a rate of 10% per annum.

   (b) What is the total money the man will have after 3 years?

[6]

7. A discount of $7.00 is offered on an item costing $28.00. What is the percentage discount?

[2]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Sample Answers to the Assignment

1. (a) (i) $\frac{15}{100} \times$ M2,300 = M345

   (ii) M345 × 12 months = M4,140

   (b) M2,300 – M345 = M1,955

2. (a) Profit = Selling Price – Cost Price

   M90 = M540 – M450

   (b) $\frac{90}{450} \times 100 = 20\%$

3.

M934.80 - 114%
X - 100%

$X = \frac{M934.80 \times 100\%}{114\%}$

X = M820

The bill before VAT is M820

4. M635 = 80%
   = 100%

$$\frac{100\% \times 635}{80\%} = \frac{63\ 500}{80\%} = M793.75$$

5.

$$\frac{1 \times R7\ 000}{1.04} = M6\ 730.77$$

6.

SI = PRT

$$\frac{9\ 500 \times 10 \times 3}{100} = M2\ 850$$

M9 500 + M2 850 = M12 350

7.

$$\frac{7}{28} \times 100 = 25\%$$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

1. **Answer All Questions.**
2. **Show all the necessary working.**

Total marks = 30 **Time**: 30 mins

1.Lerato buys a scientific calculator at M150 plus VAT at 14%. Work out the price that is actually paid.

[4]

2. Mrs Mohapi was retrenched from her job, and received a lump-sum payout of M 150 000. She has two options for investing this amount. The first option offers her simple interest at the rate of 25% per year. The second option would attract compound interest at the rate of 20% per year.

   (a) Which option will give her more interest if she wants to invest her money for two years?

   (b) Which option will give her more interest if she wants to invest her money for four years?

[7]

3. In a clearance sale, a machine was sold for M15 600. If the discount on the machine was 35% during the sale, find its original price.

[4]

4. A man is paid M65 per hour for a 40 hour week.

(a) Calculate his gross weekly pay.

(b) He pays M390 per week in income tax. Find his take home pay for the week.

[3]

5. A businessman bought 90 buckets for M16.95 each. What price must he sell each bucket for in order to make a profit of 25%?

[4]

6. Teko bought a car for M62 000. He sold it to Tšepo at a loss of 9%. After a few months, Tšepo sold the car on to Flora at a loss of 8%. How much did Flora pay for it?

[4]

7. If an item costing $150 is ordered from America by someone who lives in England. How much will it cost in pounds if the exchange rate is $1.00 = £0.62

[4]

Sample Answers to Assessment

1. (a) Simple Interest = M75,000

   Compound Interest = M66,000

   The simple interest option will give her more interest.

   (b) Simple Interest = M150,000

   Compound Interest = M161,040

   The compound interest option will give her more interest.

2. The original price is M24 000
3. Total amount = M171
4. (a) M2 600 (b) M2 210
5. M1,906.88
6. M53 047.20
7. £93

Unit Contents

Unit 9

Unit 9

Linear Inequalities

Introduction

This unit consists of 79 pages. This unit is approximately 3% of the whole course. Plan your time so that you can complete the whole course on schedule. The number of pages may feel overwhelming. However, note that a large part of the unit consists of graphs so there is less content than you might think. As reference, you will need to devote 30 hours to work on this unit, 20 hours for formal study and 10 hours for self-study and completing assessments/assignments.

In equations, one side is equal to the other side. This is indicated by the presence of an equal sign. In an inequality (inequation,) one side is greater than (or greater than or equal to) or less than (or less than or equal to) the other side. This is indicated by the signs >, ≥, <, and ≤ respectively.

In your Junior Mathematics you learned about solving inequalities with one variable. To solve an inequality, you used the same methods as you did when solving equations. When is there a difference?

Consider the inequality $6 < 15$

Multiply both sides by -1 and note what happens.

$-6 > -15$

The inequality is no longer true.

When we multiply or divide by a negative number, the direction of the inequality changes.

In this unit you are going to learn more about solving inequalities with two variables. Are you ready?

Spend a few moments reading the following learning outcomes. They are a guide to what you should focus on while studying this unit.

This Unit is Comprised of Two Lessons:

Lesson 1 Solving Linear Inequalities With Two Variables
Lesson 2 Writing the Linear Inequality Given the Drawing of the Inequality

Upon completion of this unit you will be able to:

Outcomes

- *draw* the graph of a given linear inequality
- *write* the linear inequality given the drawing of the inequality

Terminology

Inequality: A mathematical sentence in which the value of the expression on the left hand side is not equal to that on the right hand side. Symbols used with inequalities are $<, >, \leq$, and $\geq$

Expression: An expression is formed when terms are combined by either addition or subtraction

Term: It is either a single number or variable, or the product of several numbers and/or variables separated from another term by a + or – sign.

Equation: A mathematical statement connecting two expressions with an equal sign. a mathematical sentence in which the value of the expression on the left hand side is equal to that on the right hand side.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Solving Linear Inequalities With Two Variables

By the end of this subunit, you should be able to

- solve linear inequalities with two variables

In a linear inequality the exponents of the variables are "1" or "0" (no variable).

Here are examples of linear inequalities:

$x \geq 1$

$2x - 5 > 9$
These are inequalities with one variable, x.

$5x - 6y \leq 7$
This is an inequality with two variables, x and y.

Reflection

Let us take you back to the solution of inequalities with one variable.

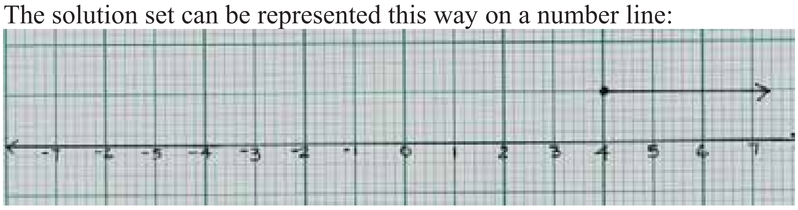
$$\begin{array}{r} 2x - 5 \geq 3 \\ \underline{\quad +5 \geq +5} \end{array}$$

$$\frac{2x}{2} \geq \frac{8}{2}$$

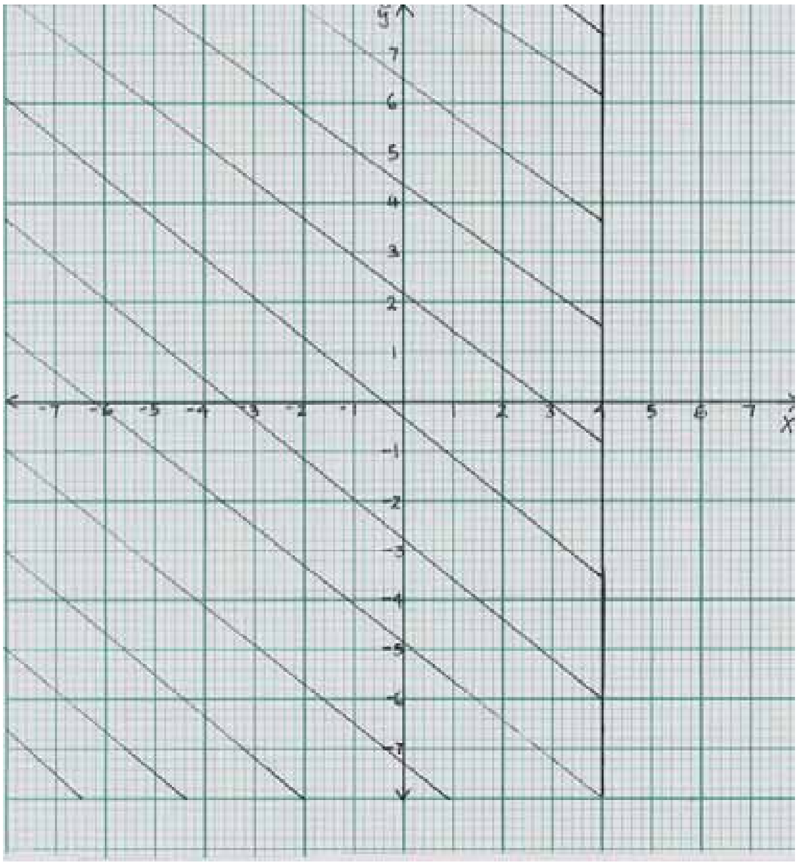
$$x \geq 4$$

Values of x can be 4, 4.0000000000001, 4.0001, 5, 5.5, 6, 6.1357, 7, or any other positive number, or any fraction in between. In other words, there are infinite solutions for an inequality.

The solution set can be represented this way on a number line:



The solution can also be represented on graphs using shading



This unshaded region contains all the points whose x coordinate is greater or equal to 4, i.e x = 4, 4.0000000000001, 4.0001, 5, 5.5, 6, 6.1357, 7, or any other positive number, or any fraction in between. In other words, there are unlimited number of points.

The line is solid because of the inequality sign "≥".

Note it!

In some countries you maybe asked to shade the side of the line that contains the solution set.

Solution of linear inequalities with two variables is best done graphically. It is done in the same way as graphing of linear equations.

These are the steps to be followed.

Step 1

Change the inequality to an equation.

Step 2

Pick the x – values that you will use. Pick whatever values you like, but it's often best to "space them out" a bit. For instance, pick $x = -3, -2, -1, 0, 1, 2, 3$. That's not a rule, but it's often a helpful method. Sometimes it is easier to choose y values and compute the corresponding x values. For example, it would be easier in an equation such as $x = 2y + 2$.

Compute the corresponding y – values.

Step 3

Plot the points.

Step 4

Then join the points with a solid or a dashed straight line. If the inequality you are working with has "<" or ">", join the points with a dashed line, or if the inequality is "≤" or "≥", join the points with a solid line.

The line divides the plane into three parts; the line itself and an area on each side of the line. The two areas are called **regions**.

The straight line is the set of points defined by an equation.

The regions are a set of points defined by inequalities.

Label the line drawn.

Step 5

To identify the region that has the coordinates that satisfies the inequality, that is, the solution:

- pick one pair of coordinates above the line. Substitute them in the inequality.
- Pick another pair of coordinates below the line. Substitute them in the inequality.
- The pair that satisfies the inequality falls in the region that has all the pairs that satisfy the inequality. **There are still many other solutions for an inequality.**

The region that will be shaded is the unwanted region, because this is a clearer way to identify the wanted region. Nothing will obscure the region, where our interest is.

Example 1

By shading the unwanted region, show the region that represents the inequality $x + y > 5$

Step 1

We start off by changing the inequality to an equation.

$x + y = 5$

Step 2

We have picked $x = -3, -2, -1, 0, 1, 2, 3$.

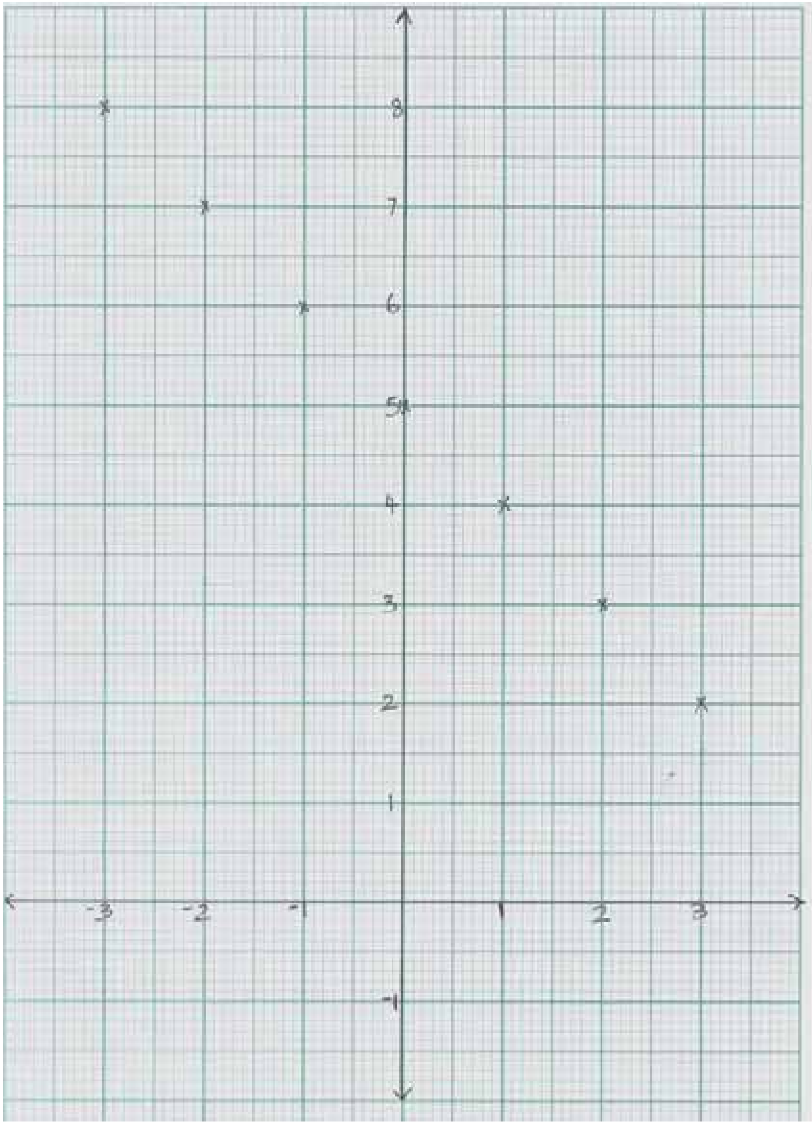
The corresponding y – values are in the table below:

x	y
–3	8
–2	7
–1	6
0	5
1	4
2	3

3	2

Step 3

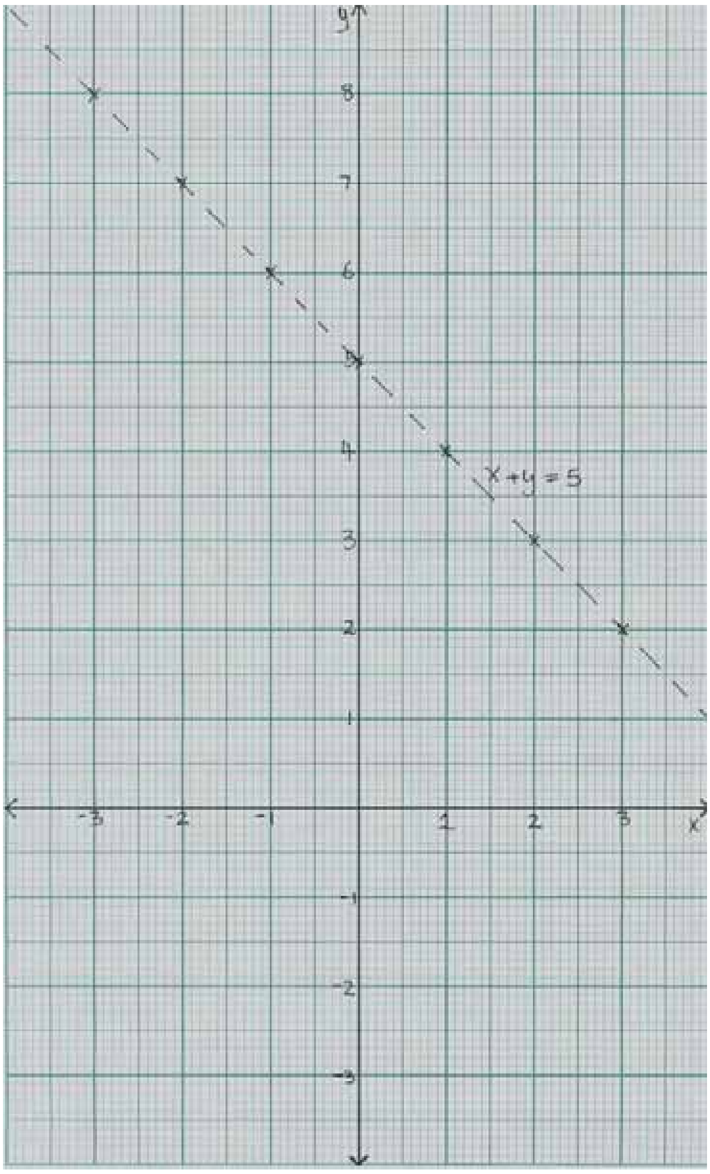
We plot the points.



Step 4

The inequality in our example is “>”. We join the points with a dashed line.

Please note that the graph below has included some of the points that are not in the table of values. This confirms that a line is made up of numerous points.



Step 5

Our inequality is $x + y > 5$

One of the many pairs of coordinates above the line is (3,6). When we substitute the pair. in the inequality we get:

$x + y > 5$

$3+ 6 > 5$

$9 > 5$

The coordinates (3,6) fall in the region that has all the coordinates that **satisfy the inequality**, for 9 is greater than 5.

One of the many pairs of coordinates below the line is (0,0). Substituting it in the inequality;

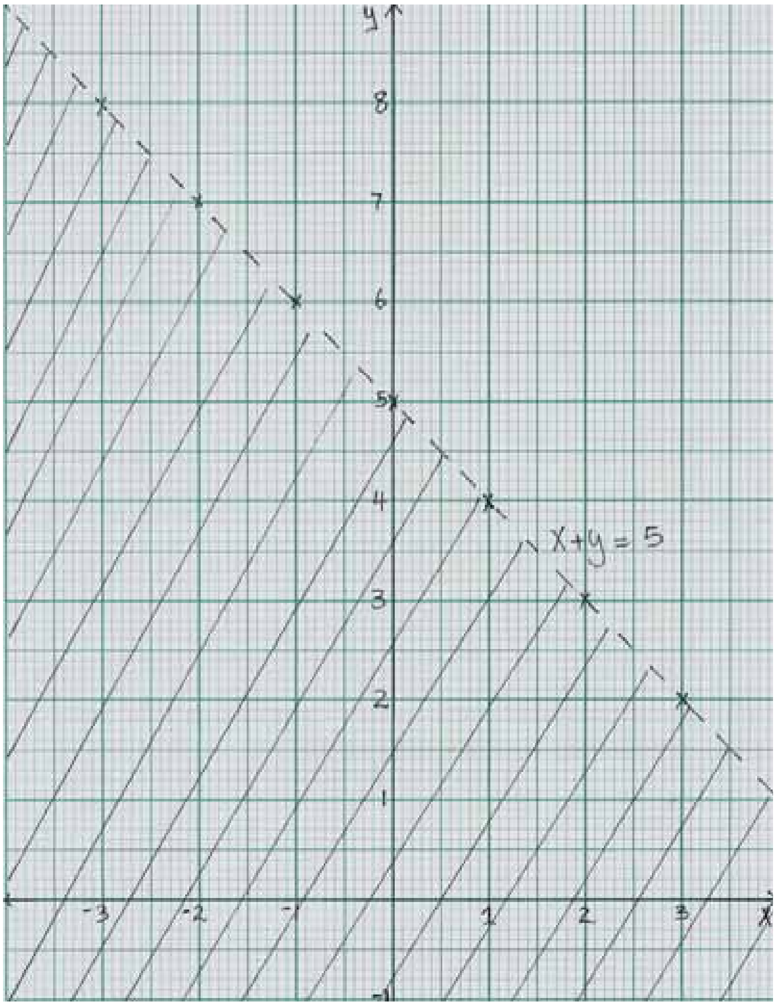
$x + y > 5$

$0 + 0 > 5$

$0 > 5$

The coordinates (0,0) fall in the region that has all the coordinates that **do not satisfy the inequality**, for 0 is less than 5.

Therefore the region shaded is the one below the line.



The line is dashed because of the symbol “>”. This means all points on it are not part of the required region. Let us test by using one point on the line.

(-2, 7)

x + y > 5

-2 + 7 > 5

5 > 5

We know that 5 is equal to 5! (-2, 7) is one of the many points on the boundary line that do not satisfy the inequality.

Example 2

Graph the inequality $y \geq x - 3$

Step 1

Change the inequality to an equation.

Compare your answer with the following:

$y = x - 3$

Step 2

Normally, you can pick any x values that you like and calculate the corresponding y values. For this exercise, the values of x will be –3, –2, –1, 0, 1, 2, and 3.

Compute the corresponding y values:

x	y
–3	
–2	
–1	
0	
1	
2	
3	

Compare your answers with the following:

x	y
–3	–6
–2	–5
–1	–4
0	–3
1	–2
2	–1
3	0

Given that this is not the first time that you are drawing graphs, you can do steps 3, 4 and 5 in one diagram. Example 1 should have reminded you.

Step 3

Plot the points.

Step 4

Determine whether the line should be solid or dashed. Draw the line.

Note that the line is solid because the inequality sign is "≥". In other words, the points in the line are included in the solution set.

Step 5

Determine which side of the line should be shaded. Shade that side of the line. To do this:

- Pick one pair of coordinates above the line. Substitute them in the inequality.

____

____

____

____

Pick another pair of coordinates below the line. Substitute them in the inequality.

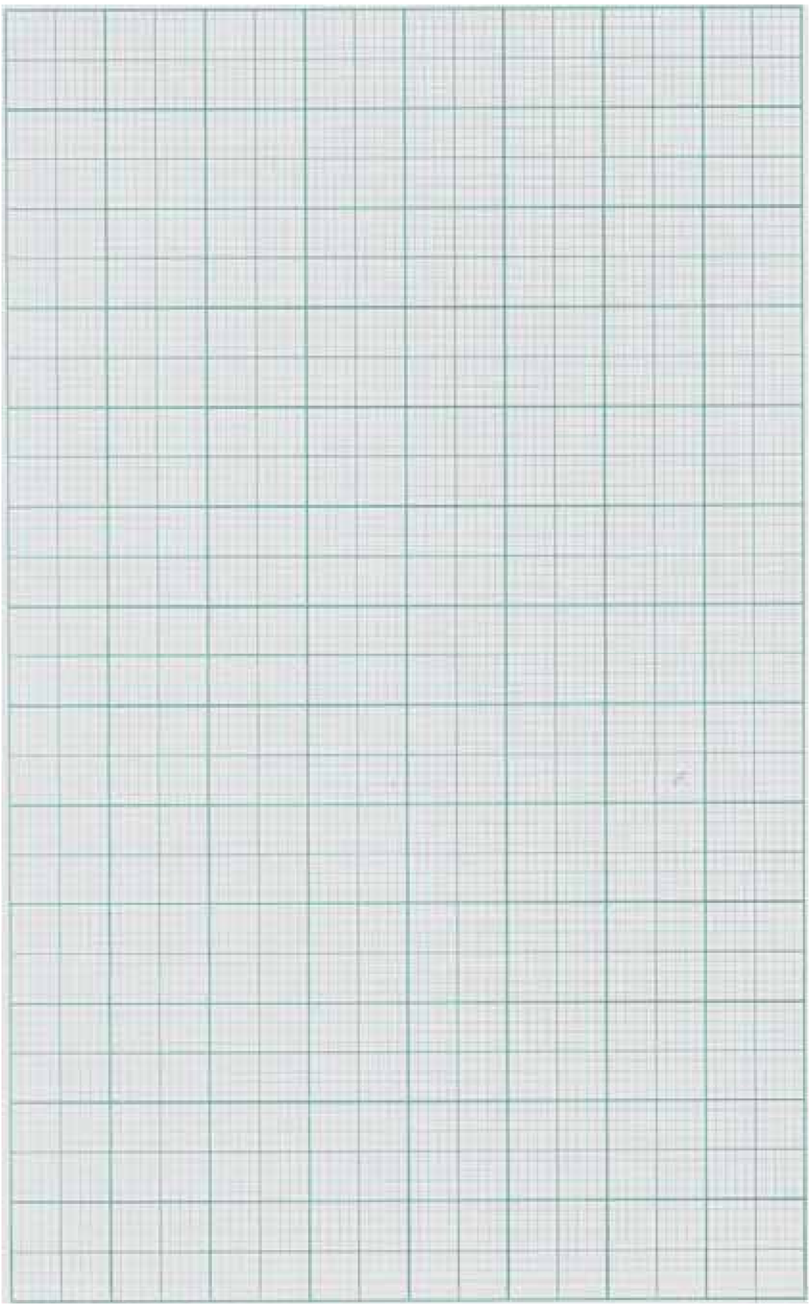
____

____

____

____

- The pair that satisfies the inequality falls in the region that has all the pairs that satisfy the inequality. Remember there are **unlimited number of points that satisfy the inequality.**
- Shade the appropriate side of the line in your graph below.



Compare your calculations and graph with the following:

You may have used different "test points" from the ones we have used. That should not in any way affect the result. Actually it is an advantage for that allows confirmation with a number of points.

Our inequality is $y \geq x - 3$

One pair of coordinates above the line is (3,1). Substitute them in the inequality.

$y \geq x - 3$

$1 \geq 3 - 3$

$1 \geq 0$

The coordinates (3,2) fall in the region that has all the coordinates that satisfy the inequality, for 1 is greater than 0.

One pair of coordinates below the line is (-2, -7). Substitute them in the inequality.

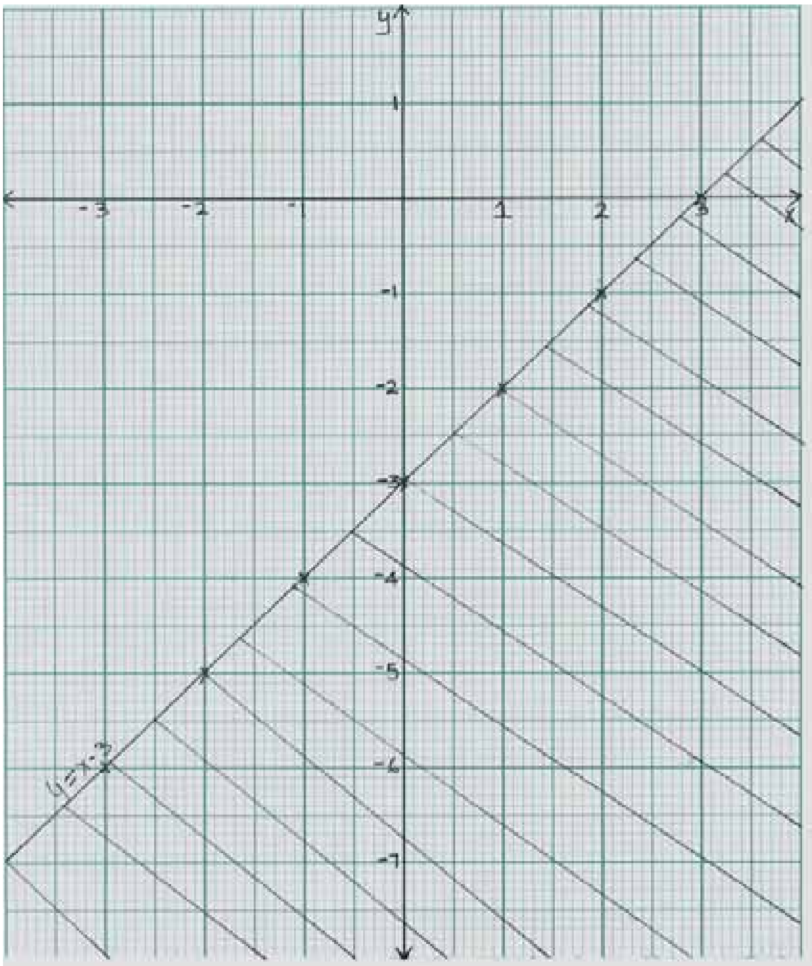
$y \geq x - 3$

$-7 \geq -2 - 3$

$-7 \geq -5$

The coordinates (-2, -7) fall in the region that has all the coordinates that **do not** satisfy the inequality, for -7 is less than -5.

Therefore the region that will be shaded will be the one below the line.



The line is solid because of the symbol "≥". This means all points on it are part of the required region. Let us test by using one point on the line, (3,0).

$y \geq x - 3$

$0 \geq 3 - 3$

$0 \geq 0$

We know that 0 is equal to 0!

Example 3

Graph the inequality $y < x$

Step 1

Change the inequality to an equation.

Compare your equation with the one below.

The result is y = x

Step 2

Pick any x values that you like and calculate the corresponding y values.

Once again we have picked $x = -3, -2, -1, 0, 1, 2, 3$.

x	y

Compare your calculations with those below.

x	y

–3	–3
–2	–2
–1	–1
0	0
1	1
2	2
3	3

You may have decided to choose the same x – values as ours. Their corresponding y – values will be the same as the one in the table above. If not, do not worry, they are still some of the many coordinates of the line y = x.

Again, given that this is not the first time that you are drawing graphs, you can do steps 3, 4 and 5 in one diagram.

Step 3

Plot the points.

Step 4

Determine whether the line should be solid or dashed. Draw the line.

Step 5

Determine which side of the line should be shaded. Shade that side of the line

____

____

____

Compare your calculations and graph with the following:

Our inequality is $y < x$

One pair of coordinates above the line is (-2, 2). Substitute them in the inequality.

$y < x$

$2 < -2$

The coordinates (-2,2) fall in the region that has all the coordinates that do not satisfy the inequality, for -2 is less than 2.

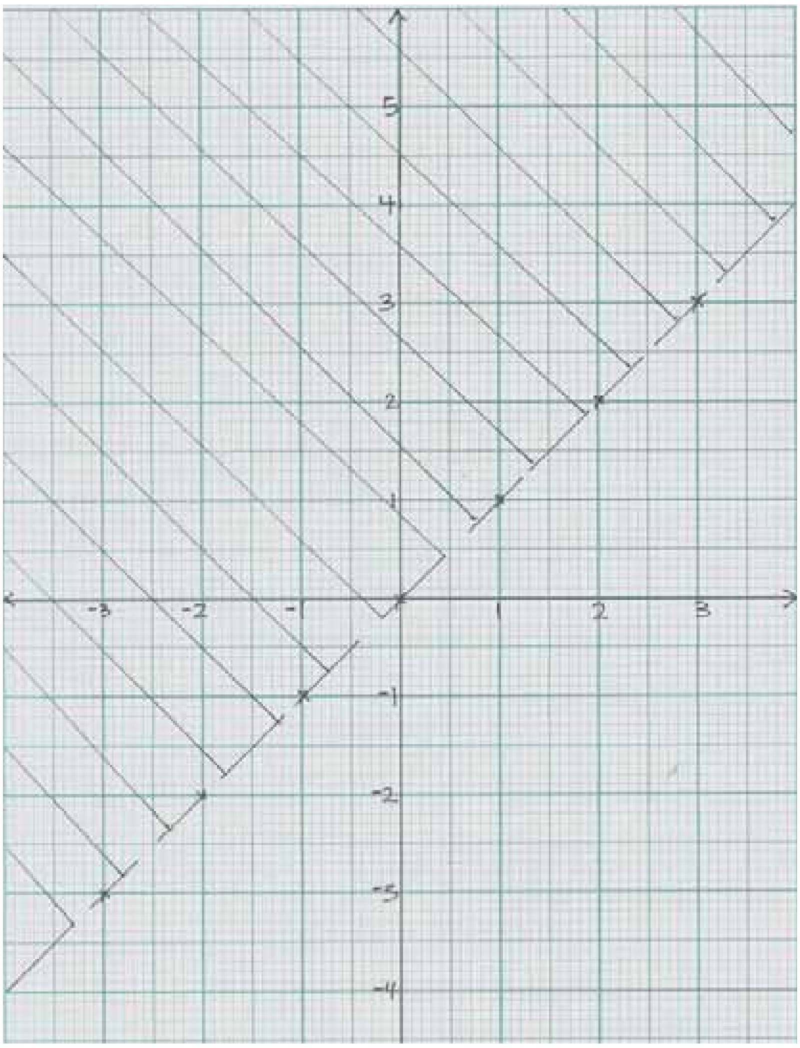
Another pair of coordinates below the line is (3, -2). Substitute them in the inequality.

$y < x$

$-2 < 3$

The coordinates (3, -2) fall in the region that has all the coordinates that satisfy the inequality, for -2 is less than 3.

Therefore the region that will be shaded will be the one above the line.



The line is dashed because of the symbol " < ". This means all points on it are not part of the required region. Let us test by using one point on the line, (1,1).

$y < x$

$1 < 1$

We know that 1 is equal to 1!

Activity 1

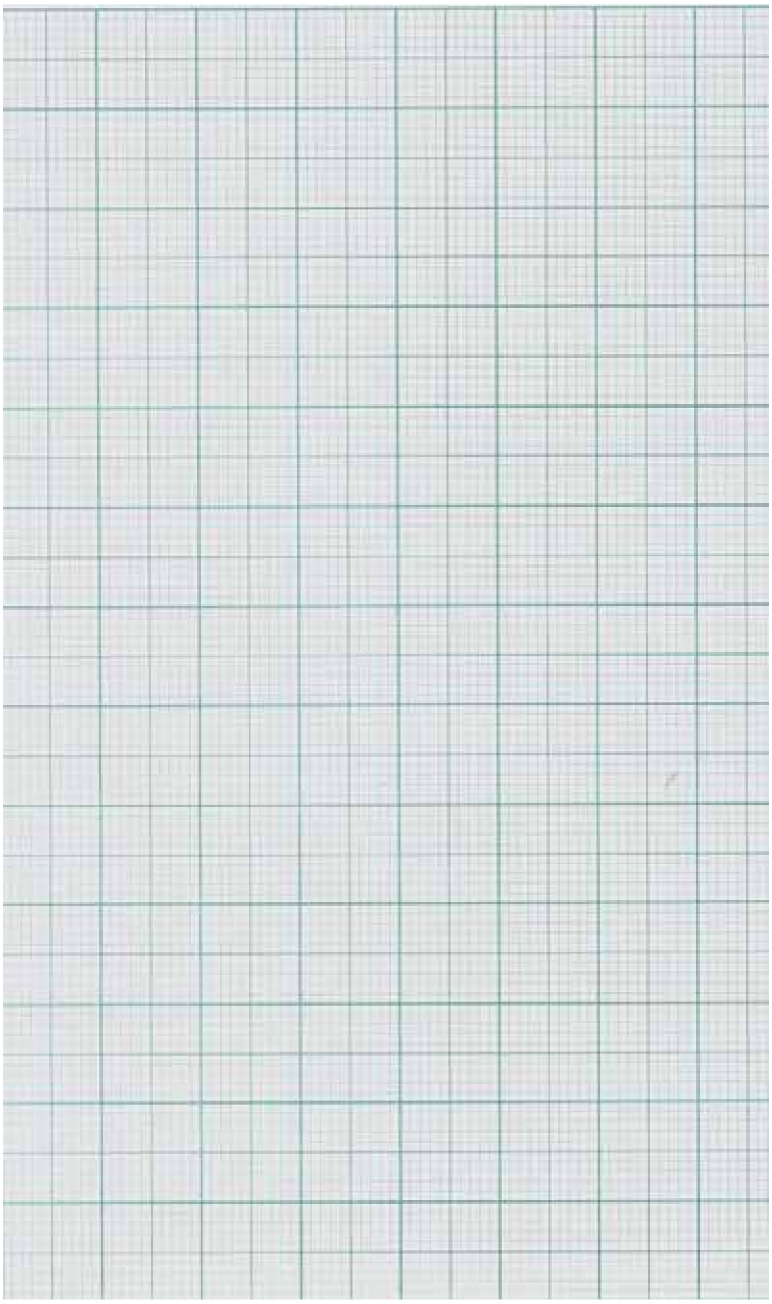
By shading the unwanted areas, indicate on graphs the regions defined by the following inequalities:

1. $x + y \leq 2$

2. $y + 2x \geq 4$

3. $x + y \leq 8$

4. $x + y \geq 1$



Check your performance against the given solutions at the end of this subunit. Continue if you are satisfied with your ability to answer the questions. If not, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on solving linear inequalities with two variables are:

- Change the inequality to an equation.

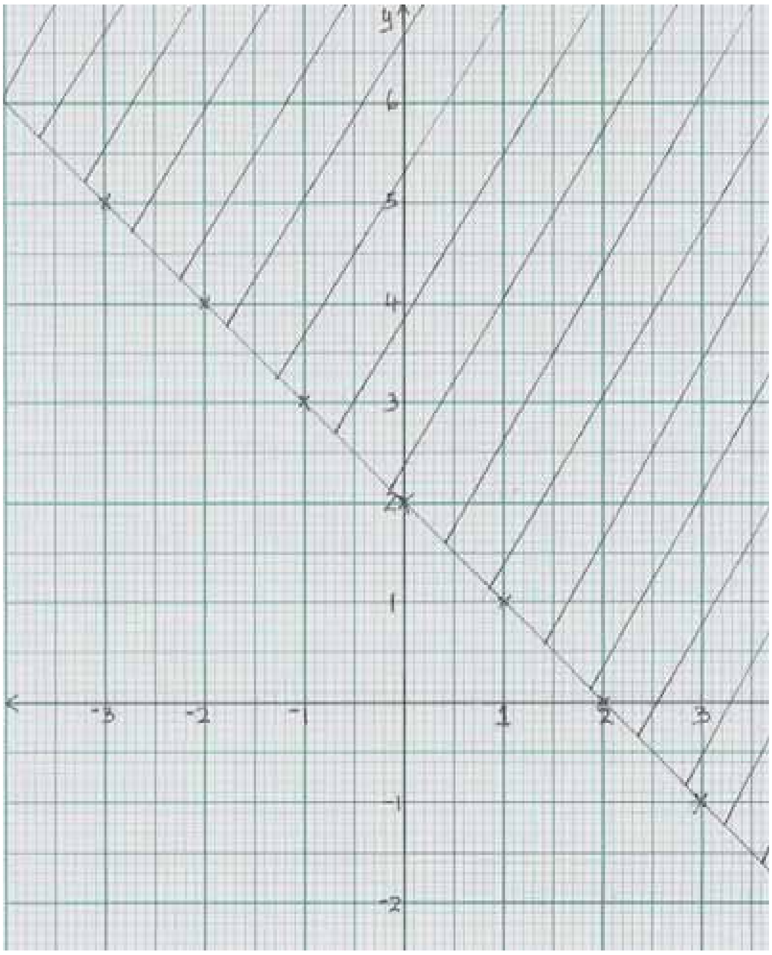
- Pick the x – values that you will use.
  Compute the corresponding y – values.

- Plot the points.

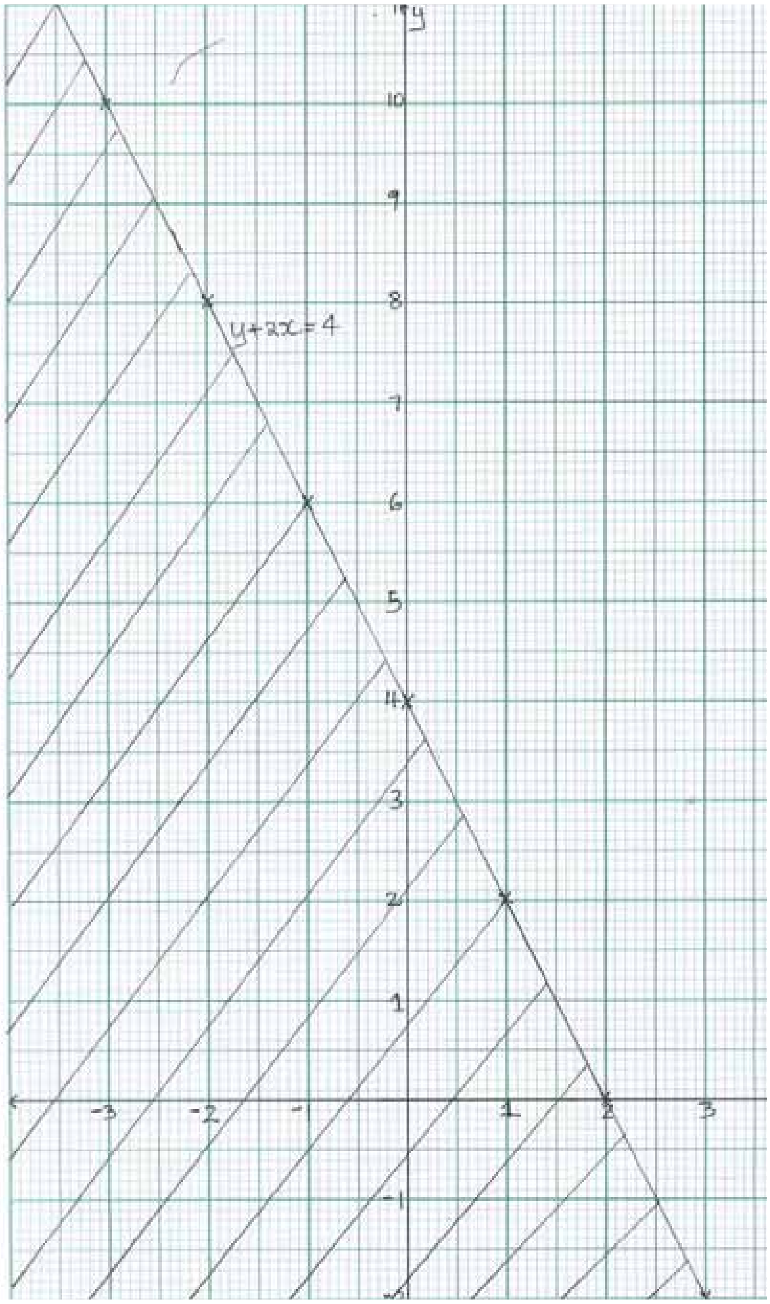
- Then join the points with a solid or a dashed straight line. If the inequality you are working with has “<” or “>”, join the points with a dashed line, or if the inequality is “≤” or “≥”, join the points with a solid line.
- Label the line drawn.
- identify the region that has the coordinates that satisfy the inequality. The pair that satisfies the inequality falls in the region that has all the pairs that satisfy the inequality. **There are still many other solutions for an inequality.**
- The region that will be shaded is the unwanted region,

Answers

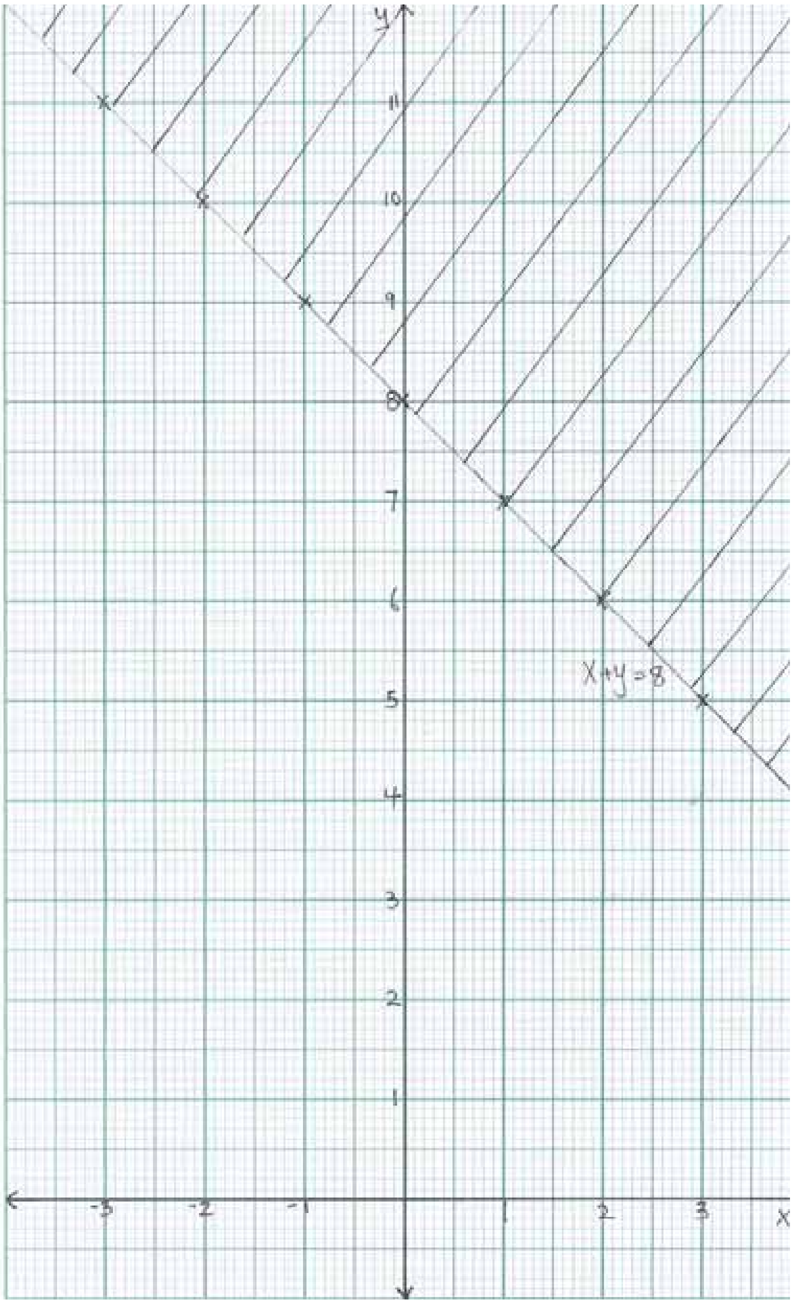
1.



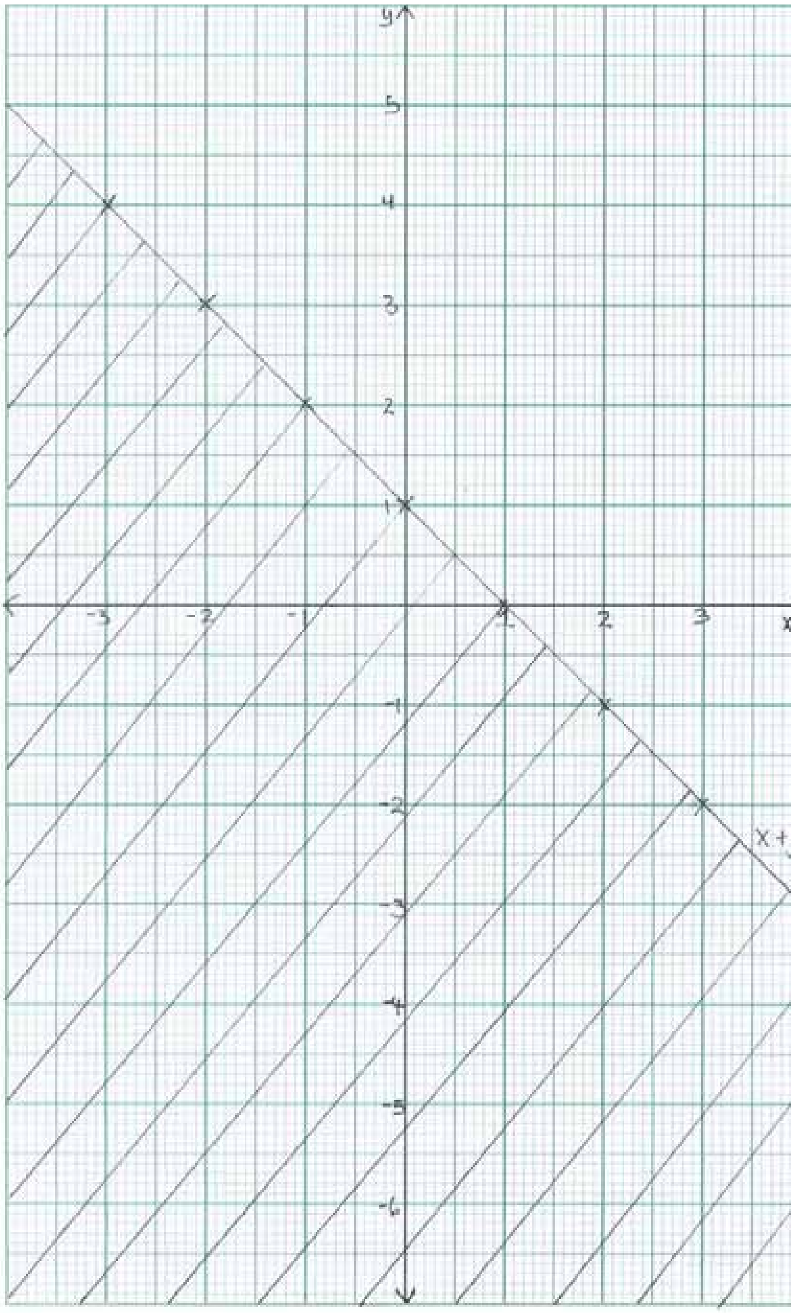
2.



3.


y
11
10
9
8
7
6
5
4
3
2
1
-3
-2
-1
1
2
3
x
X+y=8

4.



Lesson 2 Writing the Linear Inequality Given the Drawing of the Inequality

By the end of this subunit, you should be able to:

- write the linear inequality given the drawing of the inequality.

This subunit is about 4 pages in length.

In the first part of this unit, we looked at the solution of inequalities. This was done graphically. This allowed us to indicate regions where the solution set is. We can be given regions where the solution set is. We can work from the graph to determine the inequality. These are the steps to follow:

Step 1

Pick any two points on the line. The most preferred are the coordinates of the x and y intercepts.

The x – coordinate has coordinates (x,0) and the y – coordinate has coordinates (0,y).

The use of intercepts is just a preference.

Step 2

Use the points in step 1 to find the gradient. This can be done in a number of ways:

Reflection

One way of finding the gradient, "m", between two points, given by coordinates (x_1, y_1) and (x_2, y_2), we use the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$x_2 \neq x_1$

When can $x_2 = x_1$?

This will be when the line is vertical.

For example:
find the slope of a line with coordinates (3, -3) and (-2, 7)

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{7 - -3}{-2 - 3}$$

$$m = \frac{10}{-5}$$

$m = -2$

NB: It makes no difference which of the two points is called (x_1, y_1) or (x_2, y_2), the ratio will still be the same.

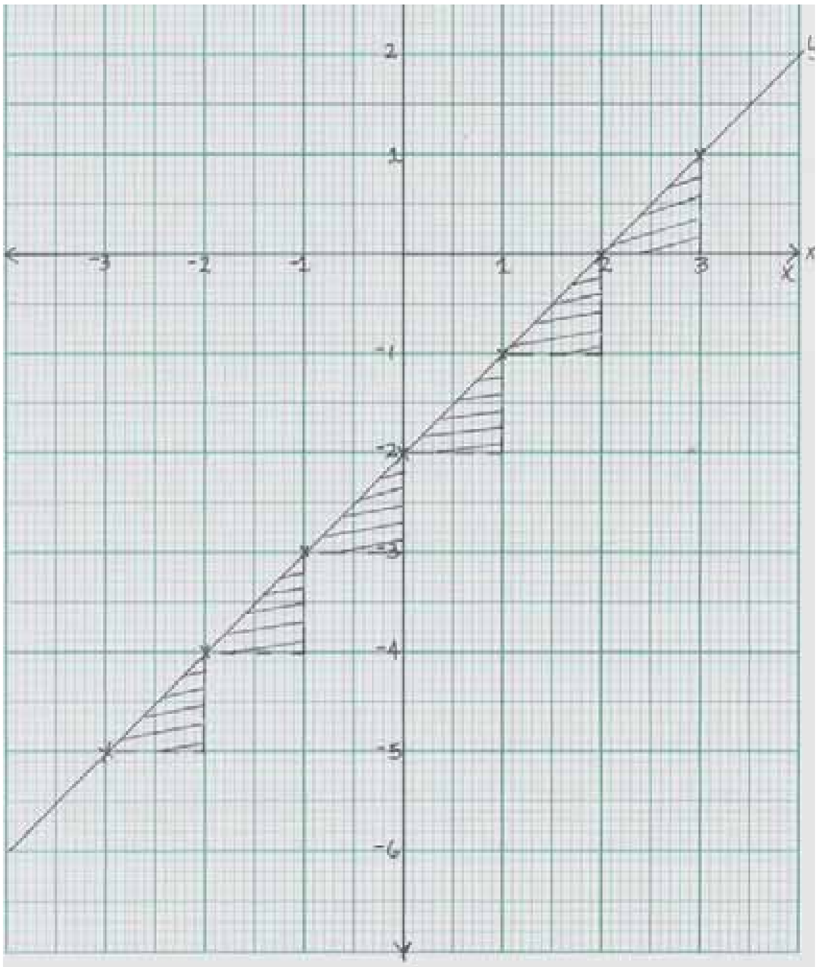
The other way of finding the gradient is by calculating the ratio of the vertical distance moved to the horizontal distance. This is also referred to as

$$\frac{\text{change in y}}{\text{change in x}}$$

which is sometimes also described as

$$\frac{\text{rise}}{\text{run}}$$

The following figure shows the graph of the linear equation y = x - 2, including the coordinates of seven specific points.



From the diagram you will see that the y – value increases 1 unit each time that the x – value increases by 1 unit. The ratio of this change in y compared to the corresponding change in x is $\frac{1}{1} = 1$

Using any coordinates on the line, say (0, -2) and (-2, -4)

$$m = \frac{-4 - -2}{-2 - 0}$$

$$m = \frac{-2}{-2}$$

m = 1

We still get our gradient as 1!

Step 3

Write down the general equation of a straight line and substitute the gradient and the y-intercept into the general equation of a straight line. The general equation of a straight line is $y = mx + c$; where

"m" is the gradient of the graph

"x" and "y" are the coordinates of any point on the graph

"c" is the y – intercept i.e. the point where the graph crosses the y – axis

To specify a straight line, we need to have the values of "m" and "c".

Step 4

To write down the inequality, change the "=" sign in the equation to:

"<" or " >" if the line is dotted.

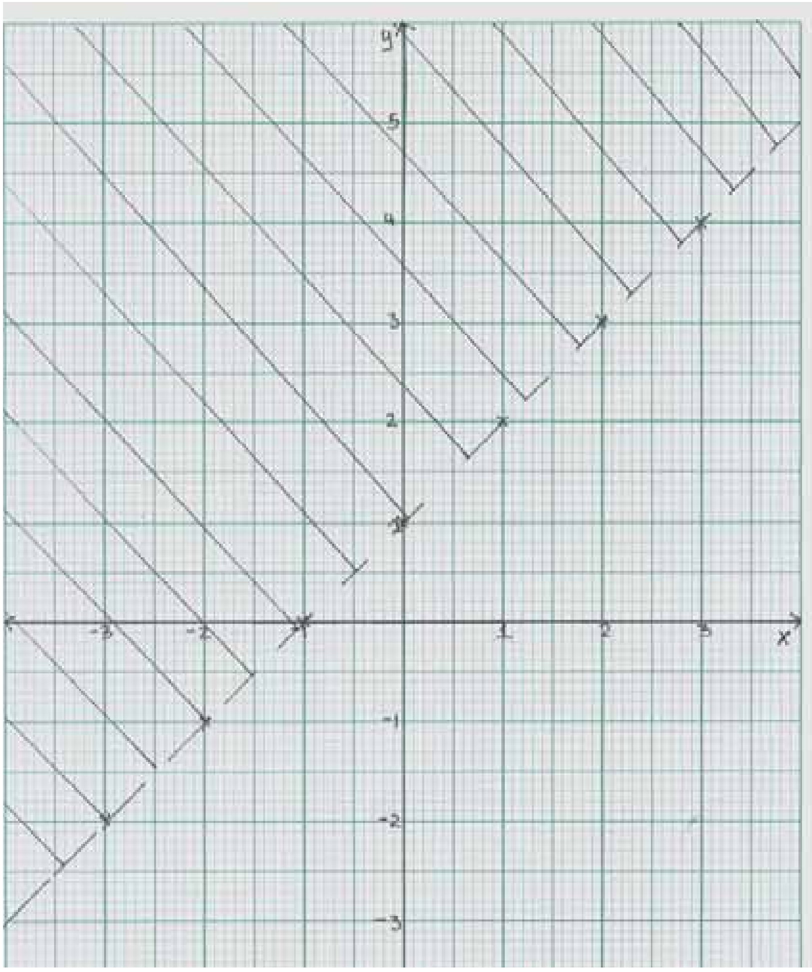
"≤ " or " ≥ " if the line is solid.

Check whether the symbol you chose is correct with "test points" from the two regions. One region has the solution set and the other does not. These are on each side of the line. These test points serve to confirm the region that we are looking for.

Let us look at the following examples to help us understand this process.

Example 1

Give the inequality that determines the unshaded region below:



Step 1

The x – intercept has coordinates (-1,0) and the y – intercept has coordinates (0,1)

Step 2

Calculate the gradient.

$$\text{Gradient} = \frac{1 - 0}{0 - (-1)}$$

$$= \frac{1}{1}$$

$$= 1$$

Step 3

The general equation of a straight line is y = mx + c

The gradient from step 2 is 1, and the y-intercept is 1

Substituting them in the equation

$y = 1x + 1$

$y = x + 1$

Step 4

To write down the inequality, change the “=” sign in the equation. The “=” sign must be changed to a “≤ ” or “ ≥ ”sign because the line is solid.

The “test point” from above the line is (3,6) .

$y = x + 1$

$6 = 3 + 1$

$6 \neq 4$

6 is greater than 4! This point is in the region that does not satisfy the inequality. This is the unwanted region. This is the reason why it is shaded.

The "test point" from below is (5,2)

$y = x + 1$

$2 = 5 + 1$

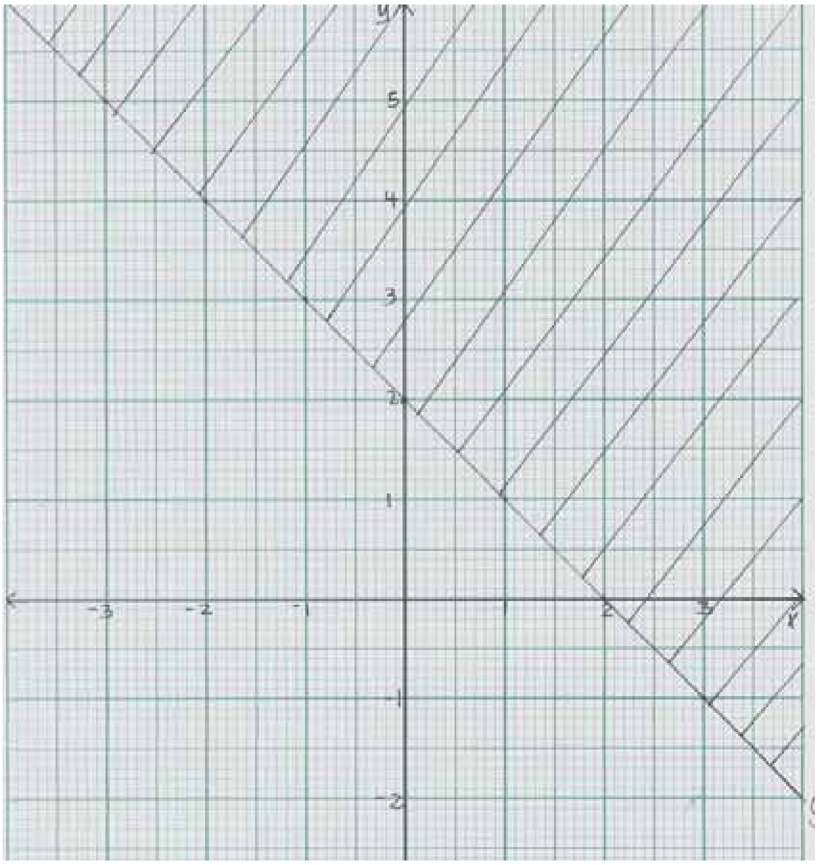
$2 \neq 6$

2 is less than 6. This point is in the region that satisfies the inequality. This is the wanted region. This is the reason why it is not shaded.

Therefore the inequality is $y \leq x + 1$

Example 2

Give the inequality that determines the unshaded region below**:**



Step 1

The x – coordinate has coordinates (2,0) and the y – coordinate has coordinates (0,2)

Step 2

Calculate the gradient.

$$\text{Gradient} = \frac{2-0}{0-2}$$

$= \frac{2}{-2}$

$= -1$

Step 3

Write down the general equation of a straight line.

y = mx + c

Step 4

Substitute the gradient and the y – intercept.

y = mx + c

y = -1x + 2

y = -x + 2

Step 5

To write down the inequality, change the "=" sign in the equation to:

y = -x + 2

The line is solid. We replace the "=" with a "≤ " or " ≥ "sign because the line is solid.

The "test point" from above the line is (4,1).

y = -x + 2

1 = -4 + 2

1 $\neq$ -2

1 is not equal to -2, it is greater than -2!

This point is in the region that does not satisfy the inequality. This is the unwanted region. This is the shaded region.

The "test point" below is (-3 ,-2)

y = -x + 2

1 = -4 + 2

1 $\neq$ -2

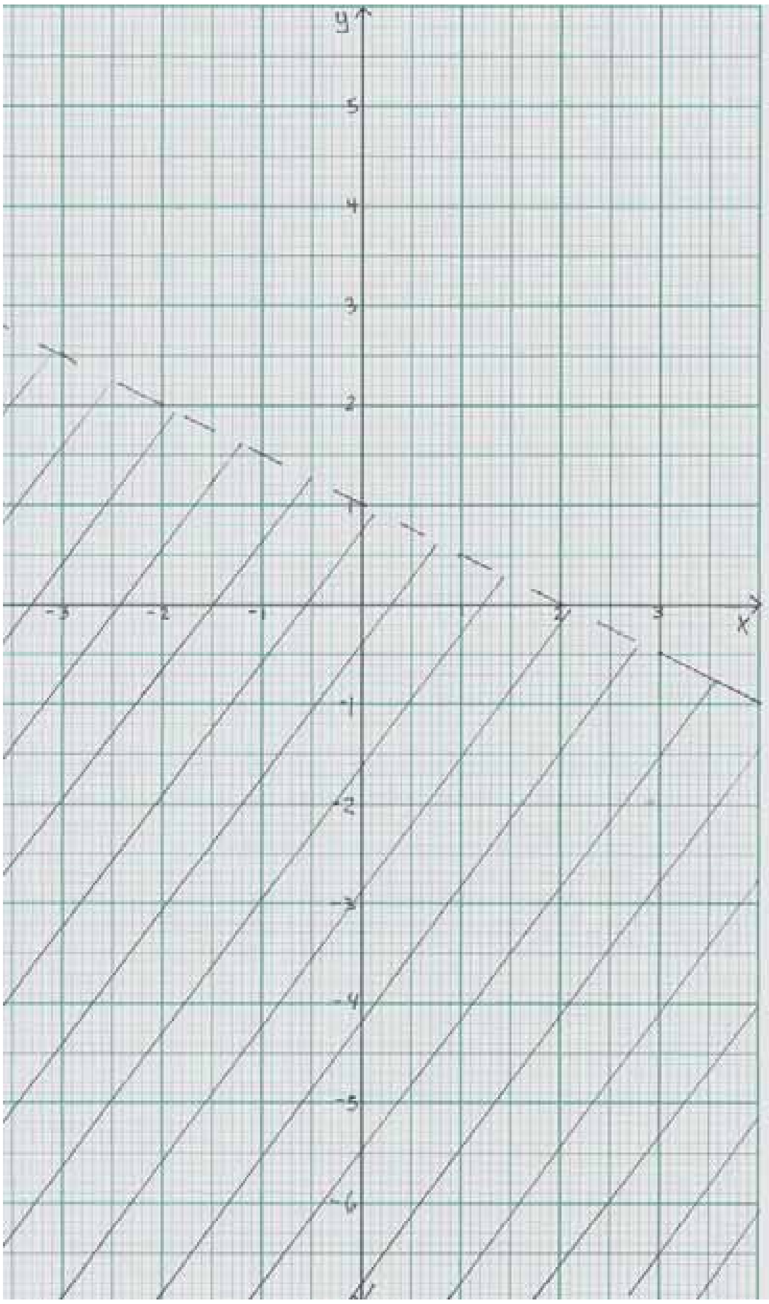
-2 $\leq$ 5

-2 is less than 5. This point is in the region that satisfies the inequality. This is the wanted region. This is the reason why it is not shaded.

Therefore the inequality is y $\leq$ -x + 2

Example 3

Give the inequality that determines the unshaded region below:



Step 1

Establish the coordinates of the x and y – intercepts.

Compare your answers with the following:

The x – intercept has coordinates (2,0) and the y – intercept has coordinates (0,1).

Step 2

Calculate the gradient.

Compare your answer with the following:

$$\text{Gradient} = \frac{0 - 1}{2 - 0}$$

$$= \frac{-1}{2}$$

$$= -\tfrac{1}{2}$$

Step 3

Write down the general equation of a straight line and substitute the gradient and the y – intercept.

Again compare your answer with the following:

$y = mx + c$

$y = -\frac{1}{2}x + 1$

Step 4

Determine which inequality sign will replace the "=" sign in the equation.

____

____

____

____

Our results are as follows:

$y = -\frac{1}{2}x + 1$

The line is broken, we replace the "=" with " $<$" or " $>$".

The "test point" from above the line is (2, 4).

$y = -\frac{1}{2}x + 1$

$4 = -\frac{1}{2}(2) + 1$

$4 = -1 + 1$

$4 \neq 0$

4 is greater than 0! This point is in the region that satisfies the inequality. This is the wanted region. This is the reason why it is not shaded.

The "test point" from below is (0. -5)

$y = -\frac{1}{2}x + 1$

$-5 = -\frac{1}{2}(0) + 1$

$-5 = 0 + 1$

$-5 \neq 1$

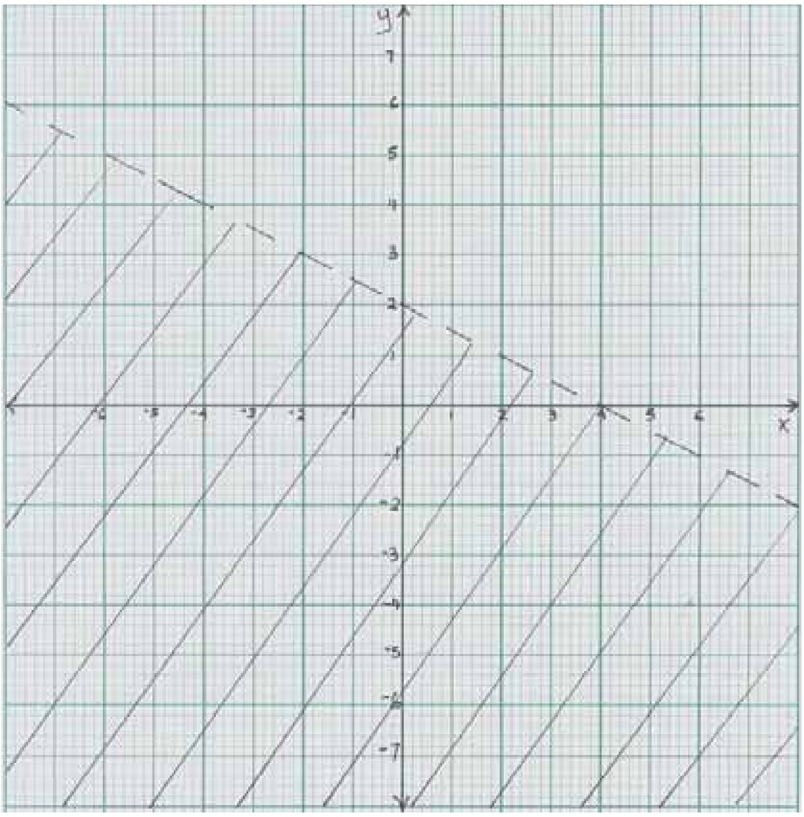
-5 is less than 1. This point is in the region that does not satisfy the inequality. This is the unwanted region. This is the reason why it is shaded.

The inequality is $y > -\frac{1}{2}x + 1$

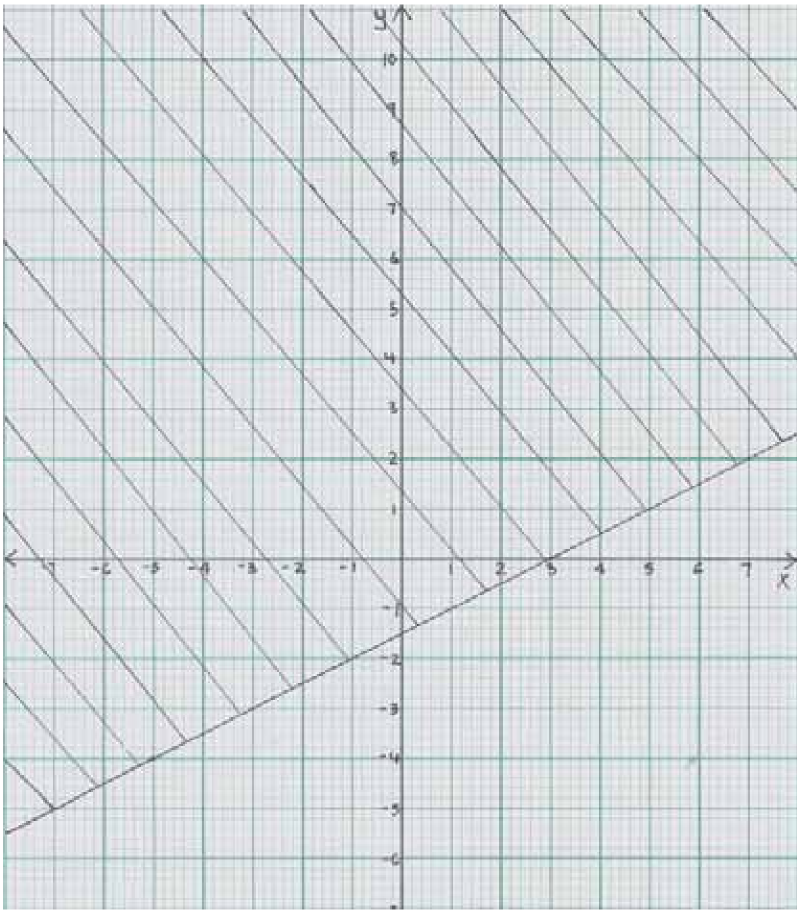
Activity 2

Give the inequalities that determine the unshaded regions below:

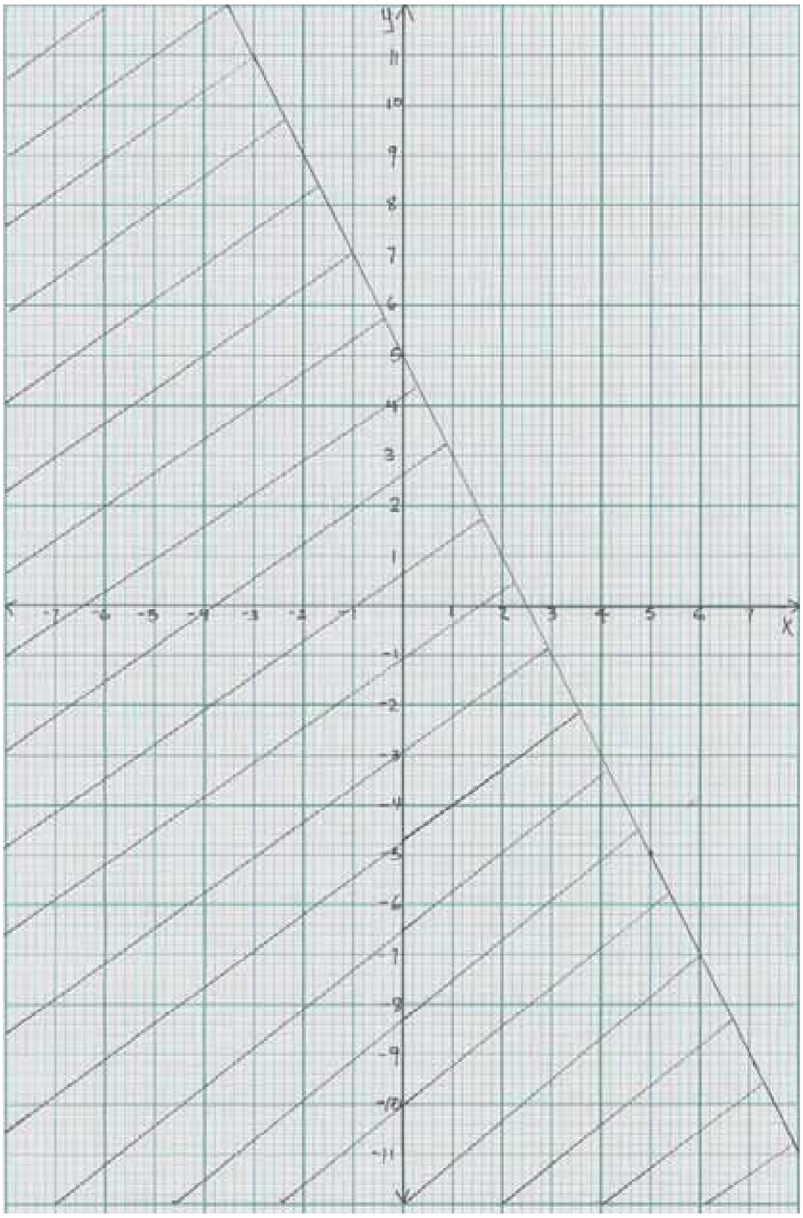
1.



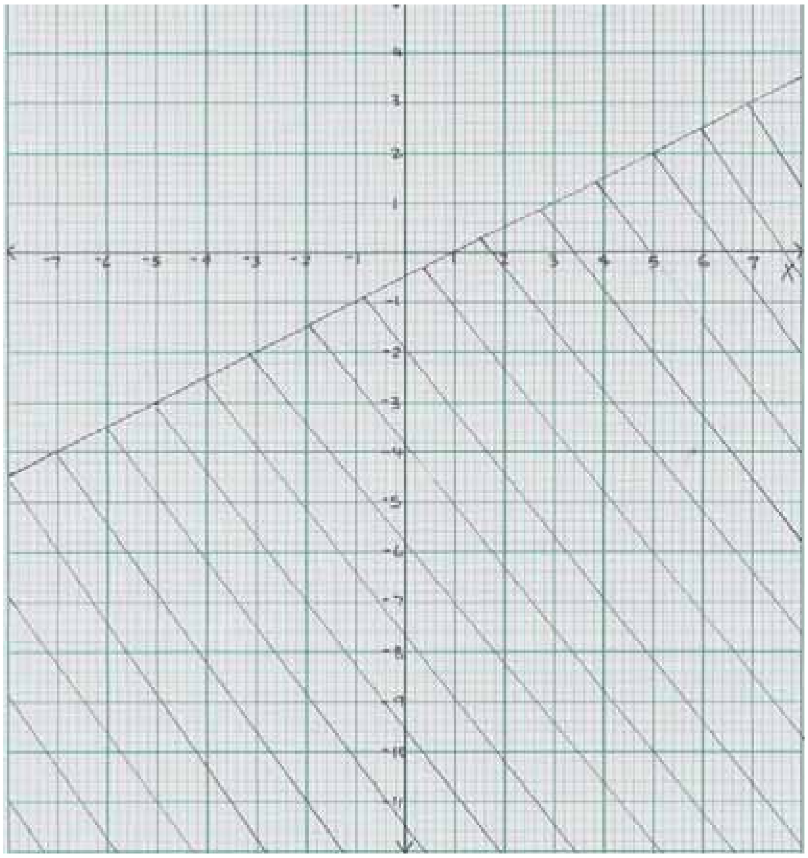
2.



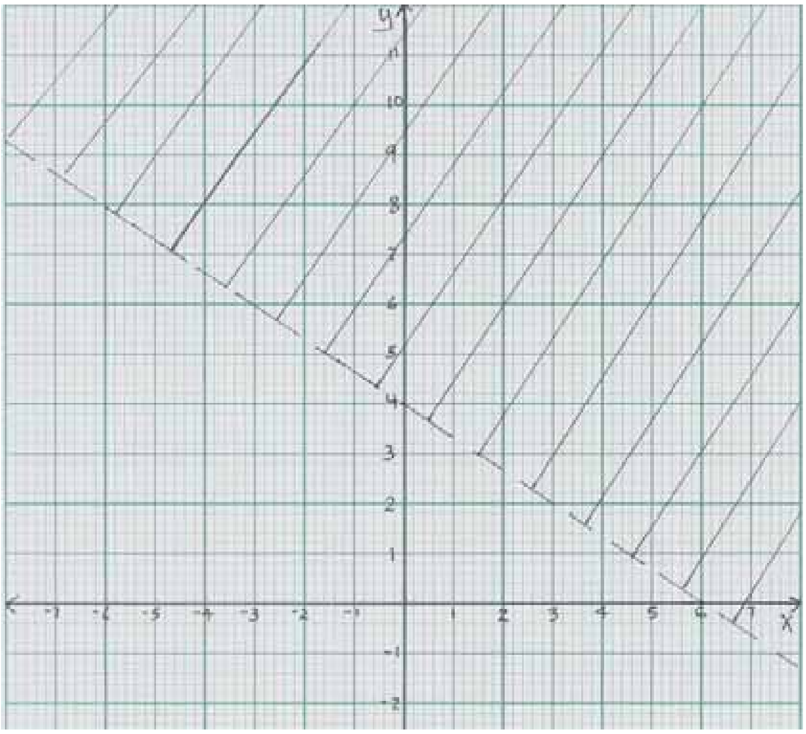
3.



4.



5.



Key Points to Remember

The key points to remember in this subunit on writing the linear inequality given the drawing of the inequality are:

- Pick any two points on the line.
- Use these two points to find the gradient.
- Write down the general equation of a straight line and substitute the gradient and the y-intercept into the general equation of a straight line. The general equation of a straight line is $y = mx + c$;
- To write down the inequality, change the "=" sign in the equation to:
- "<" or " >" if the line is dotted.
- "≤ " or " ≥ " if the line is solid.
- Check whether the symbol you chose is correct with "test points" from the two regions. One region has the solution set.

You have now completed the last subunit of this unit on solving linear inequalities. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Answers

1.

x – intercept (4,0)

y – intercept (0,2)

$$\text{Gradient} = \frac{2-0}{0-4}$$

$$= \frac{2}{-4}$$

$$= -\tfrac{1}{2}$$

$y = mx + c$

$y = -\tfrac{1}{2}x + 2$

The line is broken, we replace the "=" with " <" or " >".

The "test point" from above the line is (-2, 6)

$y = -\tfrac{1}{2}x + 2$

$6 = -\tfrac{1}{2}(-2) + 2$

$6 = 1 + 2$

$6 \neq 3$

6 is greater than 3! This point is in the region that satisfies

the inequality. This is the wanted region. This is the reason why it is not shaded.

The "test point" from below is (-2, -4)

$y = -\frac{1}{2}x + 2$

$-4 = -\frac{1}{2}(-2) + 2$

$-4 = 1 + 2$

$-4 \neq 3$

-4 is less than 3. This point is in the region that does not satisfy the inequality. This is the unwanted region. This is the reason why it is shaded.

The inequality is $y > -\frac{1}{2}x + 2$

2.

x – intercept (3,0)

y – intercept (0,-1½)

$$\text{Gradient} = \frac{-1\frac{1}{2} - 0}{0 - 3}$$

$$= \frac{-1\frac{1}{2}}{-3}$$

$$= \frac{1}{2}$$

$y = mx + c$

$y = \frac{1}{2}x - 1\frac{1}{2}$

The line is solid. We replace the "=" with a "≤ " or " ≥ "sign because the line is solid.

The "test point" from above the line is (-7, 2).

$y = \frac{1}{2}x - 1\frac{1}{2}$

$2 = \frac{1}{2}(-7) - 1\frac{1}{2}$

$2 = -3\frac{1}{2} - 1\frac{1}{2}$

$2 \neq -5$

2 is greater than -5!

This point is in the region that does not satisfy the inequality. This is the unwanted region. This is the shaded region.

The "test point" below is (0, -8)

$y = \frac{1}{2}x - 1\frac{1}{2}$

$-8 = \frac{1}{2}(0) - 1\frac{1}{2}$

$-8 = -1\frac{1}{2}$

$-8 \neq -1\frac{1}{2}$

-8 is less than -1½. This point is in the region that satisfies the inequality. This is the wanted region. This is the reason why it is not shaded.

Therefore the inequality is $y \leq \frac{1}{2}x - 1\frac{1}{2}$

3.

x – intercept (2½,0)

y – intercept (0,5)

$$\text{Gradient} = \frac{5-0}{0-2\frac{1}{2}}$$

$$= \frac{5}{-2\frac{1}{2}}$$

$$= -2$$

y = mx + c

y = -2x + 5

The line is solid. We replace the "=" with a "≤" or "≥" sign because the line is solid.

The "test point" from above the line is (0, 10).

y = -2x + 5

10= -2(0) + 5

10 = 0 + 5

$10 \neq 5$

10 is greater than 5!

This point is in the region that satisfies the inequality. This is the wanted region. This is the unshaded region.

The “test point” below is (0, -10)

y = -2x + 5

-10= -2(0) + 5

-10= 0 + 5

$-10 \neq 5$

-10 is less than 5. This point is in the region that does not satisfy the inequality. This is the unwanted region. This is the reason why it is shaded.

Therefore the inequality is $y \geq -2x + 5$

4.

x – intercept (1,0)

y – intercept (0,-½)

$$\text{Gradient} = \frac{-\tfrac{1}{2} - 0}{0 - 1}$$

$$= \frac{-\tfrac{1}{2}}{-1}$$

$$= \tfrac{1}{2}$$

y = mx + c

y = ½x - ½

The line is solid. We replace the "=" with a "≤ " or " ≥ "sign because the line is solid.

The "test point" from above the line is (6, 10).

y = ½x - ½

10 = ½ (6) - ½

10 = 3 - ½

10 ≠ 2½

10 is greater than 2½!

This point is in the region that satisfies the inequality. This is the wanted region. This is the unshaded region.

The "test point" below is (2, -8)

y = ½x - ½

-8 = ½ (2) - ½

-8 = 1 - ½

-8 ≠ ½

-8 is less than ½. This point is in the region that does not satisfy the inequality. This is the unwanted region. This is the shaded region.

Therefore the inequality is $y \geq \frac{1}{2}x - \frac{1}{2}$

5.

x – intercept (6,0)

y – intercept (0,4)

$$\text{Gradient} = \frac{4-0}{0-6} = \frac{4}{-6} = -\frac{2}{3}$$

$y = mx + c$

$y = -\frac{2}{3}x + 4$

The line is broken, we replace the “=” with “ <” or “ >”.

The “test point” from above the line is (-3,10)

$y = -\frac{2}{3}x + 4$

$10 = -\frac{2}{3}(-3) + 4$

$10 = 2 + 4$

$10 \neq 6$

10 is greater than 6! This point is in the region that does not satisfy the inequality. This is the unwanted region. This is the shaded region.

The "test point" from below is (6, -2)

$y = -\frac{2}{3}x + 4$

$-2 = -\frac{2}{3}(6) + 4$

$-2 = -4 + 4$

$-2 \neq 0$

-2 is less than 0. This point is in the region that satisfies the inequality. This is the wanted region. This is the unshaded region.

The inequality is $y < -\frac{2}{3}x + 4$

Unit Summary

Summary

In this unit you learned that

- symbols for inequalities (inequations) are <, >, ≤, and ≥
- solving inequalities is like solving equations
- linear inequalities with two variables are represented by a region in the Cartesian coordinate system
- solving an inequality involves the following steps:

  - change the inequality to an equation.

  - pick the x – values that you will use. "Space them out" a bit.

  - compute the corresponding y – values.

  - plot the points.

  - join the points with a solid or a dashed straight line.

  - identify the region that has the coordinates that satisfy the inequality, that is, the solution.

- it is conventional to shade the region **not** containing the points that satisfy the inequality.
- the region containing the points that satisfy the inequality is also called the **wanted** region or the **region of possible solutions.**
- when writing the linear inequality given the drawing of the inequality.

- establish the coordinates of the x and y intercepts.
- use these intercepts to calculate the gradient.
- substitute the gradient and the y-intercept into the general equation of a straight line, $y = mx + c$
- to write down the inequality, change the "=" sign in the equation to "<", " >", "≤ " or " ≥ "
- check whether the symbol you chose is correct with "test points

You have completed the material for this unit on solving linear inequalities. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit. It covers indices.

Assignment

Assignment

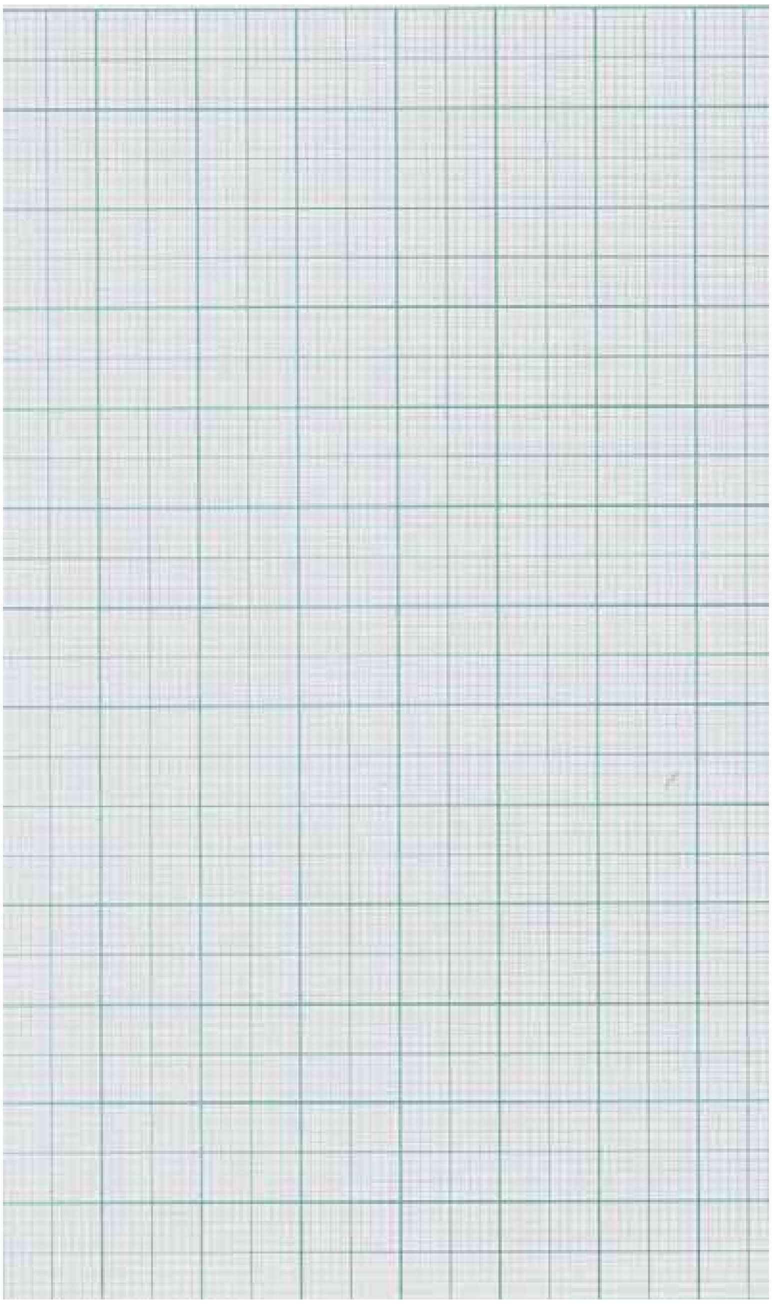
1. **Answer All Questions.**
2. **Show all the necessary working.**
3. **Draw all the graphs in the spaces provided.**
4. **Total marks = 50**
5. **Time: 1hour 30 minutes**

A. Draw the graphs of the following inequalities

1. $x + y > 3$

[8]

2. $x + y \geq 2$



[8]

3. $y + 2x < 4$

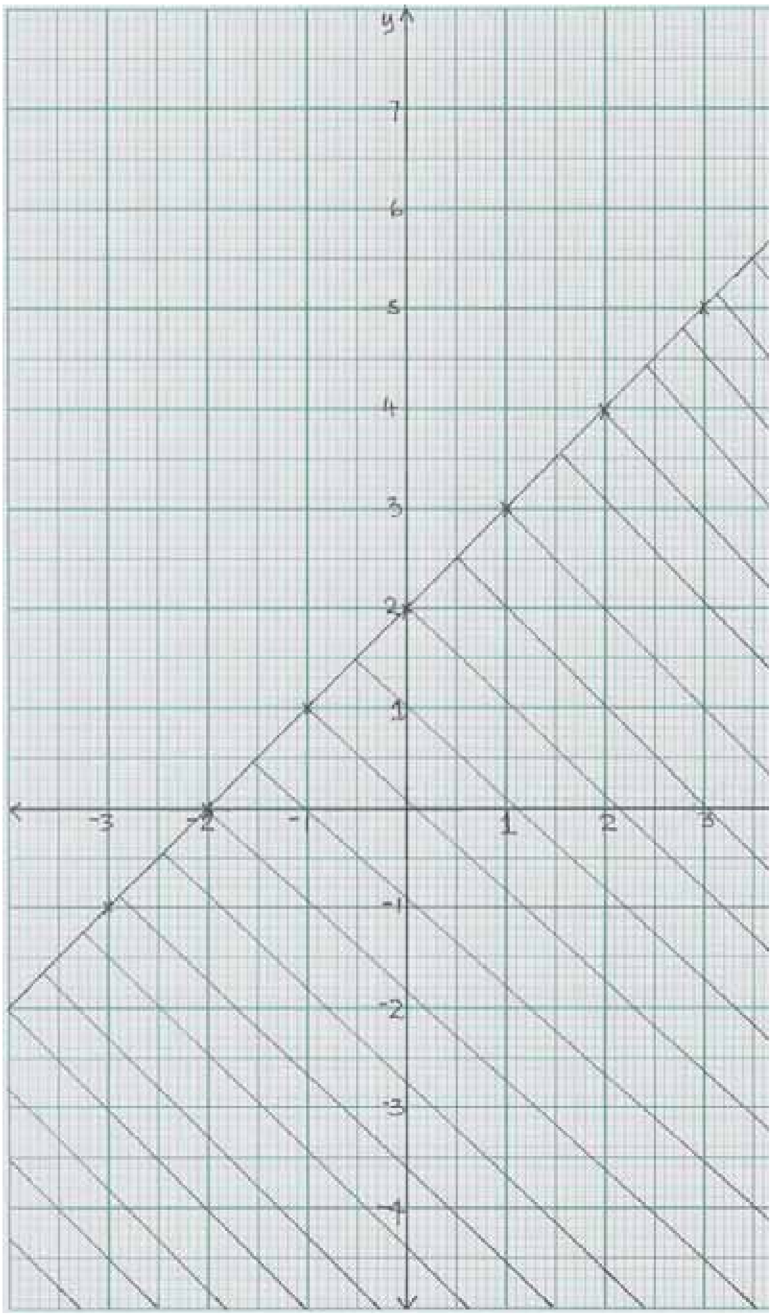
[8]

4. $y \leq x + 1$

[8]

B. Give the inequalities that determine the unshaded region

1.



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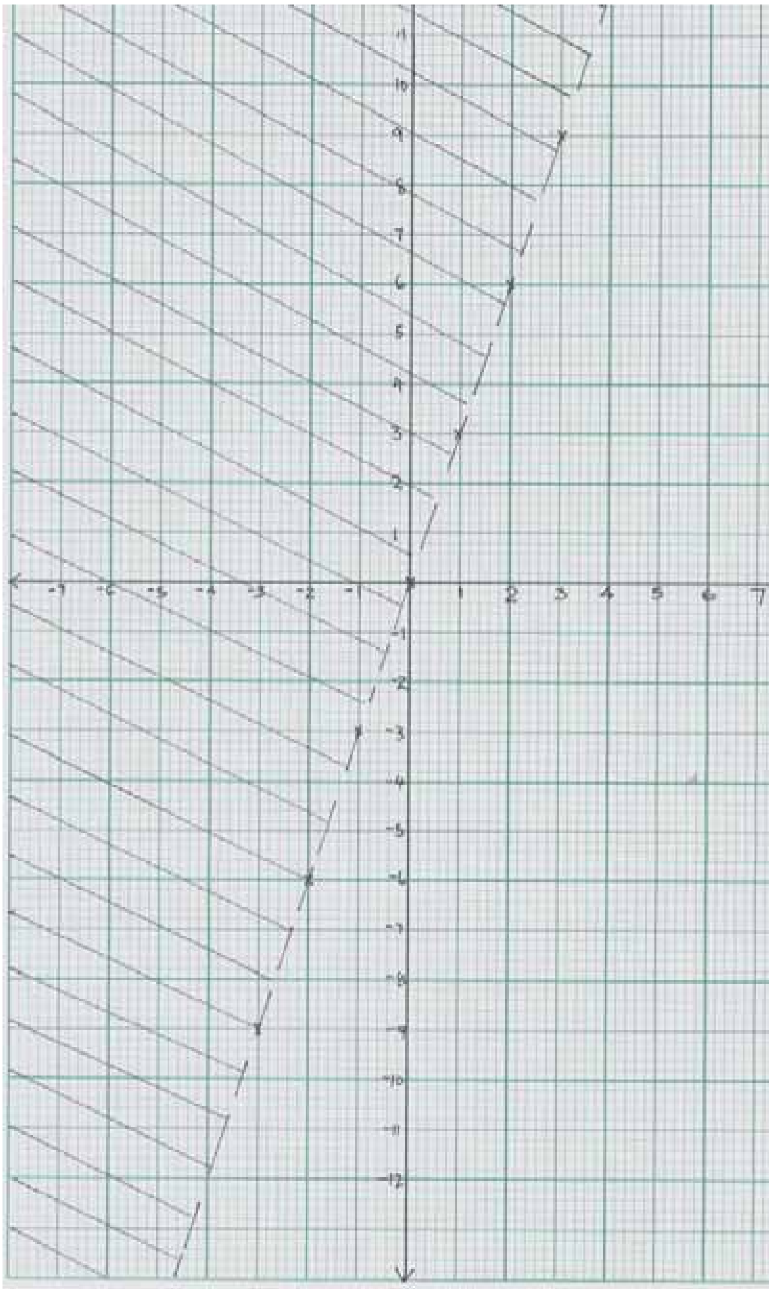
2.



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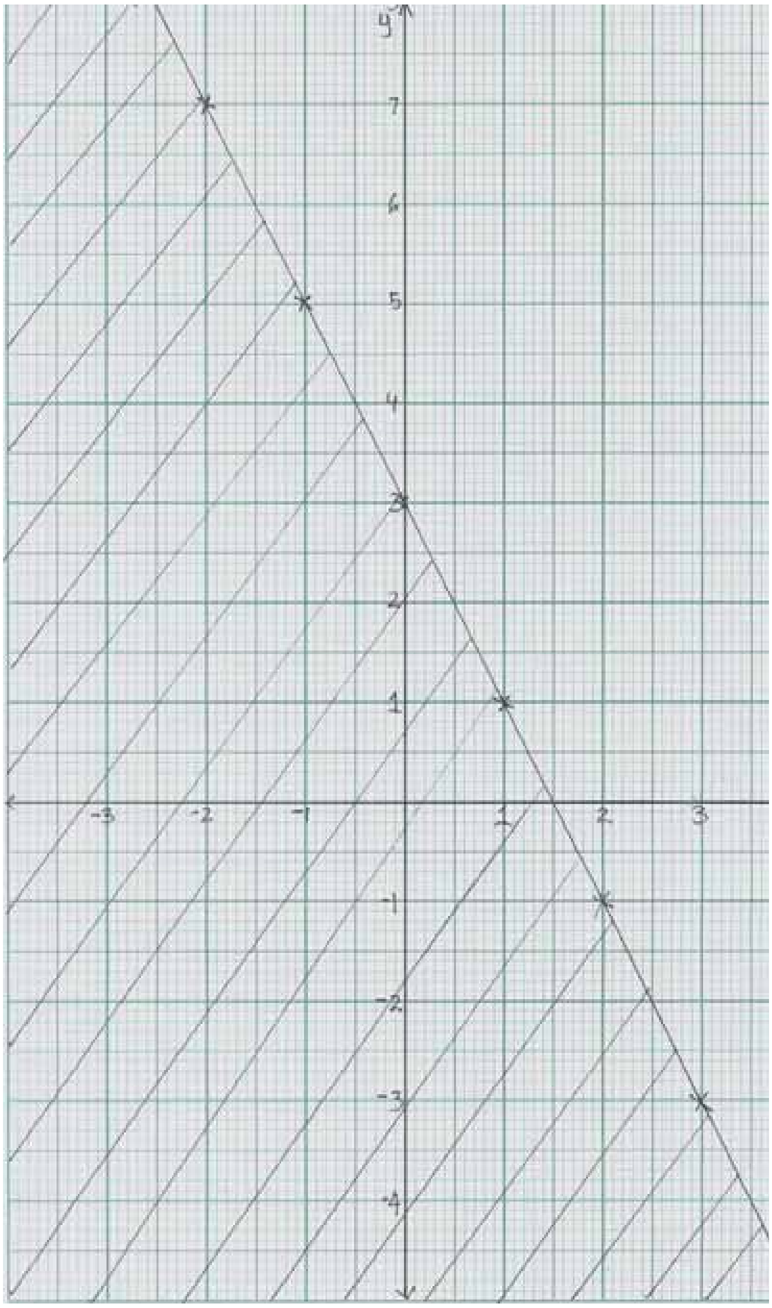
3.



[6]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1.

x	y
–3	6
–2	5
–1	4
0	3
1	2
2	1
3	0

gradient = -1

y – intercept has coordinates (0,3)

Equation of the line is $y = -x + 3$

One pair of coordinates above the line is (1,6).

$y = -x + 3$

$6 = -(1) + 3$

$6 \neq 2$

One pair of coordinates below the line is (-1, -4).

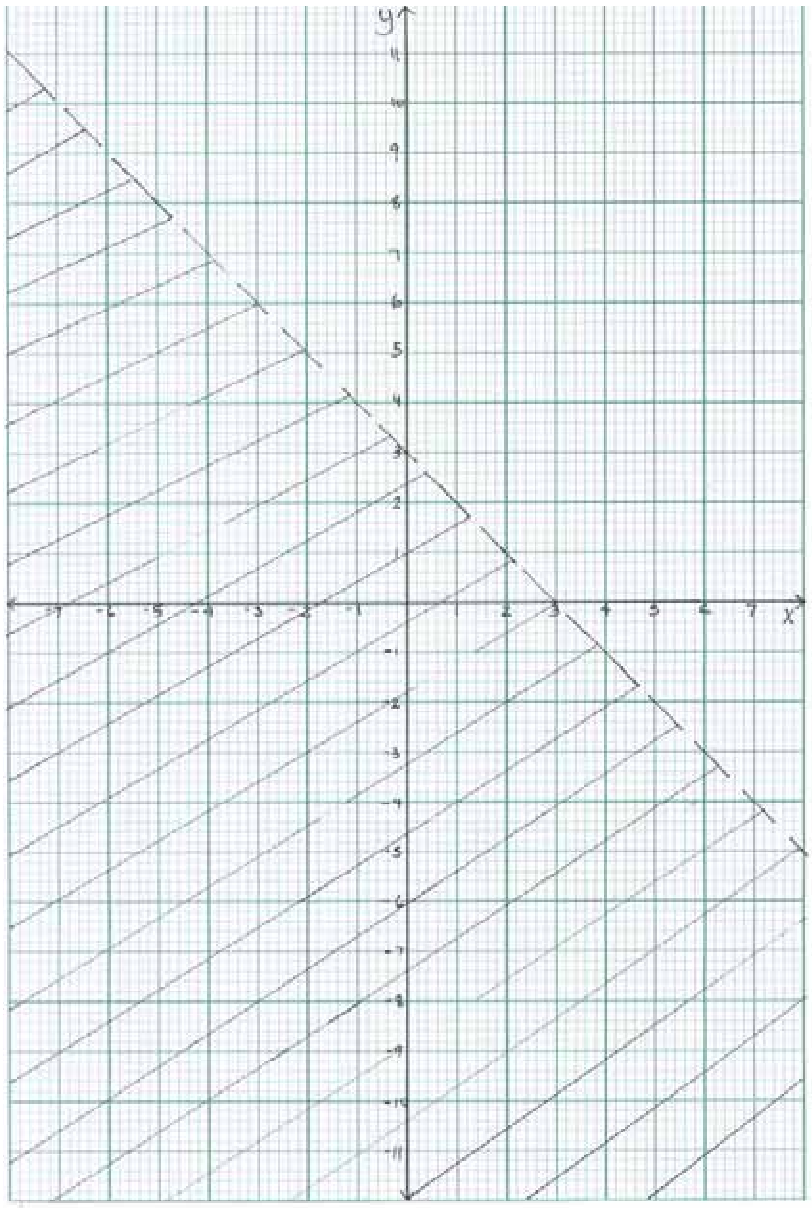
$y = -x + 3$

$-4 = -(-1) + 3$

$-4 \neq 1 + 3$

$-4 \neq 4$

y > -x + 3



2.

x	y
–3	5
–2	4
–1	3
0	2
1	1
2	0
3	-1

gradient = -1

y – intercept has coordinates (0,2)

Equation of the line is y = -x + 2

One pair of coordinates above the line is (-3, 8).

$y = -x + 2$

$8 = -(-3) + 3$

$8 \neq 6$

One pair of coordinates below the line is (3, -6).

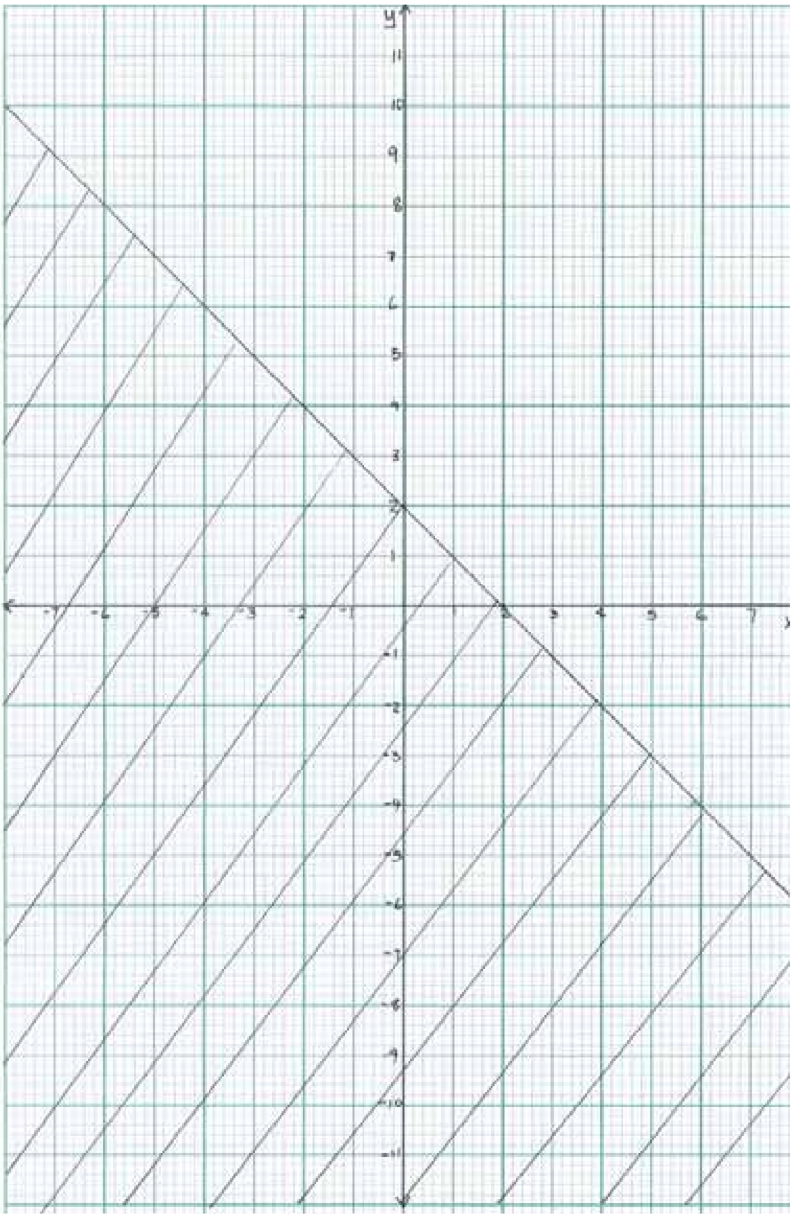
$y = -x + 3$

$-6 = -(3) + 3$

$-6 \neq -3 + 3$

$-6 \neq 0$

$y \geq -x + 2$



3.

x	y
–3	10
–2	8
–1	6
0	4
1	2
2	0
3	-2

gradient = -2

y – intercept has coordinates (0,4)

Equation of the line is $y = -2x + 4$

One pair of coordinates above the line is (4, 11).

$y = -2x + 4$

$11 = -2\,(4) + 4$

$11 \neq -4$

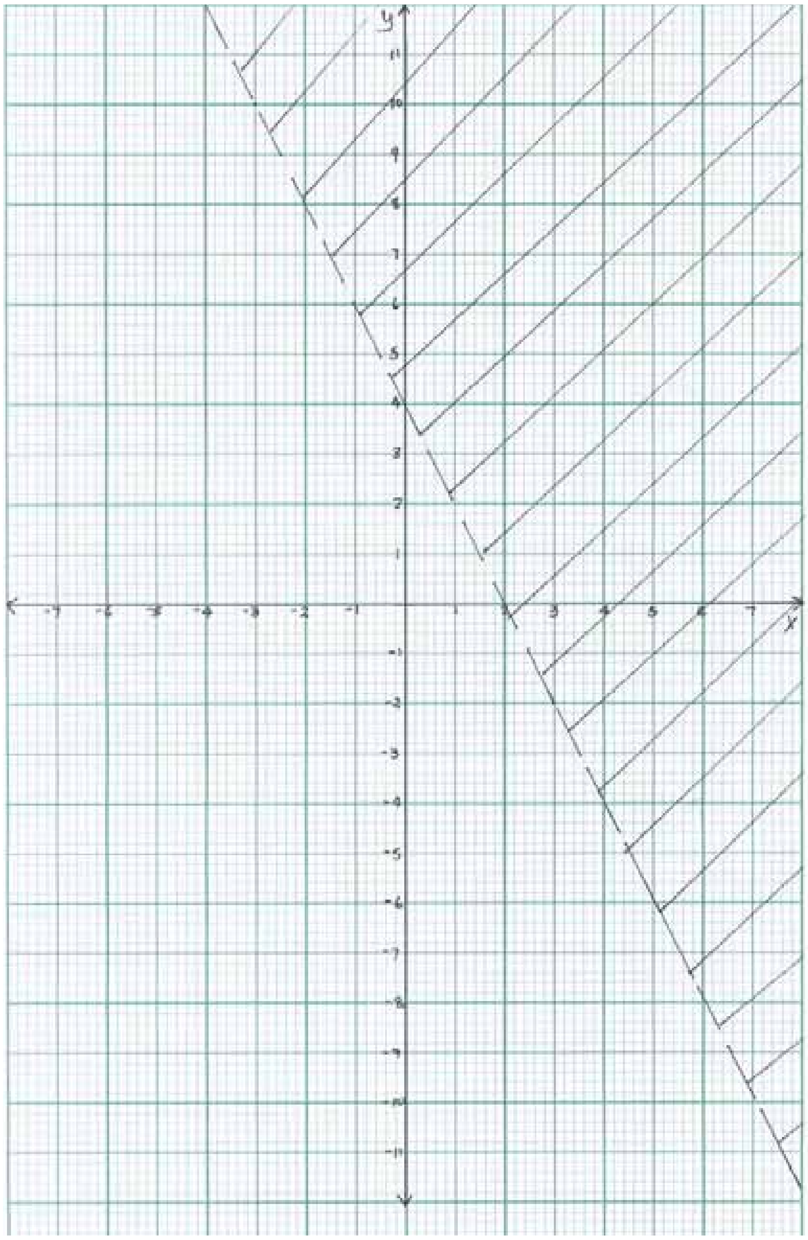
One pair of coordinates below the line is (1, -4).

$y = -2x + 4$

$-4 = -2\,(1) + 4$

$-4 \neq 2$

$y < -2x + 4$



4.

x	y
–3	-2
–2	-1
–1	0
0	1
1	2

2	3
3	4

gradient = 1

y – intercept has coordinates (0,1)

Equation of the line is $y = x + 1$

One pair of coordinates above the line is (-2, 4).

$y = x + 1$

$4 = -2 + 1$

$4 \neq -1$

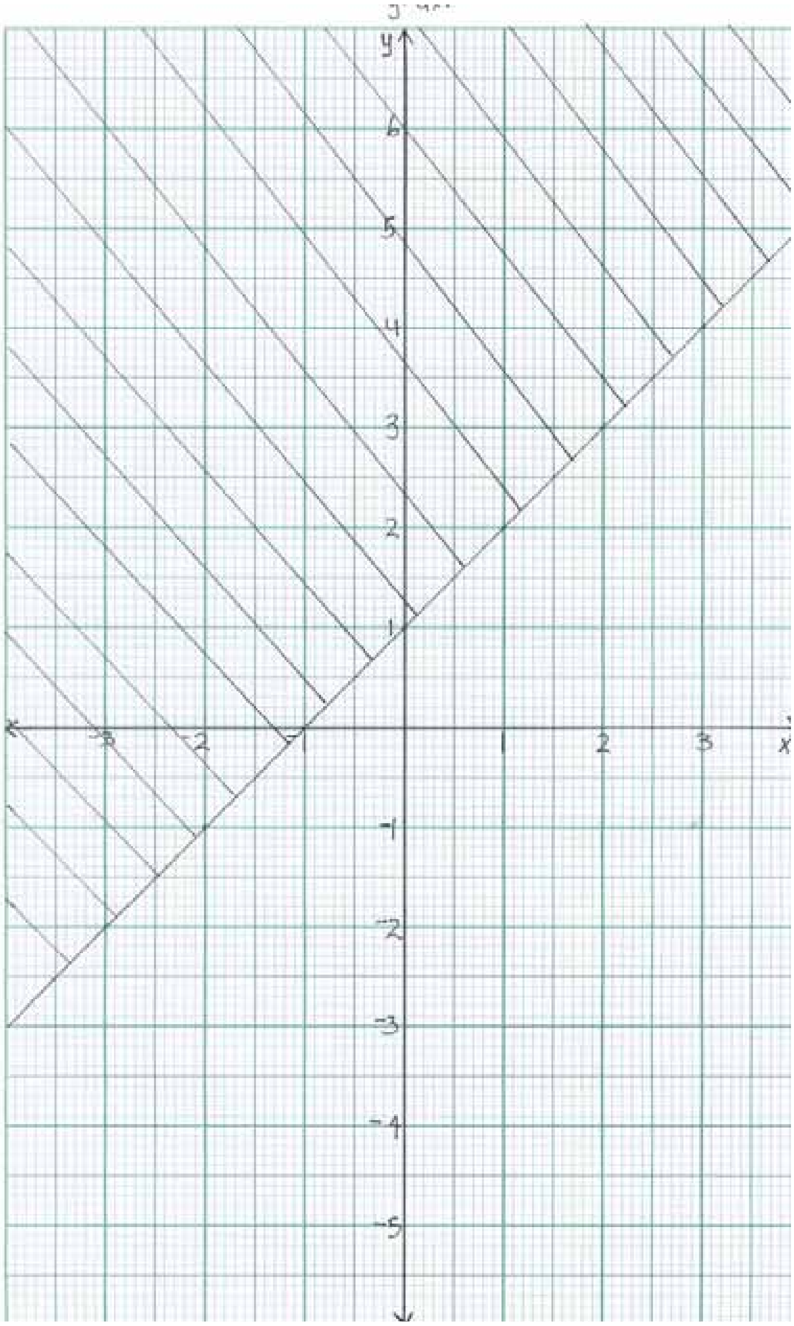
One pair of coordinates below the line is (-2, -2).

$y = x + 1$

$-2 = -2 + 1$

$-2 \neq -1$

$y \leq x + 1$


y
6
5
4
3
2
1
-3
-2
-1
1
2
3
x
-1
-2
-3
-4
-5

B.

1. y – intercept has coordinates (0,2)
using x and y – intercepts (-2, 0) and (0, 2) to calculate the gradient

$$\text{gradient} = \frac{2-0}{0-(-2)} = \frac{2}{2} = 1$$

The equation of the line y = x + 2

check points

above: (2, 7)

$y = x + 2$
$7 \neq 2 + 2$
$7 > 4$

below : (0, -4)

$y = x + 2$
$-4 \neq 0 + 2$
$-4 < 2$

wanted region

The inequality is $y \geq x + 2$

2.
y – intercept has coordinates (0,0)
using any two points (2, 6) and (3, 9) to calculate the gradient

$$\text{gradient} = \frac{9-6}{3-2} = \frac{3}{1} = 3$$

The equation of the line y = 3x

check points

above: (-4, 8)

$y = 3x$
$8 \neq 3(-4)$
$8 > -12$

below : (2, -5)

$y = 3x$
$-5 \neq 3(2)$
$-5 < 2$

wanted region

The inequality is $y < 3x$

3.
y – intercept has coordinates (0,3)
using any two points (2, -1) and (0, 3) to calculate the gradient

$$\text{gradient} = \frac{3-(-1)}{0-2}$$
$$= \frac{4}{-2}$$
$$= -2$$

The equation of the line $y = -2x + 3$

check points

above: (-1, 1)

$y = -2x + 3$
$1 \neq -2(-1) + 3$
$1 \neq 2 + 3$
$1 < 5$

below : (2, 5)

$y = -2x + 3$
$5 \neq -2(2) + 3$
$5 \neq -4 + 3$
$5 > -1$
wanted region

The inequality is $y \geq -2x + 3$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

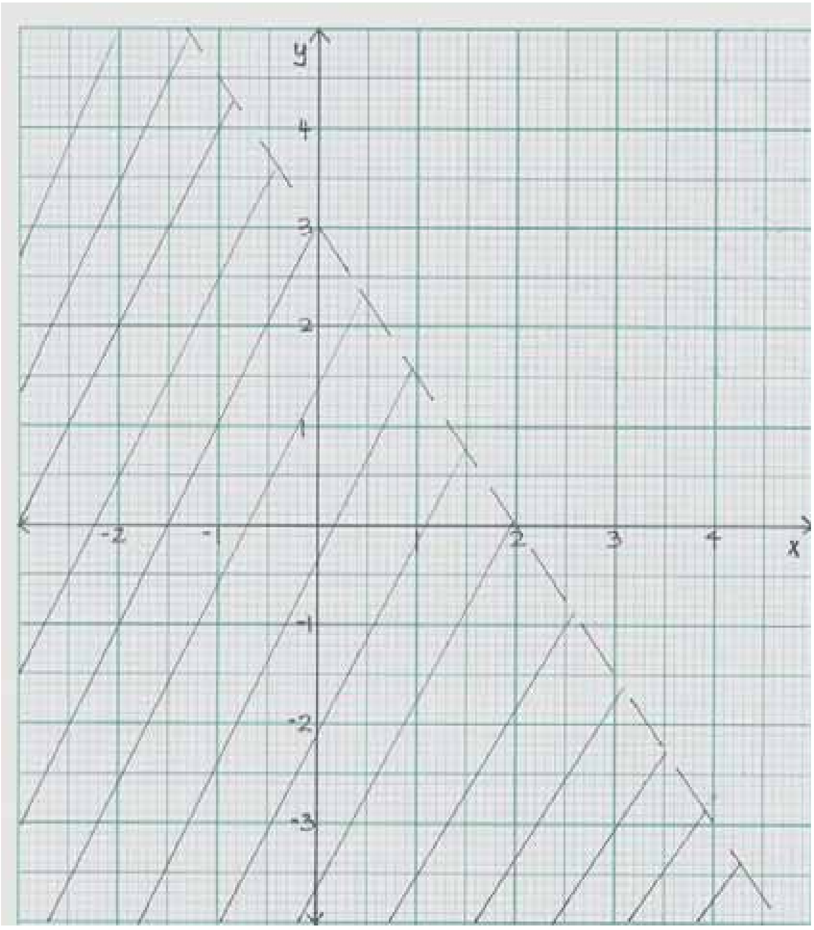
Assessment

Assessment

1. **Answer All Questions.**
2. **Show all the necessary working.**
3. **Draw all the graphs in the spaces provided.**
4. **Total marks = 24**
5. **Time: 45 minutes**

Give the inequalities that determine the unshaded region

1.



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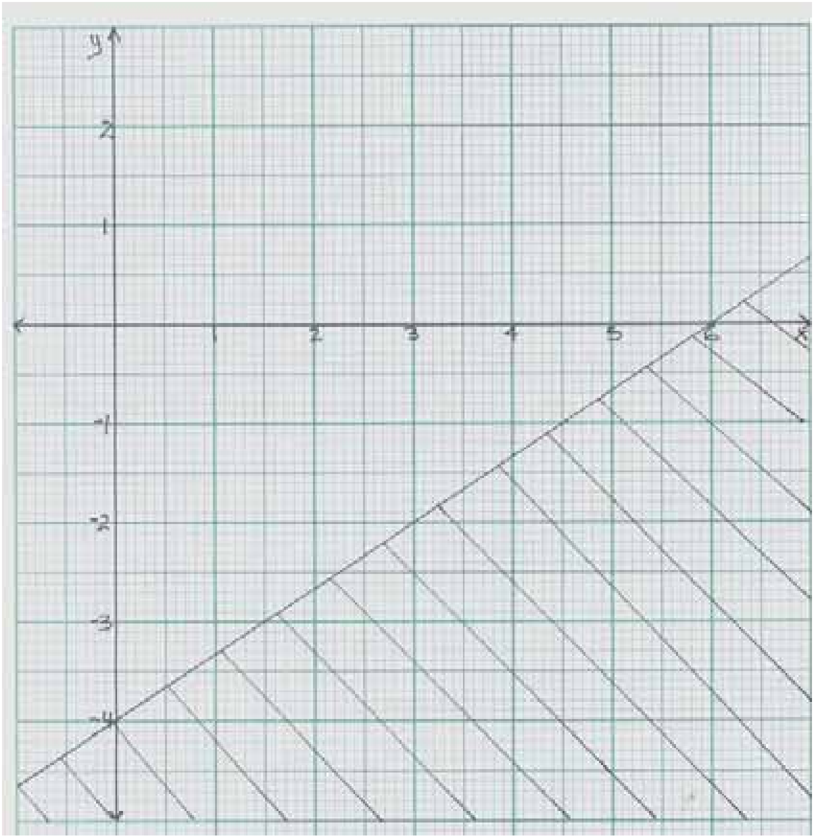
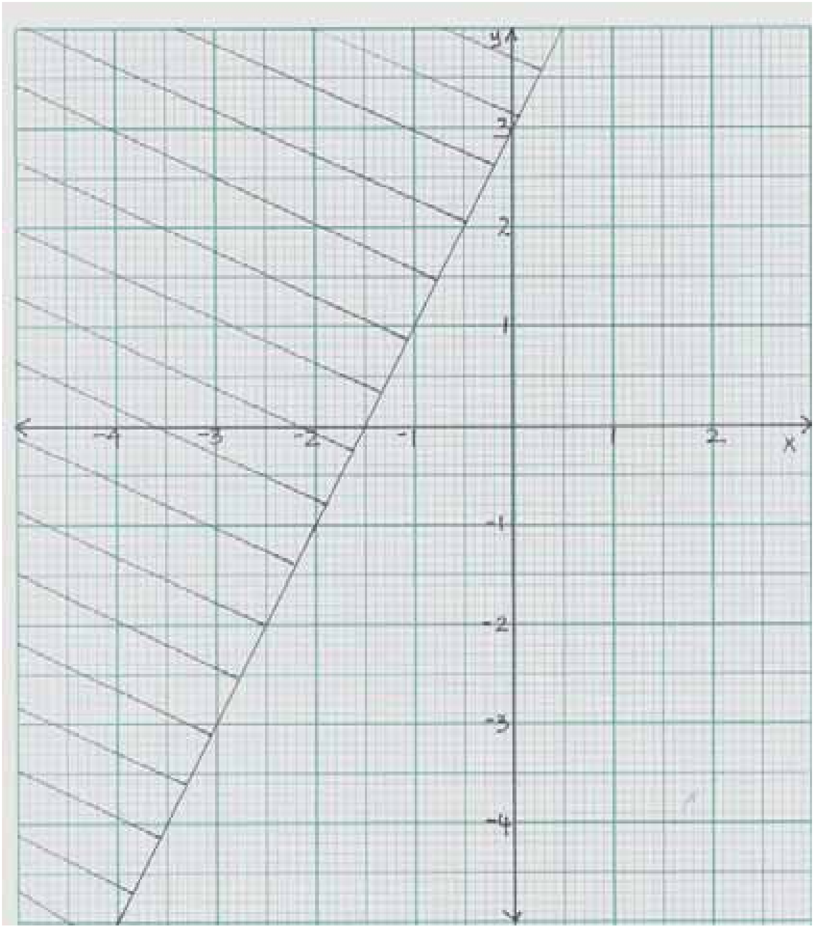
2.



[6]

3.



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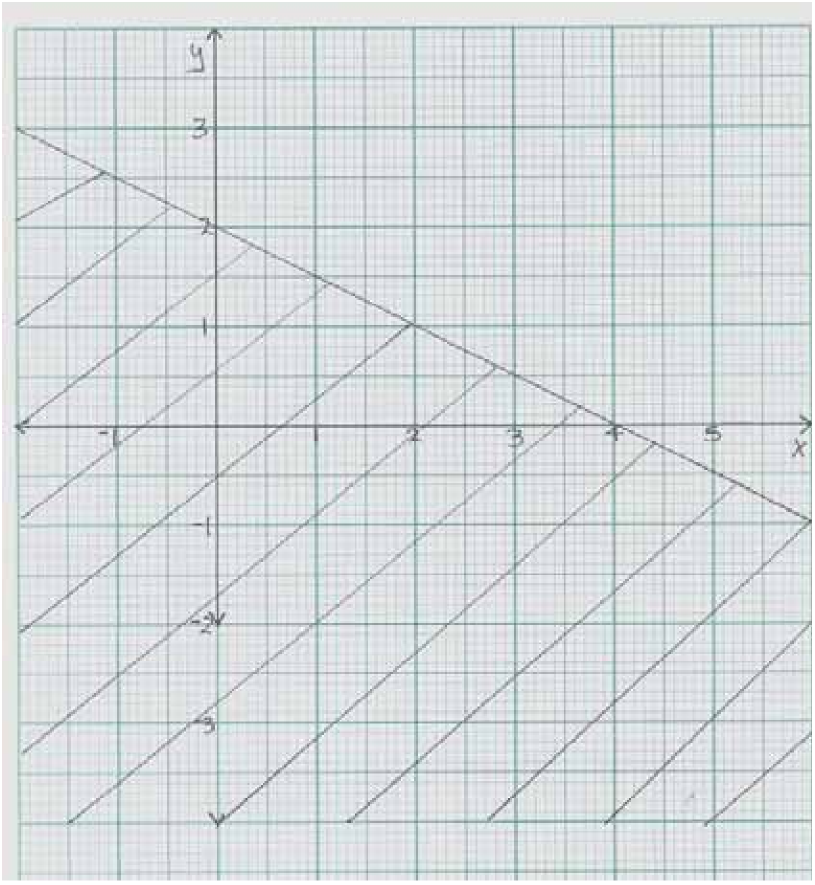
4.



[6]

Answers

1. x – intercept (2,0)

y – intercept (0,3)

Gradient = $\frac{3-0}{0-2}$

$= \frac{3}{-2}$

$= -1.5$

$y = mx + c$

$y = -1.5x + 3$

The line is dotted.

$y > -1.5x + 3$

2. x – intercept (6,0)

y – intercept (0,-4)

$$\text{Gradient} = \frac{-4 - 0}{0 - 6}$$

$$= \frac{-4}{-6}$$

$$= \frac{2}{3}$$

$y = mx + c$

$y = \frac{2}{3}x - 4$

The line is solid.

The inequality is $y \geq \frac{2}{3}x - 4$

3. x – intercept (-1½,0)

y – intercept (0,3)

$$\text{Gradient} = \frac{3 - 0}{0 - (-1\frac{1}{2})} = \frac{3}{1\frac{1}{2}} = 2$$

y = mx + c

y = 2x + 3

The line is solid.

The inequality is $y \le 2x + 3$

4. x – intercept (4,0)

y – intercept (0,2)

$$\text{Gradient} = \frac{2 - 0}{0 - 4} = \frac{2}{-4} = -\frac{1}{2}$$

y = mx + c

$y = -\frac{1}{2}x + 2$

The line is solid. The inequality is $y \geq -\frac{1}{2}x + 2$

5. x – intercept $(-2\frac{2}{3}, 0)$

y – intercept (0,2)

$$\text{Gradient} = \frac{2 - 0}{0 - (-2\frac{2}{3})}$$

$$= \frac{2}{2\frac{2}{3}}$$

$$= \frac{3}{4}$$

$y = mx + c$

$y = \frac{3}{4}x + 2$

The line is dashed.

The inequality is $y < \frac{3}{4}x + 2$

Unit Contents

Unit 10

Unit 10

Fractional Indices

Introduction

Indices play a vital role in mathematical calculations, e.g. some big numbers are simplified by expressing them in index form and operations are easily carried out in the form.

During your junior certificate course you expressed numbers in positive and negative whole number index form. We also dealt with the zero index form.

In this section we are going to express numbers in fractional index form and perform operations of multiplication, division in numbers expressed in fractional index form and having the same bases. We will also apply the rules of operations on fractional index form in simplifying expressions.

This unit consists of 22 pages. This is approximately **1%** of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 15 hours to work on this unit, 10 hours for formal study and 5 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Four Lessons:

Lesson 1 Expressing Numbers In Fractional Index Form
Lesson 2 Multiplying and Dividing Fractional Index Forms Having Same Base
Lesson 3 Simplifying Numbers Expressed in Fractional
Lesson 4 Applying Rules of Fractional Index Notation in Simplifying Expressions

Spend a few moments reading the following learning outcomes. They are a guide to what you should focus on while studying this unit.

Upon completion of this unit you will be able to:

Outcomes

- *Express* numbers in fractional index form.
- *Multiply and divide* numbers that are expressed in fractional index form and that have the same base.
- *Simplify* numbers expressed in fractional index form raised to another index.

Terminology

Index/power:	The number to which another number is raised e.g. in 5^3, 3 is the index to which 5 is raised.
indices:	The plural of the word "index".
base:	The number that is raised to a power. In "7^2", 7 is the base.
Fractional index Form:	An index form of a number whose index is a fraction.
Expression:	The sum or difference of two or more terms.
Root:	A number that multiplies itself to give another number.
Square root:	A number that multiplies itself two times to give a another number.
Cube root:	A number that multiplies itself three times to give another number.
Square:	The result when a number multiplies itself two times.
Cube:	The result when a number multiplies itself three times.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups

- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Expressing Numbers In Fractional Index Form

By the end of this subunit, you should be able to:

- Express numbers in fractional index form and vice-versa.

This subunit is about 4 pages in length.

Activity 1a

This activity is a reminder on expressing numbers in whole number indices.

Complete the table below: (The first two rows have been worked out for you as examples)

Number	**Repeating numbers**	**Index form**
25	5 x 5	5^2
64	2 x 2 x 2 x 2 x 2 x 2	2^6
(a) 256	2 x…………………	………………………
(b) 81	3 x………………..	………………………..
(c) 125	5 x………………	……………………..
(d) $\frac{1}{2^4}$		…...............................
(e)1	$\frac{6x6x6x6}{6x6x6x6}$	

Compare your answers with the ones at the end of the subunit. Proceed to the next section when you have at least 80% of this activity correct. If you score lower ensure that you go over this section to understand the concepts treated.

The above activity reminds us that

- Some numbers can be written in index form if the numbers are the products of "repeating numbers" e.g. (a) 4 = 2x2, (b) 27 = 3x3x3 etc

- A reciprocal of a number e.g. (a) $\frac{1}{3^2} = 3^{-2}$ (b) $\frac{1}{6^5} = 6^{-5}$ etc.
- Any number raised to the power of zero equals 1 e.g. (a) $5^0 = 1$, (b) $a^0 = 1$, because when a number is divided by itself the result is one or that number raised to the index zero.

Roots of Numbers

Interpreting the Root Bracket

$\sqrt[2]{25}$ represents the square root of 25 i.e. the number that multiplies itself 2 times to give 25 and that is 5.

$\sqrt[3]{27}$ represents the cube root of 27 i.e. the number that multiplies itself 3 times to give 27 and that is 3.

$\sqrt[4]{256}$ represents the fourth root of 256 i.e. the number that multiplies itself 4 times to give 256 and that number is 4.

The above examples show that the number outside the root bracket indicates the root in question. If no number is there, a 2 is assumed.

Note the following:

$$\sqrt[2]{25} = \sqrt{5^2} = 5^{\frac{2}{2}} = 5$$

$$\sqrt[3]{27} = \sqrt{3^3} = 3^{\frac{3}{3}} = 3$$

$$\sqrt[3]{81} = \sqrt{3^4} = 3^{\frac{4}{3}}$$

Note that if we are looking for a root of a number raised to a power we

Look at the following worked examples of finding roots of numbers.

(a) $\sqrt[2]{25} = \sqrt[2]{5^2} = 5^{\frac{2}{2}} = 5$. What root are we looking for here?________

(b) $\sqrt[3]{27} = \sqrt[3]{3^3} = 3^{\frac{3}{3}} = 3$. What root are we looking for here? ________

(c) $\sqrt[4]{64} = \sqrt[4]{2^6} = 2^{\frac{6}{4}}$. What root are we looking for here? ________

(d) $\sqrt[7]{8} = \sqrt[7]{2^3} = 2^{\frac{3}{7}}$. What root are we looking for here? ________

Compare your answers with the ones below:

(a) square root , (b) cube root , (c) fourth root , (d) seventh root

It is also important to be able to reverse the process:

Example

Change the following into root bracket form:

(a) $8^{\frac{1}{3}} = \sqrt[3]{8^1} = \sqrt[3]{8}$

(b) $5^{\frac{3}{5}} = \sqrt[5]{5^3}$

(c) $b^{\frac{a}{c}} = \sqrt[c]{b^a}$

Activity 1b

Activity 1

1. Express the following in fractional index form

(a) $\sqrt[3]{5^6}$ (b) $\sqrt[5]{7^2}$ (c) $\sqrt[3]{10^4}$ (d) $\sqrt[5]{9^3}$ (e) $\sqrt[4]{W^3}$

2. Change the following into root bracket form

(a) $6^{\frac{2}{5}}$

(b) $16^{\frac{2}{3}}$

(c) $4^{\frac{3}{8}}$

(d) $x^{\frac{3}{7}}$

(e) $y^{\frac{a}{b}}$

Compare your answers with the ones at the end of the subunit. Proceed to the next section when you have at least 80% of this activity correct. If you score lower ensure that you go over this section to understand the concepts treated.

Summary:

A fractional power means that you have to take a root of the number. For example, $4^{\frac{1}{2}}$ means take the square root of 4 = 2. Similarly, $x^{\frac{1}{3}}$ means take the cube root of x.

We can express roots of numbers by fractional indices as illustrated e.g cube root of $5^9 = (\sqrt[3]{5^9}) = 5^{\frac{9}{3}}$. The base is raised to a fractional index where the numerator is the power of the index form inside the root bracket and the denominator is the number that represents the root outside the bracket.

Answers to Activity 1a

Number	Repeating numbers	Index form
(a) 256	2 x 2 x 2 x 2 x 2 x 2 x 2 x 2	2^8

(b) 81	3x3x3x3	3^4
(c) 125	5x5x5	5^3
(d) $\frac{1}{2^4}$	$\frac{1}{2x2x2x2}$	2^{-4}
(e)1	$\frac{6x6x6x6}{6x6x6x6}$	6^0

Answers to Activity 1b

1. (a) $5^{\frac{6}{3}}$ (b) $7^{\frac{2}{5}}$ (c) $10^{\frac{4}{3}}$ (d) $9^{\frac{3}{5}}$ (e) $W^{\frac{3}{4}}$

2.

(a) $6^{\frac{2}{5}} = \sqrt[5]{6^2}$

(b) $16^{\frac{2}{3}} = \sqrt[3]{16^2}$

(c) $4^{\frac{3}{8}} = \sqrt[8]{4^3}$

(d) $x^{\frac{3}{7}} = \sqrt[7]{x^3}$

(e) $y^{\frac{a}{b}} = \sqrt[b]{y^a}$

Lesson 2 Multiplying and Dividing Fractional Index Forms Having Same Base

In this section we will multiply numbers that have been expressed in fractional index form that also have the same bases.

Take a moment to recall how to work out this: $\mathbf{x}^{\frac{a}{b}} \times \mathbf{x}^{\frac{c}{d}}$

Example 1

$$2^2 \times 2^3 = 2^{2+3} = 2^5$$

Example 2

$$a^b \times a^c = 2^{b+c}$$

Example 3

$$5^{\frac{2}{3}} \times 5^{\frac{4}{5}} = 5^{\frac{2}{3}+\frac{4}{5}} = 5^{\frac{10}{15}+\frac{12}{15}} = 5^{\frac{22}{15}}$$

In all examples notice that the base is the same in both numbers that are multiplied and the indices are added.

Example 4

$$2^3 \div 2^2 = 2^{3-2} = 2^1$$

Example 5

$$3^{\frac{5}{6}} \div 3^{\frac{3}{4}} = 3^{\frac{5}{6}-\frac{3}{4}} = 3^{\frac{1}{12}}$$

Example 6

$$a^b \div a^c = a^{b-c}$$

Example 7

(a) $a^2 \div a^7 = a^{2-7} = a^{-5} = \dfrac{1}{a^5}$

(b) $x^{\frac{3}{4}} \div x^{\frac{5}{6}} = x^{\frac{3}{4}-\frac{5}{6}} = x^{-\frac{1}{12}} = \dfrac{1}{x^{\frac{1}{12}}}$

Summary

When we multiply numbers in index form where their bases are the same we still add the fractional indices as we do with whole number indices.

e.g. $z^5 \times z^3 = z^{5+3} = z^8$

So $b^{\frac{1}{5}} \times b^{\frac{2}{7}} = b^{\frac{1}{5}+\frac{2}{7}} = b^{\frac{17}{35}}$

When we divide numbers with the same base we still subtract the index of the number that divides from the one that is divided.

$x^{\frac{2}{5}} \div x^{\frac{3}{8}} = x^{\frac{2}{5}-\frac{3}{8}} = x^{\frac{1}{40}}$

Sometimes the index may be negative

$x^{4} \div x^{7} = x^{4-7} = x^{-3} = \frac{1}{x^{3}}$

Activity 2

Activity 2

1. Work out the following

(a) $a^{7} \div a^{2}$

(c) $11^{2} \times 11^{5}$

(c) $8^{3} \div 8^{3}$

(c) $5^{4} \times 5^{6}$

(e) $p^{11} \div p^{6}$

(f) $r^{-11} \div r^{2}$

Compare your answers with the ones at the end of the subunit. Proceed to the next section when you have at least 80% of this activity correct. If you score lower ensure that you go over this section to understand the concepts treated.

Key Points to Remember

The key points to remember in this subunit on multiplying and dividing fractional index forms having same base are:

- When we multiply numbers in index form where their bases are the same we still add the fractional indices as we do with whole number indices.
- When we divide numbers with the same base we still subtract the index of the number that divides from the one that is divided.

Sometimes the index may be negative.

Answers to Activity 2

(a) $a^{7} \div a^{2} = a^{7-2} = a^{5}$

(b) $11^{2} \times 11^{3} = 11^{5}$

(c) $8^{3} \div 8^{3} = 8^{3-3} = 8^{0} = 1$ (also, note that $8^{3} - 8^{3} = \frac{8^{3}}{8^{3}}$)

(d) $5^{4} \times 5^{6} = 5^{4+6} = 5^{10}$

(e) $p^{11} \div p^{6} = p^{11-6} = p^{5}$

(f) $r^{-11} \div r^{2} = r^{-11-2} = r^{-13} = \frac{1}{r^{13}}$

Lesson 3 Simplifying Numbers Expressed in Fractional Index Form Raised to Another Index.

By the end of this subunit, you should be able to:

- simplify numbers expressed in fractional index form raised to another index.

This subunit is about 2 pages in length.

Example 1

$(x^2)^3 = x^2 \times x^2 \times x^2 = x^{2+2+2} = x^6$

Note the repeated addition of the exponent 2. What is the shorter process of adding the same number several times?

____

Compare your answer with the one below:

$\mathbf{2 \times 3 = 6}$

Example 2

$(x^4)^3 = x^{4\times3} = x^{12}$

Example 3

$(5x^2)^3 = 5x^{2\times3} = 5x^6$

Example 4

$(xy^4)^2 = x^{1\times2} \times y^{4\times2} = x^2y^8$

Generally,

$(x^a)^b = x^{a\times b} = x^{ab}$

Activity 3

Activity 3

Simplify the following by removing brackets

(a) $(5^2)^4$

(b) $(a^3)^3$

$(a^2c^4)^2$

Compare your answers with the ones at the end of the subunit. Proceed to the next section when you have at least 80% of this activity correct. If you score lower ensure that you go over this section to understand the concepts treated.

Key Points to Remember

The key points to remember in this subunit are:

- To simplify numbers expressed in fractional index form raised to another index we multiply the indices

Multiplying out the indices is useful in the simplification of expressions as ilustrated in the following subunit.

Answers to activity 3

(a) $(5^2)^4 = 5^{2\times4} = 5^8$

(b) $(a^3)^3 = a^{3\times3} = a^9$

(c) $(a^2c^4)^2 = a^{2\times2} \times c^{4\times2} = a^4c^8$

Lesson 4 Applying Rules of Fractional Index Notation in Simplifying Expressions

Introduction

By the end of this subunit, you should be able to:

- Apply rules of fractional indices learned in the two subunits above to simplify expressions.

This subunit is about 2 pages in length.

The index rules you learned in this unit can be used to evaluate some expressions:

Example 1

Simplify the following expression

(a) $7a^{\frac{2}{3}} \times 7a^{\frac{1}{4}}$

$$7a^{\frac{2}{3}} \times 7a^{\frac{1}{4}} = 7^{1+1} a^{\frac{2}{3}+\frac{1}{4}} = a^{\frac{11}{12}}$$

(b) $\left(-3x^{-1}y^{2}\right)^{2}$

$$\left(-3x^{-1}y^{2}\right)^{2} = \left(-3 \times \frac{1}{x} \times y^{2}\right) = \frac{-3y^{2}}{x}$$

Sometimes the rules can be used to evaluate some expressions. Look at the example below

Example 2

Evaluate $64^{-\frac{1}{3}}$

$$64^{-\frac{1}{3}} = \left(2^{6}\right)^{-\frac{1}{3}} = 2^{6\times-\frac{1}{3}} = 2^{-2} = \frac{1}{2^{2}} = \frac{1}{4}$$

Example 3

Evaluate $\left(\frac{2}{3}\right)^{-3}$

$$\left(\frac{2}{3}\right)^{-3} = \frac{2^{-3}}{3^{-3}} = \frac{1}{2^{3}} \div \frac{1}{3^{3}} = \frac{1}{8} \times \frac{27}{1} = \frac{27}{8}$$

Activity 4

Activity 4

1. Simplify the following expression

(a) $\left(3a^{4}b^{3}\right)\left(2a^{3}b\right)$

(b) $\left(-2mn^{7}p^{2}\right)^{5}$

(c) $\left(-5x^{-2}y\right)\left(-2x^{-3}y^{2}\right)$

(d) $\frac{b^{5}c^{4}}{b^{3}c}$

2. Evaluate some expressions

(a) $\left(\frac{9}{16}\right)^{-\frac{1}{2}}$

(b) $16^{\frac{5}{2}}$

(c) $81^{-\frac{3}{4}}$

(d) $125^{-\frac{2}{3}}$

(e) $\left(-8\right)^{-\frac{1}{3}}$

Compare your answers with the ones at the end of the subunit. Proceed to the next section when you have at least 80% of this activity correct. If you score lower ensure that you go over this section to understand the concepts treated.

Key Points to Remember

The key points to remember in this subunit are:

- Simplification of expression involves the application rules in simplifying numbers.
- (a) with the same base
- (b) having root bracket

You have now completed the last subunit of this unit on Applying rules of fractional index notation in simplifying expressions. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Answers to activity 4

1. Simplify the following expression

(a) $\left(3a^{4}b^{3}\right)\left(2a^{3}b\right) = 3\times 2\times a^{4+3}\times b^{3+1} = 6a^{7}b^{4}$

(b) $\left(-2mn^{7}p^{2}\right)^{5} = -2^{5}\times m^{5}\times n^{7\times 5}\times p^{2\times 5} = -32m^{5}n^{35}p^{10}$

(c) $\left(-5x^{-2}y\right)\left(-2x^{-3}y^{2}\right)=-5\times-2\times x^{-2+-3}\times y^{1+2}=10x^{-5}y^{3}$

(d) $\dfrac{b^{5}c^{4}}{b^{3}c}=b^{5-3}\times c^{4-1}=b^{2}c^{3}$

2. Evaluate some expressions

(a) $\left(\dfrac{9}{16}\right)^{-\frac{1}{2}}=\dfrac{1}{\left(\dfrac{9}{16}\right)^{\frac{1}{2}}}=\dfrac{1}{\dfrac{\sqrt{9}}{\sqrt{16}}}=1\times\dfrac{\sqrt{16}}{\sqrt{9}}=\dfrac{4}{3}$

(b) $16^{\frac{5}{2}}=\left(4^{2}\right)^{\frac{5}{2}}=4^{4\times\frac{5}{4}}=4^{5}$

(c) $81^{-\frac{3}{4}}=\left(3^{4}\right)^{-\frac{3}{4}}=3^{4\times-\frac{3}{4}}=3^{-3}=\dfrac{1}{3^{3}}=\dfrac{1}{27}$

(d) $125^{-\frac{2}{3}}=5^{3\times-\frac{2}{3}}=5^{-2}=\dfrac{1}{5^{2}}=\dfrac{1}{25}$

(e) $\left(-8\right)^{-\frac{1}{3}}=\dfrac{1}{\left(-8\right)^{\frac{1}{3}}}=-\dfrac{1}{\sqrt[3]{8}}=-\dfrac{1}{2}$

Unit Summary

Summary

In this unit you learned

Properties of indices:

(a) $2^2 \times 2^3 = (2\times 2)(2\times 2\times 2) = 2^5$ *generally* $a^m \; x \; a^n = a^{m+n}$

(b)

$$2^5 \div 2^3 = \left(\frac{\cancel{2}\times\cancel{2}\times\cancel{2}\times 2\times 2}{\cancel{2}\times\cancel{2}\times\cancel{2}}\right) = 2^2$$

Generally $a^m \div a^n = a^{m-n}$

(c) $5^4 \div 5^4 = \dfrac{5\times 5\times 5\times 5}{5\times 5\times 5\times 5} = 1 = 5^{4-4} = 5^0$

Generally

$a^0 = 1$

(d)

$$5^4 \div 5^6 = \frac{5\times 5\times 5\times 5}{5\times 5\times 5\times 5\times 5\times 5} = \frac{1}{5^2} = 5^{4-6} = 5^{-2}$$

Generally,

$$a^{-b} = \frac{1}{a^b}$$

(e)

$\sqrt[3]{\sqrt{2^6}} = \sqrt[3]{64} = 4$

$\sqrt[3]{2^6} = 2^{\frac{6}{3}} = 2^2 = 4$

Generally,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

(f)

$$(2^3)^4 = 2^3 \times 2^3 \times 2^3 \times 2^3 = 2^{3+3+3+3} = 2^{3\times4} = 2^{12}$$

Generally,

$$(x^a)^b = x^{a\times b} = x^{ab}$$

You have completed the material for this unit on fractional indices. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

Total Marks: 69 **Time:1hr**

1. Change the following into root bracket form:

(a) $16^{\frac{1}{4}}$

[2]

(b) $2^{\frac{2}{3}}$

[2]

(c) $y^{\frac{x}{z}}$

[2]

2. Change the following into fractional index form

(a) $\sqrt[3]{3^2}$

[2]

(b) $\sqrt[4]{5^3}$

[2]

(c) $\sqrt[5]{6^7}$

[2]

(d) $\sqrt[2]{9^5}$

[2]

(e) $\sqrt[a]{y^b}$

[2]

3.Work out the following

(a) $2^4 \times 2^5$

[2]

(b) $3^7 \div 3^5$

[2]

(c) $a^6 \div a^6$

[2]

(d) $x^3 \times x^7$

[2]

(e) $f^{-2} \div f^{-5}$

[3]

4. Simplify the following by removing brackets

(a) $(7^3)^2$ [2]

(b) $(5^4)^3$ [2]

(c) $(x^3y^4)^3$ [2]

5. Evaluate the following expressions

(a) $\left(25^3\right)^{\frac{1}{6}}$ [4]

(b) $\left(64^{\frac{1}{2}}\right)^{\frac{1}{3}}$ [4]

(c) $\left(4^{-\frac{1}{2}}\right)^{-\frac{3}{2}}$ [6]

6. Simplify the following expressions

(a) $\left(2x^{-4}y^{-3}\right)\left(4^2x^7y^{-2}\right)$ [3]

(b) $(4x^{-4}y^{-6})^{\frac{1}{2}}$ [4]

(c) $\left(2xy^3z^{-2}\right)^7$ [4]

(d) $\sqrt{4a^{\frac{2}{6}}b^{\frac{1}{4}}}$ [5]

(e) $\sqrt{\dfrac{a^2b^6}{a^8b^4}}$ [6]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers to Assignment

1. Change the following into root bracket form:

(a) $16^{\frac{1}{4}} = \sqrt[4]{16}$

(b) $2^{\frac{2}{3}} = \sqrt[3]{2^2}$

(c) $y^{\frac{x}{z}} = \sqrt[z]{y^x}$

2. Change the following into fractional index form

(a) $\sqrt[3]{3^2} = 3^{\frac{2}{3}}$ (b) $\sqrt[4]{5^3} = 5^{\frac{3}{4}}$ (c) $\sqrt[5]{6^7} = 6^{\frac{7}{5}}$

(b) $\sqrt[2]{9^5} = 9^{\frac{5}{2}}$ (e) $\sqrt[a]{y^b} = y^{\frac{b}{a}}$

3. Work out the following

(a) $2^4 \times 2^5 = 2^{4+5} = 2^9 = 256$

(b) $3^7 \div 3^5 = 3^{7-5} = 3^2 = 9$

(c) $a^6 \div a^6 = a^{6-6} = a^0 = 1$

(d) $x^3 \times x^7 = x^{3+7} = x^{10}$

(e) $f^{-2} \div f^{-5} = f^{-2--5} = f^3$

4. Simplify the following by removing brackets.

(a) $(7^3)^2 = 7^{3\times 2} = 7^6$

(b) $(5^4)^3 = 5^{4\times 3} = 5^{12}$

(c) $(x^3 y^4)^3 = x^{3\times 3} \times y^{4\times 3} = x^9 y^{12}$

5. Evaluate the following expressions.

(a) $\left(25^3\right)^{\frac{1}{6}} = \left(\left(5^2\right)^3\right)^{\frac{1}{6}} = 5^{6\times\frac{1}{6}} = 5$

(b) $\left(64^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(\sqrt{64}\right)^{\frac{1}{3}} = (8)^{\frac{1}{3}} = \left(2^3\right)^{\frac{1}{3}} = 2$

(c) $\left(16^{-\frac{1}{2}}\right)^{-\frac{1}{2}} = \left(\frac{1}{16^{\frac{1}{2}}}\right)^{-\frac{1}{2}} = \left(\frac{1}{\sqrt{16}}\right)^{-\frac{1}{2}} = \left(\frac{1}{4}\right)^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2} =$

6. Simplify the following expressions.

(a) $\left(2x^{-4}y^{-3}\right)\left(4^2x^7y^{-2}\right) = 32x^3y^{-5} = \frac{32x^3}{y^5}$

(b) $(4x^{-4}y^{-6})^{\frac{1}{2}} = 2x^{-2}y^{-3} = \frac{2}{x^2y^3}$

(c) $\left(2xy^3z^{-2}\right)^7 = 128x^7z^{-14}$

(d) $\sqrt{4a^{\frac{2}{6}}b^{\frac{1}{4}}} = \sqrt{4} \times a^{\frac{2}{6}\times\frac{1}{2}} \times b^{\frac{1}{4}\times\frac{1}{2}} = 2a^{\frac{1}{6}}b^{\frac{1}{8}}$

(e) $\sqrt{\frac{a^2b^6}{a^8b^4}} = \frac{\sqrt{a^2b^6}}{\sqrt{a^8b^4}} = \frac{ab^3}{a^4b^4} = \frac{1}{a^3b}$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Marks : 53 **Time:30min**

Answer all questions.

1. Evaluate the following expressions

(a) $3^2 \times 2^3$ [2]

(b) $7^0 \times 7^4 \div 7^2$ [3]

(c) $\left(\frac{5}{4}\right)^{-2}$ [4]

2. Change each of the following into fractional index form and evaluate

(a) $\sqrt{2^4}$ [3]

(b) $\sqrt[3]{(64)^{\frac{1}{2}}}$ [4]

3. Change the following into root bracket form

(a) $x^{\frac{3}{4}}$ [2]

(b) $z^{\frac{a}{b}}$ [2]

4. Evaluate the following expressions

(a) $\left(\frac{1}{8}\right)^{-\frac{1}{3}}$ [6]

(b) $3b^{\frac{1}{2}} \times 3b^{\frac{1}{3}}$ [4]

(c) $81^{\frac{3}{4}}$ [4]

(d) $\sqrt{12\frac{1}{4}}$ [3]

(e) (e) $7a^{\frac{2}{3}} \times 7a^{\frac{1}{4}} = 7^{2}a^{\frac{11}{12}}$ [2]

(e) $64^{\frac{5}{6}}$ [4]

(f) $144^{\frac{-1}{2}}$ [3]

(g) $3c^{\frac{2}{3}} \times 12c^{\frac{2}{3}}$ [3]

(h) $\left(6\frac{1}{4}\right)^{\frac{-3}{2}}$ [4]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Solutions to the Assessment

1.

(a) $3^2 \times 2^3 = 3 \times 3 \times 2 \times 2 \times 2 = 9 \times 8 = 72$

(b) $7^0 \times 7^4 \div 7^2 = 7^{0+4-2} = 7^2$

(c) $\left(\frac{5}{4}\right)^{-2} = \frac{1}{\left(\frac{5}{4}\right)^2} = 1 \times \left(\frac{4}{5}\right)^2 = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$

2. Change each of the following into fractional index form and evaluate.

(a) $\sqrt{2^4} = \left(2^4\right)^{\frac{1}{2}} = 2^2 = 4$ [3]

(b) $\sqrt[3]{(64)^{\frac{1}{2}}} = \sqrt[3]{(2^6)^{\frac{1}{2}}} = \sqrt[3]{2^3} = \left(2^3\right)^{\frac{1}{3}} = 2$ [4]

3. Change the following into root bracket form.

(a) $\mathbf{x}^{\frac{3}{4}} = \sqrt[4]{\mathbf{x}^3}$ [2]

(b) $\mathbf{z}^{\frac{a}{b}} = \sqrt[b]{\mathbf{z}^a}$ [2]

4.

Evaluate the following expressions.

(a) $\left(\frac{1}{8}\right)^{-\frac{1}{3}} = \frac{1}{\left(\frac{1}{8}\right)^{\frac{1}{3}}} = 1 \times \left(\frac{8}{1}\right)^{\frac{1}{3}} = 8^{\frac{1}{3}} = \left(2^3\right)^{\frac{1}{3}} = 2^{3 \times \frac{1}{3}} = 2$

(b) $3b^{\frac{1}{2}} \times 3b^{\frac{1}{3}} = 3^{1+1} \times b^{\frac{1}{2}+\frac{1}{3}} = 3^2 b^{\frac{5}{6}} = 9b^{\frac{5}{6}}$

(c) $81^{\frac{3}{4}} = \left(3^4\right)^{\frac{3}{4}} = 3^{4 \times \frac{3}{4}} = 3^3 = 27$

(d) $\sqrt{12\frac{1}{4}} = \sqrt{\frac{49}{4}} = \frac{7}{2} = 3\frac{1}{2}$

(e) $7a^{\frac{2}{3}} \times 7a^{\frac{1}{4}} = 7^2 a^{\frac{11}{12}}$ (g) $64^{\frac{5}{6}} = \left(2^6\right)^{\frac{5}{6}} = 2^{6\times\frac{5}{6}} = 2^5 = 32$

(f) $144^{\frac{-1}{2}} = \dfrac{1}{\left(144^{\frac{1}{2}}\right)} = \dfrac{1}{\sqrt{144}} = \dfrac{1}{12}$

(g) $3c^{\frac{2}{3}} \times 12c^{\frac{2}{3}} = 36c^{\frac{2}{3}+\frac{2}{3}} = 36c^{\frac{4}{3}}$

(h) $\left(6\frac{1}{4}\right)^{\frac{-3}{2}} = \dfrac{1}{\left(\frac{25}{4}\right)^{\frac{3}{2}}} = \dfrac{1}{\left(\sqrt{\frac{25}{4}}\right)^3} = 1\times\left(\dfrac{\sqrt{4}}{\sqrt{25}}\right)^3 = \dfrac{2^3}{5^3} = \dfrac{8}{125}$

Unit Contents

Unit 11

Unit 11

Standard Form

Introduction

Writing and reading a number like 23 000 000 000 000 000 or 0. 000 000 000 001 4 can be rather laborious, unnecessarily!

You will remember that in indices we wrote numbers in short hand form a^n.
For instance $3 \times 3 \times 3 \times 3 \times 3$ can be written as 3^5.

Numbers like these often come up in scientific measurements and calculations. For example, mass of an atom of hydrogen is approximately 1.67×10^{-24} grams and mass of an electron is approximately 9.11×10^{-31} kilograms.

You should have already learned about writing numbers in powers of ten. For example 1000 can be written as 10^3 .This form of writing numbers as powers of ten is called standard form. It is conveniently used when working with very small and very big numbers in calculations. A number in standard form is written as $\mathbf{a} \times \mathbf{10}^n$ where, **a** is one or any number, in decimal form, between one and ten; n is an integer.

This unit consists of 33 pages. This is approximately 2 % of the whole course. As reference, you will need to devote 20 hours to work on this unit, 15 hours for formal study and 5 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Four Lessons:

Lesson 1 Writing Numbers In Standard Form
Lesson 2 Multiplying Numbers In Standard Form
Lesson 3 Dividing Numbers In Standard Form
Lesson 4 Addition and Subtraction Of Numbers In Standard Form

Upon completion of this unit you will be able to:

Outcomes

- *write* numbers in standard form.
- *multiply* numbers in standard form without using a calculator.
- *divide* numbers in standard form without using a calculator
- *add and subtract* numbers in standard form without using a calculator.

Terminology

Standard form: A way of writing a number as $\mathbf{a} \times \mathbf{10}^{n}$ where, $1 \leq \mathbf{a} < 10$ and n is an integer.

Integer: A whole number, negative whole number and zero. Note that zero is neither positive nor negative.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Writing Numbers in Standard Form

Introduction
By the end of this subunit, you should be able to write numbers in standard form. This subunit is about 5 pages in length.

Guidelines for writing numbers in standard form

Writing numbers in standard form involves multiplying and dividing a number by 10^{n} so that the number changes to: $a \times 10^{n}$

Remember **a** is 1 or lies between 1 and 10, and **n** is an integer**.**

This makes the digits in the number only change their place values.

Example 1: Write 12 334 in standard form.

What is 12 334 ÷ 10 000?

____

Compare your answer to the following:

12 334 ÷ 10 000 = 1.2334

This division by 10 000 actually moves the decimal point to the left four times.

12 334. ÷ 10 000 = 1.2334 = 1.2334

Remember the decimal point in whole numbers is always to the right of the unit digit. By convention, it is not written.

Notice that all the digits in 12 334 are still there in 1.2334
All that has happened is their place values have changed.

Also notice that 1.2334 is a number between 1 and 10. This makes it **a** in $\mathbf{a \times 10^n}$.

In order to counterbalance the division by 10 000 you multiply 1.2334 by 10^4. The 10^4 is $\mathbf{10^n}$ in $\mathbf{a \times 10^n}$.

Therefore 12 334 in standard form is 1.2334×10^4

Example 2: Write 2010 in standard form

By which number can we divide 2010 to become 2.010?

Compare your answer with:

1000, which is 10^3

Notice that 2.010 is a number between 1 and 10. This makes it **a** in $\mathbf{a \times 10^n}$.

In order to counterbalance the division by 1000 you multiply 2.010 by 1000, which is 10^3. This 10^3 is $\mathbf{10^n}$ in $\mathbf{a \times 10^n}$.

Therefore 2010 in standard form is 2.010×10^3

Note that 2.010×10^3 can also be written as 2.01×10^3 as 2.010 = 2.01

Example 3: Write 0.000 231 in standard form.

What is 0.000 231 x 10 000?

Compare your answer with:

0.000 231 x 10 000 = 2.31

In what direction did the decimal point move?

____

The decimal point moved to the right.

How many times?

____

The decimal point moved four times.

0.000 231 x 10 000 = 0 0 0 0 2. 3 1 = 2.31

The decimal point moved to the right because 0.000 231 is a proper decimal fraction.

2.31 is a number between 1 and 10. This makes it **a** in $\mathbf{a \times 10^n}$.

In order to counterbalance the multiplication by 10 000 you multiply 2.31 by 10^{-4}. The 10^{-4} is $\mathbf{10^n}$ in $\mathbf{a \times 10^n}$.
It is 10^{-4} because the decimal point has been moved 4 places to the right to get the number to be 2.31

Therefore 0.000 231 in standard form is 2.31×10^{-4}

So, when writing numbers in standard form, shifting the decimal point to the right in a number results in index in the power of ten, which is equal to the negative of the number of shifts made. Shifting the decimal point to the left results in index in the power of ten, which is equal to the number of shifts made.

Activity 1

Write the following numbers in standard form:

Activity 1

(a) 124 000

____

____

____

____

____

(b) 0.025

____

____

(c) 50 000 km

(d) 5 000 000

(e) 0.04

Check your performance against the given solutions; and if you are satisfied with it continue, or otherwise review *writing numbers in standard form.*

ACTIVITY 1:

a. 124 000

124 000 ÷ 100 000 = 1.24000

This actually moves the decimal point to the left five times.

Notice that 1.24000 is a number between 1 and 10. This makes it **a** in **a** × **10^n**.

In order to counterbalance the division by 100 000 you multiply 1.24000 by 10^5. The 10^5 is **10^n** in **a × 10^n**.

Therefore 124 000 in standard form is 1.24×10^5

b. 0.025

To get **a** in **a** × **10** ^{n}, shift the decimal point, in 0.025, 2 shifts to the right.

$0.025 \times 100 = 0.02.5 = 2.5$

Shifting the decimal point 2 shifts to the right means that n = -2 in the power of ten, **10** ^{n}.

So, 0.025 in standard form is 2.5×10^{-2}

c. 50 000 **km**

To get **a** in **a** × **10** ^{n}, shift the decimal point, in 50 000, 4 shifts to the left.

$50\,000. \div 10\,000 = 5.0000 = 5$

Shifting the decimal point 4 shifts to the left means that n = 4 in the power of ten, **10** ^{n}.

So, 50 000 **km** in standard form is 5×10^4 **km**

d. 5 000 000

To get **a** in **a** × **10** ^{n}, shift the decimal point, in 5 000 000, 6 shifts to the left.

$5\,000\,000. \div 1\,000\,000 = 5.000000 = 5$

Shifting the decimal point 6 shifts to the left means that n = 6 in the power of ten, **10** ^{n}.

So, 5 000 000 in standard form is 5×10^6

e. 0.04

To get **a** in **a** × **10** ^{n}, shift the decimal point, in 0.04, 2 shifts to the right.

$0.04 \times 100 = 0.04 = 4$

Shifting the decimal point 2 shifts to the right means that n = -2 in the power of ten, $\mathbf{10}^{n}$.

So, 0.04 in standard form is 4×10^{-2}

Note it!

Remember:

When writing a number in standard form, write it as $\mathbf{a} \times \mathbf{10}^{n}$ where, **a** is one or any number, in decimal form, between one and ten; n is an integer.

Number	Correct Standard Form	Incorrect
6432	$6{\cdot}432 \times 10^{3}$	$64{\cdot}32 \times 10^{2}$
10,500	$1{\cdot}05 \times 10^{4}$	$10{\cdot}5 \times 10^{3}$
Note that values in the rows are however numerically equal!		

Lesson 2 Multiplying Numbers In Standard Form

Introduction

By the end of this subunit, you should be able to multiply any two numbers which are written in standard form.

This subunit is about 5 pages in length.

Steps for multiplying numbers in standard form

Steps are followed when multiplying numbers in standard form; and these steps are based on mathematical facts. It is true that when multiplying $\mathbf{a} \times \mathbf{10}^{n}$ by $\mathbf{b} \times \mathbf{10}^{m}$:

$$(\mathbf{a} \times 10^{\mathbf{n}}) \times (\mathbf{b} \times 10^{\mathbf{m}})$$

$$= (\mathbf{a} \times \mathbf{b}) \times (10^{\mathbf{n}} \times 10^{\mathbf{m}})$$

$$= \mathbf{ab} \times 10^{\mathbf{n+m}}.$$

Take, for example $(4 \times 10^{5}) \times (5 \times 10^{4})$:

$$(4 \times 10^{5}) \times (5 \times 10^{4})$$

Multiply 4 and 5 in one pair of brackets; 10^5 and 10^4 in the other brackets:

$(4 \times 10^5) \times (5 \times 10^4) = (20) \times (10^5 \times 10^4)$

Work out the product in the first one pair of brackets:

$(4 \times 5) \times (10^5 \times 10^4) = (20) \times (10^5 \times 10^4)$

Work out the product in the other brackets, using the rule of multiplication of powers with the same base, $10^x \times 10^y = 10^{x+y}$:

$(20) \times (10^5 \times 10^4) = (20) \times (10^{5+4})$

Now, simplify (20) x (10^{5+4}).

$(20) \times (10^{5+4}) = 20 \times 10^9$

Make 20 in 20×10^9 a number between one and ten by writing it in *standard form*:

$20 = 2 \times 10^1$

Therefore, $20 \times 10^9 = 2 \times 10^1 \times 10^9$.

$2 \times 10^1 \times 10^9 = 2 \times 10^{1+9} = 2 \times 10^{10}$, when using the rule of multiplication of powers with the same base, $10^x \times 10^y = 10^{x+y}$,on $10^1 \times 10^9$.

Let us work together on this next example:

$(4 \times 10^5) \times (5 \times 10^{-4})$

Write it as $(\mathbf{a} \times \mathbf{b}) \times (10^{\mathbf{n}} \times 10^{\mathbf{m}})$

____

____

Check your answer with:

$(4 \times 10^5) \times (5 \times 10^{-4})$

$= (4 \times 5) \times (10^5 \times 10^{-4})$

Write it as $\mathbf{ab} \times 10^{\mathbf{n+m}}$

____

____

Check your answer with:

$(4 \times 5) \times (10^5 \times 10^{-4})$

$= 20 \times 10^{5+-4}$

$= 20 \times 10^1$

Make 20 in 20×10^1 a number between one and ten by writing it in *standard form*:

Check your answer with:

$20 = 2 \times 10^1$

Therefore, $20 \times 10^1 = 2 \times 10^1 \times 10^1$.

$2 \times 10^1 \times 10^1 = 2 \times 10^{1+1} = 2 \times 10^2$, when using the rule of multiplication of powers with the same base, $10^x \times 10^y = 10^{x+y}$, on $10^1 \times 10^1$.

Activity 2

Work out the following without using a calculator:

Activity 2

(a) $(4 \times 10^6) \times (1 \times 10^7)$

(b) $(3 \times 10^2) \times (1.5 \times 10^{-3})$

(c) $\left(\frac{2}{5} \times 10^5\right) \times (5 \times 10^{-4})$

(d) $(2.4\times10^{-5})\times(5\times10^{-4})$

(e) Find the area of a square of side $2\mathbf{cm}$ in square metres, giving your answer in standard form.

(f) Find the area of a rectangle of length $12\mathbf{cm}$ and width $5\mathbf{cm}$ in square metres, giving your answer in standard form.

Check your performance against the given solutions; and if you are satisfied with it continue, or otherwise review multiplying $a\times10^{n}$ by $b\times10^{m}$.

ACTIVITY 2:

(a) $(4\times10^{6})\times(1\times10^{7})$

Write it as $(\mathbf{a}\times\mathbf{b})\times(10^{\mathbf{n}}\times10^{\mathbf{m}})$ and simplify:

$(4\times10^{6})\times(1\times10^{7})$

$=(4\times1)\times(10^{6}\times10^{7})$

$=4\times10^{6+7}$

$=4\times10^{13}$

(b) $(3\times10^{2})\times(1.5\times10^{-3})$

Write it as $(\mathbf{a}\times\mathbf{b})\times(10^{\mathbf{n}}\times10^{\mathbf{m}})$ and simplify:

$(3\times10^{2})\times(1.5\times10^{-3})$

$=(3\times1.5)\times(10^{2}\times10^{-3})$

$=4.5\times10^{2+-3}$

$=4.5\times10^{-1}$

(c) $\left(\frac{2}{5}\times10^{5}\right)\times(5\times10^{-4})$

Write it as $(\mathbf{a}\times\mathbf{b})\times(10^{\mathbf{n}}\times10^{\mathbf{m}})$ and simplify:

$\left(\frac{2}{5}\times10^{5}\right)\times(5\times10^{-4})$

$=\left(\frac{2}{5}\times5\right)\times(10^{5}\times10^{-4})$

$=2\times10^{5+-4}$

$=2\times10^{1}$

(d) $(2.4\times10^{-5})\times(5\times10^{-4})$

Write it as $(\mathbf{a}\times\mathbf{b})\times(10^{\mathbf{n}}\times10^{\mathbf{m}})$ and simplify:

$(2.4\times10^{-5})\times(5\times10^{-4})$

$=(2.4\times5)\times(10^{-5}\times10^{-4})$

$=12\times10^{-5+-4}$

$=12\times10^{-9}$

Make 12 in 12×10^{-9} a number between one and ten by writing it in *standard form*:

$12 = 1.2 \times 10^1$

Therefore, $12 \times 10^{-9} = 1.2 \times 10^1 \times 10^{-9}$.

$1.2 \times 10^1 \times 10^{-9} = 1.2 \times 10^{1+-9} = 1.2 \times 10^{-8}$, when using the rule of multiplication of powers with the same base, $10^x \times 10^y = 10^{x+y}$,on $10^1 \times 10^{-9}$.

(e) Find the area of a square of side $2\mathbf{cm}$ in square metres, giving your answer in standard form.

$2\mathbf{cm}$ is equal to $0.02\mathbf{m} = 2 \times 10^{-2}\,\mathbf{m}$

Area of square = $2\mathbf{cm} \times 2\mathbf{cm}$

Therefore,

$2\mathbf{cm} \times 2\mathbf{cm}$

$= (2 \times 10^{-2}\,\mathbf{m}) \times (2 \times 10^{-2}\,\mathbf{m})$

Write it as $(\mathbf{a} \times \mathbf{b}) \times (10^{\mathbf{n}} \times 10^{\mathbf{m}})$ and simplify:

$= (2 \times 2) \times (10^{-2} \times 10^{-2})\mathbf{m}^2$

$= 4 \times 10^{-2+-2}\,\mathbf{m}^2$

$= 4 \times 10^{-4}\,\mathbf{m}^2$

(f) Find the area of a rectangle of length $12\mathbf{cm}$ and width $5\mathbf{cm}$ in square metres, giving your answer in standard form.

$12\mathbf{cm}$ is equal to $0.12\mathbf{m} = 1.2 \times 10^{-1}\,\mathbf{m}$ and $5\mathbf{cm}$ is equal to $0.05\mathbf{m} = 5 \times 10^{-2}\,\mathbf{m}$

Area of rectangle:

Length $\times$ Width = $12\mathbf{cm} \times 5\mathbf{cm}$

Therefore,

$12\mathbf{cm} \times 5\mathbf{cm}$

$= (1.2 \times 10^{-1}\,\mathbf{m}) \times (5 \times 10^{-2}\,\mathbf{m})$

Write it as $(\mathbf{a} \times \mathbf{b}) \times (10^{\mathbf{n}} \times 10^{\mathbf{m}})$ and simplify:

$= (1.2 \times 5) \times (10^{-1} \times 10^{-2})\mathbf{m}^2$

$= 6 \times 10^{-1+-2}\,\mathbf{m}^2$

$= 6\times10^{-3}\,\mathbf{m}^2$

Note it!

Key points to remember:

1. $(\mathbf{a}\times10^{\mathbf{n}})\times(\mathbf{b}\times10^{\mathbf{m}})$

   $=(\mathbf{a}\times\mathbf{b})\times(10^{\mathbf{n}}\times10^{\mathbf{m}})$

   $=\mathbf{ab}\ \times10^{\mathbf{n+m}}$.

2. If **ab** is greater than 10 or equal to 10, write **ab** in standard form and then reduce $\mathbf{ab}\ \times10^{\mathbf{n+m}}$ to standard form:

   For example, $(8\times10^{-5})\times(5\times10^{-4})$

   Write it as $(\mathbf{a}\times\mathbf{b})\times(10^{\mathbf{n}}\times10^{\mathbf{m}})$ and simplify:

   $(8\times10^{-5})\times(5\times10^{-4})$

   $=(8\times5)\times(10^{-5}\times10^{-4})$

   $=40\times10^{-5+-4}$

   $=40\times10^{-9}$

   Since 40 is greater than 10, write 40 in *standard form*:

   $40=4\times10^{1}$

   Therefore, $40\times10^{-9}=4\times10^{1}\times10^{-9}$.

   $4\times10^{1}\times10^{-9}=4\times10^{1+-9}=4\times10^{-8}$, when using the rule of multiplication of powers with the same base, $10^{x}\times10^{y}=10^{x+y}$,on $10^{1}\times10^{-9}$.

Lesson 3 Dividing Numbers In Standard Form

Introduction

By the end of this subunit, you should be able to divide a number written in standard form, by another number which is also written in standard form.

This subunit is about 5 pages in length.

How to divide numbers in standard form

Like in multiplying numbers in standard form, steps are followed when dividing numbers in standard form. It is true that when dividing

$a\times10^{n}$ by $b\times10^{m}$:

$(\mathbf{a} \times 10^{\mathbf{n}}) \div (\mathbf{b} \times 10^{\mathbf{m}})$

$= (\mathbf{a} \div \mathbf{b}) \times (10^{\mathbf{n}} \div 10^{\mathbf{m}})$

"Notice that there is still multiplication between the double brackets".

Take, for example $(4 \times 10^5) \div (2 \times 10^4)$:

Write it as $(\mathbf{a} \div \mathbf{b}) \times (10^{\mathbf{n}} \div 10^{\mathbf{m}})$ and simplify:

$(4 \times 10^5) \div (2 \times 10^4)$

Write $4 \div 2$ in one double bracket, and multiply it by the other double bracket with $10^5 \div 10^4$ inside:

$(4 \div 2) \times (10^5 \div 10^4)$

Work out the quotient in the first double bracket:

$2 \times (10^5 \div 10^4)$

Use the rule of division of powers with the same base, $10^x \div 10^y = 10^{x\text{-}y}$ in the $(10^5 \div 10^4)$ and simplify:

$= 2 \times 10^{5-4}$

$= 2 \times 10^1$

Let us work together on this next example:

$(4 \times 10^0) \div (5 \times 10^{-4})$

Write it as $(\mathbf{a} \div \mathbf{b}) \times (10^{\mathbf{n}} \div 10^{\mathbf{m}})$

____

____

Check your answer with:

$(4 \times 10^0) \div (5 \times 10^{-4})$

$= (4 \div 5) \times (10^0 \div 10^{-4})$

Work out the quotient in the first double bracket:

____

____

Check your answer with:

$0.8 \times (10^0 \div 10^{-4})$

Use the rule of division of powers with the same base, $10^x \div 10^y = 10^{x-y}$ in the $(10^0 \div 10^{-4})$ and simplify:

Check your answer with:

$= 0.8 \times 10^{0-{}^-4}$

$= 0.8 \times 10^4$

Since 0.8 is less than 10, write 0.8 in *standard form*:

Check your answer with:

$0.8 = 8 \times 10^{-1}$

Therefore, $0.8 \times 10^4 = 8 \times 10^{-1} \times 10^4$.

Use the rule of multiplication of powers with the same base, $10^x \times 10^y = 10^{x+y}$,on $10^{-1} \times 10^4$.

Check your answer with:

$8 \times 10^{-1} \times 10^4 = 8 \times 10^{{}^-1+{}^-4} = 8 \times 10^3$.

Activity 3

Work out the following without using a calculator:

Activity 3

(a) $(4 \times 10^6) \div (1 \times 10^7)$

(b) $(3 \times 10^2) \div (1.5 \times 10^{-3})$

(c) $\left(\frac{2}{5}\times10^{5}\right)\div(5\times10^{-4})$

(d) $(2.4\times10^{-5})\div(5\times10^{-4})$

(e) Given that the speed of light in air is 3×10^{8} **m / s**, calculate the time in hours, that light takes to travel 3.6×10^{12} **m**.

Check your performance against the given solutions; and if you are satisfied with it continue, or otherwise review dividing $a\times10^{n}$ by $b\times10^{m}$.

ACTIVITY 3:

(a) $(4\times 10^{6})\div(1\times 10^{7})$

Write it as $(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div 10^{\mathbf{m}})$ and simplify:

$(4\times 10^{6})\div(1\times 10^{7})$

$=(4\div 1)\times(10^{6}\div 10^{7})$

$=\left(\frac{4}{1}\right)\times(10^{6-7})$

$=4\times 10^{6-7}$

$=4\times 10^{-1}$

(b) $(3\times 10^{2})\div(1.5\times 10^{-3})$

Write it as $(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div 10^{\mathbf{m}})$ and simplify:

$(3\times 10^{2})\div(1.5\times 10^{-3})$

$=(3\div 1.5)\times(10^{2}\div 10^{-3})$

$=\left(\frac{3}{1.5}\right)\times(10^{2--3})$

$=2\times 10^{5}$

(c) $\left(\frac{2}{5}\times 10^{5}\right)\div(5\times 10^{-4})$

Write it as $(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div 10^{\mathbf{m}})$ and simplify:

$\left(\frac{2}{5}\times 10^{5}\right)\div(5\times 10^{-4})$

$=\left(\frac{2}{5}\div 5\right)\times(10^{5}\div 10^{-4})$

$=\left(\frac{2}{25}\right)\times(10^{5--4})$

$=0.08\times 10^{9}$

Since 0.08 is less than 10, write 0.08 in *standard form*:

$0.08=8\times 10^{-2}$

Therefore, $0.08\times10^{9}=8\times10^{-2}\times10^{9}$.

Use the rule of multiplication of powers with the same base, $10^{x}\times10^{y}=10^{x+y}$,on $10^{-2}\times10^{9}$.

$8\times10^{-2}\times10^{9}=8\times10^{^-2+9}=8\times10^{7}$.

(d) $(2.4\times10^{-5})\div(5\times10^{-4})$

Write it as $(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div10^{\mathbf{m}})$ and simplify:

$(2.4\times10^{-5})\div(5\times10^{-4})$

$=(2.4\div5)\times(10^{-5}\div10^{-4})$

$=\left(\frac{2.4}{5}\right)\times(10^{-5-^-4})$

$=0.58\times10^{-1}$

Since 0.58 is less than 10, write 0.58 in *standard form*:

$0.58=5.8\times10^{-1}$

Therefore, $0.58\times10^{-1}=5.8\times10^{-1}\times10^{-1}$.

Use the rule of multiplication of powers with the same base, $10^{x}\times10^{y}=10^{x+y}$,on $10^{-1}\times10^{-1}$.

$5.8\times10^{-1}\times10^{-1}=5.8\times10^{^-1+^-1}=5.8\times10^{-2}$.

(e) Given that the speed of light in air is $3\times10^{8}\,\mathbf{m/s}$, calculate the time in hours, that light takes to travel $3.6\times10^{12}\,\mathbf{m}$.

Time = $(\mathbf{Dis}\tan\mathbf{ce})\div(\mathbf{Speed})$

$\mathbf{Time}=(3.6\times10^{12}\,\mathbf{m})\div(3\times10^{8}\,\mathbf{m/s})$

Write it as $(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div10^{\mathbf{m}})$ and simplify:

$\mathbf{Time}=(3.6\times10^{12}\,\mathbf{m})\div(3\times10^{8}\,\mathbf{m/s})$

$=(3.6\div3)\div(10^{12}\,\mathbf{m}\div10^{8}\,\mathbf{m/s})$

$=\left(\frac{3.6}{3}\right)\times10^{12-8}\mathbf{s}$

$=1.2\times10^{4}\mathbf{s}$

$=1.2\times10^{4}\mathbf{s}\times\frac{1\mathbf{hour}}{3600\mathbf{s}}$

$=\frac{12000}{3600}\times1\mathbf{hour}$

$=3\tfrac{1}{3}\mathbf{hours}$

Note it!

Key points to remember:

1. $(\mathbf{a}\times10^{\mathbf{n}})\div(\mathbf{b}\times10^{\mathbf{m}})$

   $=(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div10^{\mathbf{m}})$

2. There is still multiplication between the double brackets:

   For example, $(4\times10^{-5})\div(8\times10^{-4})$

   Write it as $(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div10^{\mathbf{m}})$ and simplify:

   $(4\times10^{-5})\div(8\times10^{-4})$

   $=(4\div8)\times(10^{-5}\div10^{-4})$

   $=\left(\frac{4}{8}\right)\times(10^{-5-{}^{-}4})$

   $=0.5\times10^{-1}$

   Since 0.5 is less than 10, write 0.5 in *standard form*:

   $0.5=5\times10^{-1}$

   Therefore, $0.5\times10^{-1}=5\times10^{-1}\times10^{-1}$.

   Use the rule of multiplication of powers with the same base, $10^{x}\times10^{y}=10^{x+y}$, on $10^{-1}\times10^{-1}$.

   $5\times10^{-1}\times10^{-1}=5\times10^{{}^{-}1+{}^{-}1}=5\times10^{-2}$.

Lesson 4 Addition and Subtraction Of Numbers In Standard Form

Introduction

By the end of this subunit, you should be able to add numbers in standard form, and also subtract numbers written in standard form.

This subunit is also about 5 pages in length.

Steps for adding and subtracting numbers in standard form

Addition and subtraction of numbers in standard form can be done in the following steps:

1. Rewrite the number with smaller power of ten so that the two numbers have the same power of ten.
2. Factorise the power of ten from the terms.
3. Work out the sum or difference inside the brackets.

For example,

$2.3\times10^{3}+4.1\times10^{4}$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$2.3\times10^{3}=0.23\times10^{4}$

Now, substitute 0.23×10^{4} for 2.3×10^{3} in $2.3\times10^{3}+4.1\times10^{4}$ and factorise the power of ten:

$2.3\times10^{3}+4.1\times10^{4}$

$=0.23\times10^{4}+4.1\times10^{4}$

$=(0.23+4.1)\times10^{4}$

Work out the sum inside the brackets:

$(0.23+4.1)\times10^{4}$

$=4.33\times10^{4}$

Let us work together on this next example:

$2.3\times10^{7}-4.1\times10^{4}$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

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Check your answer with:

$4.1\times10^4 = 0.0041\times10^7$

Now, substitute 0.0041×10^7 for 4.1×10^4 in $2.3\times10^7 - 4.1\times10^4$ and factorise the power of ten :

Check your answer with:

$2.3\times10^7 - 4.1\times10^4$

$= 2.3\times10^7 - 0.0041\times10^7$

$= (2.3 - 0.0041)\times10^7$

Work out the difference inside the brackets:

Check your answer with:

$(2.3 - 0.0041)\times10^7$

$= 2.2959\times10^7$

Activity 4

Work out the following without a calculator:

Activity 4

(a) $4\times10^6 + 1\times10^7$

(b) $3\times10^2 - 1.5\times10^{-1}$

(c) $2.4\times10^{-5}+5\times10^{-4}$

(d) Mass of the Earth is $5.978\times10^{24}\,\mathbf{kg}$ and that of Mercury is $3.301\times10^{23}\,\mathbf{kg}$.

Calculate, in standard form, the mass difference between the planets.

(e) The distance of the Earth from the sun is $1.496\times10^{11}\,\mathbf{m}$ and that of Neptune from the sun is $4.498\times10^{12}\,\mathbf{m}$. What is the distance between Earth and Neptune when both are in a straight line with the Sun. (Give the answer in standard form).

Compare your answers with those at the end of the subunit. Be sure that you understand each answer before continuing.

Key points to remember:

Note it!

When adding or subtracting numbers in standard form:

1. Rewrite the number with smaller power of ten so that the two numbers have the same power of ten.
2. Factorise the power of ten and work out the sum or difference inside the brackets.

You have now completed the last subunit of this unit about standard form. Do a quick review of the entire content of this unit and then continue on to the unit summary.

ANSWERS TO ACTIVITY 4:

(a) $4\times10^6+1\times10^7$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$4\times10^6=0.4\times10^7$

Now, substitute 0.4×10^7 for 4×10^6 in $4\times10^6+1\times10^7$ and factorise the power of ten:

$4\times10^6+1\times10^7$

$=0.4\times10^7+1\times10^7$

$=(0.4+1)\times10^7$

Work out the sum inside the brackets:

$(0.4+1)\times10^7$

$=1.4\times10^7$

(b) $3\times10^2-1.5\times10^{-1}$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$1.5\times10^{-1}=0.0015\times10^2$

Now, substitute 0.0015×10^2 for 1.5×10^{-1} in $3\times10^2-1.5\times10^{-1}$ and factorise the power of ten:

$3\times10^2-1.5\times10^{-1}$

$=3\times10^2-0.0015\times10^2$

$=(3-0.0015)\times 10^{2}$

Work out the difference inside the brackets:

$(3-0.0015)\times 10^{2}$

$=2.9985\times 10^{2}$

(c) $2.4\times 10^{-5}+5\times 10^{-4}$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$2.4\times 10^{-5}=0.24\times 10^{-4}$

Now, substitute 0.24×10^{-4} for 2.4×10^{-5} in $2.4\times 10^{-5}+5\times 10^{-4}$ and factorise the power of ten:

$2.4\times 10^{-5}+5\times 10^{-4}$

$=0.24\times 10^{-4}+5\times 10^{-4}$

$=(0.24+5)\times 10^{-4}$

Work out the sum inside the brackets:

$(0.24+5)\times 10^{-4}$

$=5.24\times 10^{-4}$

(d) Mass of the Earth is $5.978\times 10^{24}\,\textbf{kg}$ and that of Mercury is $3.301\times 10^{23}\,\textbf{kg}$.

Calculate, in standard form, the mass difference between the planets.

Mass of Earth minus mass of Mercury:

$5.978\times 10^{24}\,\textbf{kg}-3.301\times 10^{23}\,\textbf{kg}$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$3.301\times 10^{23}\,\textbf{kg}=0.3301\times 10^{24}\,\textbf{kg}$

Now, substitute $0.3301\times 10^{24}\,\textbf{kg}$ for $3.301\times 10^{23}\,\textbf{kg}$ in $5.978\times 10^{24}\,\textbf{kg}-3.301\times 10^{23}\,\textbf{kg}$ and factorise the power of ten :

$5.978\times 10^{24}\,\textbf{kg}-3.301\times 10^{23}\,\textbf{kg}$

$=5.978\times 10^{24}\,\textbf{kg}-0.3301\times 10^{24}\,\textbf{kg}$

$=(5.978-0.3301)\times 10^{24}\,\textbf{kg}$

Work out the difference inside the brackets:

$(5.978 - 0.3301)\times 10^{24}\,\mathbf{kg}$

$= 5.6479\times 10^{24}\,\mathbf{kg}$

(e) The distance of the Earth from the sun is $1.496\times 10^{11}\,\mathbf{m}$ and that of Neptune from the sun is $4.498\times 10^{12}\,\mathbf{m}$. What is the distance between Earth and Neptune when both are in a straight line with the Sun. (Give the answer in standard form).

Distance of Neptune from the sun, minus distance of the Earth from the sun:

$4.498\times 10^{12}\,\mathbf{m} - 1.496\times 10^{11}\,\mathbf{m}$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$1.496\times 10^{11}\,\mathbf{m} = 0.1496\times 10^{12}\,\mathbf{m}$

Now, substitute $0.1496\times 10^{12}\,\mathbf{m}$ for $1.496\times 10^{11}\,\mathbf{m}$ in $4.498\times 10^{12}\,\mathbf{m} - 1.496\times 10^{11}\,\mathbf{m}$ and factorise the power of ten :

$4.498\times 10^{12}\,\mathbf{m} - 1.496\times 10^{11}\,\mathbf{m}$

$= 4.498\times 10^{12}\,\mathbf{m} - 0.1496\times 10^{12}\,\mathbf{m}$

$= (4.498 - 0.1496)\times 10^{12}\,\mathbf{m}$

Work out the difference inside the brackets:

$(4.498 - 0.1496)\times 10^{12}\,\mathbf{m}$

$= 4.3484\times 10^{12}\,\mathbf{m}$

Unit Summary

Summary

In this unit you learned that very small and very big numbers can be written in standard form: **a** × **10** ^{n} where **a** is one or a number between one and ten**;** **n** is an integer

The important rule is that the number we are multiplying by a power of ten must itself lie between one and ten or be one:

Number	Correct Standard Form	Incorrect
6432	$6{\cdot}432 \times 10^{3}$	$64{\cdot}32 \times 10^{2}$
10,500	$1{\cdot}05 \times 10^{4}$	$10{\cdot}5 \times 10^{3}$

Note that values in the rows are however numerically equal!

When multiplying $a \times 10^n$ by $b \times 10^m$.

It is true that :

$$(\mathbf{a} \times 10^{\mathbf{n}}) \times (\mathbf{b} \times 10^{\mathbf{m}})$$

$$= (\mathbf{a} \times \mathbf{b}) \times (10^{\mathbf{n}} \times 10^{\mathbf{m}})$$

$$= \mathbf{ab} \times 10^{\mathbf{n+m}}.$$

When dividing $a \times 10^n$ by $b \times 10^m$.

It is true that :

$$(\mathbf{a} \times 10^{\mathbf{n}}) \div (\mathbf{b} \times 10^{\mathbf{m}})$$

$$= (\mathbf{a} \div \mathbf{b}) \times (10^{\mathbf{n}} \div 10^{\mathbf{m}}).$$

Addition and subtraction of numbers in standard form can be done in the following steps:

1. Rewrite the number with smaller power of ten so that the two numbers have the same power of ten.
2. Factorise the power of ten from the terms.
3. Work out the sum or difference inside the brackets.

You have completed the material for this unit on Standard Form. You should now spend some time reviewing the content. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

When you work on this assignment, please observe the time allocated and show your working or a reason for your answer.

TOTAL MARKS: 25

TIME: 30 minutes

Assignment

Work out the following without using a calculator:

1. Write the numbers in standard form.

(a) 234.56 (2 marks)

(b) 0.000 203 (2 marks)

(c) $\frac{179}{100}$ (3 marks)

(d) $79\frac{4}{25}$ (3 marks)

2. Work out, without using a calculator, and give the answer in standard form.

(a) $(2\times10^4)\times(9\times10^3)$ (3 marks)

(b) $(2.5\times10^{4})\div(8\times10^{3})$ (3 marks)

(c) $3.25\times10^{5}-3.25\times10^{4}$ (3 marks)

(d) $3.25\times10^{5}+3.25\times10^{3}$ (3 marks)

(e) $3.25\times10^{5}-1.25\times10^{3}$ (3 marks)

Check your performance against the given solutions; and if you scored 80% or more, then go on to the next unit; otherwise review *section(s) on which unsatisfactory performance occurred.*

ASSIGNMENT:

1. Write the numbers in standard form:

(a) 234.56

234.56 ÷ 100 = 2.345 6

The division by 100 moves the decimal point to the left two times.

Notice that 2.345 6 is a number between 1 and 10. This makes it **a** in **a** × **10^n^**.

In order to counterbalance the division by 100 you multiply 2.345 6 by 100, which is 10^2. The 10^2 is $\mathbf{10^n}$ in $\mathbf{a \times 10^n}$.

Therefore 234.56 in standard form is $2.345\,6 \times 10^2$

(b) 0.000 203

0.000 203 x 10 000 = 2.03

This actually moves the decimal point to the right four times.

Notice that 2.03 is a number between 1 and 10. This makes it **a** in $\mathbf{a \times 10^n}$.

In order to counterbalance the multiplication by 10 000 you divide 2.03 by 10 000, which is the same as multiplication of 2.03 by 10^{-4}. The 10^{-4} is $\mathbf{10^n}$ in $\mathbf{a \times 10^n}$.

Therefore 0.000 203 in standard form is 2.03×10^{-4}

(c) $\frac{179}{100}$

= 1.79

1.79 is a number, in decimal form, between one and ten; so multiply it by a power of ten which is equal to 1; that is 10^0.

Therefore $\frac{179}{100}$ in standard form is 1.79×10^0

(d) $79\frac{4}{25}$

= 79.16

79.16 ÷ 10 = 7.916

Dividing by 10 moves the decimal point to the left once.

Notice that 2.345 6 is a number between 1 and 10. This makes it **a** in $\mathbf{a \times 10^n}$.

In order to counterbalance the division by 10 you multiply 2.345 6 by 10. Remember that 10 is equal to 10^1. The 10^1 is $\mathbf{10^n}$ in $\mathbf{a} \times \mathbf{10^n}$.

Therefore $79\frac{4}{25}$ in standard form is 7.916×10^1

2. Work out, without using a calculator, and give the answer in standard form.

(a) $(2\times10^4)\times(9\times10^3)$

Write it as $(\mathbf{a}\times\mathbf{b})\times(10^{\mathbf{n}}\times10^{\mathbf{m}})$ and simplify:

$(2\times10^4)\times(9\times10^3)$

$=(2\times9)\times(10^4\times10^3)$

$=18\times10^{4+3}$

$=18\times10^7$

Make 18 in 18×10^7 a number between one and ten by writing it in *standard form*:

$18=1.8\times10^1$

Therefore, $18\times10^{-9}=1.8\times10^1\times10^7$.

$1.8\times10^1\times10^7=1.8\times10^{1+7}=1.8\times10^8$, when using the rule of multiplication of powers with the same base, $10^x\times10^y=10^{x+y}$, on $10^1\times10^7$.

(b) $(2.5\times10^4)\div(8\times10^3)$

Write it as $(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div10^{\mathbf{m}})$ and simplify:

$(2.5\times10^4)\div(8\times10^3)$

$=(2.5\div8)\times(10^4\div10^3)$

$=\left(\frac{2.5}{8}\right)\times(10^{4-3})$

$=0.3125\times10^1$

Since 0.3125 is less than 10, write 0.3125 in *standard form*:

$0.3125=3.125\times10^{-1}$

Therefore, $0.3125\times10^1=3.125\times10^{-1}\times10^1$.

Use the rule of multiplication of powers with the same base, $10^x \times 10^y = 10^{x+y}$, on $10^{-1} \times 10^1$.

$3.125 \times 10^{-1} \times 10^1 = 3.125 \times 10^{-1+1} = 3.125 \times 10^0$.

(c) $3.25 \times 10^5 - 3.25 \times 10^4$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$3.25 \times 10^4 = 0.325 \times 10^5$

Now, substitute 0.325×10^5 for 3.25×10^4 in $3.25 \times 10^5 - 3.25 \times 10^4$ and factorise the power of ten:

$3.25 \times 10^5 - 3.25 \times 10^4$

$= 3.25 \times 10^5 - 0.325 \times 10^5$

$= (3.25 - 0.325) \times 10^5$

Work out the difference inside the brackets:

$(3.25 - 0.325) \times 10^5$

$= 2.925 \times 10^5$

(d) $3.25 \times 10^5 + 3.25 \times 10^3$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$3.25 \times 10^3 = 0.0325 \times 10^5$

Now, substitute 0.0325×10^5 for 3.25×10^3 in $3.25 \times 10^5 + 3.25 \times 10^3$ and factorise the power of ten:

$3.25 \times 10^5 + 3.25 \times 10^3$

$= 3.25 \times 10^5 + 0.0325 \times 10^5$

$= (3.25 + 0.0325) \times 10^5$

Work out the sum inside the brackets:

$(3.25 + 0.0325) \times 10^5$

$= 3.2825 \times 10^5$

(e) $3.25 \times 10^5 - 1.25 \times 10^3$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$1.25\times10^{3} = 0.0125\times10^{5}$

Now, substitute 0.0125×10^{5} for 1.25×10^{3} in $3.25\times10^{5} - 1.25\times10^{3}$ and factorise the power of ten:

$3.25\times10^{5} - 1.25\times10^{3}$

$= 3.25\times10^{5} - 0.0125\times10^{5}$

$= (3.25 - 0.0125)\times10^{5}$

Work out the difference inside the brackets:

$(3.25 - 0.0125)\times10^{5}$

$= 3.2375\times10^{5}$

Assessment

Assessment

Attempt all the questions.

TOTAL MARKS: 35

TIME: 40 minutes

1) Write 234 800 in standard form. (2 marks)

2) Write 0.000 307 00 in standard form. (2 marks)

3) Write 911.307 in standard form. (2 marks)

4) Work out $(2.1\times10^{4})\times(1.1\times10^{3})$, without using a calculator, and give the answer in standard form. (3marks)

5) Work out $(2\times10^{0})\times(7\times10^{4})$, without using a calculator, and give the answer in standard form. (3 marks)

6) Work out $(1.5\times10^{4})\div(2\times10^{3})$, without using a calculator, and give the answer in standard form. (3 marks)

7) Work out $(1\times10^{4})\div(2.5\times10^{5})$, without using a calculator, and give the answer in standard form. (3 marks)

8) Work out $7.2\times10^{5}+3.25\times10^{6}$, without using a calculator, and give the answer in standard form. (3 marks)

9) Work out $4.21\times10^{4}-1.2\times10^{3}$, without using a calculator, and give the answer in standard form. (3 marks)

10) Mass of Neptune is approximately $1.02 \times 10^{26} kg$ and that of Uranus is approximately $8.68 \times 10^{25} kg$. Find the approximate difference between the masses of the two planets. (5 marks)

11) If the volume of Jupiter is $1.43 \times 10^{12} km^3$ and the volume of Saturn is $8.27 \times 10^{11} km^3$, what is their total volume? (5 marks)

12) The density of a substance is found by dividing the mass of the substance by its volume. If the mass of the sun is $1.99 \times 10^{30} kg$ and its volume is $1.4 \times 10^{16} km^3$, what is the density of the sun, correct to 1 decimal place? (6 marks)

Solutions to the assessment questions:

1) 234 800 in standard form.

234 800 ÷ 100 000 = 2.348 00

The division by 100 000 moves the decimal point, five times to the left. In order to counterbalance the division by 100 000 you multiply 2.348 00 by 100 000, which is 10^5. This 10^5 is $\mathbf{10^n}$ in $\mathbf{a} \times \mathbf{10^n}$, where **a** =2.348 00 .

Therefore 234 800 in standard form is 2.348 00 x 10^5 = 2.348 x 10^5

2) 0.000 307 00 in standard form.

0.000 307 00 × 10 000 = 3.070 0

The multiplication by 10 000 moves the decimal point, four times to the right.
In order to counterbalance the multiplication by 10 000 you divide 3.070 0 by 10 000, which is 10^4. The division by 10^4 is the same as multiplying by 10^{-4}. This 10^{-4} is $\mathbf{10^n}$ in $\mathbf{a} \times \mathbf{10^n}$, and **a** =3.070 0 .

Therefore 0.000 307 00 in standard form is 3.070 0 x 10^{-4}

3) 911.307 in standard form.

911.307 ÷ 100 = 9.11307

The division by 100 moves the decimal point, two times to the left.
In order to counterbalance the division by100 you multiply 9.11307 by 100, which is 10^2. This 10^2 is $\mathbf{10^n}$ in $\mathbf{a} \times \mathbf{10^n}$, and **a** =9.113 07.

Therefore 911.307 in standard form is 9.113 07 x 10^2

4) $(2.1 \times 10^4) \times (1.1 \times 10^3)$

Write it as $(\mathbf{a} \times \mathbf{b}) \times (10^{\mathbf{n}} \times 10^{\mathbf{m}})$ and simplify:

$(2.1\times10^4)\times(1.1\times10^3)$

$=(2.1\times1.1)\times(10^4\times10^3)$

$=2.31\times10^{4+3}$

$=2.31\times10^7$

5) $(2\times10^0)\times(7\times10^4)$

Write it as $(\mathbf{a}\times\mathbf{b})\times(10^{\mathbf{n}}\times10^{\mathbf{m}})$ and simplify:

$(2\times10^0)\times(7\times10^4)$

$=(2\times7)\times(10^0\times10^4)$

$=14\times10^{0+4}$

$=14\times10^4$

Make 14 in 14×10^4 a number between one and ten by writing it in *standard form*:

$14=1.4\times10^1$

Therefore, $14\times10^4=1.4\times10^1\times10^4$.

$1.4\times10^1\times10^4=1.4\times10^{1+4}=1.4\times10^5$, when using the rule of multiplication of powers with the same base, $10^x\times10^y=10^{x+y}$,on $10^1\times10^4$.

6) $(1.5\times10^4)\div(2\times10^3)$

Write it as $(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div10^{\mathbf{m}})$ and simplify:

$(1.5\times10^4)\div(2\times10^3)$

$=(1.5\div2)\times(10^4\div10^3)$

$=\left(\frac{1.5}{2}\right)\times(10^{4-3})$

$=0.75\times10^1$

Since 0.75 is less than 10, write 0.75 in *standard form*:

$0.75=7.5\times10^{-1}$

Therefore, $0.75\times10^1=7.5\times10^{-1}\times10^1$.

Use the rule of multiplication of powers with the same base, $10^x \times 10^y = 10^{x+y}$,on $10^{-1} \times 10^1$.

$7.5 \times 10^{-1} \times 10^1 = 7.5 \times 10^{-1+1} = 7.5 \times 10^0$

7) $(1 \times 10^4) \div (2.5 \times 10^5)$

Write it as $(\mathbf{a} \div \mathbf{b}) \times (10^{\mathbf{n}} \div 10^{\mathbf{m}})$ and simplify:

$(1 \times 10^4) \div (2.5 \times 10^5)$

$= (1 \div 2.5) \times (10^4 \div 10^5)$

$= \left(\frac{1}{2.5}\right) \times (10^{4-5})$

$= 0.4 \times 10^{-1}$

Since 0.4 is less than 10, write 0.4 in *standard form*:

$0.4 = 4 \times 10^{-1}$

Therefore, $0.4 \times 10^{-1} = 4 \times 10^{-1} \times 10^{-1}$.

Use the rule of multiplication of powers with the same base, $10^x \times 10^y = 10^{x+y}$,on $10^{-1} \times 10^{-1}$.

$4 \times 10^{-1} \times 10^{-1} = 4 \times 10^{-1+-1} = 4 \times 10^{-2}$

8) $7.2 \times 10^5 + 3.25 \times 10^6$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$7.2 \times 10^5 = 0.72 \times 10^6$

Now, substitute 0.72×10^6 for 7.2×10^5 in $7.2 \times 10^5 + 3.25 \times 10^6$ and factorise the power of ten:

$7.2 \times 10^5 + 3.25 \times 10^6$

$= 0.72 \times 10^6 + 3.25 \times 10^6$

$= (0.72 + 3.25) \times 10^6$

Work out the sum inside the brackets:

$(0.72 + 3.25) \times 10^6$

$=3.97\times10^{6}$

9) $4.21\times10^{4}-1.2\times10^{3}$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$1.2\times10^{3}=0.12\times10^{4}$

Now, substitute 0.12×10^{4} for 1.2×10^{3} in $4.21\times10^{4}-1.2\times10^{3}$ and factorise the power of ten:

$4.21\times10^{4}-1.2\times10^{3}$

$=4.21\times10^{4}-0.12\times10^{4}$

$=(4.21-0.12)\times10^{4}$

Work out the difference inside the brackets:

$(4.21-0.12)\times10^{4}$

$=4.09\times10^{4}$

10) Difference between the masses of the two planets

= mass of Neptune − mass of Uranus

$1.02\times10^{26}\,kg-8.68\times10^{25}\,kg$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$8.68\times10^{25}\,kg=0.868\times10^{26}\,kg$

Now, substitute $0.868\times10^{26}\,kg$ for $8.68\times10^{25}\,kg$ in $1.02\times10^{26}\,kg-8.68\times10^{25}\,kg$ and factorise the power of ten :

$1.02\times10^{26}\,kg-8.68\times10^{25}\,kg$

$=1.02\times10^{26}\,kg-0.868\times10^{26}\,kg$

$=(1.02-0.868)\times10^{26}\,kg$

Work out the difference inside the brackets:

$(1.02-0.868)\times10^{26}\,kg$

$=0.152\times10^{26}\,kg$

Since 0.152 is less than 10, write it in *standard form*:

$0.152=1.52\times10^{-1}$

Therefore, $0.152\times10^{26}\,kg = 1.52\times10^{-1}\times10^{26}\,kg$

$1.52\times10^{-1+26}\,kg = 1.52\times10^{25}\,kg$.

11) Total volume

= volume of Jupiter + volume of Saturn

$1.43\times10^{12}\,km^3 + 8.27\times10^{11}\,km^3$

Rewrite the number with smaller power of ten so that the two numbers have the same power of ten:

$8.27\times10^{11}\,km^3 = 0.827\times10^{12}\,km^3$

Now, substitute $0.827\times10^{12}\,km^3$ for $8.27\times10^{11}\,km^3$
in $1.43\times10^{12}\,km^3 + 8.27\times10^{11}\,km^3$ and factorise the power of ten :

$1.43\times10^{12}\,km^3 + 8.27\times10^{11}\,km^3$

$= 1.43\times10^{12}\,km^3 + 0.827\times10^{12}\,km^3$

$= (1.43 + 0.827)\times10^{12}\,km^3$

Work out the sum inside the brackets:

$(1.43 + 0.827)\times10^{12}\,km^3$

$= 2.257\times10^{12}\,km^3$

$= 2.26\times10^{12}\,km^3$,correct to two decimal places.

12) Density = mass ÷ volume:

$Density = 1.99\times10^{30}\,kg \div 1.4\times10^{16}\,km^3$

Write it as $(\mathbf{a}\div\mathbf{b})\times(10^{\mathbf{n}}\div10^{\mathbf{m}})$ and simplify:

$(1.99\times10^{30})\div(1.4\times10^{16})kg/km^3$

$= (1.99\div1.4)\times(10^{30}\div10^{16})kg/km^3$

$= \left(\frac{1.99}{1.4}\right)\times(10^{30-16})kg/km^3$

$= 1.4\times10^{14}\,kg/km^3$ to 1 decimal place.

Unit Contents

Unit 12

Sequences

Numbers can form interesting patterns.

Given the numbers 1, 2, 3, 4, 5, how can you say they are related?

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- These are the first five counting numbers.
- To go from one term to the next, you add 1, every time.

You are also given the numbers 2, 4, 8, 16, 32, how can you say they are related?

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- These are some of the positive numbers.
- To go from one term to the next, you multiply by 2, every time.

We say these numbers have formed patterns.

These patterns are called **sequences**.

A sequence may not go on forever. This is a **finite** sequence.

1, 2, 3, 4, 5 is a finite sequence.

A sequence may go on forever. This is an **infinite** sequence.

2, 4, 8, 16, 32, is an infinite sequence.

In this unit we are going to do work on sequences.

This unit consists of 54 pages. This is 2% of the entire course, so plan your time accordingly. As reference, you will need to devote 15 hours to work on this unit, 10 hours for formal study and 5 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Three Lessons:

Lesson 1 Arithmetic Sequence
Lesson 2 Geometric Sequence
Lesson 3 Sigma Notation

Outcomes

Upon completion of this unit you should be able to:

- *find* the common difference for arithmetic sequences;
- *find* the common ratio for geometric sequences;
- *find* missing numbers in a sequence of number patterns;
- *determine* the n^{th} term of an arithmetic sequence;
- *determine* the n^{th} term of a geometric sequence;
- *solve* practical problems involving number sequences;
- *Interpret and apply* the Sigma notation;
- *find* the sum to infinity of a geometric sequence using the formula.

Terminology

Sequence:	An ordered list of numbers which follow a certain rule.
Term:	Each of the numbers in a list forming a sequence.
Arithmetic sequence:	A kind of a sequence in which the difference between any two consecutive terms is a constant value.
Geometric sequence:	A kind of a sequence in which the relationship between any two consecutive terms is a constant ratio.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations

- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Arithmetic Sequence

At the end of this sub-unit you should be able to:

- *find* the common difference for the arithmetic sequences.
- *find* missing numbers in a sequence of number patterns.
- *determine* the n-th term of an arithmetic sequence.

What special patterns do you notice in the following sequences of numbers?

2, 4, 6, 8, 10

____

____

____

$4 = 2 + 2$

$6 = 4 + 2$

$8 = 6 + 2$

$10 = 8 + 2$

To go from one term to the next, we added 2 every time.

We can also relate these numbers this way:

$4 = 2 - -2$

$6 = 4 - -2$

$8 = 6 - -2$

$10 = 8 - -2$

To go from one term to the next, we subtracted -2 every time.

-4, 1, 6, 11, 16

____

____

____

1 = -4 + *5*

6 = 1 + *5*

11 = 6 + *5*

16 = 11 + *5*

To go from one term to the next, we added 5 every time.

We can also relate these numbers this way:

1 = -4 - -*5*

6 = 1 - -*5*

11 = 6 - -*5*

16 = 11 - -*5*

To go from one term to the next, we subtracted -5 every time.

These are examples of arithmetic sequences. A sequence is said to be arithmetic if going from one term to the next is by always adding or subtracting the same number.

Sequences follow a certain **rule.**

In the sequence 2, 4, 6, 8, 10, the rule is: start at 2 and continue to add 2.

In the sequence -4, 1, 6, 11, 16, the rule is: start at -4 and continue to add 5.

Activity 1

What is the rule for finding the next term in each of the following sequences?

(a) 1, 2, 3, 4, 5...

(b) 2, 5, 8, 11, 14…

(c) 3, 7, 11, 15, 19...

(d) 6, 11, 16, 21, 26...

(e) 17, 13, 9, 5, 1...

(f) 34, 29, 24, 19, 14...

(g) 28, 20, 12, 4, −4…

(h) 11, 4, −3, −10, -17...

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

(a) The rule is: start at 1 and continue to add 1 or continue to subtract -1

(b) The rule is: start at 2 and continue to add 3 or continue to subtract -3

(c) The rule is: start at 3 and continue to add 4 or continue to subtract -4

(d) The rule is: start at 6 and continue to add 5 or continue to subtract -5

(e) The rule is: start at 17 and continue to subtract 4 or continue to add -4

(f) The rule is: start at 34 and continue to subtract 5 or continue to add -5

(g) The rule is: start at 28 and continue to subtract 8 or continue to add -8

(h) The rule is: start at 11 and continue to subtract 7 or continue to add -7

The set of counting numbers forms the simplest arithmetic sequence. 1, 2, 3, 4, 5, 6..... t_n

1, the first term may be labelled t_1

2, the second term may be labelled t_2

3, the third term may be labelled t_3

4, the fourth term may be labelled t_4

5, the fifth term may be labelled t_5

6, the sixth term may be labelled t_6

Therefore, any two successive terms may be represented as

t_n and t_{n+1} (or as t_{n-1} and t_n)

For example,

1	2	3	4
t_1	t_2	t_3	t_4
	t_{1+1}	t_{2+1}	t_{3+1}

or

1	2	3	4
t_1	t_2	t_3	t_4
t_{2-1}	t_{3-1}	t_{4-1}	

Example 1

What is the common difference in the sequence 4, 9, 14, 19…?

Solution

$d = t_2 - t_1$ where $t_1 = 4, t_2 = 9$

$= 9 - 4$

$= 5$

Example 2

a) What is the common difference in the sequence 8, 5, 2, -1,…?

Solution

$d = t_3 - t_2$

$= 2 - 5$

$= -3$

Activity 2

Find the common difference for the following sequences.

(a) 1, 3, 5, 7, 9,

(b) 5, 9, 13, 17, 21....

(c) 12, 23, 34, 45, 56...

(d) 4, 26, 48, 70, 92...

(e) 17, 13, 9, 5, 1...

(f) 28, 22, 16, 10, 4...

(g) 33, 22, 11, 0, -11...

(h) 25, 12, $-1, -14, -27$

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

(a) $d = t_2 - t_1$

$= 3 - 1$

$= 2$

(b) $d = t_3 - t_2$

$= 13 - 9$

$= 4$

(c) $d = t_2 - t_4$

$= 56 - 45$

$= 11$

(d) $d = t_4 - t_3$

$= 70 - 48$

$= 22$

(e) $d = t_2 - t_1$

$= 13 - 17$

$= -4$

(f) $d = t_3 \quad t_2$

$= 16 - 22$

$= -6$

(g) $d = t_4 - t_3$

$= 0 - 11$

$= -11$

(h) $d = t_3 - t_4$

$= -27 - 14$

$= -13$

Note it!

In these arithmetic sequences, the common difference is found by finding the difference between any two consecutive terms. Therefore,

Common difference (d) = 2^{nd} term – 1^{st} term, or, 3^{rd} term – 2^{nd} term, or 4^{th} term – 3^{rd} term and so on and so forth

$d = t_2 - t_1, or\ d = t_3 - t_2, or, d = t_3 - t_4$ so on and so forth.

From the above, in general,

Common difference (d) = $t_n - t_{n-1}$

or $d = t_{n+1} - t_n$

Note that d may be positive or negative depending on the sequence.

Finding the n^{th} term in an Arithmetic Sequence

It is not always easy to see what the next terms of a sequence are. In this section we are going to develop a general form that makes it possible to find the n^{th} term of any arithmetic sequence.

Consider this example of an arithmetic sequence:

2 5 8 11 14 .....

If you were asked to find the 100^{th} term (t_{100}), how would you go about?

Let us look at the possible ways of finding the solution to this task.

One way of doing it is to find the common difference, then adding it from the first term up to the 100^{th} term.

However, adding 3 to each term up to the 100^{th} term is likely to be a very long and tiring job.

Let us look at the other ways to find the 100^{th} term in this sequence.

Consider the same sequence written in a different way like this:

2	5	8	11	14	…
1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	n^{th} term
2	$2+1\times 3$	$2+2\times 3$	$2+3\times 3$	$2+4\times 3$	…

Notice that a pattern is formed here. 2 is always added to 3 times one less than the number of the term.

That is,

The first term is 2: to get 2, add 2 to 3 times 1 minus 1,

$2 + 3(1 - 1) = 2 + 3(0) = 2$

The second term is 5: to get 5, add 2 to 3 times 2 minus 1,

$2 + 3(2 - 1) = 2 + 3(1) = 5$

The third term is 8: to get 8, add 2 to 3 times 3 minus 1, ...

$2 + 3(3 - 1) = 2 + 3(2) = 8$ ...

So the 100^{th} term would be:

Add 2 to 3 times 100 minus 1

$2 + 99 \times 3 = 2 + 297 = 299$

Therefore for this sequence, the n^{th} term (t_n) is always

$$t_n = 2 + 3(n - 1)$$

Where, 2 is the first term (t_1) in this sequence,

3 is the common difference (**d**) of the sequence

n is the position of the term.

Note it!

The rule for finding the n^{th} term in an arithmetic sequence is

$$t_n = t_1 + d(n-1)$$

where t_1 is the first term and **d** is the common difference.

Activity 3

1. Write the formula for the n^{th} term (t_n) for the following arithmetic sequences:

(a) 6, 9, 12, 15, ...

(b) 19, 15, 11, 7, ...

(c) $\frac{4}{3}, \frac{1}{3}, \frac{-2}{3}, \frac{-5}{3}, \ldots$

2. Find the indicated term in the following arithmetic sequences.

(a) 2, 5, 8, 11, t_{50}

(b) 24, 19, 14, 9, t_{20}

(c) 3 , -1, -5, -9, t_{13}

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1.

(a) $d = t_4 - t_3$

$= 15 - 12$

$= 3$

$t_1 = 6$

Therefore, $t_n = 6 + 3(n - 1)$

(b) $d = t_3 - t_2$

$= 11 - 15$

$= -4$

$t_1 = 19$

Therefore, $t_n = 19 - 4(n - 1)$

(c) $d = t_2 \quad t_1 = \frac{1}{3} \quad \frac{4}{3} = \quad 1$

$t_1 = \frac{4}{3}$

Therefore, $t_n = \frac{4}{3} - 1(n - 1)$

2.

(a) $d = t_3 - t_2$

$= 8 - 5$

$= 3$

$t_1 = 2$ and $n = 50$

therefore,

$t_{50} = 2 + 3(50 - 1)$

$= 149$

(b) $d = t_4 \quad t_3$

$= 9 - 14$

$= -5$

$t_1 = 24$ and $n = 20$

therefore,

$t_{20} = 24 - 5(20 - 1)$

$= -71$

(c) $d = t_3 - t_2$

$= -5 - -1$

$= -4$

$t_1 = 3$ and $n = 13$

therefore,

$t_{13} = 3 - 4(13 - 1)$

$= -45$

The method of finding the n^{th} term using $t_n = t_1 + d(n - 1)$ works for sequences of the form y = mx + c. If the sequence is not of this linear form, this will not work. Therefore, let us look at the method which can be used for other types of sequences.

This means when you put the values of a, b, and c in.

Up to now, you have been introduced to ways of finding the terms of an arithmetic sequence.

Writing a sequence given the 1st term and the common difference, d.

So far we have worked with given sequences. In this section, we are going to write a sequence given the 1st term and the common difference.

Example 1

Write down the first five terms of a sequence with $t_1 = 7$ and $d = 5$

Solution

1st term	2nd term	3rd term	4th term	5th term
7	7+5	12+5	17+5	22+5
7	12	17	22	27

Therefore the first five terms in this sequence are 7, 12, 17, 22, 27…

Example 2

Write down the first six terms of a sequence with $t_1 = 20$ and $d = -4$

Solution

1st term	2nd term	3rd term	4th term	5th term	6th term
20	20 + -4	16 + -4	12 + -4	8 + -4	4 + -4
20	16	12	8	4	0

Therefore, the sequence is 20, 16, 12, 8, 4, 0…

Activity 4

Write the first five terms of an arithmetic sequence if:

1. $t_1 = 13 \text{ and } d = -4$

2. $t_1 = 6 \text{ and } d = 5$

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1.

1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term
13	13 + -4	9 + -4	5 + -4	1 + -4
13	9	5	1	-3

Therefore the first five terms of this sequence are 13, 9, 5, 1 and -3

2.

1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term
6	6 + 5	11 + 5	16 + 5	21 + 5
6	11	16	21	26

Therefore, the sequence is 6 11 16 21 26

The Sum of the first *n* terms of an Arithmetic Sequence

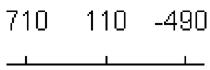
Addition of terms in a finite sequence having, say, 6 terms may not be much work. When many terms are to be added, say 2011 terms, a lot of time and effort will be required to do that.

Fortunately there is a way of findings sums of n terms in an arithmetic sequence.

Let us look at the first three terms of the sequence,

710, 110, -490, ...

You should note the following: the second term 110 is the middle term. Why? First, it occupies the middle place of the three terms. Second, it lies halfway between 710 and -490. They are shown on the number line below:



Note that if you add

710 + 110 + -460 = (110 - 600) + 110 + (110 + 600)

= 3 x 110

= 330

Check if this is true for any arithmetic sequence.

In all arithmetic sequences, the sum of three successive terms is

3 x middle term.

In general, the common difference is denoted by d and the middle term is denoted by t, he first term is t – d and the third term is t + d. Therefore, their sum can be written as,

$$(t - d) + t + (t + d) = 3t$$

What is the relationship between the sum of the first term and the last term?

____

The sum of the first term and the last term in the odd sequence is two times bigger than the middle term of the sequence. Therefore, the middle term is half the sum of the first and the last term.

When this is translated into algebraic expressions, it becomes

$$t = \frac{1}{2}(a + l)$$

Where, *a* is the first term and *l* is the last of the three terms.

We have,

Sum of three terms (S_3) = 3t

Now substituting for t you get

$$S_3 = 3 \times \frac{1}{2}(a + l)$$

In general, the sum of any number *n* of the first terms of an Arithmetic sequence

$$\boxed{S_n = \frac{n}{2}(a + l)}$$

Remember, that the n^{th} term

$$t_n = a + d(n - 1).$$

Hence, $l = a + d(n - 1)$

Substituting *l* in S_n to get

$$S_n = \frac{n}{2}(a + a + d(n - 1))$$

$$= \frac{n}{2}(2a + d(n - 1))$$

$$\boxed{S_n = \frac{n}{2}(2a + d(n - 1))}$$

Note it!

The formula $S_n = \frac{n}{2}(2a + d(n - 1))$ can be used to find the sum of any arithmetic sequence for any number of terms.

Where n is the number of terms whose sum is required

a is the first term

d is the common difference

Activity 5

1. Find the sum of the first 30 even numbers,
   2, 4, 6, ...,

2. Use the general formula to find the sum of
   1 + 2 + 3 + ... + 50

3. The first three terms of the Arithmetic sequence are 6, 10, 14, ...

   Find the common difference, the 20th term and the sum of the first 32 terms.

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1. $S_{30} = \frac{30}{2}(2(2) + 2(30 - 1))$

   $= 15\ (4 + 2\ (29))$

   $= 15\ (62)$

   $= 930$

2. $S_{50} = \frac{30}{2}(2(1) + 1(50 - 1))$

   $= 25\ (2 + 49)$

   $= 25\ (51)$

   $= 1275$

3. Common difference = 4

   $t_{20} = 6 + 4(20 - 1) = 82$

   $S_{32} = \frac{32}{2}(2(6) + 4(32 - 1))$

   $= 16\ (12 + 4(31))$

   $= 16\ (12 + 124)$

   $= 16\ (136)$

   $= 2176$

4. The first three terms of the Arithmetic sequence are 6, 10, 14, ...

   Find the common difference, the 20^{th} term and the sum of the first 32 terms.

Key Points to Remember

The key point to remember in this subunit on arithmetic sequence are:

1. To get the next term you add or subtract the same number called a common difference.
2. Common difference is found by finding the difference between any two consecutive terms.
3. Finding the n^{th} term in an arithmetic sequence is done by using the formula, $t_n = t_1 + d(n-1)$,

   where t_1 is the first term and **d** is the common difference and n is the position of the term.

4. $S_n = \frac{n}{2}(2a + d(n-1))$ is used to find the sum of any arithmetic sequence for any number of terms.

   Where n is the number of terms whose sum is required, a is the first term and d is the common difference.

You have now completed the sub-unit of arithmetic sequences, do a quick review of the entire content of this sub-unit and then continue on to the next sub-unit of Geometric sequence.

Lesson 2 Geometric Sequence

At the end of this sub-unit you should be able to:

- *find* missing numbers in a sequence of number patterns;
- *find* the common ratio for geometric sequences;
- *determine* the n^{th} term of a geometric sequence;
- *find* the sum to infinity of a geometric sequence using the formula.

What special patterns do you notice in the following sequences of numbers?

1, 2, 4, 8, …

____

____

____

2 = 1 × *2*

4 = 2 × *2*

8 = 4 × *2*

After the first term, each successive term can be obtained by multiplying the previous term by 2.

We can also relate these numbers this way:

2 = 1 ÷ ½

4 = 2 ÷ ½

8 = 4 ÷ ½

After the first term, each successive term can be obtained by dividing the previous term by ½.

This is an example of a geometric sequence. A sequence is said to be geometric if going from one term to the next is by always multiplying or dividing by the same number.

Example 1

Write the missing number in the sequence 3, 6, 12, _____, 48, 96, …

Solution

1st term	2nd term	3rd term	4th term	5th term	6th term
3	3×2	6×2	12×2	24×2	48×2
3	6	12	**24**	48	96

Therefore the missing term is 24.

Activity 6

1. Write the missing number in each of these sequences.

(a) 7, 28, 112, ______...

(b) 108, 36, 12, _____...

(c) 250, 50, _____ , 2...

2. What is the rule for finding the next term in each case?

(a)

(b)

(c)

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1. (a) 448 (b) 4 (c) 10

2.

(a) start at 7 and continue to multiply by 4

(b) start at 108 and continue to multiply by $\frac{1}{3}$

(c) start at 250 and continue to multiply by $\frac{1}{5}$

The ratio between any two successive terms is always constant as you have seen in the above sequence. This is called a **common ratio**.

To find the common ratio (r) we say

$$r = t_{n+1} \div t_n \text{ or } t_n \div t_{n-1}$$

Where, t_n is the n^{th} term

r is the common ratio of the sequence

n is the position of the term.

Activity 7

1. Find the common ratio for each of the sequences given below.

(a) 3, 6, 12, _____ 48, 96, ...

(b) 7, 28, 112, _______...

(c) 108, 36, 12, _____...

(d) 250, 50, __, …

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

a) $r = t_n \div t_{n-1}$

$= \frac{6}{3} = 2$

b) $r = \frac{28}{7} = 4$

c) $r = \frac{36}{108} = \frac{1}{3}$

d) $r = \frac{50}{250} = \frac{1}{5}$

Finding the n^{th} term in a geometric sequence

Let a_1 be the first term in a geometric series.

Let a_n be the n^{th} term in a geometric series.

The common multiplier, which is also called the common ratio, is denoted *r*.

Here are the first six terms:

1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term
a_1	a_2	a_3	a_4	a_5	a_1
a_1	$a_1 r$	$a_2 r$	$a_3 r$	$a_4 r$	$a_5 r$
		$(a_1 r) r$	$(a_1 r^2) r$	$(a_1 r^3) r$	$(a_1 r^4) r$
		$a_1 r^2$	$a_1 r^3$	$a_1 r^4$	$a_1 r^5$

You should notice the relationship between the position of the term and the exponent. The exponent is one less than the position of the term.

This observation allows us to write the n^{th} term as follows:

$$a_n = a_1 r^{n-1}$$

where a_1 is the first term and *r* is the common ratio.

Example 2

Find the 10^{th} term for the geometric sequence below.

5, 15, 45, 135, 405,

Solution

$r = 15 \div 5$

$=$

$a_1 = 5$

$a_n = a_1 r^{n-1}$

$a_{10} = a_1 r^{10-1}$

$a_{10} = 5 \times 3^9$

Note it!

The rule for finding the n^{th} term of a geometric sequence is

$$t_n = t_1 r^{n-1}$$

Activity 8

1. Find the n^{th} terms for the following sequences.

(a) 1, 4, 16, 64, …

(b) 4, 12, 36, 108, …

(c) 500, 100, 20, 4, …

2. Find the indicated term in the following sequences.

(a) 3, 6, 12, 24, …, t_{11}

(b) 2, -10, 50, -250, …, t_8

(c) 256, 64, 16, 4, …, t_6

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1. a) $r = t_2 + t_1$

$= \frac{4}{1}$

$= 4$

$t_1 = 1$ therefore,

$t_n = 1 \times 4^{n-1}$

b) $r = t_3 \div t_2$

$= \frac{36}{12}$

$= 3$

$t_1 = 4$ therefore,

$t_n = 4 \times 3^{n-1}$

c) $r = t_3 \div t_2$

$= \frac{20}{100}$

$= \frac{1}{5}$

$t_1 = 500$ therefore,

$t_n = 500 \times \left(\frac{1}{5}\right)^{n-1}$

2. a) $t_{11} = 3072$

b) $t_8 = 156\ 250$

c) $t_6 = \frac{1}{4}$

The sum to infinity of a geometric sequence using a formula

In the previous sections, you learned how to find the common ratio and how to determine the n^{th} term of a geometric sequence. In this section you will use them to help you find the sum of a geometric sequence to infinity. This means going on and on without ending.

The common ratio (r) is found by using

$$r = \frac{t_n}{t_{n-1}}$$

The n^{th} term is found by using

$$t_n = t_1 r^{n-1}$$

Now, let us introduce the symbol S_n to represent the sum of the geometric sequence.

So

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} \qquad (1)$$

Multiply (1) by r to get:

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^n \qquad (2)$$

If you subtract (1) from (2) you get,

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^n$$

-

$$\underline{S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}}$$

$$rS_n - S_n \quad = ar^n - a$$

Equating LHS = RHS to get:

Which simplifies to:

$$S_n(r-1) = a(r^n 1)$$

Dividing by (r – 1) to get:

$$S_n = \frac{a(r^n-1)}{r-1}$$

If you subtract (2) from (1) you get:

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1}$$

-

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^n$$

$$S_n - rS_n = a - ar^n$$

Which simplifies to:

$$S_n(1 - r) = a(1 - r^n)$$

Dividing both sides by (1 – r) to get:

$$S_n = \frac{a(1-r^n)}{1-r}$$

Note it!

$$S_n = \frac{a(r^n-1)}{r-1}$$ **(type 1)**

This is used when the common ratio (r) is greater than 1 and positive.

$$S_n = \frac{a(1-r^n)}{1-r}$$ **(type 2)**

This is used when the common ratio (r) is less than 1 or is negative.

One more thing to note in relation to type 2 equation is that when number of terms (*n*) in a sequence is very great, we say that *n* approaches infinity(∞).Under these circumstances , the term r^n will result in a value which is very near zero (we say the value of r^n approaches zero).

These expressions are written as $n \to \infty$ and $r^n \to 0$.

This means that:

$$S_\infty = \frac{a(1-0)}{1-r}$$

$$S_\infty = \frac{a}{1-r}$$

Which is the type 3 equation.

To find the sum S_n to infinity you use the Type 3 equation.

When you use the Type 3 equation, you need to have any two of the following three:

the first term of the sequence *a*,

the common ratio *r* or

the sum of the sequence S_n.

As you did when changing the subject of the formula, you can find any of the three (*a, r* or S_n) if you know the other two.

Let us go to the example below and see how this works.

Example 3

Find the sum to infinity of this geometric sequence.

$$\frac{8}{3}, \frac{4}{9}, \frac{2}{27}, \frac{1}{81}, \ldots$$

Solution

Remember, the sum to infinity is found by using the Type 3 equation:

$$S_\infty = \frac{a}{1-r}$$

a. Value for first term $= \frac{8}{3}$

Value for r $= \frac{\frac{4}{9}}{\frac{8}{3}} = \frac{4}{9} \times \frac{3}{8} = \frac{1}{6}$

Substituting into equation type 3

$$S_\infty = \frac{\frac{8}{3}}{1-\frac{1}{6}} = \frac{\frac{8}{3}}{\frac{5}{6}} = \frac{8}{3} \times \frac{6}{5} = \frac{16}{5}$$

Activity 9

Find the sum to infinity of these geometric sequences.

(a) $5, -1, \frac{1}{5}, \frac{-1}{25}, \ldots$

(b) $9, -6, 4, \frac{-8}{3}, \ldots$

Answers

Remember, the sum to infinity is found by using the Type 3 equation:

$$S_\infty = \frac{a}{1-r}$$

(a) Value for first term = 5

Value for r = $\frac{-1}{5}$

Substituting into equation type 3

$$S_\infty = \frac{5}{1-\frac{-1}{5}} = \frac{5}{\frac{6}{5}} = 5 \times \frac{5}{6} = \frac{25}{6}$$

(b) Value for first term = 9

Value for r = $\frac{-2}{3}$

Substituting into equation type 3

$$S_\infty = \frac{9}{1-\frac{-2}{3}} = \frac{9}{\frac{5}{3}} = 9 \times \frac{3}{5} = \frac{27}{5}$$

Key Points to Remember

The key points to remember in this sub-unit on geometric sequences are:

1. To get the next term you multiply or divide by the same number called a common ratio.

2. The rule for finding the n^{th} term of a geometric sequence is

$$t_n = t_1 r^{n-1}$$

3. The formulae used to find the sum to infinity of a geometric sequence are

$$S_n = \frac{a(r^n-1)}{r-1} \quad \textbf{(type 1)}$$

This is used when the common ratio (r) is greater than 1 and positive.

$$S_n = \frac{a(1-r^n)}{1-r} \quad \textbf{(type 2)}$$

This is used when the common ratio (r) is less than 1 or is negative.

$$S_\infty = \frac{a}{1-r} \quad \textbf{(type 3)}$$

This is used when there are many number of terms (n) in a sequence .

You have now completed the sub-unit on geometric sequences. Go back to the beginning of the sub-unit and carry out a quick review of the entire content. Only if you are confident that you have mastered the material should you continue on to the next sub-unit on Sigma notation.

Lesson 3 Sigma Notation

At the end of this sub-unit you should be able to:

- *Interpret and apply* sigma notation.

There are eight pages in this subunit.

In the previous sub-units of this unit you have found an n^{th} term for a number of sequences. The n^{th} term makes it easy to find the next term from the preceding term. Sometimes we need to find the sum of many terms of an infinite sequence. In order to write the sum in an easy way we use the sigma (summation) notation.

For example, if the infinite sequence is:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

We use the symbol:

$$\sum_{n=1}^{k} a_n$$

to represent the sum of the first k terms.

The summation in general is:

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \dots + a_n$$

where the Greek letter sigma $\sum$ means "the sum of" and the symbol a_n represents the n^{th} term of the sequence. The terms will start at:

$$n = 1 \text{ to } n = k$$

Sigma notation simply means summation notation (the sum of the following terms). In other words, the sigma symbol is a shorthand way of representing the sum of a sequence.

Note it!

- To write a sum in Sigma notation, try to find the n^{th} term of the sequence first.
- The first term can be obtained by setting n=1, the second term n=2 and so on.

Rules for use with sigma notation

There are a number of useful results that can be obtained when using Sigma notation.

Example 1

Suppose we had a sum of five constant terms. The sum can be written easily in sigma notation as follows for the constant term 3.

$$\sum_{n=1}^{5} 3 \quad .$$

This summation is short and easy, but if you write it out in full you generate a general result that is useful in solving similar problems. Let us do it in the solution below.

Solution

If we write this out in full we get:

$$\sum_{n=1}^{5} 3 = 3+3+3+3+3$$

$$= 5 \times 3$$

$$= 15$$

Therefore, in general this can be written as:

$$\boxed{\sum_{n=1}^{k} c = c+c+c+\cdots+c = nc}$$

Where c is a constant and k is the number of term in the sequence.

Example 2

Suppose we had the sum of a constant time *n*

$$\sum_{n=1}^{5} 3n$$

As in the above example, if you write out this sum in full, you get a general result that can be used for similar problems. Let us look at it in the solution below.

Solution

$$\sum_{n=1}^{5} 3n = (3 \times 1) + (3 \times 2) + (3 \times 3) + (3 \times 4) + (3 \times 5)$$

$$= 3(1 + 2 + 3 + 4 + 5)$$

$$= 3 \times 15$$

$$= 45$$

The result can also be equal to:

$$3(1 + 2 + 3 + 4 + 5) = 3\sum_{n=1}^{5} n$$

Therefore:

$$\sum_{n=1}^{5} 3n = 3\sum_{n=1}^{5} n$$

In general we say that:

$$\sum_{n=1}^{k} cn = c\sum_{n=1}^{k} n$$

Example 3

Suppose we had a sum of *n* plus a constant. As in the examples above, the sum can be written in full to get:

$$\sum_{n=1}^{4} n + 7 = (1+7) + (2+7) + (3+7) + (4+7)$$

$$= (1+2+3+4) + (4 \times 7)$$

$$= 10 + 28$$

$$= 38$$

The result can also be equal to:

$$(4 \times 7) + (1+2+3+4) = (4 \times 7) + \sum_{n=1}^{4} n$$

Therefore:

$$\sum_{n=1}^{4} n + 7 = (4 \times 7) + \sum_{n=1}^{4} n$$

In general we can say that:

$$\boxed{\sum_{n=1}^{k} n + c = nc + \sum_{n=1}^{k} n}$$

Now, let us apply the general results in the next activity.

Activity 10

1. Find the value of

a)

$$\sum_{n=1}^{7} 4$$

b)

$$\sum_{n=2}^{4} 8$$

c)

$$\sum_{n=1}^{10} 15$$

2. Find the value of

a)

$$\sum_{n=1}^{6} 12n$$

b)

$$\sum_{n=1}^{3} \frac{3}{8}n$$

c)

$$\sum_{n=1}^{8} -2n$$

3. Work out

a)

$$\sum_{n=1}^{3} n + 24$$

b)

$$\sum_{n=3}^{9} n - 5$$

4. Work out

$$\sum_{n=7}^{12} (3n - 2)$$

5. Work out

$$\sum_{n=1}^{6} \frac{n(n+1)}{2}$$

Answers

1. a)

$$\sum_{n=1}^{7} 4 = 4 \times 7$$

$$= 28$$

b)

$$\sum_{n=2}^{4} 8 = 8 + 8 + 8$$

$$= 24$$

c)

$$\sum_{n=1}^{10} 15 = 10 \times 15$$

$$= 150$$

2.a)

$$\sum_{n=1}^{6} 12n = 12 \sum_{n=1}^{6} n = 12(1 + 2 + 3 + 4 + 5 + 6)$$

$$= 12 \times 21$$

$= 252$

b)

$$\sum_{n=1}^{5} \frac{3}{5} n = \frac{3}{5} \sum_{n=1}^{5} n = \frac{3}{5}(1 + 2 + 3 + 4 + 5)$$

$$= \frac{3}{5} \times 15$$

$$= 9$$

c)

$$\sum_{n=1}^{8} -2n = -2 \sum_{n=1}^{8} n = -2(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)$$

$= -2 \times 36$

$= -72$

3.a)

$$\sum_{n=1}^{3} n + 24 = 3 \times 24 + \sum_{n=1}^{3} n$$

$= 72 + (1 + 2 + 3)$

$= 72 + 6$

$= 78$

b)

$$\sum_{n=3}^{9} n - 5 = 7 \times -5 + \sum_{n=3}^{9} n$$

$= -35 + (3 + 4 + 5 + 6 + 7 + 8 + 9)$

$= -35 + 42$

$= 7$

4.

$$\sum_{n=7}^{12} (3n - 2) = \sum_{n=7}^{12} (3 \times 7 - 2)$$

$= 19 + 19 + 19 + 19 + 19 + 19 + 19 + 19 + 19 + 19 + 19 + 19$

$= 228$

The solution can also be written like the following.

$$\sum_{n=7}^{12} 3(n - 2) = \sum_{n=7}^{12} (3 \times 7 - 2)$$

$= 19 \times 12$

$= 228$

5.

$$\sum_{n=1}^{6}\frac{n(n+1)}{2}=\sum_{n=1}^{6}\frac{1(1+1)}{2}$$

$$=\frac{2}{2}+\frac{2}{2}+\frac{2}{2}+\frac{2}{2}+\frac{2}{2}+\frac{2}{2}$$

$$=\frac{12}{2}$$

$$=6$$

The solution can also be written like the following,

$$\sum_{n=1}^{6}\frac{n(n+1)}{2}=\sum_{n=1}^{6}\frac{1(1+1)}{2}=6\times\frac{2}{2}=6$$

You have now completed the last sub-unit of this unit on Sequences. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Unit Summary

Summary

In this unit you learned that:

- For arithmetic sequences, the common difference is found by finding the difference between any two consecutive terms.
- **Common difference (d)** = $t_n - t_{n-1}$

  (or $d = t_{n+1} - t_n$)
- The rule for finding the n^{th} term in an arithmetic sequence is:

  $t_n = t_1 + d(n-1)$,

  where t_1 is the first term and **d** is the common difference
- The rule for finding the n^{th} term of a geometric sequence is:

  $t_n = t_1 r^{n-1}$
- To write a sum in Sigma notation try to find the n^{th} term of the sequence.
- The sum to infinity of a geometric sequence is found by using the formula:

  $S_\infty = \frac{a}{1-r}$

You have completed the material for this unit on Sequences. You should now spend some time reviewing the content in detail.

Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers against those provided and clarify any misunderstandings that you have.

Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Instructions

1. Answer all of the questions.
2. The marks for each question are shown.
3. Calculators may be used.
4. Show all the necessary working.

Total marks = 64 **Time:** 1hr 30 minutes

Good luck!

1. Given the following sequences,
   a. 3, 7, 11, 15, 19, ...
   b. 8, 15, 22, 29, 36, ...
   c. 30, 25, 20, 15, 10, ...

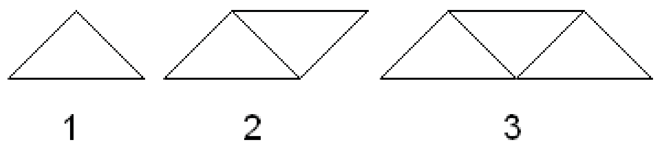
   Find the following for each sequence.
   i. The common difference (6)

   ii. The rule for finding the next term (6)

iii. The next two terms (6^{th} and 7^{th}) (6)

iv. Write the n^{th} term of the sequences. (6)

2. A sequence of equilateral triangles is made by placing them side by side as shown below.



Number of triangles	Perimeter
1	3
2	
3	
4	
5	

i. Complete the table. (4)

ii. Write the n^{th} term for the perimeter. (2)

iii. Write the 10^{th} term for the perimeter. (2)

3. Given the following sequences,
   a. 2, 6, 18, 54, ...
   b. 2, -4, 8, -16, ...
   c. 3, 15, 75, 375, ...
   d. a, ax^2, ax^4, ax^6, ...
      i. The common ratio. (8)

      ii. The rule for finding the next term. (8)

      iii. The next two terms (5^{th} and 6^{th}). (8)

iv. Write the n^{th} term of the sequences. (8)

Model Answers to the Assignment

1.

i. a) 4 b) 7 c) -5

ii. a) start at 3 and continue to add 4

b) start at 8 and continue to add 7

c) start at 30 and continue to subtract 5

iii. a) 23, 27

b) 43, 50

c) 5, 0

iv. a) $3 + 4(n - 1)$

b) $8 + 7(n - 1)$

c) $30 + -5(n - 1)$

2.

ii. 3, 4, 5, 6, 7,

iii. $3 + (n - 1)$

iv. 10^{th} term $t_{10} = 10 + 2 = 12$

3.

ii. a) 3

b) -2

c) 5

d) x^2

ii. a) start at 2 continue to multiply by 3

b) start at 2 continue to multiply by -2

c) start at 3 continue to multiply by 5

d) start at a continue to multiply by x^2

iii. a) 162, 486

b) 32, -64

c) 1875, 9375

d) ax^8, ax^{10}

iv. a) $2(3)^{n-1}$

b) $2(-2)^{n-1}$

c) $3(5)^{n-1}$

d) $ax^{2(n-1)}$

Assessment

Assessment

Instructions

1. Answer all of the questions.
2. The marks for each question are shown.
3. Calculators may be used.
4. Show all the necessary working.

Total marks = 22 **Time:** 30 minutes

Good luck!

1. Write down the next two terms in the sequence.
   a. 12, 11, 9, 6, ... (2)

   b. 4, 9, 16, 25, ... (2)

      i. Write down the n^{th} term for the sequence in (b) (2)

2. Write down the first six terms of the sequence whose first term is 3 and common ratio is 4. (6)

3. If the first term in a Geometric sequence is 2 and the 4^{th} term is 54, what is the common ratio? (2)

4. If the first term of the geometric sequence is 37, and the common ratio is 3. Find the n^{th} term. (3)

5. Find S_∞ of the sequence $\frac{3}{10}, \frac{3}{100}, \frac{3}{1000}$ (5)

Model Answers to Assessment

1.
   a. 2, -3
   b. 36, 49
      i. $t_n = (n+1)^2$

2. 3, 7, 11, 15, 19, 23

3.

$$54 = 2r^3$$
$$\frac{54}{2} = r^3$$
$$27 = r^3$$
$$3 = r$$

4. $t_n = 37(3)^{n-1}$

5. $$S_\infty = \frac{a}{1-r}$$

Value of a = $\frac{3}{100}$

Value of $r = \frac{\frac{3}{1000}}{\frac{3}{100}} = \frac{3}{1000} \times \frac{100}{3} = \frac{1}{10}$

Substituting,

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{\frac{3}{100}}{1-\frac{1}{10}} = \frac{\frac{3}{100}}{\frac{9}{10}} = \frac{3}{100} \times \frac{10}{9} = \frac{1}{30}$$

Unit Contents

Unit 13

Unit 13

Ratio and Proportion

Introduction

While studying mathematics at the junior secondary level, you learned how to:

- compare like quantities by ratios,
- find missing quantities in a ratio, and
- compare unlike quantities by rates.

You also did some work on direct and indirect proportion. Proportion is used with quantities that are seemingly unrelated.

In this unit, we are going to continue work on ratio, rates, and proportion. Have you ever thought of why ratios, proportions, and rates are important?

Let us look at the following situations:

- If you have ever travelled by public transport, you have definitely paid the bus or taxi fare.
- If you have ever bought groceries for your family, you have definitely paid the money for each item you buy.
- If you have attended a party or feast and helped in the preparations, especially food preparations for the guests, you probably diluted some kind of concentrated drink.
- If you have ever been involved in building a house, you will definitely need the knowledge from this unit.

These are some examples where the knowledge of how to use ratios, proportions, and rates can help you make informed decisions in everyday life especially in commercial math (unit 9).

This unit consists of 45 pages. This is 1% of the whole course, so plan your time accordingly. As reference, you will need to devote 15 hours to work on this unit, 10 hours for formal study and 5 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Three Lessons:

Lesson 1 Ratios
Lesson 2 Rate
Lesson 3 Proportion

Upon completion of this unit you will be able to:

Outcomes

- *divide* a quantity in a given ratio;
- *increase and decrease* a quantity by a given ratio;
- *calculate* average speed;
- *formulate* equations from direct and inverse proportion situations, and use these equations to solve for one quantity if given others;
- *apply* the ideas and notation of ratio, proportion, and rate to practical situations.

Terminology

Ratio: The comparison of two or more like quantities, that is, quantities measured in the same units.

Rate: A special type of ratio that is used when comparing quantities measured in different units.

Proportion: Statement of equality between two ratios or rates.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Ratios

In Lesotho, children are often given sweets or another treat when visitors come to their homes or when they accompany their elders to visit another household. Sometimes, the sweets must be shared among a number of children. When I was growing up, we had to share everything, usually with the biggest portion going to the youngest and so on all the way up to the eldest. Often, the eldest child got only a small share or nothing at all.

I also felt this was terribly unfair, except while I was the youngest. But at the time, I didn't understand mathematics. The fairest and most common way of sharing would be to use a ratio to divide the sweets. How do we do this? Keep your eyes open for the answer down below.

At the end of this subunit you should be able to:

- define the term 'ratio';
- write a ratio in fraction form;
- present examples of how ratios are used in everyday life;
- divide a quantity in a given ratio.

There are 6 pages on this sub-unit on ratios.

Simple Ratios

Do you remember how to do fractions from your previous studies? If not, this example might help to revise these concepts.

Imagine that you have gone to your neighbour's house and a peach pie is brought out to share among everyone there. One family has four members, the other family has three members. Each member is to get an equal share.

The **first step** here is to find how much is going to be shared

One pie

The **second step** is to find how many shares are needed. The total number of people who are going to be sharing the pie is the sum of the members from the two families:

$4 + 3 = 7$

So, we need to cut the pie into seven equal slices so that each person gets an equal share. We can say that the pie has been cut up in proportion to the number of people who will be sharing it. If we compare the slice that each person gets to the whole pie, this can be shown as:

$\frac{1}{7}$, which can also be written as 1 : 7.

1 is the first term of the ratio and 7 is the second term of the ratio.

Other relationships could also be expressed as ratios. For example, if we had a bag full of pies of different flavours, we could show the ratio between one flavour and another using a ratio.

A ratio expresses a relationship between two or more like quantities.

When we say 'like quantities', we mean things that are alike in some way. In order to have a ratio, the quantities of the two things must be expressed in the same unit of measurement. If you need to revise this concept, go back to Unit 1 of this course and read through the materials again.

Example

Bohlokoa has three apples and two oranges; these can't be compared directly with each other in a ratio because they are different things. Apples are not exactly the same as oranges – they are not 'like quantities'.

However, if we think of them as two different types of the same thing – fruit – then a comparison can be made in the form of a ratio.

3 apples (fruit) compared to 2 oranges (fruit)

Which can also be written as $\frac{3}{2}$ or 3: 2

In our everyday life we make a lot of comparisons. If there are 16 orange trees and 12 guava trees in an orchard, then the ratio of orange trees to guava trees is:

$$\frac{16}{12}$$

However, ratios need to be reduced to their simplest form. To do this you use the same process that you learned for simplifying fractions. You look for common factors and divide the terms both above and below the line by the same factor.

In this example, the figures both above and below the line can be divided by a factor of four. So, when this operation is complete our ratio looks like this in its simplest form:

$\frac{4}{3}$, which can also be written 4 : 3

Ratios with more than two parts

Let's go back to the example of building a house, which usually involves mixing cement, sand and gravel to make concrete. However, depending on what the concrete will be used for, the amount of each ingredient can be varied in the mix. If you look at the instructions on the back of a pocket or bag of cement you will see a table of numbers similar to the one below (*the numbers may differ from one brand of cement to another):*

Purpose of mix	**Cement**	**Sand**	**Gravel**
Home concrete	1	$3\frac{1}{2}$	$3\frac{1}{2}$
Medium strength	1	$2\frac{1}{2}$	$2\frac{1}{2}$
Reinforced concrete	1	2	2

Watertight concrete	1	$1\frac{1}{2}$	$1\frac{1}{2}$
Mortar for bricklaying and plastering	1	6	-
Brick and block mix	1	8	-

The numbers inside the table show the amounts of each material used to make concrete for different purposes. The numbers have been compared to help the builder mix the concrete correctly. If you use a shovel to measure, then you will use one shovel of cement and six shovels of sand to make mortar.

If you use a wheelbarrow to measure, you will use one wheelbarrow full of cement and six wheelbarrows of sand to make mortar. Notice that, when we say 'like quantities', we mean the same units of measurement; if you use a shovel to measure one ingredient, you must use a shovel for all the others. It would be the same if you use a wheelbarrow, a bucket or anything else to measure the different materials in your mix. However, no matter what measure you use, the ratio between the different materials stays the same.

For home concrete, the mix requires a different ratio; it should have one measure of cement, three and a half measures of sand and three and a half measures of gravel. This can be expressed as a ratio between the three ingredients as follows:

$$1 : 3\frac{1}{2} : 3\frac{1}{2}$$

The ratio of cement to gravel in our home concrete mix is $1 : 3\frac{1}{2}$

However, if we want to calculate the ratio of cement to total mixture, we need to carry out a process of addition to find the total amount in the mix:

The total number of 'like quantities' (i.e. parts, shovels, wheelbarrows, buckets, etc.) in the mixture is the sum of all the quantities:

$$1 + 3\frac{1}{2} + 3\frac{1}{2} = 8$$

Since only one measure of cement is needed, the ratio of cement to the total mixture is

1: 8 which can also be written as $\frac{1}{8}$ in fraction form.

Now, try writing the ratio for medium strength concrete mix:

The ratio for medium strength concrete mix is $1 : 2\frac{1}{2} : 2\frac{1}{2}$

What is the ratio of cement to sand?

The ratio of cement to sand $1 : 2\frac{1}{2}$

What is the ratio of cement to the total amount in the mix?

The total number of like quantities is

$1 + 2\frac{1}{2} + 2\frac{1}{2} = 6$

The ratio of cement to the total amount in the mix is 1 : 6 which can also be written as $\frac{1}{6}$ in fraction form.

Activity

Activity 13.1

1. A box contains 20 red balls and 30 black balls. Write down the ratio of the red balls to the black balls.

2. A family's income is M8 000 per month, which is spent as follows:

Accommodation (Rent)	M 1 500
Food	M 2 400
Utilities	M 500
Education Expenses	M 1 200
Transport	M 800
Savings & Investment	M 400
Debt Repayments	M 500
Other Expenses	M 700

Write as ratios in their simplest forms:

a) the ratio of the amount spent on grocery to the total income;

b) the ratio of the money spent on transport to rent;

c) the ratio of money spent on savings to other expenses;

d) the ratio of money spent on school fees to rent.

Compare your answers to those given at the end of the sub-unit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Model Answers

Activity 13.1

1. 20 : 30 *(divide by 10)*

   2 : 3 or $\frac{2}{3}$

2.
   a. 2400 : 8000 *(divide by 800)*

      3 : 10 or $\frac{3}{10}$

   b. 800 : 1500 *(divide by100)*

      8: 15 or $\frac{8}{15}$

   c. 400 : 700 *(divide by 100)*

      4 : 7 or $\frac{4}{7}$

   d. 1200 : 1500 (divide by 300)

      4 : 5 or $\frac{4}{5}$

Dividing a quantity in a given ratio

You have just done some work on writing ratios and simplifying them where possible. These two skills will enable you to use a ratio to divide a given quantity in a ratio.

Consider the following case.

Example 1

A basket contains brown eggs and white eggs in a ratio of 3 : 4. If there are 42 eggs, find the number of eggs of each colour.

Solution

The ratio $3:4$ has already been reduced to its simplest form. In order to find out the total number of 'like measures' there are in this example, we add the two terms of the ratio together, as follows:

$$3+4=7$$

Therefore the ratio of brown eggs to the total is $\frac{3}{7}$.

To find the number of brown eggs in the basket, we multiply the ratio by the total number of eggs:

$$=\frac{3}{7}\times 42$$

= 18 brown eggs

The ratio of white eggs to the total is $\frac{4}{7}$.

Using the same process as we used above, the number of white eggs

$$=\frac{4}{7}\times 42$$

= 24 white eggs

We can check our answers by adding them together to see whether they equal the total number of eggs in the basket.

Tip

$$18+24=42$$

Alternative way of working

There is another way of solving this problem, which you might find easier to understand. Run through the following steps for the example above and see whether if makes more sense to do it this way.

STEP 1 Simplify the ratio, if this is possible. (This may not always be required, but calculations are much easier to perform when the ratio is in its simple form.)

3 : 4 is already in simple form

STEP 2 Add the numbers in the ratio together to get total number of shares or units of 'like measurement':

3 + 4 = 7

STEP 3 Divide the total number of items (in these case, eggs) by the total number of shares (from Step 2 above) to get the amount for a single share.

$\frac{42}{7} = 6$ (this means one share is equal to 6)

STEP 4 Multiply each of the numbers in the ratio by the answer in Step 3

3 x 6 = 18

The number of brown eggs is 18.

4 x 6 = 24

The number of white eggs is 24.

Example 2

The ratio between the sizes of the three angles of a triangle is $3 : 2 : 1$

Calculate the size of each angle of the triangle.

Solution

Let's start with the alternative method for solving problems such as this. In order to find the number of degrees represented by each of the terms or numbers in the ratio:

STEP 1 Write ratio in simplest form, if possible. Here it is already in simplest form:

3 : 2: 1

STEP 2 Add the numbers in the ratio together. This gives $3 + 2 + 1 = 6$. This is the total number of shares or units of 'like measurement' in the ratio.

STEP 3 Now, we need the total number of degrees you get when you add up the internal angles of a triangle. Can you remember how many degrees that is?

I hope you remembered that the sum of the angles in a triangle is 180°. Now, you need to divide the 180° by 6 to get the number of degrees in one share.

$$\frac{180^\circ}{6} = 30^\circ$$

STEP 4 Multiply the amount for each share by each of the terms or numbers in the ratio to get the number of degrees for each angle.

The number of degrees for 3 shares is:

$30^\circ \times 3 = 90^\circ$

The number of degrees in the angle with 2 shares is:

$30^\circ \times 2 = 60^\circ$

The number of degrees for the angle with a single share is:

$30^\circ \times 1 = 30^\circ$

Check our solution by adding the answers for the three angels:

$90^{\circ} + 60^{\circ} + 30^{\circ} = 180^{\circ}$

Look at the following set of equations which show another way of working out the number of degrees using the same ratio as in the example above. Why do they give the same result?

Angle A $= \frac{3}{6} \times 180^{\circ} = 90^{\circ}$

Angle B $= \frac{2}{6} \times 180^{\circ} = 60^{\circ}$

Angle C $= \frac{1}{6} \times 180^{\circ} = 30^{\circ}$

Example 3

The ratio of a man's mass to that of his wife is $7 : 5$

What is the man's mass if his wife has a mass of 75kg?

Solution

The mass of the man is calculated as follows:

STEP 1 Write the ratio in its simplest form, if possible. (Here it is already in simplest form.)

7 : 5

STEP 2 There is no need to find the total of the terms or numbers in the ratio as we can get the amount for each share by using the wife's mass and the number in the ratio for the wife's mass. (step 3)

STEP 3 Divide the $75\ kg$ by 5 to get the amount representing one share.

$\frac{75\ kg}{5} = 15\ kg$

STEP 4 Multiply the amount for each share by the term or number in the ratio representing the mass' mass.

$15 \times 7 = 105\ kg$

The solution can also be worked out as follows. Why does it give same result?

$$\frac{5}{12} \times total\ mass = 75kg$$

Simplify the above equation by making the total mass the subject:

$$total\ mass = 75kg \times \frac{12}{5}$$

$$= 180kg$$

The man's mass = *total mass – wife's mass*

$$= 180kg - 75kg$$

$$= 105kg$$

Note it

A ratio expresses a relationship between two or more like quantities.

When there are only two quantities being compared, the ratio *m* to *n* can be written in the form $m : n$ or $\frac{m}{n}$

When more than two quantities are compared, the ratio should be written as *m : n : o: p* (and so forth).

A ratio is usually expressed in its simplest form.

Activity 13.2

1. Thabiso was born in 1950 and Clarke was born in 1960. In his will, their father said that a sum of M11 700 should be divided between them in the ratio of their ages at the time of his death. If their father died in 2005, how much did each son receive?

2. Divide 81 sweets between three children in the ratio $2 : 3 : 4$

3. The ratio of Tumo's money to Manti's money is $5 : 4$. If Tumo has Pula 45, how much has Manti?

Model Answers

Activity 13.2

1.

STEP 1 Find the ratio between Thabiso's age and Clarke's age at the time of their father's death.

Thabiso'age: Clarke's age

$(2005 - 1950) : (2005 - 1960)$

$55 : 45$

$11 : 9$ (simplest form)

STEP 2 Add the terms or numbers in the ratio together to find the total of shares:

$11 + 9 = 20$

STEP 3 Divide the sum of money by the total to get the size of each share

$\frac{M11\ 700}{20} = M585$

STEP 4 Multiply the size of a single share by the number in the ratio representing each son:

Thabiso's amount $= M585 \times 11 = M6435$

Clarke's amount $= M585 \times 9 = M5265$

Have a look at the following working. Why does it give the same result?

Thabiso received $\frac{55}{100} \times M11700 = M6435$

Clarke received $\frac{45}{100} \times M11700 = M5265$

2.

STEP 1 Write the ratio in its simplest form, if possible:

2:3:4 (already in simplest form)

STEP 2 Add the terms or numbers in the ratio to get the total number of shares:

$2 + 3 + 4 = 9$

STEP 3 Divide the total number of sweets by the total of ratio numbers to get the number in one share:

$\frac{81}{9} = 9$

STEP 4 Multiply each of the terms or numbers in the ratio to get the share for each child.

For the child with 2 shares:

$2 \times 9 = 18\ sweets$

For the child with 3 shares:

$3 \times 9 = 27\ sweets$

For the child with 4 shares:

$4 \times 9 = 36\ sweets$

Check you solution by adding up all the answers:

18 + 27 +36 = 81 sweets

Once more, can you suggest why the following working gives the same result?

$81\ sweets \times \frac{2}{9} = 18\ sweets$

$81\ sweets \times \frac{3}{9} = 27 sweets$

$81\ sweets \times \frac{4}{9} = 36\ sweets$

3. Remember the ratio of mass of a man to mass of his wife. This is a similar problem where you are already given the fact that Tumi has 5 shares and are asked to find Manti's shares.

STEP 1 Write the ratio in simplest form, if possible:

Tumi's money: Manti's money

$5:4$ (already in its simplest form)

STEP 2 There is no need to find total of the terms or numbers in the ration. You can get it from Tumi's money. (Step 3)

STEP 3 To find the amount for one share, use the amount of money Tumi has in his five shares:

$\frac{P45}{5} = P9$

STEP 4 To find how much money Manti has, multiply the amount for one share by the number representing Manti's money in the ratio:

$P9 \times 4 = P36$

Compare the answers below with the ones found above. Can you explain why you get the same result even though you went about it a different way?

Tumo's money is calculated as follows:

$$\frac{5}{9} \times total\ money = P\ 45$$

Therefore:

$$total\ money = P\ 45 \times \frac{9}{5}$$

$$= P\ 81$$

Manti has $P\ 81 - P\ 45 = P\ 36$

How would you go about checking your solution to this problem?

Key Points to Remember

The key points to remember in this subunit on ratios are:

- Simplify the ratio, if possible, by dividing the terms or numbers both about and below the line by the same factor.
- Add the numbers in the ratio together to get the total number of shares.
- Divide the given quantity by the total number of shares in the ratio. (Or write each of the terms/numbers in the ratio as a fraction of the total).
- Multiply the answer from 3 above by each term/number in the ratio. (Or multiply the given quantity by the fraction from 3 above for all the terms/numbers in the ratio).

You have now completed the subunit on ratios. Do a quick review of the content of this subunit and then continue on to the next subunit of Rate.

Lesson 2 Rate

Ratios compare two or more like quantities. In other words, the quantities must be expressed in the same units in order to forma ratio. But, what can we use when we need to relate two things which have different units.

Recall the example of travelling by public transportation that I raised at the beginning of this unit. The bus or taxi fare you have to pay is related to the distance you travel. There is no question that distance and money are measured in different units. For this kind of relationship, a special word is used; that word is **rate**. Can you think of other examples of rates used when talking about transport, travel and vehicles?

How much do you have to pay for transporting goods in a hired van? How much do you have to pay for a litre of petrol or diesel? How many kilometres does your vehicle travel in one hour?

The answers to all these questions must be stated in terms of two quantities with different units of measurement, either Maloti per kilometre, Maloti per litre or kilometres per hour. These are all examples of the mathematical expression called rate.

At the end of this subunit you should be able to:

- *define* the term 'rate.'
- *change* the units of measurement to equivalent units.
- *calculate* average speed.

There are 3 pages in this subunit.

Meaning of the term 'rate'

We talk about ratio when the comparison or relationship is between two or more quantities which are alike. However, there are cases where we do compare two or more quantities which are completely different, as in the comparison between the quantity of paraffin you get and the price you pay for it. If you buy two litres, you get two times the amount of paraffin and you also pay twice as much money.

In the introduction to this subunit, I introduced the term **rate** to express a relationship between two or more unlike quantities or things measured in different units.

There are some similarities between ratios and rates. First, rates can be written in a form very similar to what we have been using to write ratios. For example, if a car travels sixty kilometres in one hour, this can be written as:

$$60\ km\ per\ hour = \frac{60\ km}{1\ hour}$$

Notice that $\frac{60\ km}{1\ hour}$ is in the form of a fraction, but that the units above and below the line are different. This form is useful during calculations when you have to work out the units for the calculated answers. Let us move to equivalent rates.

When we talk about equivalent rates, using the same example above, you can write

$$\frac{60\ km}{1\ hour} = \frac{120\ km}{2\ hours} = \frac{180\ km}{3\ hours}$$

All of the above expressions are equivalent – you can replace one of them with any of the others without changing the units of measurement. All of them are expressed in kilometres per hour.

Changing measurement units to equivalent units

Another similarity between ratios and rates is the ability to change the units of measurement by using conversation factors. What does changing measurement units mean? Let's return to the example above.

If you want to express the above speed in kilometres per minute, you can substitute 60 minutes for 1 hour. The rate will then look like this:

$$\frac{60\,kilometres}{60\,\min utes}$$

You then need to divide both the top and the bottom of this fraction by a factor of 60 to reduce it to its simplest form:

$\frac{1\,kilometre}{1\,\min ute}$, or one kilometre per minute.

Suppose, you want to change the distance units from kilometres to metres and the time units from hours to seconds. How do you do this?

Remember, how you convert kilometres to metres and hours to seconds? The same knowledge is used here.

1 km = 1000 m 1 hour = 3600 seconds

$$\frac{60\ km}{1\ hour} = \frac{60 \times 1\ km}{1 \times 1\ hour} = \frac{60 \times 1000\ m}{1 \times 3600\ s} = \frac{60\ 000\ m}{3\ 600\ s}$$

$$= \frac{600\ m}{36\ s} = \frac{16\frac{24}{36}\ m}{1\ s} = \frac{16\frac{2}{3}\ m}{1\ s} = 16\frac{2}{3}\ m/s$$

So, 60 kilometres per hour is the same as one kilometre per minute or 16⅔ metres per second. The rate remains the same, only the units of measurement change.

Activity

Activity 13.3

This activity serves as a reminder to what you have learned in your Mathematics course at junior secondary level. Therefore, in this activity you are going to find average speed, distance travelled, and time taken to complete a journey.

1. Find the speed of a car that travels 45 km in 30 minutes.

2. Find the distance in km travelled by an aeroplane moving at 80 m/s for 15minutes

3. How long will it take a cyclist to make 10 laps of a 400 m cycle track cycling at an average speed of 40 km/h?

I hope you got all of them correct, then continue and compare with the answers given below. If not, you should go through the ones you missed carefully and then compare with the given solutions.

Model Answers

Activity 13.3

1. You should write 30 minutes as $\frac{1}{2}$ hour

$\textit{The average speed} = 45km \div 30\,min$

Change 30 minutes into hours:

$= 45\text{km} \div \frac{1}{2}\text{h}$

Change from division to multiplication by inverting the second number:

$= 45\text{km} \times \frac{2}{h}$

$= 90\ km/h$

2. 15 minutes = $15 \times 60\ s = 900\ seconds$

$The\ distance\ travelled = 80\ m/s \times 900\ s$

$= 72\ 000\ m$

Converting 72 km to metres

1 km = 1000 m

$72\ km = 72 \times 1\ km = 72 \times 1000m = 72000m$

3. Total distance cycled = $10 \times 400\ m$

$= 4000\ m$

$Time\ taken = \frac{4000\ m}{40\ m/s}$

$= 100\ s$

You have now completed the subunit on Rate, do a quick review of the content of this subunit on Rate and then continue on to the next subunit on decrease and increase a quantity by a given ratio.

Decrease and increase a quantity by a given ratio

In this subunit we will look at how to decrease and increase a quantity by a given ratio. You have already been introduced to this concept in Unit 8, when learning about discounts in commercial mathematics. When a shop-owner decides to have a sale, prices are usually reduced by the same percentage for all goods in the shop. This idea is not restricted to shop-keepers but can be used in a variety of situations.

At the end of this subunit you should be able to:

- *increase and decrease* a quantity by a given ratio.

There are 8 pages in this subunit.

A quantity can be increased or decreased in a given ratio by being multiplied by a suitable fraction. For example,

if a price increases in the ratio 5: 3, then it is multiplied by the fraction $\frac{5}{3}$ and if a price is decreased in the ratio 4: 9, then it is multiplied by $\frac{4}{9}$.

When Christmas approaches, shopkeepers often charge the full price because demand among consumers is very high. After the Christmas period, shopkeepers want to get rid of any stock they didn't sell. The sale starts. Put items tend to increase their prices from the sale price to the full non-sale price.

Can you see any comparisons being made in this situation?

One can compare the full price before the sale and the price after the sale period.

Typically, prices drop during a sale as shopkeepers try to attract additional customers. This means that, compared to the normal price, the price during the sale is lower.

Figure 13.1 : Some of the common advertisements.

I'm sure you have come across advertisements (like those in figure 13.1) in shop windows and on pamphlets. Did you realise that there is a connection between these advertisements and the type of mathematics you are studying in this course?

The M500 above is reduced to M200.

The price is reduced in the ratio $\frac{200}{500} = \frac{2}{5}$ which we normally write as 2:5

M500.00 x $\frac{2}{5}$ = M200.00 (decreasing a quantity by a ratio)

Note that the seller does not put all of the information in the advertisement, so that it will not resemble school mathematics. As the buyer, you have to fill in the missing information to realise the mathematics involved. But if you know how to do the mathematics for yourself, you can make informed choices about whether to buy or not.

Now let us look at the following examples to emphasise the ideas of decreasing or increasing a quantity by a given ratio.

Example 4

Increase R450 in the ratio 7 : 3

Solution

$R450 \times \frac{7}{3} = R1050$

R450 is increased to R1 050, it is not increased by R 1050.

Example 5

Decrease 70 kg in the ratio 2: 5

Solution

$70\ kg \times \frac{2}{5} = \frac{140}{5} kg$

$= 25 kg$

70 kg is decreased to 25 kg; it is not decreased by 25kg.

Activity

Activity 13.4

1. Increase M2.10 in the ratio 9:7.

2. Increase 4.50 km in the ratio 7:5.

3. Decrease 750 ml in the ratio 10:25.

4. Decrease 96 apples in the ratio 5:6.

Model Answer

Activity 13.4

1. $M2.10 \times \frac{9}{7} = M2.70$

2. $4.50\ km \times \frac{7}{5} = 6.3\ km$

3. $750\ ml \times \frac{10}{25} = 300\ ml$

4. $96\ apples \times \frac{5}{6} = 80\ apples$

Note that the units in each question are the same. We use the idea of ratio only when sets or quantities of the same type are compared.

You have now completed the subunit on how to decrease and increase a quantity by a given ratio, do a quick review of the content of this subunit and then continue on to the next one.

Lesson 3 Proportion

When studying mathematics at junior secondary level, you learned about direct and inverse proportion.

Mr and Mrs Williams have 3 children. They have only one in school this year. The annual fees are M3 400. The second child is due for school next year. How much will they pay for the two children next year, assuming there will be no fees increase?

____

____

Mr and Mrs Williams will pay M 3 400 + M3 400 = M6 800.

The number of children in school next year will have increased. It seems even the money to be paid has increased.

The number of children has doubled, so has the fees. This is an example of direct proportion. In direct proportion, increasing one quantity increases another.

What happens in indirect or inverse proportion?

____

____

In indirect proportion, increasing one quantity decreases another.

At the end of this sub-unit you should be able to:

- *formulate* equations from direct and inverse proportion situations, and use the equations to solve for one quantity given others.

There are 13 pages on this subunit on variation and proportion.

Direct proportion

Proportion is a way of relating quantities.

Two items, seemingly unrelated, will be said to be directly proportional if increasing one increases another, or if decreasing one also decreases another.

Example 1

A bar of soap costs M7.95. How much will Tefo pay for 4 bars?

Solution

The more bars of soap are bought, the more money Tefo will have to pay.

Number of bars of soap	Cost
1	M7.95
2	M15.90
3	M23.85
4	M31.80

The money to be paid is said to be directly proportional to the bars of soap to be bought. This is normally written, Y α X, where in this case Y is the money to be paid, and X is the bars of soap.

We normally say

1 bar = M 7.95

2 bars = M15.90

3 bars = M23.85

4 bars = M31.80

To get the amounts on the right side of the equation, each of the quantities on the left has to be multiplied by M7.95

We can say:

1 bar x M7.95 = M 7.95

2 bars x M7.95 = M15.90

3 bars x M7.95 = M23.85

4 bars x M7.95 = M31.80

The M7.95 is a constant of proportionality. This is the constant of one bar.

This leads to the equation of direct proportion which is Y = kX, where k is the constant of proportionality.

Example 2

Bus fare for 15 children going on a school trip is M900. Mpho has two children on that trip. How much is she going to pay for them?

Solution

This is a case of direct proportion.

M900 = k15

$$k = \frac{900}{15}$$

k = 60

Each child is paying M60

Mpho will pay M60 × 2 children = M120

Example 3

Tebello sells milk for his uncle. The milk is sold in litres. Each litre of milk costs M5.00. Tebello is paid 8 lisente for each litre sold.

The table below shows the number of litres of milk that Tebello sold from Monday to Friday.

Number of litres of milk(x)	5	10	20	30	8
Tebello's income in lisente(y)	40	80	160	240	64

a) Find the ratio of Tebello's income to the number of litres sold.

b) If the number of litres of milk sold is 15, find Tebello's income.

c) If Tebello's income is 200 lisente, what is the number of litres of milk sold?

d) How is the value of y (Tebello's income) related to x (the number of litres of milk)?

Solution

a) 8 lisente: 1 litre

8:1

b) In direct proportion, the quantities increase or decrease by the same factor

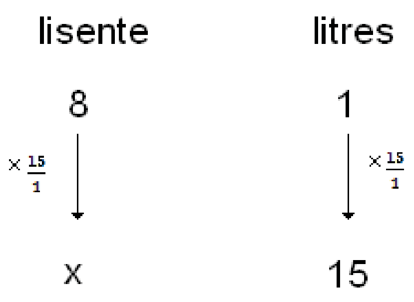
8 : 1

X: 15

1 has increased by the factor 15. 8 will have to change by the factor 15.

$8 \times 15 = 120$

Tebello's income is 120 lisente

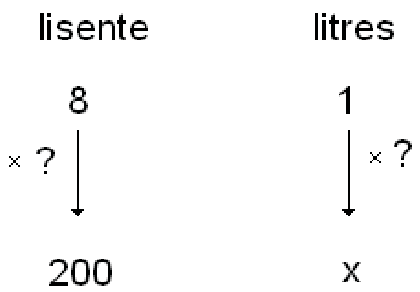


From the above ratio because 1x 15 = 15, then 8 x 15 = x.

Therefore, x = 120.

Tebello's income is 120 lisente

c)



From the information given you can write (to find the multiplying factor):

8 x ? = 200,

$? = \frac{200}{8}$

? = 25

Therefore, to get the number of litres sold you use the same multiplying factor found above. Thus,

1 x 25 = x

That is, x = 25

The number of litres of milk sold is 25.

d) $y = 8x$

If you were able to get at least three of them correct you can go through the next activity.

Note it

In the above activity we say the income is **directly proportional** to the number of litres of milk sold, or we can say the income **varies directly** as the number of litres of milk sold. Therefore, the ratio $\frac{y}{x} = 8$, which is a constant and represented by letter k.

The symbol $\propto$ is used to represent variation.

y **varies directly** as *x* is written as $y \propto x$.

$y \propto x$ means that $y = kx$, where *k* is constant and $k \neq 0$.

Therefore the above activity is an example of **direct proportion** or **direct variation.**

If the number of litres of milk sold increases, the income also increases.

If the number of litres of milk sold decreases, the income also decreases.

Inverse or Indirect Proportion

In indirect proportion, as one quantity increases the other decreases.

Example 1

10 men can do a certain piece of work in 4 days. Two fall sick. How long will it take the 8 men to complete the work, assuming they work at the same rate?

It goes without saying that the 8 men will take more than 4 days.

Fewer men will do the work in more than 4 days.

This is a case of indirect proportion.

Two quantities Y and X, are in inverse proportion if a factor k, that changes Y, changes X by the reciprocal of k, which is $\frac{1}{k}$.

We can say 4 days = 10 men

X days = 8 men

This is the question we ask ourselves. What changed 10 to 8?

The number of men is decreased by $\frac{8}{10}$

$10 \times \frac{8}{10}$

$= 8$ men

This is a case of indirect proportion. So the change to the number of days should be brought by the inverse of $\frac{8}{10}$ which is $\frac{10}{8}$.

So the number of days is increased in the ratio $\frac{10}{8}$

So time taken by 8 men $= 4 \times \frac{10}{8}$

$= 5$ days

Number of men	Days taken to do the work
10	4
8	5

The number of men, Y, is indirectly proportional to the number of days, X, taken to do the work. This is normally written, $Y \alpha \frac{1}{X}$.

This leads to the equation of indirect proportion which is $Y = k\frac{1}{X}$.

$$Y = \frac{k}{X}$$

Example 2

The given table shows the number of days that can be taken in a boarding school to feed pupils with the remaining food.

Number of pupils(x)	5	20	25	60
Number of days(y)	120	30	24	10

a) Multiply each pair of numbers in the table above. What do you realise?

b) Find the number of pupils that the school would feed in 40 days.

c) Find the number of days the school will take to feed 100 pupils.

d) How long would the food last to feed 50 pupils?

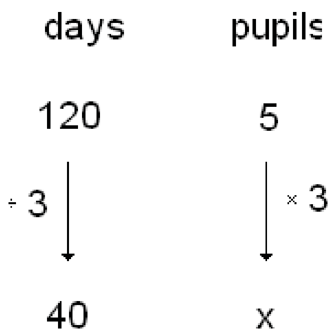
e) How many pupils would the school be able to feed for 85 days?

Solution

a) $5 \times 120 = 600, 20 \times 30 = 600, 25 \times 24 = 600, 60 \times 10 = 600$

When multiplying each pair the result is always 600.

b)



From the provided information you can write:

120 x ? = 40 (to find the ratio)

$? = \frac{40}{120}$ $(= \frac{1}{3})$ (this is the ratio).

Then, to find the number of pupils to be fed in 40 days we use the inverse of the ratio (i.e. divide by $\frac{40}{120}$ or $\frac{1}{3}$ or multiply by $\frac{120}{40}$ or 3). Therefore, you get

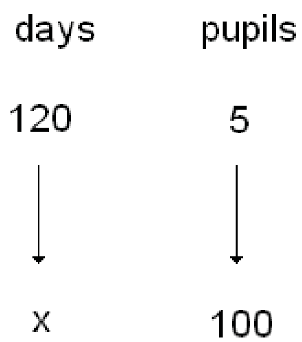
$5 \div \frac{1}{3} = x$ (or alternatively $5 \times 3 = x$)

$5 \div \frac{1}{3} = 15$ (or alternatively $5 \times 3 = 15$)

X = 15

Hence, the number of pupils that can be fed in 40 days is 15.

c)



From the information given you can write:

5 × ? = 100   (to find the multiplying ratio)

$? = \frac{100}{5}$ (this is the ratio (also = 20))

Then, to find the number of days 100 pupils will be fed you use (multiply by) the inverse ratio or divide by the same ratio.

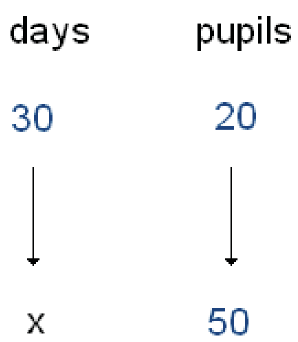
Since the food will last 5 pupils for 120 days, you must decrease the number of days in the ratio 5:100.

$y = 120 \times \frac{5}{100}$

$= 6 \text{ days}$

The school would be able to feed 100 pupils for 6 days.

d)



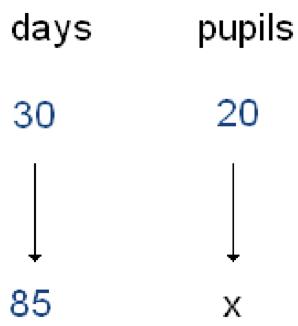
Since the food will last 20 pupils for 30 days, you must decrease the number of days in the ratio 20:50.

$y = 30 \times \frac{20}{50}$

$= 12 \text{ days}$

The school would be able to feed 50 pupils for 12days.

e)



Since the food will last 20 pupils for 30 days, you must decrease the number of pupils in the ratio 30:85.

$X = 20 \times \frac{30}{85}$

$= 7.05 \approx 7$ (no pupils fractions)

Therefore, the school will be able to feed 7 pupils for 85 days.

Note that yx = 600, the number 600 is a constant.

We can also say $y = \frac{600}{x}$

Or say ,

$$y = 600\ \frac{1}{x}$$

This is **inverse variation** or **indirect proportion**. In this example the number of pupils is **inversely proportional** to the number of days.

In conclusion, note that we say *y* is **inversely proportional** to *x*. This can be written as $y \propto \frac{1}{x}$.

Therefore, $y \propto \frac{1}{x}$ **means** $y = \frac{k}{x}$ or $yx = k$, the number *k* is a constant.

Where $k \neq 0$

Activity

Activity 13.11

Answer all the questions.

1. Write an equation for each of the following statements:

a) For a given distance, the time, t, taken on a journey is inversely proportional to *v*, the speed.

b) The weight of an object *w*, varies inversely as the square of *r*, its distance from the centre of the earth.

2. (a) y is inversely proportional to x, and $y = 6$ when $x = 3$. Find the value of y when $x = 4$.

b) y varies as $\frac{1}{x^2}$, and when $x = 4, y = \frac{3}{4}$. Find the value of y when $x = 1$.

3. A car travelling at 80 km/h takes 6 hour to complete a journey.

a) How long would it take a car travelling at 120 km/h?

b) At what speed would a car finish a journey in 12 hours?

4. A quantity of water will last 20 chickens for 4 days. How long would it last for

a) 40 chickens? b) 80 chickens?

Model Answers

Activity 13.11

1.a) $t \propto \frac{1}{u}$, the equation is $t = k\frac{1}{u}$ or $t = \frac{k}{u}$

b) $w \propto \frac{1}{d^2}$, the equation is $w = \frac{k}{d^2}$

2.a) $y \propto \frac{1}{x}$, the equation is $y = \frac{k}{x}$ or $yx = k$

Substituting the values of x and y

$6 \times 3 = k$

$18 = k$ or $k = 18$

Now $y = \frac{18}{x}$

Substituting the value of x

$y = \frac{18}{4}$

$= 4\frac{1}{2}$

b) $y \propto \frac{1}{x^2}$, the equation is $y = \frac{k}{x^2}$ or $yx^2 = k$

Substituting the values of x and y

$\frac{3}{4} \times 4^2 = k$

$k = 12$

Now $y = \frac{12}{x^2}$

Substituting the value of x

$y = \frac{12}{1}$

$y = 12$

3.a) $k = 80\frac{km}{h} \times 6\ hours$

$= 480\,km$

$s \propto \frac{1}{t}$ therefore, $s = \frac{k}{t}$

Substituting the values of k and s

$120 = \frac{480}{t}$

$t = \frac{480}{120}$

$= 4\ hours$

The journey would take 4 hours.

b) $s \propto \frac{1}{t}$ therefore, $s = \frac{k}{t}$

Substituting the values of k and t

$s = \frac{480}{12}$

$= 40 km/h$

A car would take 40 km/h to finish the journey.

4.a) $k = 20 \times 4 = 80$

Number of chickens (c) varies indirectly as number of days (d).

$c \propto \frac{1}{d}\ or\ c = \frac{k}{d}$

Substituting the values of k and c

$40 = \frac{80}{d}$

$d = 2$

The water would last for 2 days.

b) $k = 20 \times 4 = 80$

Number of chickens varies indirectly as number of days.

$c \propto \frac{1}{d} \; or \; c = \frac{k}{d}$

Substituting the values of k and c

$80 = \frac{80}{d}$

$d = 1$

The water would last for 1 day.

You have now completed work on this unit on Ratio, Rate, and Proportion. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Unit Summary

Summary

In this unit you learned that:

1. A ratio is a fraction that expresses a relationship between two like quantities.
2. The ratio m to n can be written in the form $m : n$ or $\frac{m}{n}$
3. A ratio between two quantities is usually expressed in its simplest form, like a fraction.
4. Ratios can also be used to compare more than two like quantities. In these cases, the ratio is normally written in the form $m : n : o : p$.
5. When a quantity is increased in a given ratio, it becomes more than the original quantity.
6. When a quantity is decreased by a given ratio, it becomes less than the original quantity.
7. The symbol $\propto$ is used to represent proportion.
8. y **varies directly** as x is written as $y \propto x$

$y \propto x$ means that $y = kx$, where k is constant and $k \neq 0$

9. y is **inversely proportional** to x. This can be written as $y \propto \frac{1}{x}$

Therefore, $y \propto \frac{1}{x}$ **means** $y = \frac{k}{x}$ or $yx = k$, the number k is a constant.

Where $k \neq 0$.

You have completed the material for this unit on Ratio, Proportion, and Rate. You should spend some time reviewing the content in detail.

Once you are confident that you can successfully write an examination on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings.

Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit, which looks at different properties of the Circle.

Assignment

Assignment

1. **Answer all questions.**
2. **Show all of the work you do to arrive at an answer.**

Total marks = 30 **Time**: 60 mins

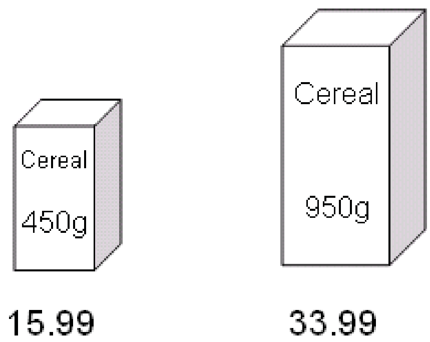
.

The marks allotted to each question or part of a question is shown in (parentheses).

1. (a) Write $\frac{30}{72}$ in its simplest form. (2)

(b) Write 25:40:15 in its simplest form. (2)

2. Breakfast cereal is available in two different sizes from our shops.



(a) Work out the cost of 100g of Cereal when bought in the 450g box (to the nearest cent). (2)

(b) Work out the cost of 450g when bought in the 950g box (to the nearest cent). (2)

(c) Which is the better buy? Give one reason in support of each size. (2)

3. Three men can build a wall in 10 hours. How many men will be needed to build the wall in 6 hours? (3)

4. Two boys Thabo and Lebo share the cost of a meal in the ratio 3:4. Lebo pays M37.40. What is the total cost of the meal? (3)

5. Given that the quantity y is inversely proportional to another quantity x, and that $y = 10$ when $x = 20$; find y when $x = 0.01$
(4)

6. Two brothers Khoali and Khaka bought shares costing M140 000.00 in the stock market. Khoali paid M60 000.00 and Khaka paid M80 000.00. They decided to sell the shares for M168 000.00. How much of M168 000.00 will Khaka get? (4)

7. The cost of making a telephone call is proportional to the time of call. If the telephone company charges 63 lisente for a 7 minutes call, how much will a 3 minutes call be charged? (6)

Model Answers to Assignment

Question 1

(a) $\frac{30}{72} = \frac{5}{12}$

(b) 25 : 40: 15

5 : 8: 3

Question 2

$\frac{100 \times 15.99}{450} = 3.55$

100g costs 3.55

$\frac{450g \times 33.99}{950g} = \quad 16.10$

450g costs 16.10

Buying 450g for 15.99 is a better deal

Question 3

$\frac{3 \times 10}{6} = 5$ men

Question 4

Thabo pays 3 parts and Lebo pays 4 parts of the peal.
4 parts of the meal cost M37.40.
So, 1 part costs M37.40 ÷ 4 = M9.35
Therefore, total cost of the meal (3+4 =7) = M9.37 × 7 = M65.45

Question 5

$y = \frac{k}{x}$

$$10 = \frac{k}{20}$$

k = 200

When x = 0.01; $y = \frac{200}{0.01}$

$y = 20\,000$

Question 6

Ratio of shares	Khoali : Khaka M60 000.00 : M80 000.00 3 : 4
Ratio of profit	Khoali : Khaka 3:4
Total profit	3 + 4 = 7
Khaka share of profit	$\frac{4}{7} x M168\,000.00 = M96\,000.00$

Question 7

Let Cost of call be C

Let Time of call be t

So, $C \propto t$

That is, $C = Kt$

To find the value of K

C = Kt

63 = K (7)

$\frac{63}{7} = K$

$9 = K$

This means C = 9t

Charge of 3 minutes call

$C = 9(3)$

$C = 27$

3 minutes call is charged 27 lisente

Assessment

Assessment

1. **Answer all questions.**
2. **Show all of the work you do to arrive at an answer.**

Total marks = 24 **Time**: 80 mins

The marks allotted to each question or part of a question are shown in (parentheses).

1. A bag contains a number of balls, some are blue and some are black. Three quarters of the balls are blue. What is the ratio of the blue to black balls? (2)

2. (a) Write each ratio in its simplest form

   (i) $\frac{4}{9} : \frac{5}{4} : \frac{3}{5}$

   (ii) 3 : 0.24 : 0.6

   (4)

   (b) Find the value of x and y in the following ratios:

   (i) 6 : 8 = 15: x

   (ii) $\frac{2}{3}$: 1 = y:4

   (4)

(c) If 12 men can plough a field in 8 days, how many men can do the job in 2 days if they work at the same rate?

(2)

3. (a) 100 g of spaghetti contains 3.6 g of fibre.
Express mass of fibre : mass of spaghetti as a ratio of two integers in its simplest form.
(2)

4. Lineo bought 5 maize cobs for M5.00 each. At the same price, how would 10 maize cobs costs? (2)

5. Calculate
   a. A packet of apples is sold for M12.99 and there were 9 apples in the packet. What is the cost per apple? (2)

   b. If 1.3 Kg of mince meat costs M42.84. What is the price per Kilogram? (2)

c. If you used 22 Kilolitre (Kl) of water in June and the cost was M144.98. What is the price of water per kilolitre?
(2)

d. If petrol costs M8.24 per litre, how much would it cost to fill a 50 litre tank?

(2)

Answers

1. Three quarters of balls are blue. So one quarter of balls are black

$\frac{3}{4} : \frac{1}{4}$

Multiply by 4 to simplify the ratio to get

$3 : 1$

2.

(a)

(i) $180 \times \frac{4}{9} \; : \; \frac{5}{4} \times 180 \; : \; \frac{3}{5} \times 180$

80: 225 : 108

(ii)

$3 \times 100 : 0.24 \times 100 : 0.6 \times 100$

300: 24: 60
25 : 2 : 5

(b)

(i) 6 : 8 = 15: x

$$\frac{6}{8} = \frac{15}{x}$$

$$x = \frac{15 \times 8}{6}$$

$= 20$

(ii) $\frac{2}{3} : 1 = y:4$

$$\frac{\frac{2}{3}}{1} = \frac{y}{4}$$

$$y = \frac{\frac{2}{3} \times 4}{1}$$

$$y = \frac{8}{3}$$

(c) If 12 men can plough a field in 8 days, how many men can do the job in 2 days if they work at the same rate?

$$x = \frac{12 \times 8}{2}$$

$= 48$ men

3.

3.6 g : 100g
3.6 : 100
1 : 27.78
1 : 27

4. 5 maize cobs – M5.00
10 maize cobs - ?

$? = \frac{10}{5}$ x M5.00
$=$ M10.00

5. (a) 9 apples – M12.99
1 apple - ?

$? = \frac{1}{9}$ x M12.99
$=$ M 1.44

(b) 1.3 Kg of mince meat – M42.84
1 Kg of mince meat - ?

$? = \frac{1}{1.3} x M42.84$
$= M32.95$

(c) 22 Kilolitre of water – M144.98
1 Kilolitre - ?

$? = \frac{1}{22} x\, M144.98$
$= M6.59$

(d) 1 litre – M8.24
50 litre - ?

? = 50 x M8.24
= M412.00

Unit Contents

Unit 14

Unit 14

Circle

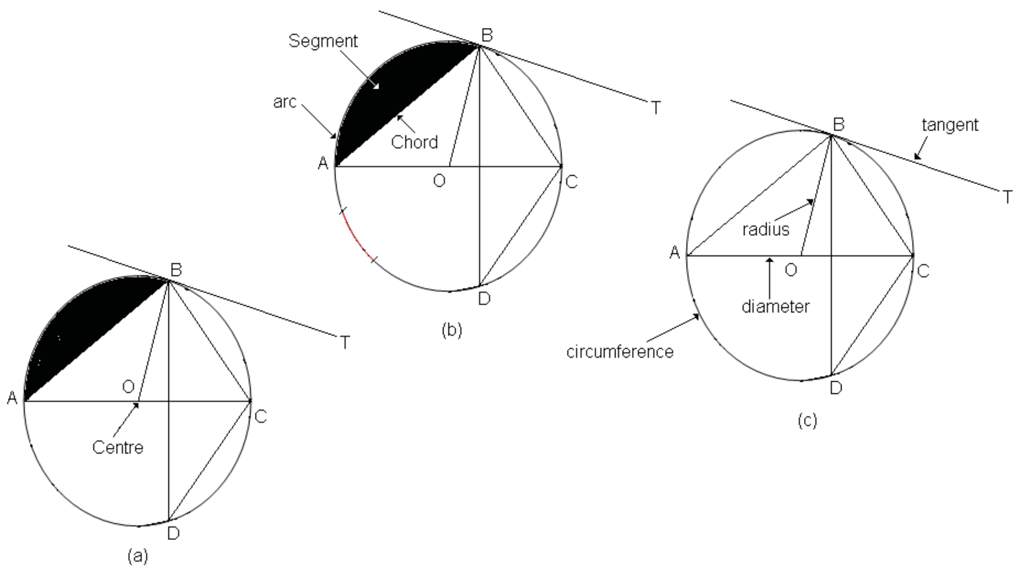
Introduction

In Junior Certificate we learned how to measure and draw all types of lines and angles. We also learned about properties and relationships of two - dimensional shapes. In this unit we are going to learn about angle properties of a circle and other shapes.

Basically, you will compare lines and angles found in circles. The question may be *why circles*? There are a number of reasons which can be put forward. For example, a circle is very important in the construction of all machinery. In vehicles we have wheels, gears, and pistons. Also, the bolt and the nut will allow easy circular motion when the circles between them are nicely round with no dents or small corners. These, and many other designs, remind us everyday the very importance of the circle in real life situations.

The idea of a circle has fascinated man for a very long time, since the beginning of recorded time. It is not surprising to find a number of words which are used when talking about the circle or its parts. Some of the words which you will meet in this unit include: arc, chord, radius, diameter, segment, centre and tangent (Figure 14.1.)

Figure 14.1: Parts of a circle



It is essential that you are able to identify and recognise each one of the parts of a circle.

You are about to have an enjoyable experience. You will soon find yourself pleasantly acquiring knowledge about this fascinating shape; a circle.

This unit consists of 89 pages. It covers approximately 4% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 35 hours to work on this unit. 20 hours for formal study and 15 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Six Lessons:

Lesson 1 Theorems of the Geometry of Circles
Lesson 2 Chords
Lesson 3 Properties of Angles in a Circle
Lesson 4 Cyclic Quadrilaterals
Lesson 5 Properties of Tangents
Lesson 6 Symmetry and Angle Properties of Circles

Upon completion of this unit you will be able to:

Outcomes

- *prove* that a perpendicular line from the centre of a circle to a chord bisects the chord.
- *prove* that equal chords of a circle are equidistant from the centre and that chords equidistant from the centre are equal.
- *prove* that angles in the same segment are equal.
- *prove* that the angle at the centre of a circle is equal to twice the angle at any point on the remaining part of the circumference, if both angles stand on the same arc or chord.
- *prove* that the angle in the semi-circle is a right angle.
- *prove* that the opposite angle of a cyclic quadrilateral are supplementary.
- *prove* that a tangent to a circle is perpendicular to the radius at the point of contact and tangents from a point to a circle are equal.
- *solve* practical problems that involve symmetry and angle properties of circles

Terminology

Arc: is part of the circumference.

Chord: is a line that cuts the circle in to two parts.

Perpendicular line: Is a line that cuts another line at right angles.

Line bisector: Is a line that cuts another line into two equal parts.

Line of symmetry: Is a line that cuts a figure into two equal parts; one part is a reflection of the other

Have you ever looked around your home to see what surrounds you? Look at the containers, utensils ,and toys in your home. Which of them have some kind of a circle?

Object	Does it have any type of circle ?
a cup	
a plate	
a bucket	
a bowl	
a three legged pot	

a frying pan	
a drinking glass	
a fork	
a dust pan	

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Theorems of the Geometry of Circles

A theorem is a general statement that is not self-evident but is proved by reasoning.

Once it has been proved, the theorem is used when solving geometry problems just by referring to it when it applies to the problem.

In this unit, on circles, you will be introduced to a number of proofs of theorems applicable when dealing with the geometry of a circle. In the introduction it was mentioned that man has always been interested in the circle for a number of reasons. Further examples include everyday experiences. For example, during the day man has always looked at the sun for a number of useful purposes such as it being a timer. Although it appears as spherical the sun is in fact a circle. At night there is the moon with a similar circular look. The earth is also circular. With time, there came the philosophers who suggested circular motions of planets. Therefore, the study of a circle and its properties is indeed very important.

The properties of a circle are written as theorems. In Mathematics circle theorems are used to determine the sizes (measures) of angles and lengths (distances) in a circle by doing calculations, rather than actual measuring of the angles or lengths with relevant instruments.

In this unit, first we will look at theorems about chords. We will then look at theorems about angles in a circle. We will then move to properties of tangents. And, finally, we will look at practical problems involving symmentry and angle properties of circles.

Lesson 2 Chords

An interesting point about the train is the construction of its wheel. Have you ever wondered how this is done? The early engineering pioneers may have tried a number of options before they settled for a solution.

The steel cylinder is cut, and then a hole is drilled into the centre of the circular disk.

However, there is still a question; how can the centre of the disk be located? Well, do you still remember the chord of a circle? The Chord Perpendicular Bisector Theorem suggests a method for finding the centre using a property of the circle chords.

Now, what does the theorem actually say?

Chord Perpendicular Bisector Theorem The perpendicular bisector of a chord of a circle passes through the centre of the circle

At the end of this sub-unit you should be able to:

- *prove* that a perpendicular line from the centre of a circle to a chord bisects the chord.
- *prove* that equal chords of a circle are equidistant from the centre and that chords equidistant from the centre are equal.

Note that there can be various ways of putting the theorem statement. For our purposes here, we will use the following statement:

A perpendicular line from the centre of a circle to a chord bisects the chord

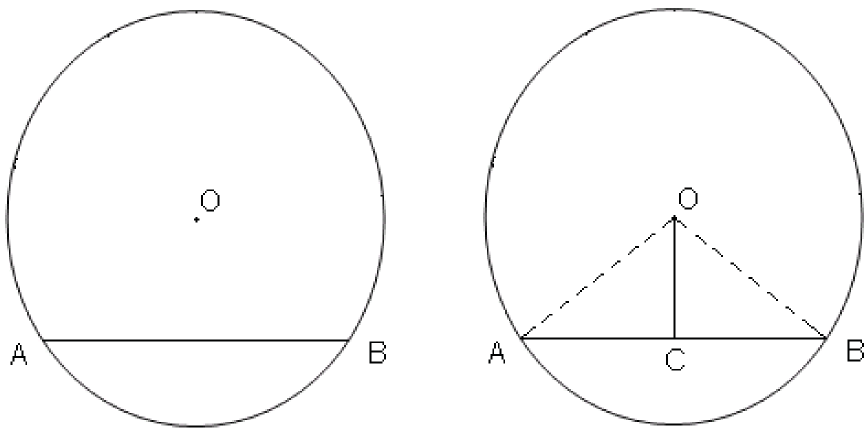
Did you get that? Well do not be afraid, we will go slowly in explaining the theorem statement. This will be done in theorem 1 below.

However, before you use a theorem it has to be proved to show that it is a true statement. Once it has been proved, you will use it as knowledge to work problems where it is applicable. There will be no need to prove it again.

The theorems about the chords of a circle and their proofs are in one place here.

Theorem 1. *A perpendicular line from the centre of a circle to a chord bisects the chord.*

Proof



(a) (b)

In the statement above you are told that you have a chord of a circle and a centre O, see (a.)

You are required to show that (to prove):

> if you can draw a line from the centre O to the chord, making a right angle (90^0) with the chord AB, it will cut the chord AB into two equal halves (bisects the chord.)
>
> So, the **aim is to show that** $AC = BC$ $(\ or\ AC = CB)$

What you will do to build step by step work to show this (what reasoning are you going to use to show that this is true) is:

1. Draw the radii OA and OB, see (b) above. By drawing these radii, you will show that they are equal as all radii of the same circle are. This is written as $OA = OB$ (the reason for this statement is because they are radii of the same circle.)
2. Also, from diagram (b), the two radii and the perpendicular line drawn from centre to chord AB forms two right-angled triangles, $OAC\ and\ OBC$.
3. Now, use the well known Pythagoras theorem on both triangles to get the following statements:
   a. from $\Delta\ OAC$ you have $OA^2 = AC^2 + CO^2$
   b. from $\Delta\ OBC$ you have $OB^2 = BC^2 + CO^2$

If you remember that from statement 1 you have $OA = OB$ then you can write the following statements:

i. $OA = OB$
(from statement 1)

ii. $OA^2 = OB^2$
(follows from statement i)

iii. $AC^2 + CO^2 = BC^2 + CO^2$
(replacing by statement a and b in 3 above)

iv. $AC^2 = BC^2$
(CO^2 is common on both sides)

v. $AC = BC$
(follows from statement iv, if squares are equal then square roots will be equal)

Statement v (above) means line OC bisects chord AB.

Note that, here statement v is the same as what you wanted to show in the aim statement.

The proof written above can be written in a shorter form where a statement and a reason for the statement are given in short. This is shown below. Refer to the diagram to see the radii, the triangles, the common side, and the corresponding sides.

Statement	Reason
1. AO = BO	radii
2. $O\hat{C}A = O\hat{C}B = 90^0$	right-angled triangles

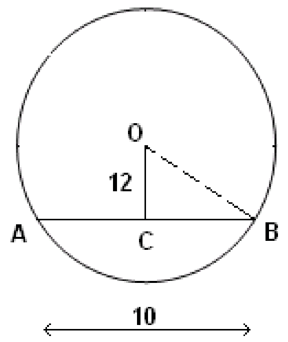
3. $OC = OC$	**common side to both triangles**
4. $CA = CB$	**corresponding sides of congruent** $\triangle$ OAC and $\triangle$ OBC
5. Point C bisects the chord AB	**by statement 4**

When you use a theorem to solve a given problem where it applies, you need not show the proof, but show which applicable theorem is being used. This will be illustrated by examples below.

Example 1

A circle has a chord of 10cm and its perpendicular distance from the centre is 12cm.
What is the radius of the circle?

Solution



If a radius is perpendicular to a chord through the centre, it bisects the chord.
This is what the theorem states. Here it is put in a way to relate to the problem:

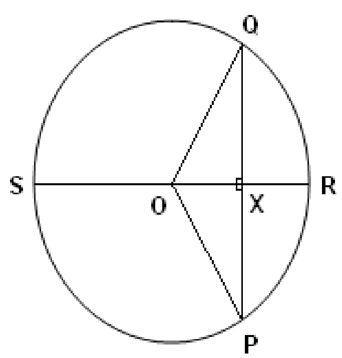
$$AC = \frac{1}{2}AB \qquad \text{(the chord AB is bisected into AC = CB)}$$
$$= \frac{1}{2}(10\ cm)$$
$$= 5\ cm$$

Using the Pythagoras theorem,

$$OB^2 = BC^2 + CO^2$$
$$= (5\ cm)^2 + (12\ cm)^2$$
$$= 25\ cm^2 + 144\ cm^2$$
$$OB = \sqrt{169\ cm^2}$$
$$= 13\ cm$$

The radius of the circle is 13cm.

Example 2



In this figure, PQ is 30cm, and OX = 8cm.

Calculate:

a) the radius of the circle,

b) the length SX,

c) the area of ΔOPQ.

Solution

a) If a line through the centre O is perpendicular to a chord, the line bisects the chord.

RS goes through the centre and at right angle to PQ, so it bisects the chord PQ (this is how the theorem and the problem are related.)

That is,

$$PX = XQ = \frac{1}{2}(PQ)$$

(how the chord is bisected)

$$= \frac{1}{2}(30\ cm)$$

$$= 15\ cm$$

Using Pythagoras theorem in Δ OQX

$$OQ^2 = QX^2 + XO^2$$

$$= (15\ cm)^2 + (8\ cm)^2$$

$$= 225^2 + 64\ cm^2$$

$$OQ = \sqrt{289\ cm^2}$$

$$= 17\ cm$$

The radius of circle is 17cm.

b) Length of $SX = SO + OX$
(SO = OQ radii of same circle)

$$= 17\ cm + 8\ cm$$

$$= 25\ cm$$

c) Area of Δ OPQ $= \frac{1}{2}bh$
(remember area of triangle!)

$$= \frac{1}{2}(30\ cm)(8\ cm)$$

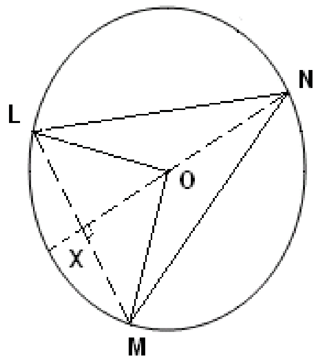
$$= 120\ cm$$

Can you put what you have learned from the examples above in activity 14.1 below? Enjoy your work!

Activity 14.1

Activity

1.

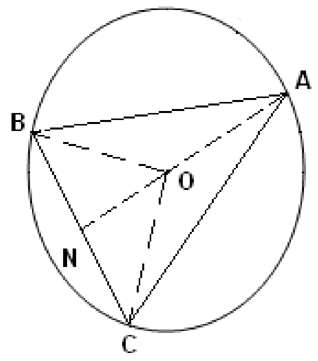


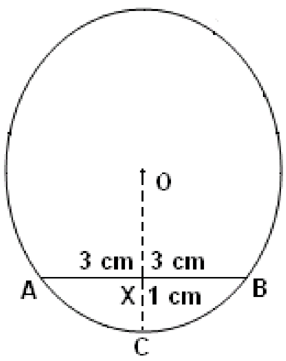
The radius of the circle is $7\frac{1}{2}\ cm$, $OX = 6\ cm$,

Calculate:

a) length of LM
b) length of NX
c) area of ΔLMO
d) area of kite LOMN

2. Δ ABC is isosceles of side BC = 6cm. The radius is 5cm.
Calculate:
a) height AN,
b) area of ΔABC.

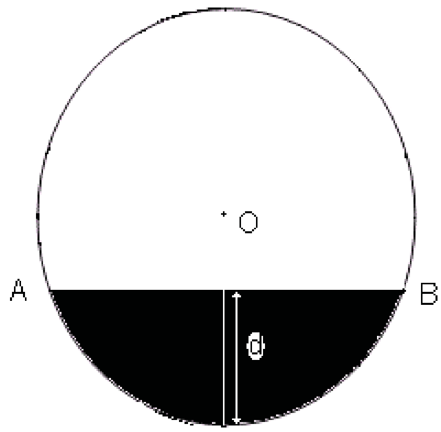




$CX = 1\ cm,\ \ AB = 6\ cm$

Calculate the radius of the circle.

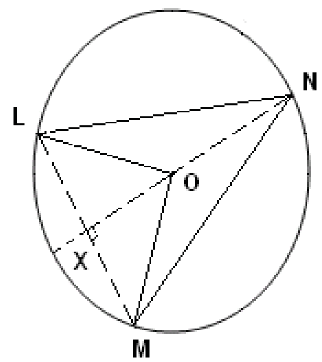
3. The figure below shows the cross-section of a water pipe of radius 25cm. If the width of the surface of the water AB = 48cm, calculate the depth of the water, d in the pipe.



After attempting the questions compare your answers to the ones provided below.

Answers to Activity 14.1

1.



The radius of the circle is $7\frac{1}{2}\ cm$, $OX = 6\ cm$,

Calculate:

a) length of LM
b) length of NX
c) area of ΔLMO
d) area of kite LOMN

Solution

a) If a line through centre O, is perpendicular to chord, it bisects the chord.
XN is perpendicular to chord LM, so it bisects LM.
Thus,

$$LX = XM$$

To find length of LM, first find LX (or XM), why?
Using Pythagoras theorem in ΔLOX (or ΔMOX)

$$LO^2 = OX^2 + XL^2$$

$$(7\tfrac{1}{2}\ cm)^2 = (6\ cm)^2 + XL^2$$

$$\tfrac{225}{4} cm^2 - 36\ cm^2 = XL^2$$

$$\tfrac{81}{4} cm^2 = XL^2$$

$$\sqrt{\tfrac{81}{4} cm^2} = XL$$

$$\tfrac{9}{2} cm = XL$$

Length of LM $= LX + XM$

$= 2(LX)$ (LX = XM)

$= 2(\tfrac{9}{2} cm)$

$= 9\ cm$

b) Length of $NX = NO + OX$ (NO = LO = MO, radii of same circle)

$= 7\tfrac{1}{2} cm + 6\ cm$

$= 13\tfrac{1}{2}\ cm$

c) Area of $\Delta LOM = \tfrac{1}{2} bh$

$= \tfrac{1}{2}(9\ cm)(6\ cm)$

$= 27\ cm^2$

d) Area of kite LOMN $=$ *area of* ΔLMN $-$ *area of* ΔLOM

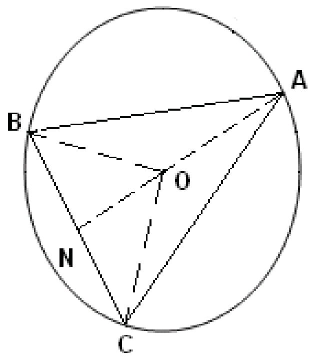
$= \tfrac{1}{2} bh - 27\ cm^2$

$= \tfrac{1}{2}(9\ cm)\left(13\tfrac{1}{2}\ cm\right) - 27\ cm^2$

$= 60\tfrac{3}{4}\ cm^2 - 27\ cm^2$

$= 33\tfrac{3}{4}\ cm^2$

2. Δ ABC is Isosceles of side BC = 6cm. The radius is 5cm.
   Calculate:
   a) height AN,
   b) area of Δ ABC.



Solution

a) If a line through centre O, is perpendicular to chord, it bisects the chord.

ON is through the centre O, and Δ ABC is isosceles, so ON bisects BC.
Thus,

$$BN = NC = 3\ cm$$

To find height AN, first find ON.
Using Pythagoras theorem to find ON:

$$OB^2 = BN^2 + NO^2$$

$$(5\ cm)^2 = (3\ cm)^2 + NO^2$$

$$(5\ cm)^2 - (3\ cm)^2 = NO^2$$

$$25\ cm^2 - 9\ cm^2 =$$

$$\sqrt{16\ cm^2} = NO$$

$$4\ cm = NO$$

Height $AN = AO + ON$

$$= 5\ cm + 4\ cm$$

$$= 9\ cm$$

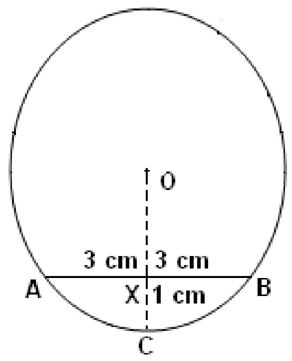
b) Area of $\Delta ABC = \frac{1}{2}bh$

$$= \frac{1}{2}(BC)(AN)$$

$= \frac{1}{2}(6\ cm)(9\ cm)$

$= 27\ cm^2$

3.

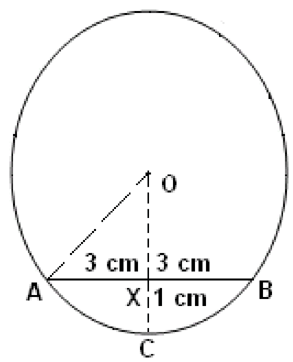


$CX = 1\ cm, \quad AB = 6\ cm$

Calculate the radius of the circle.

Solution

Join A to O (or B to O) to get



The radius OC bisects the chord AB, so it is perpendicular to AB. To find radius of circle use Pythagoras theorem to find AO.

$AO^2 = OX^2 + XA^2$

Let radius be *r*,

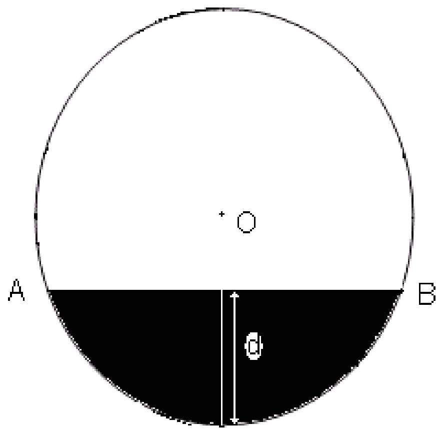
So, $OX = r - 1$

Substituting, into equation (Pythagoras)

$$AO^2 = OX^2 + XA^2$$
$$(r\ cm)^2 = ((r-1)\ cm)^2 + (3\ cm)^2$$
$$r^2 = r^2 - 2r + 1 + 9$$
$$r^2 - r^2 + 2r = 10 \text{ (collect like terms)}$$
$$2r = 10 \text{ (divide by 2 both sides)}$$
$$r = \frac{10}{2}$$
$$r = 5$$

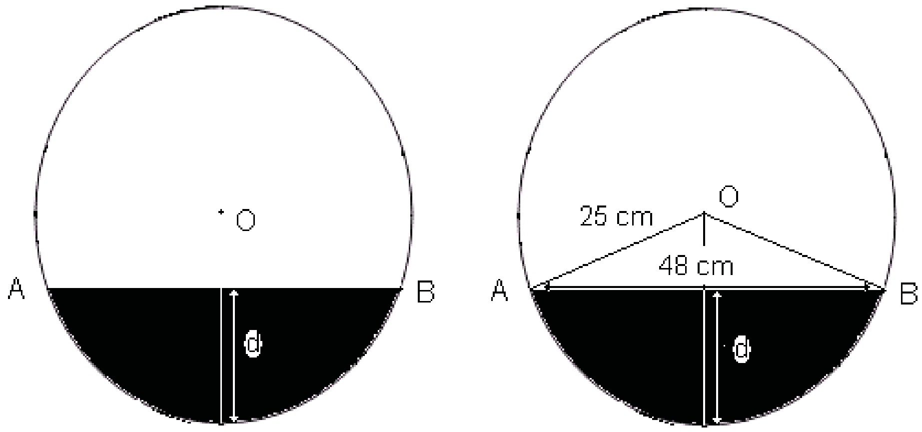
Radius is 5cm.

4. The figure below shows the cross-section of a water pipe of radius 25cm. If the width of the surface of the water PQ = 48cm, Calculate the depth of the water, d in the pipe.



Solution

Since AB is a chord, then Perpendicular Bisector theorem will apply. It says a line (radius) from centre O perpendicular to chord AB will bisect the chord AB. If you call the midpoint of AB, M then AM = 24cm and MB = 24cm. Looking at the diagram, if you join centre O with A and with B you get an Isosceles triangle. Note that the perpendicular line bisects the triangle into two right angled triangles. You can apply Pythagoras theorem in one of them to find the depth of the water! How? First find the distance from water surface to centre, O then the difference of this distance and the radius will be the depth, d! See the diagram below.



Applying Pythagoras,

$AO^2 = OM^2 + MA^2$

$(25\ cm)^2 = OM^2 + (24\ cm)^2$

$625\ cm^2 - 576\ cm^2 = OM^2$

$49\ cm^2 = OM^2$

$\sqrt{49\ cm^2} = \sqrt{OM^2}$

$7\ cm = OM$

This means that from centre, O to the surface of the water is 7cm.

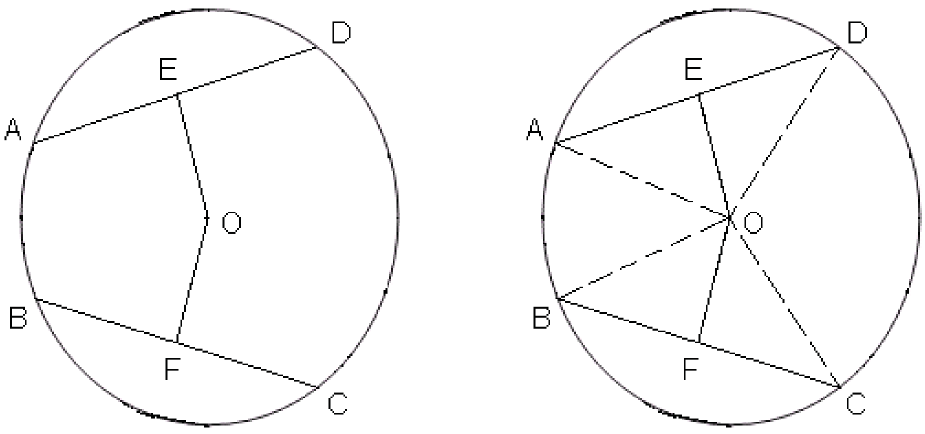
Therefore, the depth of the water in the pipe

$= radius - distance\ from\ centre\ to\ water\ surface$

$= 25\ cm - 7\ cm$

$= 18\ cm$

Theorem 2. *Equal chords of a circle are equidistant from the centre.*



(a) (b)

Proof

In the statement you are told that you have two equal chords in a circle and a centre O, see (a) above. Also, remember from theorem 1, you have the perpendicular lines OE and OF from the centre, O to the two chords AD and BC respectively.

You are required to show that the distance from the centre, O to each of the two chords is equal (equidistant from the centre of the circle.)

So, the **aim is to show that** $OE = OF$

What you will do to build step by step work to show this (what reasoning are you going to use to show that this is true) is:

1. $AD = BC$ (given in theorem statement)

2. $E \text{ and } F$ are the midpoints of $AD \text{ and } BC$ respectively (by theorem 1 proved)

3. $AE = ED = BF = FC$ (equal chords bisected by perpendiculars)

4. Look at triangles, $OEA \text{ and } OFC$
   a. $OA = OC$ (radii of same circle are equal)

   b. $AE = FC$ (proved in theorem 1)

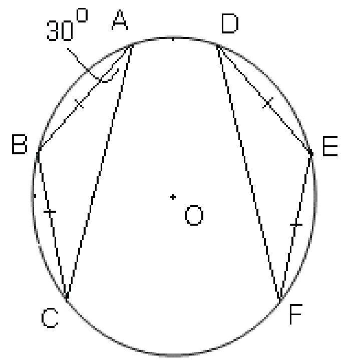
c. $\sphericalangle OEA = \sphericalangle OFC = 90^0$ (proved in theorem 1)

5. Triangles OEA and OFC are same in size and shape (by statement 4a, 4b, 4c)

6. $OE = OF$ (by statement 5)

Example

1. AB, BC, DE, and EF are equal chords of a circle with centre, O and $\sphericalangle BAC = 30^0$.



Find the angles:

a) $\sphericalangle DEF$

b) $\sphericalangle DFE$

Solution

The chords are equal, this means they are equidistant from the centre. But, note that AB = BC, therefore Δ ABC is Isosceles, so base angles are equal.

That is,

$\sphericalangle BAC = \sphericalangle BCA = 30^0$

$\sphericalangle ABC = 180^0 - (2 \sphericalangle BAC)$ (angles in Isosceles Δ)

$= 180^0 - 2(30^0)$

$= 180^0 - 60^0$

$= 120^0$

Since the chords are equal, AB = DE and BC = EF, follows that Δ ABC is similar to Δ DEF, therefore corresponding angles are equal.

That is,

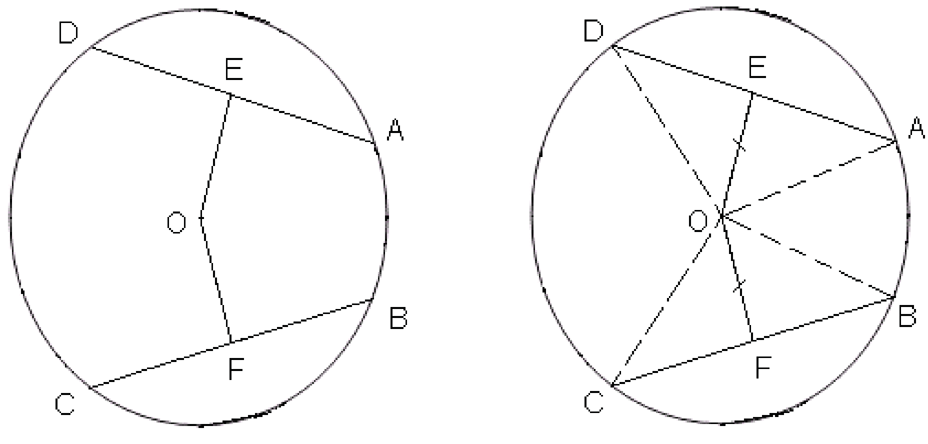
$\measuredangle DEF = \measuredangle ABC = 120^0$

and $\measuredangle DFE = \measuredangle BCA = 30^0$

(remember that the chords are also equidistant from centre, O)

The questions for theorem 2 are grouped together with questions for theorem 3 as they are a theorem and its converse.

Theorem 3. *Chords equidistant from the centre are equal.*



(a) (b)

Proof

In the statement you are told that you have two chords in a circle equidistant from centre O, see (a) above. But, from theorem 1, you also know that if,

$OE = OF$ (chords are equidistant from centre) then $OE \perp DA$ and $OF \perp CB$ you are required to show that (to prove) the chords DA and CB are equal.

So, the **aim is to show that** $DA = CB$

What you will do to build step by step work to show this (what reasoning are you going to use to show that this is true) is:

1. Look at triangles, OED and OFC

   a. $OE = OF$ (given, equidistant chords from centre)

   b. $OD = OC$ (radii of same circle are equal)

   c. $\angle OED = \angle OFC = 90^0$ (given, chords and bisectors)

   d. Triangles OED and OFC are same in size and shape (by statement 1a, 1b, 1c)

2. $ED = FC$ (sides of similar triangles by statement 1d)

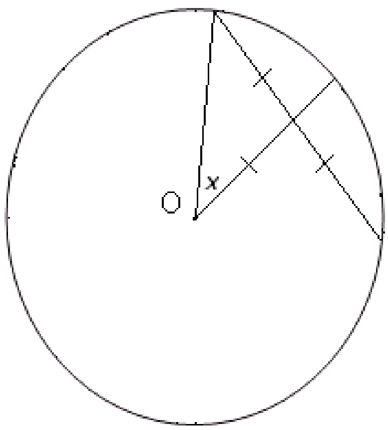
3. $ED = \frac{1}{2}DA$ and $FC = \frac{1}{2}CB$ (by theorem 1, perpendicular bisector)

4. $DA = CB$ (by statement 3)

Example

1. Find the value of x in the following:

   a)



Solution

The chord has been bisected by a radius, the angle they form is a right angle. The triangle formed is an Isosceles, hence, the base angles are equal. But, the angles must add up to 180^0 .

That is,

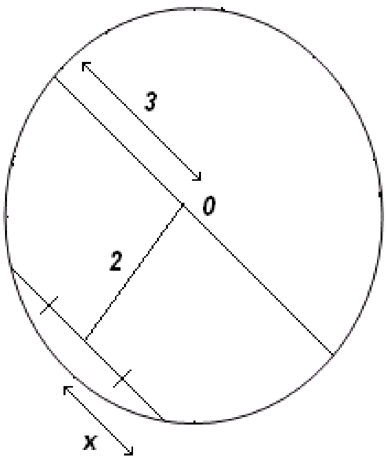
$$90^0 + 2x = 180^0$$

$$2x = 180^0 - 90^0$$

$$x = \frac{90^0}{2}$$

$$x = 45^0$$

b)



Solution

The radius is 3, the distance between the diameter and the chord is 2, this line is going through the centre. What is the angle formed between the diameter and the chord?

By Pythagoras theorem,

$$3^2 = 2^2 + x^2$$

$$3^2 - 2^2 = x^2$$

$$9 - 4 = x^2$$

$$5 = x^2 \qquad (4 < 5 < 9)$$

$$\sqrt{5} = \sqrt{x^2} \qquad (\sqrt{4} < \sqrt{5} < \sqrt{9})$$

$$\sqrt{5} = x \qquad (2 < \sqrt{5} < 3)$$

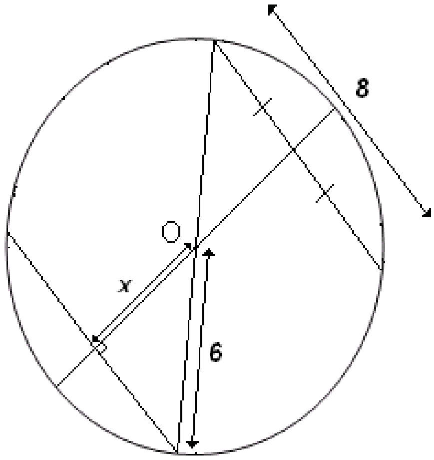
$$2.236 = x \qquad (2 < 2.236 < 3)$$

$$2.24 \approx x \qquad (to\ 3\ s.f.)$$

Activity

The chord has been bisected, the angle between the chord and the radius is a right angle.

c)



Solution

The chord has been bisected by radius, the angle between them is a right angle.

Using the Pythagoras theorem

$$6^2 = 4^2 + x^2$$

$$6^2 - 4^2 = x^2 \quad \text{(subtract } 4^2 \text{ both sides)}$$

$$36 - 16 = x^2$$

$$20 = x^2 \qquad (16 < 20 < 25)$$

$$\sqrt{20} = \sqrt{x^2} \qquad (\sqrt{16} < \sqrt{20} < \sqrt{25})$$

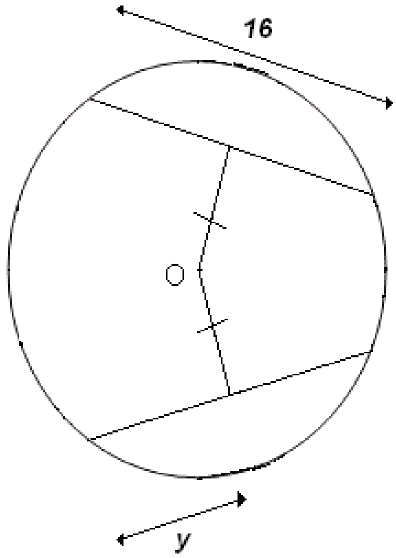
$$4.472 = x \qquad (4 < \sqrt{20} = 5)$$

$$4.47 \approx x \qquad (to\ 3\ s.f.)$$

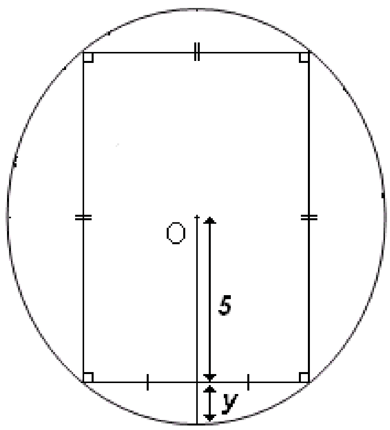
Activity 14.2

1. Find the value of y in the following

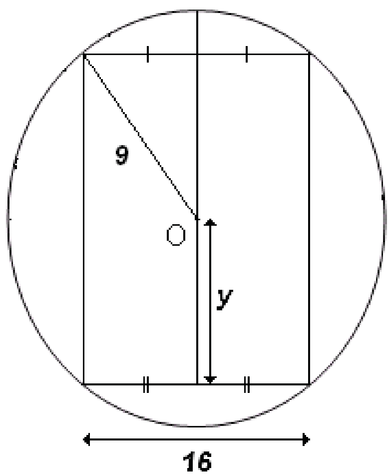
a)



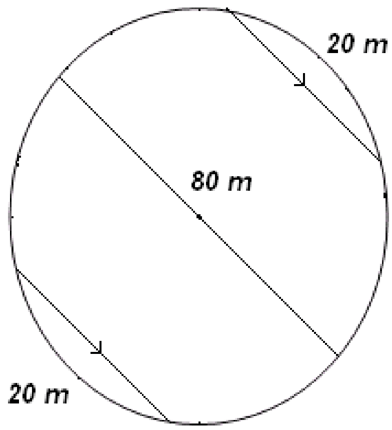
b)



c)



2. The base of a large hemispherical dome is a circle of diameter 80m. How far apart are two 20m parallel support beams which form "chords" of the circular base?



Answers to Activity 14.2

1.

a) Two theorems are going to apply here:

   i. Perpendicular Bisector theorem, because the line from the centre which is perpendicular to the chord is bisecting the chord.

   ii. Equidistant chords from the centre are equal
   by Equidistant Chords theorem, you can be able to say the chord of 16 is equal to the chord where z is to be found.
   By perpendicular bisector theorem, you can be able to say

   $y = \frac{1}{2}(16)$

   $= 8$

b) To find y you will need to use Pythagoras theorem, because the radius is (5 + Y). The other two sides you are given when you are told that it is 5 from centre to chord and the chords are all equal, which means they are all equidistant from the centre by theorem. So, by Pythagoras theorem

$(5+y)^2 = 5^2 + 5^2$

$(5+y)^2 = 50$

$\sqrt{(5+y)^2} = \sqrt{50}$

$5 + y = \pm\sqrt{50}$

$y = -5 \pm \sqrt{50}$

$y = -5 + 7.071 \; or \; y = -5 - 7.071$

$y = 2.071 \; or \; y = -12.071$

Now, looking at the two values for y, the only reasonable value for y in this situation is $y = 2.071$, -12.071 is not reasonable because it is negative, radius is length, there cannot be a negative length!

c) Looking at the diagram, you can see that y is perpendicular to the chord and the chord is bisected, therefore you can use Pythagoras since you have right angled triangles. So, you can have

$9^2 = 8^2 + y^2$ (subtract 64 both sides)

$81 - 64 = y^2$

$17 = y^2$ (take square roots both sides)

$$\sqrt{17} = \sqrt{y^2}$$
$$4.1 = y$$

2. Since the chords are equal, then they must be the same distance from the centre by theorem. The line from the centre to the chords must bisect the chords so you can apply the famous Pythagoras theorem since there will be right angled triangles.
   Note that the base of the triangle is the 20m chord, which will be bisected to equal lengths of 10m each. The hypotenuse, will be the radius of the dome, since you are told the diameter is 80m then radius is 40m. Now, applying Pythagoras theorem

$$hypotenuse^2 = side^2 + side^2$$
$$(40m)^2 = (10m)^2 + side^2$$

(subtract $(10m)^2$ both sides)

$$1600m^2 - 100m^2 = side^2$$
$$1500m^2 = side^2$$

(square roots both sides)

$$\sqrt{1500m^2} = \sqrt{side^2}$$
$$38.7\ m = side$$

So, distance between the 20m parallel support beams is

$$38.7\ m + 38.7\ m = 77.4\ m$$

Double the calculated distance because the beams are at equal distance on both sides of the centre!

The point to keep in mind is that a chord in a circle suggests that a diameter or radius will bisect it, into two equal parts, also the angle between the chord and the diameter or radius is a right angle. You can use Pythagoras theorem to find any of the sides of the right angled triangle if you are given any two sides!

Lesson 3 Properties of Angles in a Circle

In our everyday life the use of the circle is a necessity in simple tools and many designs in art and architecture. The examples are many. They include arches of buildings, and patterns for various manufactures such as, Arabic patterns, which are based on the circle.

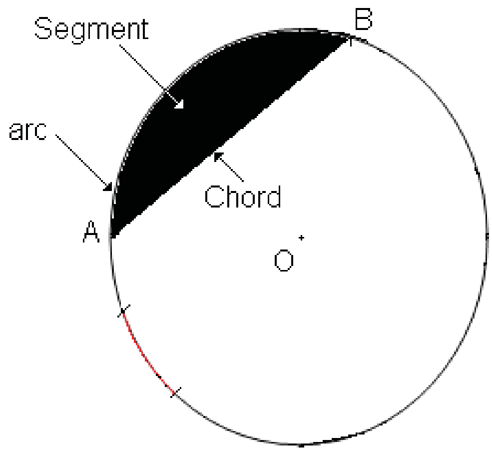
At the end of this sub-unit you should be able to:

- *define* the term ‘segment’
- *prove* that angles in the same segment are equal
- *prove* that the angle at the centre of a circle is equal to twice the angle at any point on the remaining part of the circumference, if both angles stand on the same arc or chord

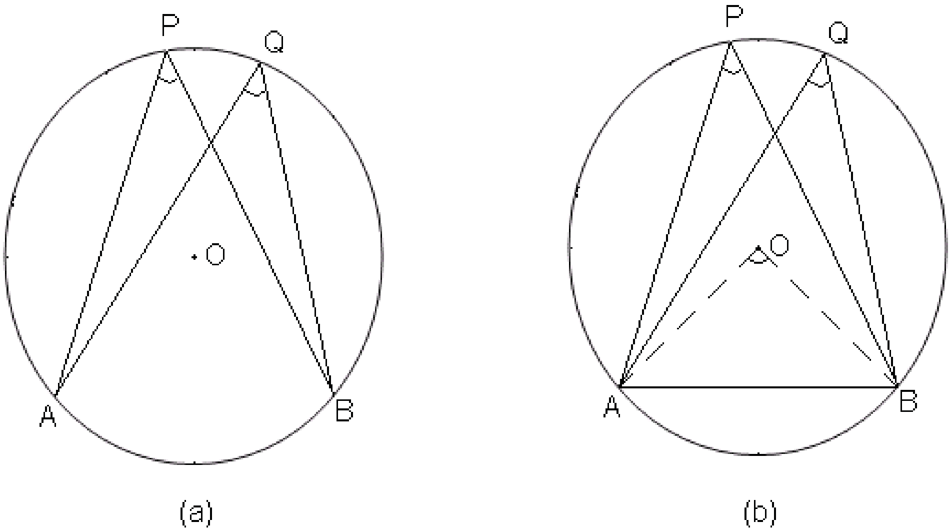
- *prove* that the angle in the semi-circle is a right angle

The theorems about the properties of angles in a circle and their proofs are in one place here.

A segment is an area in a circle enclosed by a chord (AB) and an arc (AB) sharing two end points (AB).



Theorem 4: *angles in the same segment are equal.*



Proof

In this statement you are told that you have angles in the same segment. The angles are the same size if they all stand on the same chord. The same segment means they will be on the same side of the centre of the circle, O, see (a).

You are required to show that the angles are all equal in size.

So, **the aim is to show that $\angle APB = \angle AQB$.**

What you will do to build step by step work to show this (reasoning you are going to use to show that this is true) is:

1. Draw the chord AB, see (b) above

2. Draw the radii OA and OB, see (b) above

3. $\angle AOB = 2(\angle AQB)$
 (angle at centre is twice angle at circumference)

4. $\angle AOB = 2(\angle APB)$
 (angle at centre is twice angle at circumference)

5. But,

   i. $\angle AOB = \angle AOB$
   (by statement 3 and 4)

   ii. $2(\angle APB) = 2(\angle AQB)$
   (by statement 3 and 4)

   iii. $\angle APB = \angle AQB$
   (follows from statement ii)

By statement iii, the angles at the circumference are equal if they all stand on the same chord and are in the same segment.

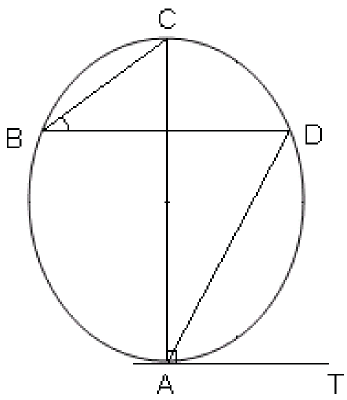
(Note that here statement iii is the same as what you wanted to show in the aim.)

Example

a. TA is a tangent to the circle with AC a diameter.
   $\angle DAT = 54^0$
   Calculate

   a. $\angle CAD$

   b. $\angle CBD$



a. $\angle CAT = 90^0$

$\angle CAD = \angle CAT - \angle DAT$

$= 90^0 - 54^0$

$= 36^0$

b. By theorem, $\angle CBD$ and $\angle CAD$ are both standing on chord CD and the same side of the chord. Therefore, they are equal. Thus,

$\angle CAD = \angle CBD$

$= 36^0$

Activity

Activity 14.3

1. The sides BA and CD of a cyclic quadrilateral ABCD are extended to meet at P.

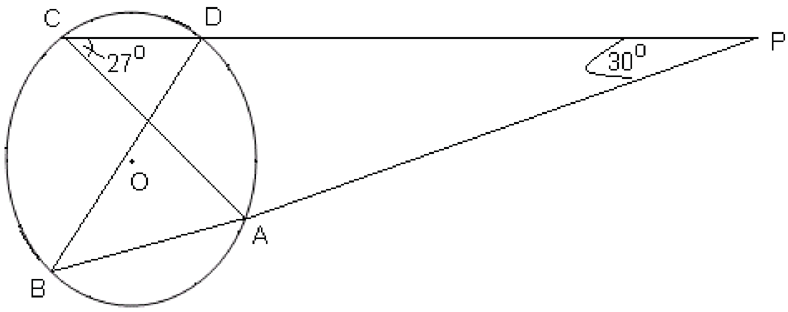
   $\angle ACD = 27^0$, and $\angle APD = 50^0$

   Calculate:

   a. $\angle DBA$

   b. $\angle BDP$

   c. $\angle CDB$



Answers to Activity 14.3

a. By theorem, $\sphericalangle DBA = \sphericalangle ACD = 27^0$ because the two angles are standing on the same chord AD and are the same side of the chord.

b. $\sphericalangle BDP + \sphericalangle DPA + \sphericalangle PBD = 180^0$

$\sphericalangle BDP + 30^0 + 27^0 = 180^0$

$\sphericalangle BDP = 180^0 - (30^0 + 27^0)$

$= 180^0 - 57^0$

$= 123^0$

c. Note, that $\sphericalangle CDP = 180^0$

So,

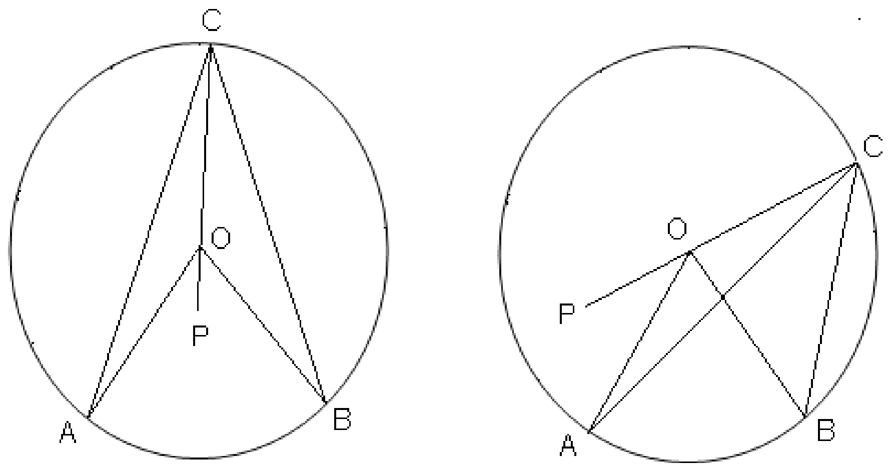
$\sphericalangle CDB + \sphericalangle BDP = 180^0$

$\sphericalangle CDB = 180^0 - 123^0$

$= 57^0$

You are about to have an enjoyable experience. You will soon find yourself pleasantly acquiring knowledge about the fascinating shape of a circle. Why is this shape so fascinating?

Theorem 5: *the angle at the centre of a circle is equal to twice the angle at any point on the remaining part of the circumference, if both angles stand on the same arc or chord.*



(a) (b)

Proof

In this statement you are told that the size of the angle at the centre of the circle is equal to twice (two times) the size of the angle at any point on the circumference if both angles stand on the same chord (or arc), see (a) and (b) above.

You are required to show that (to prove):

> If you can draw any two angles, one at the centre and the second one at the circumference then the angle at the centre of the circle is equal to twice the angle at the circumference if both stand on the same chord and are on the same side of that chord.

So, **the aim is to show that** $\angle AOB = 2(\angle ACB)$

What you will do to built a step by step work to show this (what reasoning are you going to use to show that this is true)

1. $OC = OA = OB$ (radii of same circle)
2. $\angle ACO = \angle CAO$ (Isosceles Δ OAC)
3. $\angle AOP = \angle OAC + \angle OCA$ (Isosceles Δ OAC)
4. $\angle OAC = \angle OCA$ (angles of Isosceles Δ OAC)
5. $\angle BOP = \angle OBC + \angle OCB$ (Isosceles Δ BCO)
6. $\angle OBC = \angle OCB$
7. Case (i)

$$\angle AOB = \angle AOP + \angle BOP$$
$$= 2(\angle OCA) + 2(\angle OCB)$$
$$= 2(\angle OCA + \angle OCB)$$
$$= 2(\angle ACB)$$

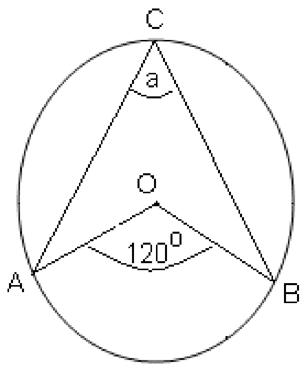
Case (ii)

$$\angle AOB = \angle POB - \angle POA$$
$$= 2(\angle OCB) - 2(\angle OCA)$$
$$= 2(\angle OCB - \angle OCA)$$
$$= 2(\angle ACB)$$

Note that in both case (i) and (ii) the last statement is the same as the aim statement, meaning that is what you wanted to show that the angle at centre is twice angle at circumference.

Example

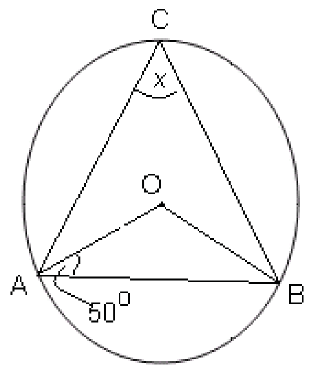
1. Calculate angle a in the diagram below.



By theorem, the angle at centre is twice the angle at circumference,

$$\angle AOB = 2(\angle ACB)$$
$$\frac{1}{2}(\angle AOB) = \angle ACB$$
$$\frac{1}{2}(120^0) = a$$
$$60^0 = a$$

2. In the diagram below



Find

a. $\measuredangle AOB$

b. x

Consider Δ AOB, it is Isosceles because OA = OB, both radii of same circle. Therefore, base angles are equal;

$\measuredangle OAB = \measuredangle OBA = 50^0$

$\measuredangle AOB = 180^0 - (\measuredangle OAB + \measuredangle OBA)$

$= 180^0 - 100^0$

$= 80^0$

Note that x is an angle at the circumference standing on same chord AB as $\measuredangle AOB$,

Therefore,

$x = \frac{1}{2}(\measuredangle AOB)$

$= \frac{1}{2}(80^0)$

$= 40^0$

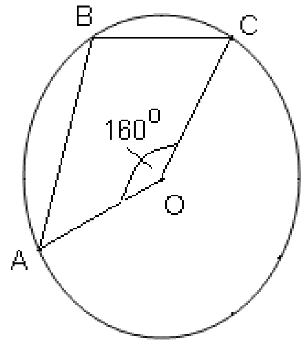
Activity 14.4

Activity

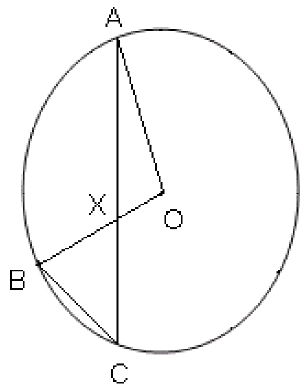
1. In the figure below, O is the centre of the circle and $\measuredangle AOC = 160^0$.

   a. What is the reflex angle $\measuredangle AOC$?

   b. What is $\measuredangle ABC$?



2. A, B, C are points on a circle, centre O, such that AC intersects OB at X. $\sphericalangle ACB = 43^0$ and $\sphericalangle BXC = 70^0$.



Calculate

a. $\sphericalangle AOB$

b. $\sphericalangle AXO$

c. $\sphericalangle XAO$

Answers to Activity 14.4

1.

a.

Reflex angle

$\sphericalangle AOC = 360^0 - 160^0$

$= 200^0$

b. $\sphericalangle ABC$ is angle at circumference, standing on same chord AC as reflex angle $\sphericalangle AOC$.

\`

So,

$\sphericalangle ABC = 100^o$

(angle at circumference is half angle at centre)

2.

a. AB is a chord, both $\sphericalangle AOB$ and $\sphericalangle ACB$ are standing on AB! I hope you were able to see that although the chord is not drawn! Otherwise you can see the arc AB. By theorem, the angle at centre is twice the angle at the circumference.

$\sphericalangle AOB = 2(\sphericalangle ACB)$

$= 2(45^0)$

$= 86^0$

b. $\sphericalangle BXC = 70^0$

$\sphericalangle AXO = \sphericalangle BXO$ (alternate angles)

$= 70^0$

c. $\sphericalangle AXO + \sphericalangle XOA + \sphericalangle OAX = 180^0$

(angles of a triangle ΔAOX)

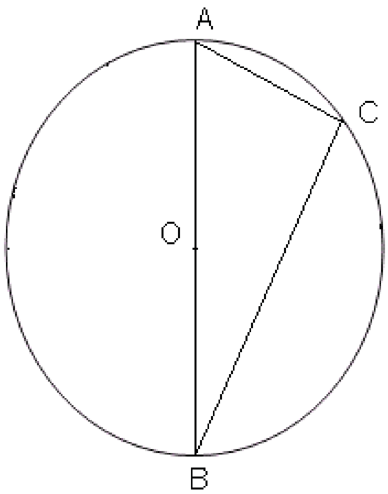
$\sphericalangle XAO = 180^0 - (\sphericalangle AOX + \sphericalangle OXA)$

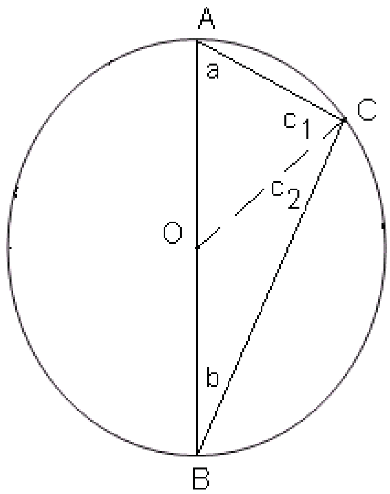
$= 180^0 - (86^0 + 70^0)$

$= 180^0 - 156^0$

$= 34^0$

Theorem 6: *the angle in the semi-circle is a right angle.*





Proof

In the statement you are told that you have a semi-circle of a circle and a centre O, see (a) above.

You are required to show that (to prove):

if you can draw a chord through the centre O, the angle at the circumference standing on this chord will be (90^0) .

So, the **aim is to show that** $\sphericalangle ACB = 90^0$

What you will do to build step by step work to show this (what reasoning are you going to use to show that this is true)

1. $AO = BO = CO$
   (radii of the same circle)
2. $\sphericalangle OAC = \sphericalangle ACO \text{ and } \sphericalangle OCB = \sphericalangle OBC$
   (Isosceles Δ OAC and OBC)
3. $\sphericalangle OAC + \sphericalangle OBC + \sphericalangle ACB = 180^0$
   (angles of Δ ABC add up to 180^0)
4. $a + b + c_1 + c_2 = 180^0$
   (replacing by small letters, look at diagram)
5. $2a + 2b = 180^0$
   (by statement 2, $a = c_1$, and $b = c_2$)
6. $a + b = 90^0$
   (divide all equation 5 by 2)
7. $c_1 + c_2 = 90^0$
   (by statement 2)
8. $\sphericalangle ACB = 90^0$

By last statement the angle in the semi-circle is a right angle.

This same theorem can be proved by using theorem 5.

Look at the diagram. AOB is a diameter, so what is the angle at the centre?

Well, you have thought correctly, it must be 180^0.

Note, also AOB subtends $\sphericalangle ACB$ at the circumference, so by theorem 5, you can safely say the angle at the centre is twice the angle at the circumference, if they both stand on the same chord and are on the same side of the chord.

That is,

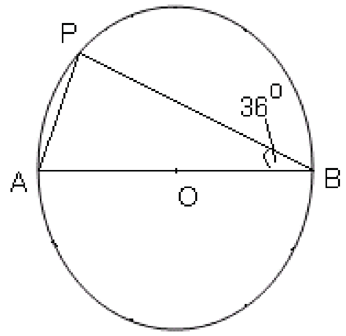
$$\angle ACB = \frac{1}{2}(\angle AOB)$$
$$= \frac{1}{2}(180^0)$$
$$= 90^0$$

The last statement shows that the angle at the circumference of the circle subtended by a diameter of the circle is a right angle ($= 90^0$).

Example

1. AB is a diameter of a circle and P is another point on the circumference such that $\angle PBA = 36^0$.

What is $\angle BAP$?



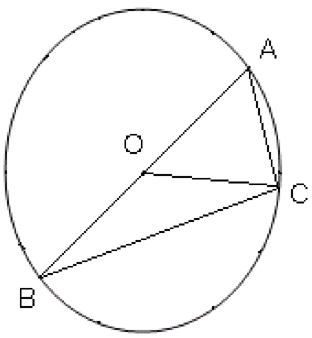
2. O is the centre of the circle and $\angle OCA = 72^0$.

Calculate

a. $\angle OAC$

b. $\angle BOC$

c. $\angle OCB$



Solution

1. Note, $OA = OC$ both radii. Therefore, triangle OAC is Isosceles. So, base angles are equal.

   $\angle OCA = \angle CAO = 72^0$

2. $\angle BOC = 2(\angle OCA)$ (exterior angle of Δ OAC)

   $= 2(72^0)$

   $= 144^0$

3. $\angle OCB = \angle ACB - \angle ACO$

   $(\angle ACB = 90^0, \text{angle in semi} - \text{circle})$

   $= 90^0 - 72^0$

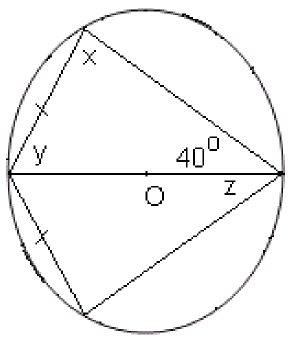
   $= 18^0$

Activity

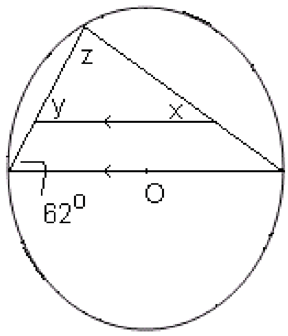
Activity 14.5

Find the angles shown by the letter *x, y* and *z* in the following diagrams.

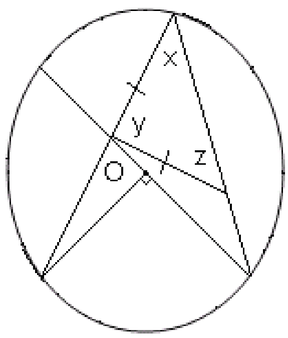
a.



b.

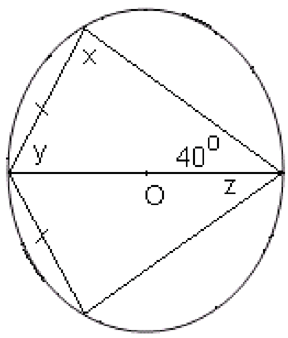


c.



Answers to Activity 14.5

a.



Solution

x is an angle in a semi-circle. By theorem, angle in a semi-circle is a right angle.

$x = 90^0$

To find y, use the fact that the angles of a triangle add up to 180^0.

$y + 90^0 + 40^0 = 180^0$

$$y = 180^0 - (90^0 + 40^0)$$

$$y = 180^0 - 130^0$$

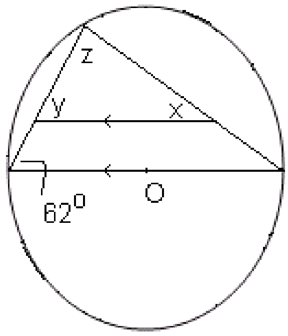
$$= 50^0$$

To find z:

Looking at the diagram, the diameter is a line of symmetry, also it is the common side to the two triangles, angles at the circumference are equal because they are angles in a semi-circle, and the sides at angle y

are equal. Therefore, the two two triangles are similar in shape and size, hence $z = 40^0$.

b.



Solution

z the angle in a semi-circle. By theorem, angle in a semi-circle is a right angle.

So, $z = 90^0$

To find *y:*

Note that the small triangle is similar to the big triangle because their sides are parallel. They share angle *z* . This means that angles y and z of the small triangle will be the same as the corresponding angles in the big triangle.

y correspond to $angle\ of\ 62^0$, therefore they are equal.

$y = 62^0$

To find *x*:

Angles of a triangle add up to 180^0 .

$$x + y + z = 180^0$$

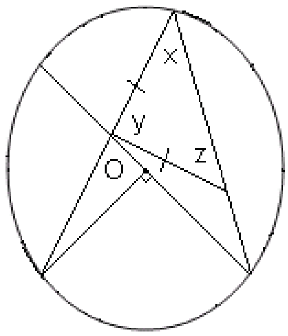
$$x + 62^0 + 90^0 = 180^0$$

$$x = 180^0 - (62^0 + 90^0)$$

$$x = 180^0 - 152^0$$

$$x = 28^0$$

c.



Solution

x is an angle at circumference, angle at centre is 90^0. By theorem, angle at circumference is half the angle at the centre if they both stand on the same chord.

Therefore,

$$x = \frac{1}{2}(90^0)$$

$$= 45^0$$

Note that *z and x are both base angles of the Isosceles triangle.*

So, the are equal.

$$z = 45^0$$

$$x = 45^0$$

To find *y*:

Angles of a triangle add up to 180^0

$$y + x + z = 180^0$$

$$y = 180^0 \quad (x + z)$$

$$y = 180^0 - 90^0 \qquad (45^0 + 45^0)$$

$$y = 90^0$$

Lesson 4 Cyclic Quadrilaterals

At the end of this sub-unit you should be able to:

- *prove* that the opposite angle of a cyclic quadrilateral are supplementary.

The theorems about the cyclic quadrilaterals and their proofs are in one place here.

Theorem 7. *opposite angles of a cyclic quadrilateral are supplementary.*

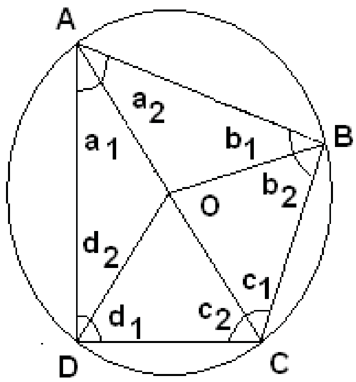
Proof

In the statement you are told that you have a chord of a circle and a centre O, see (a) above.

You are required to show that (to prove):

> if you have a cyclic quadrilateral ABCD drawn in a circle centre O, then the opposite angles add up to 180^0.
>
> So, the **aim is to show that** $\sphericalangle ABC + \sphericalangle CDA = 180^0$
>
> $\sphericalangle DAB + \sphericalangle BCD = 180^0$

What you will do to build step by step work to show this (what reasoning are you going to use to show that this is true):



Statement	Reason
1. OA = OB = OC = OD	radii
2. $\hat{a}_2 = \hat{b}_1, \hat{b}_2 = \hat{c}_1, \hat{c}_2 = \hat{d}_1, \hat{d}_2 = \hat{a}_1$	angles in Isosceles triangles
3. $2(a_2 + b_2 + c_2 + d_2) = 360^0$ $(a_2 + b_2) + (c_2 + d_2) = \frac{1}{2}(360^0)$ $= 180^0$	sum of angles in a quadrilateral = 360^0 and statement 2
4. $a_2 + b_2 = A\hat{B}C$; $c_2 + d_2 = A\hat{D}C$	
5. $A\hat{B}C + A\hat{D}C = 180^0$	by statement 2
6. This is the same for $B\hat{A}D + B\hat{C}D = 180^0$ Opposite angles of a cyclic quadrilateral are supplementary.	

Example

a. ABCD is a cyclic quadrilateral and X is a point on BA extended.

If $\sphericalangle BCD = 106^0$ and 68^0, what are $\sphericalangle ABC$ and $\sphericalangle XAD$?

Solution

$\measuredangle ABC$ is opposite to $\measuredangle CDA$ in a cyclic quadrilateral. Therefore, $\measuredangle ABC = 85^0$.

$\measuredangle XAD$ is an exterior angle in a cyclic quadrilateral. So, it is equal to interior opposite angle in a cyclic quadrilateral.

That is,

$$\measuredangle XAD = \measuredangle BCD$$

$$= 108^0$$

b. $\measuredangle ADB = 32^0$, $\measuredangle BDC = 25^0$, and $\measuredangle DBC = 50^0$.

Calculate

a. $\measuredangle BCD$

b. $\measuredangle BAD$

c. $\measuredangle ABD$

Solution

1. $\measuredangle BCD = 180^0 - (50^0 + 25^0)$ (angles in a triangle BCD)

$$= 180^0 - 75^0$$

$$= 105^0$$

2. $\measuredangle BAD = 180^0 - \measuredangle BCD$

$$= 180^0 - 105^0$$

$$= 75^0$$

3. $\measuredangle\measuredangle ABD = 180^0 - (32^0 + 75^0)$

$$= 180^0 - 107^0$$

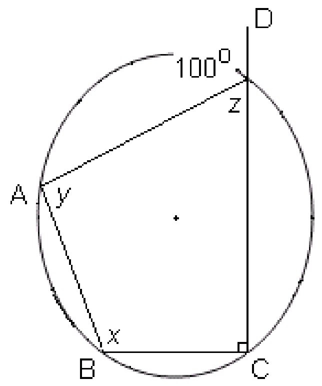
$$= 73^0$$

Activity 14.6

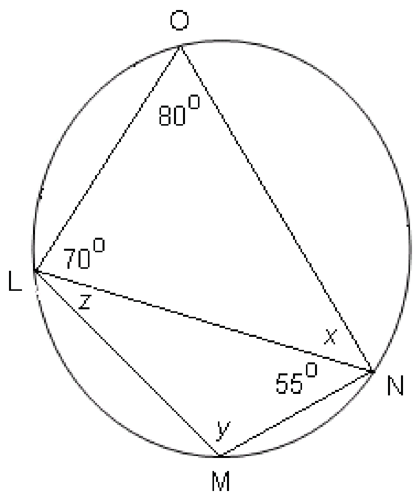
Activity

Find the values of *x, y, and z.*

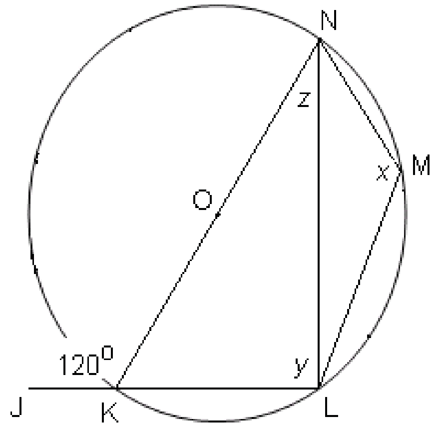
a)



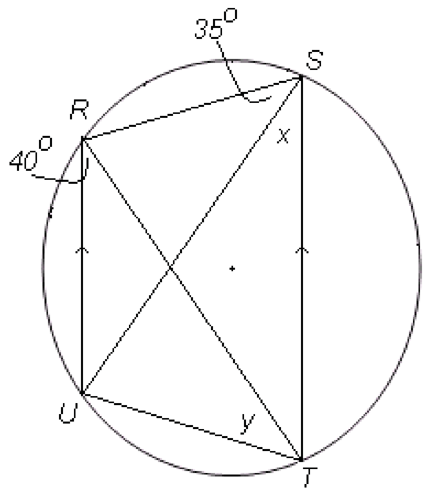
b)



c)



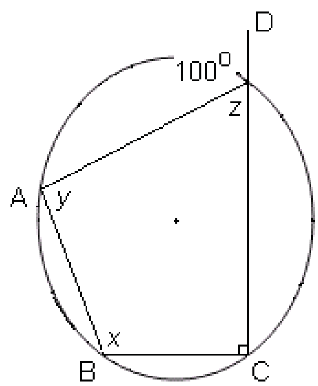
d)



Answers to Activity 14.6

Find the values of *x, y, and z.*

a)



Solution

$\measuredangle C = 90^0$

$\measuredangle C + \measuredangle A = 180^0$

$\measuredangle A = 180^0 - 90^0$

(subtract $\measuredangle C$ *from both sides*)

$\measuredangle A = 90^0$

$y = 90^0$

In a cyclic quadrilateral, exterior angle equal interior opposite angle.

So,

$x = 100^0$

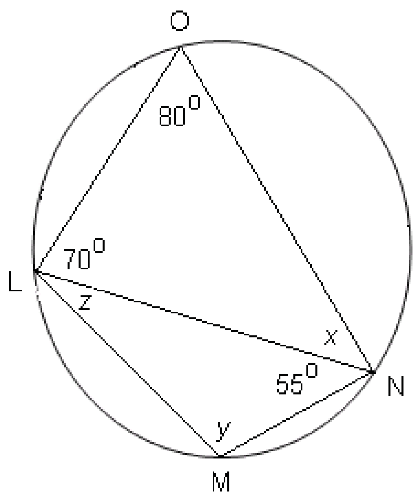
$x + z = 180^0$ (opposite angles are supplementary)

$z = 180^0 - x$

$z = 180^0 - 100^0$

$z = 80^0$

b)



Solution

$y + \angle O = 180^0$

(opposite angles are supplementary)

$y = 180^0 - 80^0$

$y = 100^0$

$z + y + 55^0 = 180^0$ (angles of a Δ LMN)

$z = 180^0 - 155^0$ $(100^0 + 55^0 = 155^0)$

$z = 25^0$

$25^0 + 70^0 + (55^0 + x) = 180^0$

(opposite angles are supplementary)

$95^0 + 55^0 + x = 180^0$

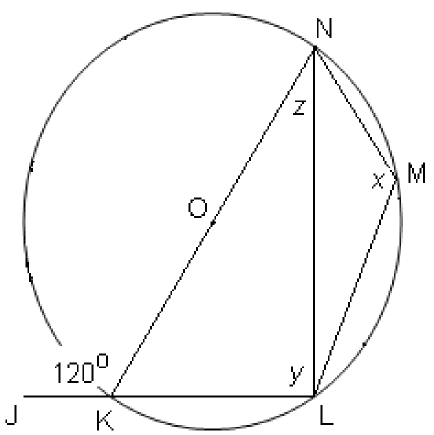
$150^0 + x = 180^0$

$x = 180^0 - 150^0$

$x = 30^0$

(Note that x can also found by using the angles of Δ LNO.)

c)



Solution

Exterior angle is equal to interior opposite in cyclic quadrilateral.

So,

$x = 120^0$

KON is a chord through the centre, *y* is angle in the semi-circle.

So,

$y = 90^0$ (angle in semi-circle is a right angle)

JKL is a straight angle,

$\sphericalangle OKL = 180^0 - 120^0 = 60^0$

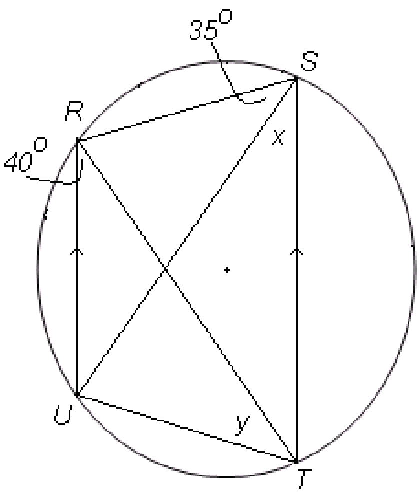
(Angles of a triangle KLN add up to 180^0.)

$z = 180^0 - (90^0 + 60^0)$

$z = 180^0 - 150^0$

$z = 30^0$

d)



Solution

Chord TU is where the angles $\sphericalangle URT$ and $\sphericalangle UST$ are both standing. Therefore, by same segment theorem they are equal.

$$\angle URT = 40^0$$

$$x = 40^0$$

By the same segment theorem, angles $\angle RSU$ and $\angle RTU$ both stand on chord RU, therefore they are equal.

$$\angle RSU = 35^0$$

$$y = 35^0$$

Lesson 5 Properties of Tangents

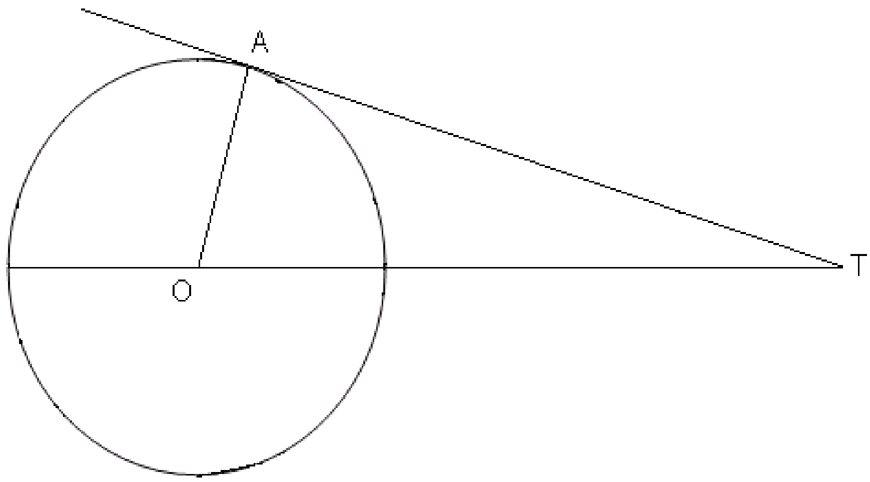
In Biblical times, when a small boy, David, fought a giant, Goliath, which war weapon did David use? How does this weapon work?

At the end of this sub-unit you should be able to:

- *prove* that a tangent to a circle is perpendicular to the radius at the point of contact, and tangents from a point to a circle are equal.

The theorems about the properties of tangents and their proofs are in one place here.

Theorem 8. *a tangent to a circle is perpendicular to the radius at the point of contact.*



Proof

In the statement you are told that you have a tangent to a circle and a centre O, and radius OA above.

You are required to show that (to prove):

> if you can draw a line from the centre, O,to the circumference, and another line outside the circle, which touches the circle at the circumference where the line from the centre meet, the circumference the two lines will form an angle equal to 90^0 (they will be perpendicular.)
>
> So, the **aim is to show that $OA \perp AT$ (the lines are perpendicular)**

What you will do to build step by step work to show this (what reasoning are you going to use to show that this is true) is:

1. A tangent AT is outside the circle

   (AT intersect the circle at one point)

2. $OT > OA$

   (because if they are equal they will coincide, or, be on one line)

   (OA is radius)

3. $In\ fact,\ OT > OA$ for all points on AT

4. $OA \perp AT$

   (by statement 3)

Statement 3 says that for the two lines OA and OT to be equal, they have to coincide, (angle between them has to be zero.) Once the angle increases between them, the angle where they both meet at the circumference is 90^0. This will be true for all radius and tangent lines.

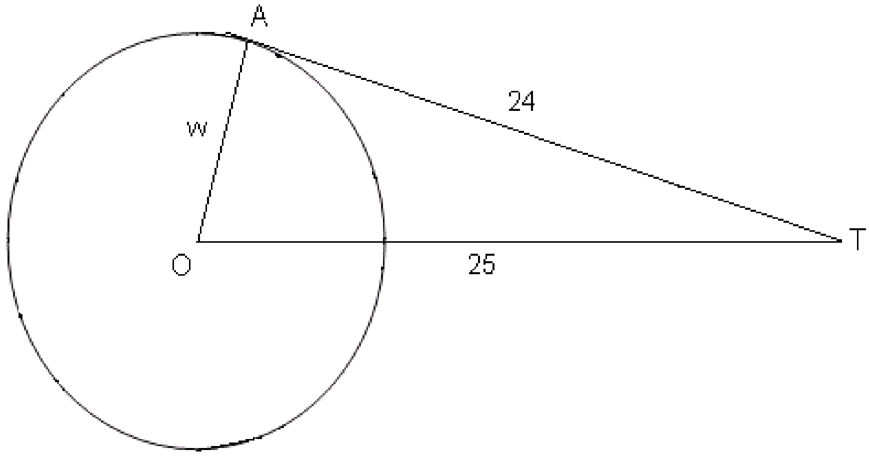
Note that from this tangent theorem, you have to remember the point:

- A tangent to the circle, AT, is perpendicular to the radius of the circle, OA at the point of contact.

Exampe

1. Find the value of *w* in the following:

   a)



Solution

Here you are looking for one of the sides of the Δ AOT.
What kind of triangle is it? Yes, it is a right angled triangle, because a tangent and a ridius are perpendicular at the point of contact.
This will allow the use of Pythagoras theorem to find the side.

$$OT^2 = TA^2 + AO^2 \quad \text{(Pythagoras)}$$

$$25^2 = 24^2 + AO^2 \quad \text{(put in numbers)}$$

$$625 - 576 = AO^2 \quad \text{(subtract 576 from both sides)}$$

$$49 = AO^2$$

$$\sqrt{49} = \sqrt{AO^2}$$

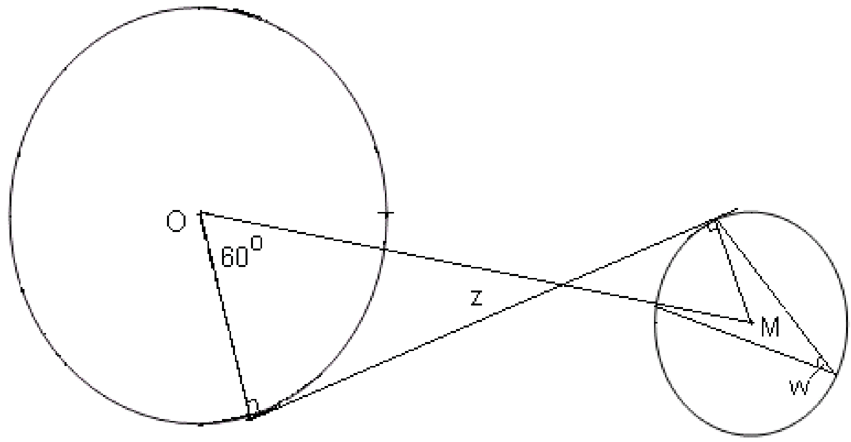
(take the square root of both sides)

$$7 = AO$$

Therefore, $w = AO = 7$

2. Find the value of *z* and *w*. *O* and *M* mark the centres of the circles.

a)



Solution

The diagram may look some how different from what you have seen from the theorem and the previous example. However, this must not prevent you from using what you already know in situations which look new!

If you look at the bigger triangle formed from the bigger circle centre with the tangent to the circle up to where z is, what type of a triangle is it? Once again, it is a right angled triangle because a radius of the circle is perpendicular to the tangent at the point of contact. Since, here the part you have to find is an angle z, you use the fact that sum of the angles of a triangle add up to 180^0.

$60^0 + 90^0 + z = 180^0$ (90^0 because tangent perpendicular to radius)

$z = 180^0 - (60^0 + 90^0)$

$z = 180^0 - 150^0$

$z = 30^0$

If you look at the two triangles, they are formed by two crossing lines, the alternate angles must be equal. So, the triangles are similar because the angle alternate to z is equal to $z = 30^0$. The angles formed at the contact of tangent and radius are right angles (90^0), this leaves the angles at the centres of the triangles! What is their size?

They, have no way of being different, they must be equal to 60^0.

This just lead you into another theorem; since w is at the circumference and you have just found the angle at the centre in the same circle also, they are both standing on the same arc (or chord, the legs of angle at centre, and M and at w are standing on the same place at the circumference.)

Therefore, the theorem says angle at M is twice angle w.

So,

$$w = \frac{1}{2}(angle\ at\ centre, M)$$

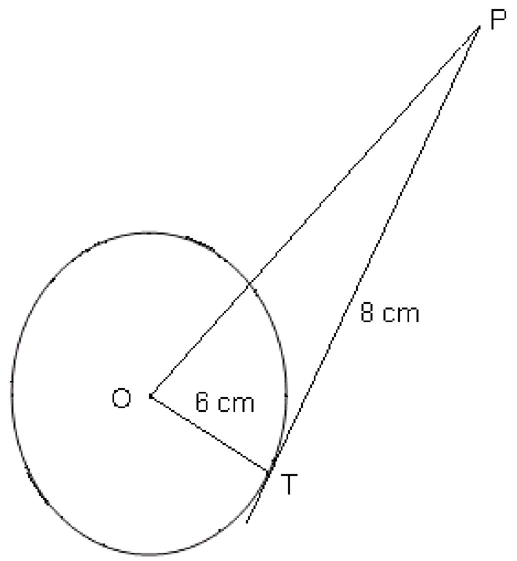
$$w = \frac{1}{2}(60^0)$$

$$w = 30^0$$

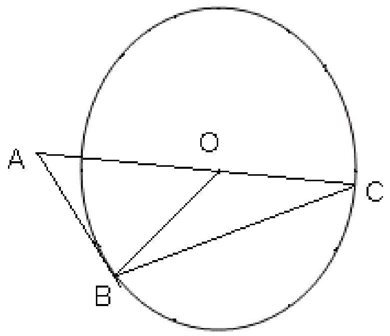
Activity

Activity 14.7

1. A tangent from point P touches a circle at T and PT is 8cm long. The radius of the circle, centre O, is 6cm. Calculate OP.



2. In the figure below, AB is tangent to the circle, centre O and $\sphericalangle OCB = 25^{0}$



Calculate:

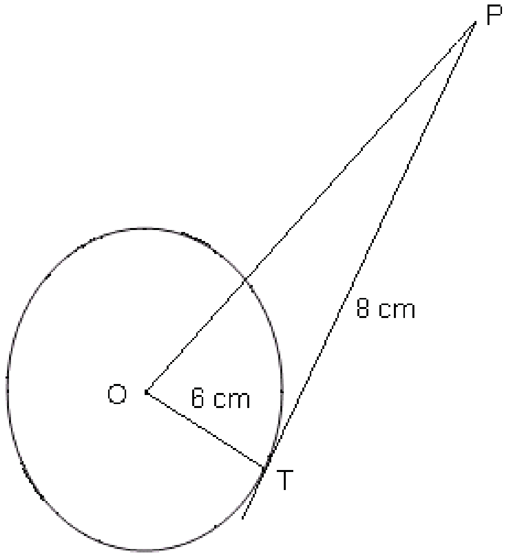
a. $\sphericalangle OBC$

b. $\sphericalangle AOB$

c. $\sphericalangle OAB$

Answers to Activity 14.7

1. A tangent from point P touches a circle at T, which makes PT is 8cm long. The radius of the circle, centre O, is 6cm. Calculate OP.



Solution

A radius of a circle, centre O, is perpendicular to a tangent PT at point T (contact point.)

Since ΔOPT is a right angled triangle, you can use Pythagoras theorem.

$PT^2 = TO^2 + OP^2$ (Pythagoras)

$(8\ cm)^2 = (6\ cm)^2 + OP^2$

$64\ cm^2 - 36\ cm^2 = OP^2$

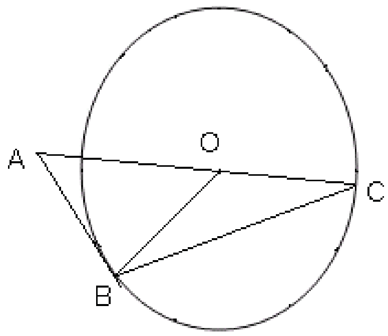
$28\ cm^2 = OP^2$

$\sqrt{28\ cm^2} = \sqrt{OP^2}$ $(\sqrt{25} < \sqrt{28} < \sqrt{36})$

$5.291\ cm = OP$ $(5 < \sqrt{28} < 6)$

$5.29\ cm = OP$ (to 3 s.f.)

2. In the figure below, AB is tangent to the circle, centre O and $\angle OCB = 25^0$



Calculate:

a. $\angle OBC$

b. $\angle AOB$

c. $\angle OAB$

Solution

a) Notice that Δ OBC is Isosceles because OB = OC, are both radii of same circle. So, base angles are also equal, that is,

$\angle OCB = \angle OBC$

$= 25^0$

b) Note here that $\angle AOB$ is an exterior angle of triangle OBC. Therefore,

$\angle AOB = \angle OBC + \angle OCB$

$= 25^0 + 25^0$

$= 50^0$

c) Here you look at Δ OAB, it is a right angled triangle because tangent AB meets the radius of a circle at point B.

So,

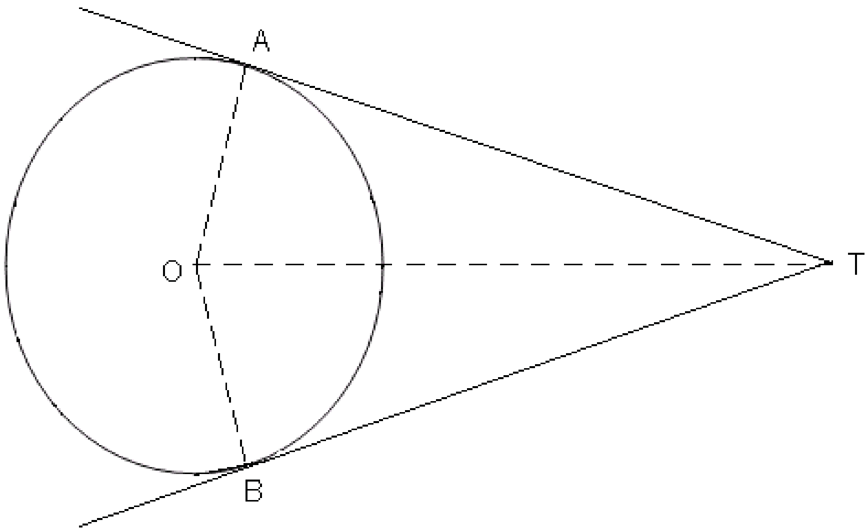
$\measuredangle OAB + \measuredangle AOB + \measuredangle ABO = 180^0$

$\measuredangle OAB = 180^0 - (50^0 + 90^0)$

$= 180^0 - 140^0$

$= 40^0$

Theorem 9. *tangents from a point to a circle are equal.*



Proof

In the statement you are told that you have a chord of a circle and a centre O, see (a) above.

You are required to show that (to prove):

you can draw only two lines from the point T to the circle, which touch the circle and these lines are equal in length.

So, the **aim is to show that $AT = BT$**

What you will do to build step by step work to show this (what reasoning are you going to use to show that this is true) is:

In Δ OAT and Δ OBT

1. $AO = BO$ (radii of same circle)
2. $\angle OAT = \angle OBT$ ($AO \perp AT$ and $BO \perp BT$ radii and tangents at contact points)

   $= 90^0$
3. OT is common side to both triangles.
4. $\Delta OAT = \Delta OBT$ (by side, angle, side property which are radii (AO and BO), right angle (90^0), hypotenuse(OT) in statement 1, 2, and 3) $AT = BT$ (by statement 4)

By statement 5, tangents from a point T from outside a circle are equal. This is true for any two tangents.

The points to remember for this theorem are:

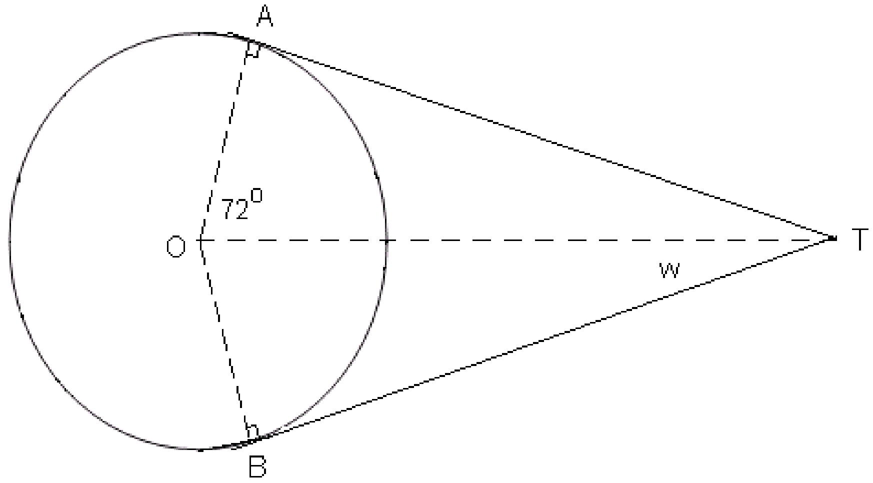
1. The tangents drawn from the point outside the circle are equal in length, ($AT = BT$)
2. Tangents subtend equal angles at the centre ($\angle AOT = \angle BOT$)
3. The line AT bisects angle between the tangents

   ($\angle ATO = \angle BTO$)

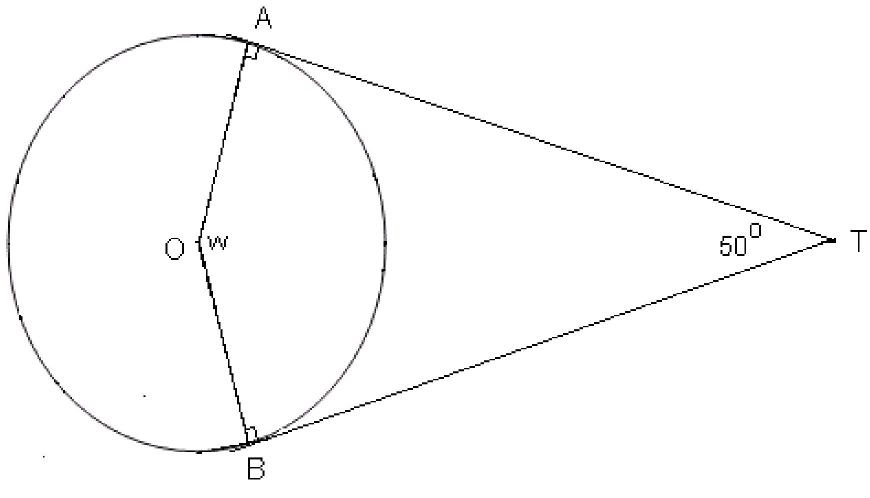
Example

1. Find the value of w in the following:

   a)

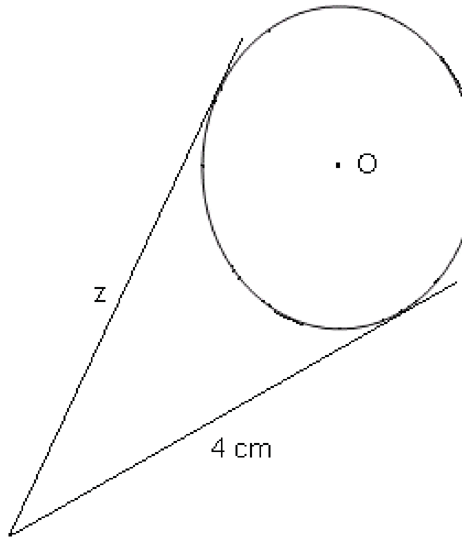


b)

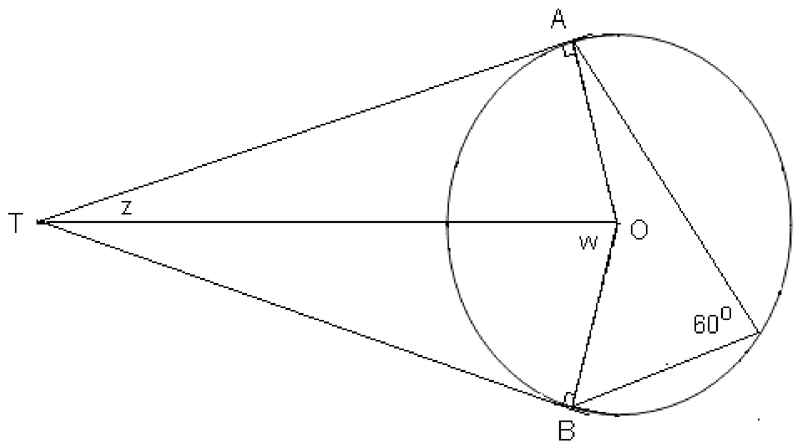


2. Find the value of z and w. O marks the centre of the circle.

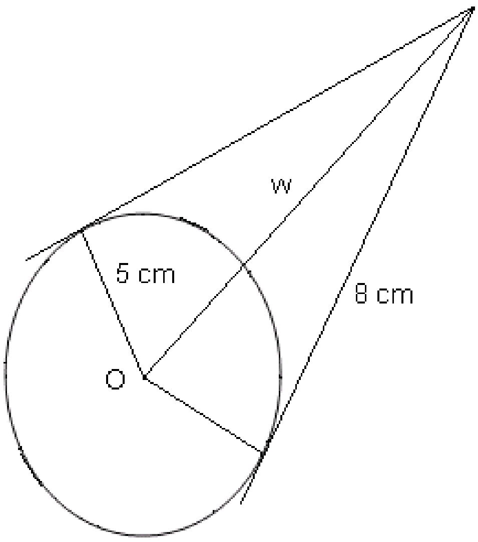
a)



b)



c)



Activity

Activity 14.8

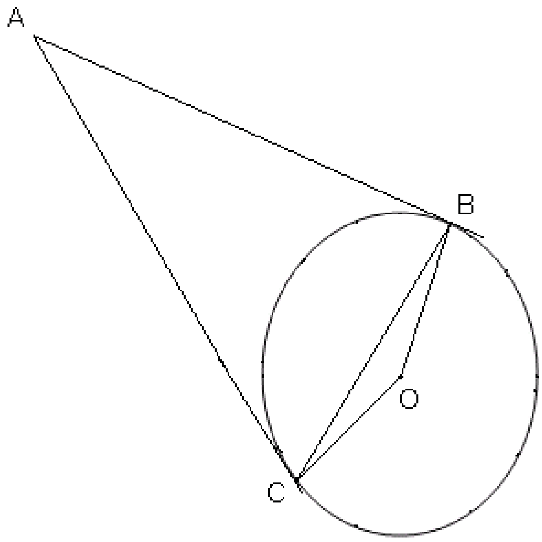
1. In the diagram below, AB and AC are tangents to the circle, centre O, and $BAC = 40^0$.

   Calculate:

   a) $\sphericalangle ABC$

   b) $\sphericalangle OBC$

   c) $\sphericalangle BOC$



Answers to Activity 14.8

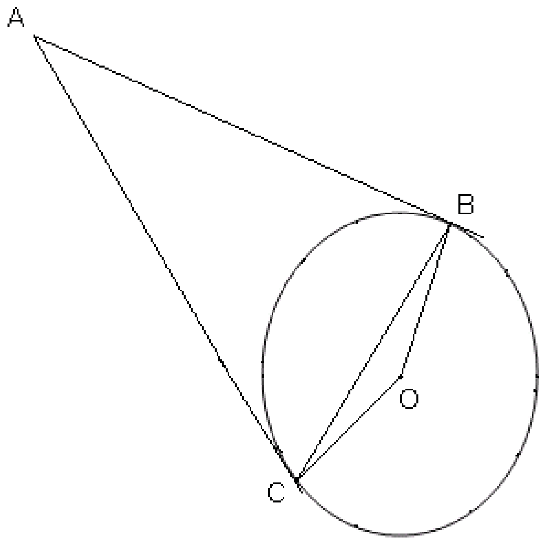
1. In the diagram below, AB and AC are tangents to the circle, centre O, and $BAC = 40^0$.

   Calculate:

   a) $\sphericalangle ABC$

   b) $\sphericalangle OBC$

   c) $\sphericalangle BOC$



Solution

$\measuredangle BAC = 40^0$

Notice that AB is a tangent, OB is a radius, therefore,

$\measuredangle ABO = 90^0$

Consider a quadrilateral ABOC,

$\measuredangle OBA + \measuredangle COB + \measuredangle ACO + \measuredangle BAC = 360^0$

$\measuredangle COB = 360^0 - (\measuredangle OBA + \measuredangle ACO + \measuredangle BAC)$

$\measuredangle COB = 360^0 - (90^0 + 90^0 + 40^0)$

$\measuredangle COB = 360^0 - (220^0)$

$\measuredangle COB = 140^0$

Once again, notice that OA is a bisector of the angles $\measuredangle BAC \text{ and } \measuredangle COB$.

Therefore,

$\measuredangle BAO = \measuredangle CAO$

$= 20^0$

and $\measuredangle BOA = \measuredangle COA$

$= 70^{0}$

$\sphericalangle ABC = \sphericalangle ABO - \sphericalangle GBC$

$= 90^{0} - 20^{0}$

$= 70^{0}$

Lesson 6 Symmetry and Angle Properties of Circles

You have seen how the proofs of the theorems are done and how they can be used to find measures of angles, and lengths (of distances) without actually doing the measurements with the instruments.

What other things have you seen (observed) in these proofs or the diagrams which were used ?

Is there any relationship between arcs, chords, angles, and bisectors of chords?

For the answers to these and other related questions, you have to look back at the theorems again! This time you look at the lines and angles. For instance, do all bisectors pass through the centre of the circle?

At the end of this sub-unit you should be able to:

- *define* the term 'symmetry'
- *identify* line of symmetry in a given diagram
- *solve* practical problems that involve symmetry and angle properties of circles

Symmetry properties of a circle

For a start, let us return to the theorems and categories as we did in the proofs part.

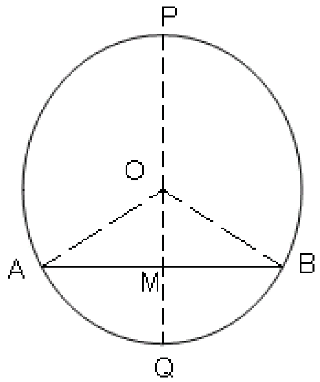
Chords

The theorem about chords is:

Theorem 1. *A perpendicular line from the centre of a circle to a chord bisects the chord. To get a clear picture of the line of symmetry from what we have done, let us do the following activity:*

a) Draw a circle, centre O.
b) Draw a diameter PQ perpendicular to a chord AB, see figure below.
c) Label the point M, be an intersection of diameter PQ and chord AB.

d) Join centre O, to point A and to point B.



Observations from the activity

i. What type of a triangle is Δ ABO?

ii. Measure AO and AB.

AO =

AB =

What do you notice about these distances?

What does a diameter PQ do to the chord AB?

iii. Measure $\sphericalangle OMA$ and $\sphericalangle OMB$.

$\sphericalangle OMA =$

$\sphericalangle OMB =$

What do you notice about the size of these angles?

Complete the sentence using your observations and own words.

In a circle, if the diameter (PQ) (or radius (OQ)) is perpendicular (90^0) to the chord (AB), it ______________________.

Which theorem is this?

Which line is dividing the circle into two equal parts?

From the activity above, you must have noticed that:

i. The diameter PQ cuts the chord into two equal parts and the circle is also divided into two equal parts (semi-circles.) We say a diameter is a line of symmetry in a circle.

ii. A line of symmetry is a line which cuts an object or figure into two identical parts which are mirror images of each other.

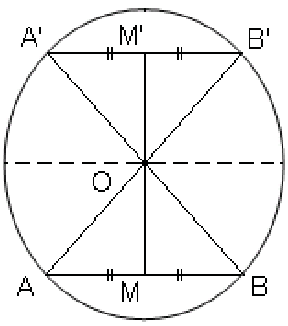
Equal Chords

The theorem about equal chords is:

Theorem 2. *Equal chords of a circle are equidistant from the centre. Now let us do the activity below.*

a) Draw a circle, with centre O.

b) Draw an Isosceles triangle ABO and A'B'O.



Observations from the activity

i. Measure OM and OM'

OM =

OM' =

What do you notice about the distances?

Which theorem is this?

Which line is dividing the circle into two equal parts?

Solution

From the activity, you must have seen that equal chords are the same distance from the centre, O. Again, here the diameter is the line of symmetry along the dotted line to map ABO onto A'B'O.

Note that the line of symmetry MM' is the same as in activity 1!

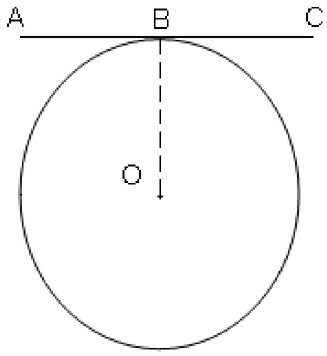
Angle properties of circles

Tangent to a circle:

a) Draw a circle with centre, O.

b) Draw a tangent to the circle.

c) Draw a radius to a point of contact with the tangent.



Observations

Measure the angle between the tangent and the radius.

Size of angle $\sphericalangle ABO$ (or CBO) is

Which theorem is this?

Solution

From the activity you should have noticed that the angle formed by the radius and the tangent at the pont of contact divides the semi-circle into two equal parts. The radius is a line of symmetry on the semi-circle.

Unit Summary

Summary

In this unit you learned that:

- the angle at the centre of a circle is equal to twice the angle at any point on the remaining part of the circumference, if both angles stand on the same arc or chord.
- angles in the same segment are equal.
- the angle in the semi-circle is a right angle. A diameter suggests equal semi-circles and right angles are angles in semi-circles.
- the opposite angles of a cyclic quadrilateral are supplementary. A quadrilateral drawn inside a circle, with all its corners on the circumference, suggests supplementary angles.
- a perpendicular line from the centre of a circle to a chord bisects the chord. A diameter (or a radius) suggests bisected chord (or arc.)
- a tangent to a circle is perpendicular to the radius at the point of contact, and tangents from a point to a circle are equal. A tangent suggests a right angle where a radius meets the circumference and tangent. Two tangents suggest equal lines and equal angles.
- equal chords of a circle are equidistant from the centre and that chords equidistant from the centre are equal. Equal chords suggest equal distance from the centre to the chords. Also, equal distance from the centre to the chords suggests equal chords.
- practical problems that involve symmetry and angle properties of circles. The practical problems include calculations of areas, and volumes of fluids.

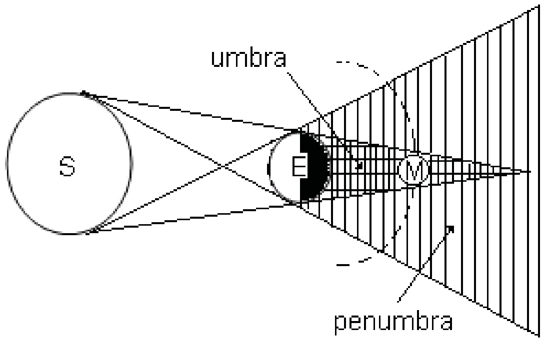
Below are some applications in other fields. Read about them and think about the questions that follow.

Practical applications of a circle

1. In Sciece (Physics, and Geography.)

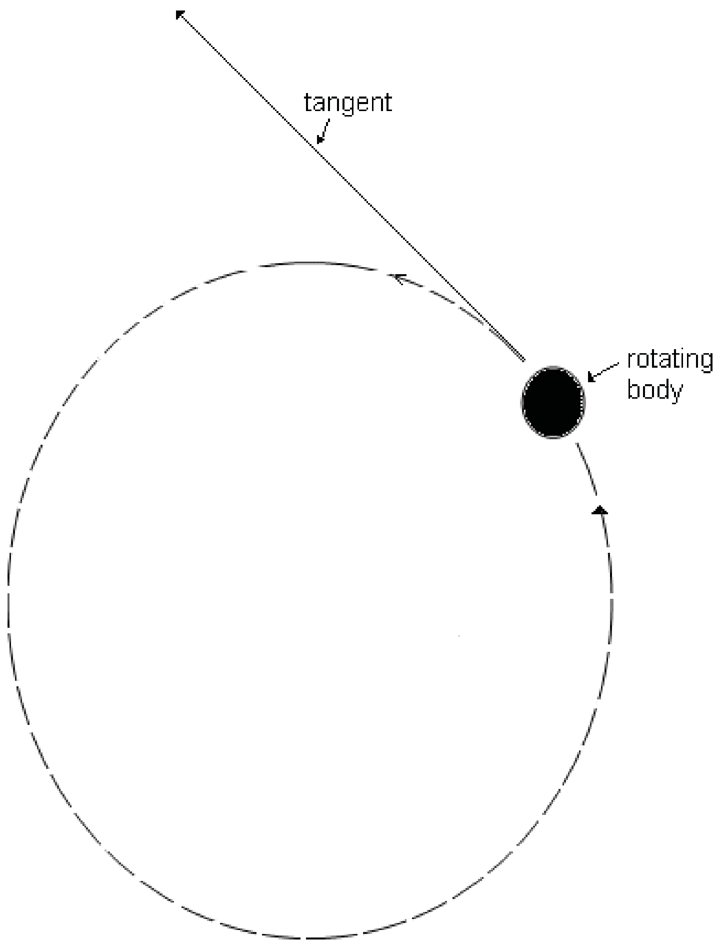
   The sun, S, shining upon the earth, E, causes the earth to cast a shadow, consisting of two parts called the umbra and penumbra. The umbra, or space from which all light is cut off, is the part of the shadow included between the common external tangents; the penumbera, or space from which the light is partially cut off, is included between the common internal tangents.

   What happens when the moon, M, passes through the earth's umbra? When it passes through the penumbra?



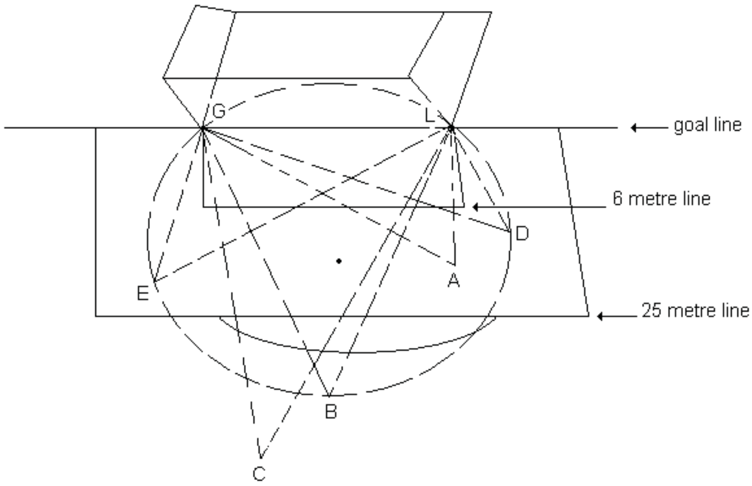
2. In Science (Physics.)

   A body moving in a circular path has a tendency to break away from its circular path and move along the straight line, which is a tangent to the circle. This is the centrifugal tendency.



3. In sports (soccer, rugby, snooker (pool).)

   In soccer, you have goal posts, GL, on a goal line, the six metre line, the twenty-five metre line, and the arc on the twenty-five metre line. If there is a foul, a team is awarded a free kick. Which position, A or B, gives the kicker a better angle to score? Which position, E, B, or D, is best to score? Why? What principles of angle measurement are being used here?


G
L
goal line
6 metre line
D
A
E
25 metre line
B
C

Assignment

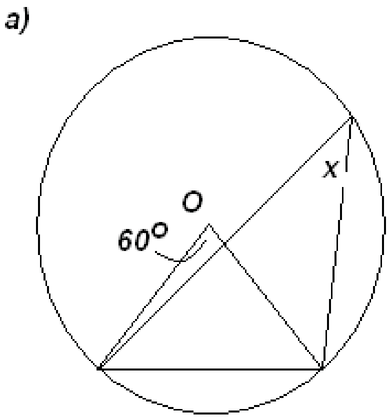
Assignment

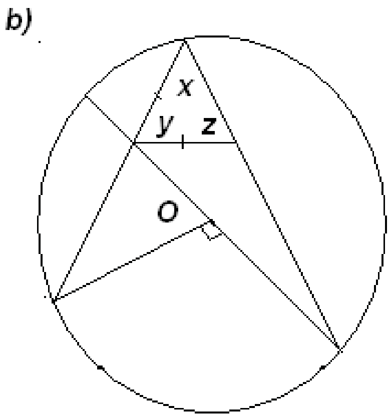
Assignment 14

You should be able to complete this assignment on the Circle in 60 minutes.

[Total marks:29]

1. Find the value of *x, y and z* in the following





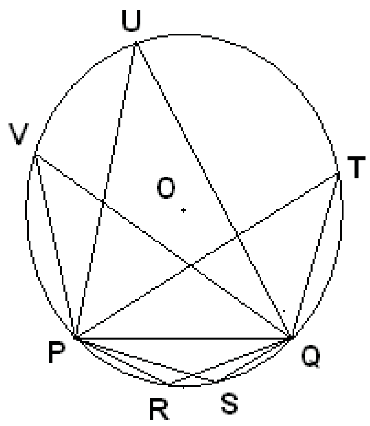
a. x = ____(2)

b. x = ____________ (2)

y = ____________ (2)

z = ____________ (2)

2. With the help of the diagram of the circle centre O, name the following angles



a. Five angles standing on (subtended by) chord PQ on the circumference of the circle.

____________(5)

b. Two angles standing on (subtended by) chord PQ on the circumference, and are on the same side of PQ as $P\hat{T}Q$.

____________(2)

c. Two angles standing on (subtended by) chord PQ on the circumference, and are on opposite sides of PQ.

____________(2)

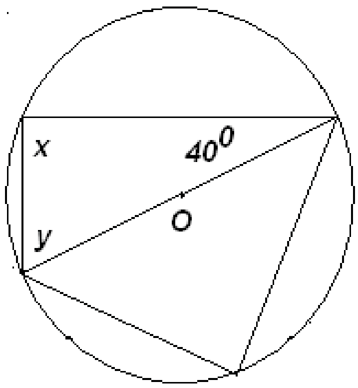
d. An angle standing on (subtended by) chord PQ on the circumference, and on the same side as $P\hat{S}Q$.

____(1)

e. An angle standing on (subtended by) chord PQ at the centre of the circle.

____(1)

3. Find the value of x and y in the following



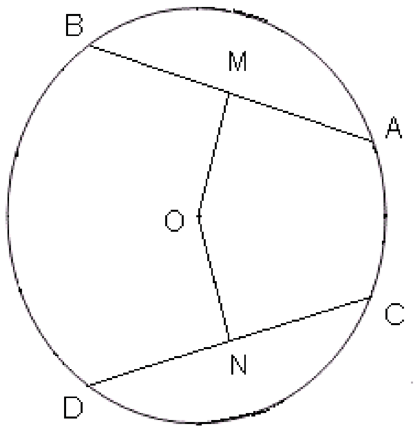
x =

____(2)

y =

___(2)

4. O is the centre; AB = 6cm; OM = 4cm; ON = 3cm.



Calculate

a) The radius of the circle. (3)

b) The length of CD. (3)

Answers to Assignment 14

1.

a. If the angle at the centre and the angle at the circumference are standing on (subtended by) the same chord and are on the same side of that chord, then the angle at the centre is twice the size of the angle at the circumference.

Angle at centre O = 60^0

Angle at circumference = x

By theorem,

$$2x = 60^0$$

$$x = \frac{60^0}{2}$$

$$x = 30^0$$

b. Using same theorem as in a)

Angle at centre O = 90^0

Angle at circumference = x

By theorem,

$$2x = 90^0$$

$$x = \frac{90^0}{2}$$

$$x = 45^0$$

Note that x, y and z are angles in an Isosceles triangle because two sides are equal.

Therefore, the base angles are also equal.

That is, $x = z$

$$z = 45^0$$

$x + y + z = 180^0$ (angles of a triangle)

$$y = 180^0 - (x + y)$$

$$y = 180^0 - (45^0 + 45^0)$$

$$y = 180^0 - 90^0$$

$y = 90^0$

2.

a. $P\hat{R}Q$, $P\hat{S}Q$, $P\hat{T}Q$, $P\hat{U}Q$, $P\hat{V}Q$.

b. $P\hat{U}Q$, $P\hat{V}Q$

c. $P\hat{V}Q$ and $P\hat{R}Q$ or $P\hat{U}Q$ and $P\hat{S}Q$ or $P\hat{T}Q$ and $P\hat{S}Q$ or $P\hat{U}Q$ and $P\hat{R}Q$

d. $P\hat{R}Q$

e. $P\hat{O}Q$

3. If the angle on the circumference of a circle is standing on (subtended by) the diameter of the circle, then that angle is a right angle.

Angle x is subtended by the diameter of the circle.

Therefore,

$x = 90^0$

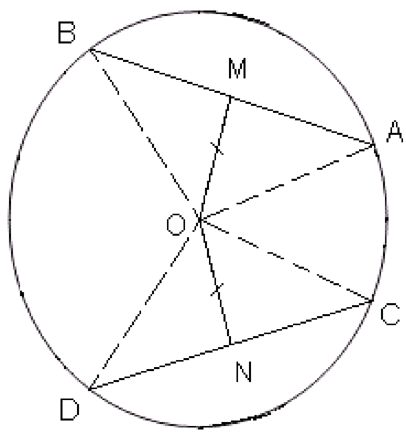
$x + y + 40^0 = 180^0$ (angles of a triangle)

$y = 180^0 - (x + 40^0)$

$y = 180^0 - 130^0$

$y = 50^0$

5.



OM is perpendicular bisector, AB ia a chord. Therefore,

$\measuredangle OMB$ is a right angle. This means you can use the

Pythagoras theorem to find the radius as the missing side of the right angled triangle OAB.

$$OA^2 = AN^2 + MO^2$$

$$= (3\ cm)^2 + (4\ cm)^2$$

$$= 9\ cm^2 + 16\ cm^2$$

$$= 25\ cm^2$$

$$\sqrt{OA^2} = \sqrt{25\ cm^2}$$

$$OA = 5\ cm$$

Radius of the circle is 5 cm.

b) Notice that $OA = OD = OC$ all radii of the same circle.

ON is a bisector of chord CD, CN = ND. You can use Pythagoras because you have two sides in triangle OCN.

$$OC^2 = CN^2 + NO^2$$

$$(5\ cm)^2 = CN^2 + 3\ cm^2$$

$$25\ cm^2 - 9\ cm^2 = CN^2$$

$$16\ cm^2 = CN^2$$

$$\sqrt{16\ cm^2} = \sqrt{CN^2}$$

$$4\ cm = CN$$

The length of CD $= CN + ND$

$$= 4\ cm + 4\ cm$$

$$= 8\ cm$$

Assessment

Assessment

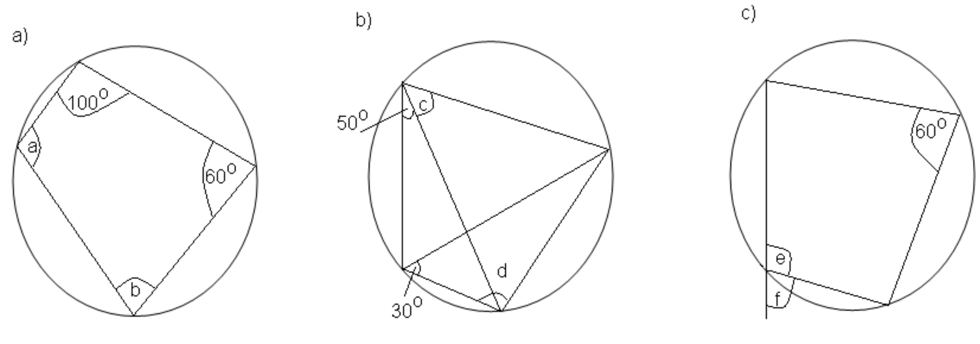
Assessment 14

You should be able to complete this assessment on Circle in 90 minutes.

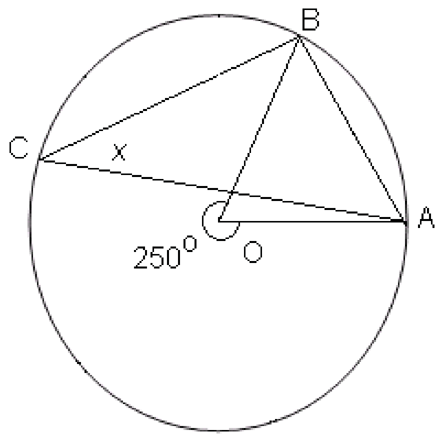
[Total marks: 32]

1. Calculate the size of the angles marked by letters a, b, c, e and f.

(6)

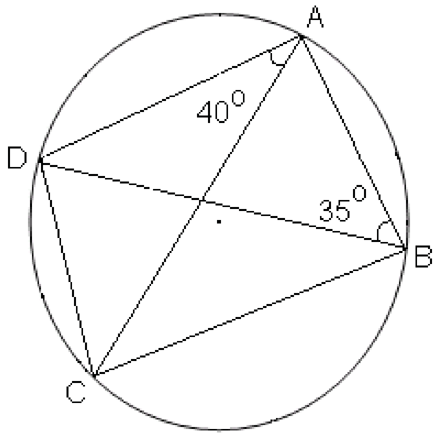


2. Find the value of angle *x*. *(2)*

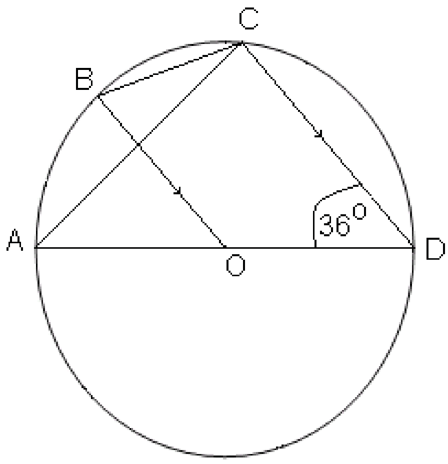


3. A quadrilateral ABCD is drawn inside a circle and AD is parallel to BC. If $\measuredangle ABD = 35^0$ and $\measuredangle DAC = 40^0$.

Find $\measuredangle ADC$. *(2)*



4. The diagram shows a circle, centre O, passing through A,B,C and D. AOD is a straight line, BO is parallel to CD and $\measuredangle CDA = 36^0$.



Find:

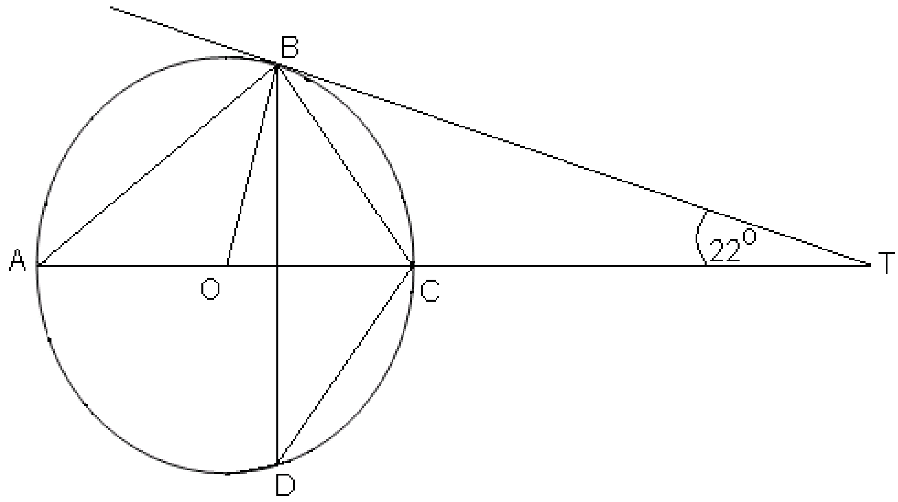
a) $\measuredangle BOA$
(2)

b) $\measuredangle BCA$
(2)

c) $\measuredangle DCB$
(2)

d) $\measuredangle OBC$
(2)

5. In the diagram below, TB is a tangent to the circle, centre O. TO meets the circle at C and A. D is another point on the circle.
$\measuredangle BTC = 22^0$.



a) Use the letters in the diagram to name two right angles.
(2)

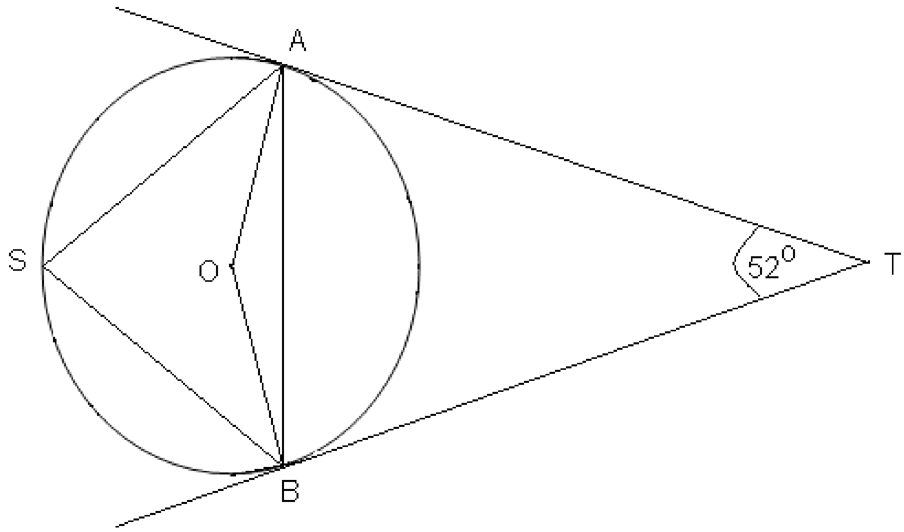
b) Find:

i. $\sphericalangle OAB$
(2)

ii. $\sphericalangle ABT$
(2)

iii. $\sphericalangle BDC$
(2)

6. A, B and S are points on a circle, centre O. TA and TB are tangents.



Calculate:

a) $\measuredangle AOB$
(2)

b) $\measuredangle OBA$
(2)

c) $\sphericalangle ASB$
(2)

Work on your own. Then, check the provided answers below to compare with your responses to the assessment questions.

Answers to Assessment 14

1.

Opposite angles are supplementary in a cyclic quadrilateral.

$a + 60^0 = 180^0$

$a = 180^0 - 60^0$

$a = 120^0$

$b + 100^0 = 180^0$

$b = 180^0 - 100^0$

$b = 80^0$

a)

Angles in a same segment are equal if they stand on the same chord.
Angle *c* is standing on the same chord as angle of 30^0 .
Therefore,

$c = 30^0$.

To find angle *d:*

Opposite angles are supplementary in a cyclic quadtrilateral.

$d + 80^0 = 180^0$

$d = 180^0 - 80^0$

$d = 100^0$

b)

Opposite angles are supplementary in a cyclic quadrilateral.

$e + 60^0 = 180^0$

$e = 180^0 - 60^0$

$e = 120^0$

To find angle *f*:

Exterior angle is equal to interior opposite angle in a cyclic quadrilateral.
So,
$f = 60^0$ (opposite interior is 60^0.)

Note, *f* can also be calculated using the straight angle!

2.

$\measuredangle AOB + 250^0 = 360^0$ (angles at a point add to 360^0)

$$\measuredangle AOB = 360^0 - 250^0$$
$$= 110^0$$

$$\measuredangle ACB = \frac{1}{2} \measuredangle AOB$$

(angle at centre of circle is twice the size of angle at circumference if they are standing on same chord (or arc) AB.)

$$\measuredangle ACB = \frac{1}{2}(110^0)$$
$$= 55^0$$

Therefore, $x = 55^0$

Answer:

$$\measuredangle DAC = \measuredangle DBC = 40^0$$

(angles in same segment standing on chord (arc) DC)

$$\measuredangle ABC = \measuredangle ABD + \measuredangle DBC$$
$$= 55^0 + 40^0$$
$$= 75^0$$

$$\measuredangle ADC + \measuredangle ABC = 180^0$$

(opposite angles of a cyclic quadrilateral ABCD)

$$\measuredangle ADC = 180^0 - \measuredangle ABC$$

$$\measuredangle ADC = 180^0 - 75^0$$
$$= 105^0$$

3.

$$\measuredangle DAC = \measuredangle DBC = 40^0$$

(angles in same segment standing on chord (arc) DC)

$$\measuredangle ABC = \measuredangle ABD + \measuredangle DBC$$
$$= 55^0 + 40^0$$
$$= 75^0$$

$$\measuredangle ADC + \measuredangle ABC = 180^0$$

(opposite angles of a cyclic quadrilateral ABCD)

$\sphericalangle ADC = 180^0 - \sphericalangle ABC$

$\sphericalangle ADC = 180^0 - 75^0$

$= 105^0$

4.

a) $\sphericalangle BOA = 36^0$ (BO is parallel to CD)

b)

$\sphericalangle BCA = \frac{1}{2}(\sphericalangle BOA)$

(angle at centre is twice angle at circumference if both are standing on same chord (arc) AD)

$\sphericalangle BCA = \frac{1}{2}(36^0)$

$= 18^0$

c) $\sphericalangle DCB = \sphericalangle ACD + \sphericalangle BCA$

$= 90^0 + 18^0$

($\sphericalangle ACD$ is angle at circumference subtented by diameter)

$= 108^0$

d) $\sphericalangle OBC + \sphericalangle DCB = 180^0$ (BO is parallel to CD)

$\sphericalangle OBC = 180^0 - \sphericalangle DCB$

$= 180^0 - 108^0$

$= 82^0$

5.

a) Two right angles are $\sphericalangle OBT$ and $\sphericalangle ABC$

$\sphericalangle OBT$ (point of contact between tangent TB and radius OB)

$\sphericalangle ABC$ (angle at circumference standing on diameter AOC)

b)

i. $\sphericalangle BOT = 180^0 - (\sphericalangle OBT + \sphericalangle BTC)$
(angles of Δ BOT)

$\sphericalangle BOT = 180^0 - (90^0 + 22^0)$

$= 180^0 - 112^0$

$= 68^0$

$\measuredangle OAB = \frac{1}{2}(\measuredangle BOT)$

(exterior angle of triangle ABO, $\measuredangle A = \measuredangle B$)

$= \frac{1}{2}(68^0)$ (△AOB is Isosceles, OA= OB; radii)

$= 34^0$

$\measuredangle ABT = \measuredangle OBT + \measuredangle ABO$ ($\measuredangle OAB = \measuredangle ABO$ Δ ABO Isosceles)

$= 90^0 + 34^0$

$= 124^0$

ii. $\measuredangle BDC = \measuredangle BAC$ (angles in same segment standing on chord AB)

$= 34^0$

6.

a) $\measuredangle AOB = 2(\measuredangle AOT)$

$\measuredangle AOT = 180^0 - (90^0 + \frac{1}{2}(52^0))$

$= 180^0 - (90^0 + 26^0)$

$= 180^0 - 116^0$

$= 74^0$

Therefore, $\measuredangle AOB = 2(74^0)$

$= 148^0$

b) $\measuredangle AOB + \measuredangle OBA + \measuredangle OAB = 180^0$

Note that Δ OAB is Isosceles, so base angles are equal;

$\sphericalangle OBA = \sphericalangle OAB$

$\sphericalangle AOB + 2(\sphericalangle OBA) = 180^0$

$148^0 + 2(\sphericalangle OBA) = 180^0$

$2(\sphericalangle OBA) = 180^0 \quad 148^0$

$\sphericalangle OBA = \frac{1}{2}(32^0)$

$\sphericalangle OBA = 16^0$

To find $\sphericalangle ASB$;

The angle at centre is twice the angle at circumference, if both angles are standing on the same chord.

$\sphericalangle ASB$ *is angle at circumference,*

$\sphericalangle AOB$ *is angle at centre*

$\sphericalangle ASB = \frac{1}{2}(\sphericalangle AOB)$

$\sphericalangle ASB = \frac{1}{2}(148^0)$

$= 74^0$

Unit Contents

Unit 15

Unit 15

Trigonometry

Introduction

The word trigonometry comes from two Latin words: *'trigon'* meaning triangle and *'metria'* meaning a measure. Thus, trigonometry is that branch of mathematics concerned with the measurement of triangles.

Trigonometry involves the measurement of angles and sides of triangles. Formerly, this branch of mathematics was used widely in surveying and construction projects, as well as in navigation by ships at sea. Trigonometry enables us to determine lengths and distances when other forms of measurement are difficult or impossible.

During your Junior Certificate course you used the *sine*, *cosine* and *tangent* ratios to calculate sides and angles of right-angled triangles. In this unit that work will be extended to cover calculations of sides and angles in other triangles that are not right-angled.

This unit consists of 79 pages. It covers approximately 3% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 25 hours to work on this unit, 20 hours for formal study and 5 hours for self-study and completing assessments/assignments.

Take a moment to read the following learning outcomes. They are a guide to what you should focus on while studying this unit.

This Unit is Comprised of Six Lessons:

Lesson 1 Sine, Cosine and Tangent

Lesson 2 The Sine Rule

Lesson 3 Deriving the Sine Formula

Lesson 4 The Cosine Formula

Lesson 5 Area of a Triangle

Lesson 6 Angles of Elevation and Angle of Depression

Upon completion of this unit you will be able to:

Outcomes

- *Solve* problems using sine and cosine rules.
- *Calculate* the area of a triangle using trigonometric formulae.
- *Solve* trigonometric problems in two dimensions involving angles of elevation and depression.

Terminology

Sine: Sine* of an angle = the ratio of the side opposite the angle divided by the hypotenuse.

Cosine: Cosine* of an angle = the ratio of the side adjacent to the angle divided by the hypotenuse.

tangent: Tangent* of an angle = ratio of the side opposite the angle divided by the side adjacent to the angle.

Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin c}$

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$

Convention: What has been agreed upon globally by Mathematicians.

An angle of elevation: An angle measured from the horizontal plane upward.

An angle of depression: An angle measured from the horizontal plane downward.

Quadrilateral: Any four-sided shape, e.g. rectangle, square parallelogram.

Perpendicular: An adjective meaning ‘meeting at right angles with”. Two lines are perpendicular if they form an angle of 90° where they meet.

*NOTE: When used in equations, these terms are abbreviated as sin, cos and tan.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Sine, Cosine and Tangent

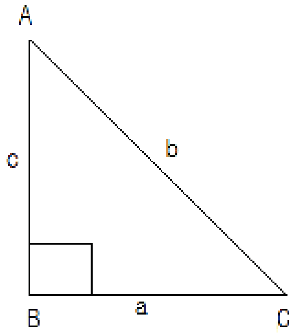
In this section we will remind ourselves of some principles of basic trigonometry and the use of sine, cosine, and tangent ratios that you learned at the Junior Secondary Certificate level. These ratios are used in calculating the size of sides and angles of a right-angled triangle.

I hope you can remember the definition of a right-angled triangle, which you learned during your Junior Secondary Certificate Mathematics course.

Did you remember something like this:

> *A right-angled triangle is a figure with three sides that has a right or 90 degree angle as one of its three angles.*

Did you also recall that, if we have a right-angled triangle with points or vertices at A, B and C, it will have two short sides a and b, while the longest side c is called the hypotenuse.



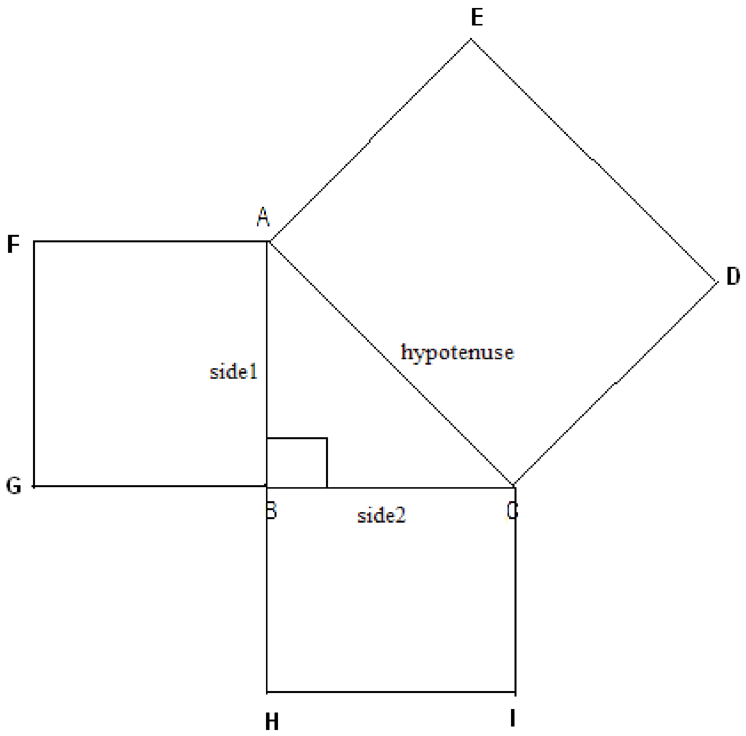
In Triangle ABC above:

$$\sin A = \frac{a}{c}$$

$$\cos A = \frac{b}{c}$$

$$\tan A = \frac{a}{c}$$

Do you still remember how it is derived? Look at the derivation below:



When the squares are drawn accurately, and their areas are calculated it will be found that the area of a square on the hypotenuse is equal the sum of the areas of the squares on side1 and side2. This fact is used to find the sides of the right-angled triangle in the middle. The theorem can be used even when the squares are absent.

$\text{hypotenuse}^2 = \text{side1}^2 + \text{side2}^2$

Pythagoras' theorem: $c^2 = a^2 + b^2$

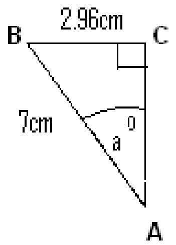
NOTE: The letters a, b, and c represent the sides of the triangle, while A, B, and C refer to both the vertices and the angles formed at them. These labels will be used throughout this unit.

Let's remind ourselves of how these formulae work by finding the sides and angles marked with a letter in the following triangles:

Example 1

If we want to know the size of angle formed at A, and we know the length of the side opposite it and the length of the hypotenuse, we can use the sine to find angle A. The sine of angle A is equal to the side opposite angle A divided by the hypotenuse. You can then use your mathematical tables or a calculator to find the angle that corresponds to the value of the sine.

Calculate angle A in the diagram

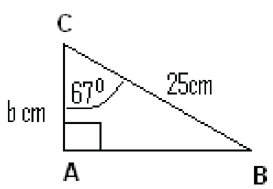


$$\sin A° = \frac{2.96}{7}$$
$$\sin A° = 0.4229$$
$$A = 25°$$

Example 2

If you know how many degrees there are in one of the other angles and the length of the hypotenuse, you can find the length of the other side b that forms the angle with the hypotenuse. Side b is referred to as adjacent to the angle.

Calculate AC in the following diagram



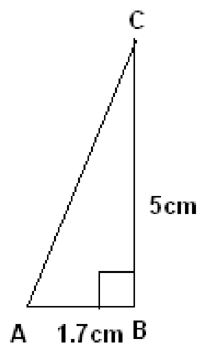
Remember that the cosine equals the adjacent side divided by the hypotenuse. Using your mathematical tables or a calculator you can then find the value for the cosine of the angle you know, then substitute this in the formula.

$$\cos 67° = \frac{b}{25}$$
$$25\cos 67° = b$$
$$9.8cm = b$$

Example 3

If we know the lengths of the two sides adjacent to the right angle, we can find how many degrees are in the other angles by using the tangent formula. Dividing the side opposite by the side adjacent gives us a figure for the tangent of angle C. Again, you need to go to your mathematical tables or use your calculator to find the size of the angle that corresponds with the figure you have calculated for its tangent.

Calculate angle C in the following diagram



$$\tan C = \frac{1.7}{5}$$
$$\tan C = 0.34$$
$$C = 18.8°$$

Can you figure out how to find the size of the third angle in the triangle above?

Remember that the three angles in every triangle add up to 180°. Thus, we can just subtract the two angles we know from 180° to find the answer. Can you do this in your head?

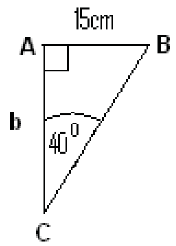
Or

You can reverse the values in the tangent formula above to find the value of angle A.

$$\tan A = \frac{5.0}{1.7}$$
$$\tan A = 2.94$$
$$A = 71.2°$$

Example 4

Which formula should we use in this example? Try figuring it out on your own before looking at the answer below:

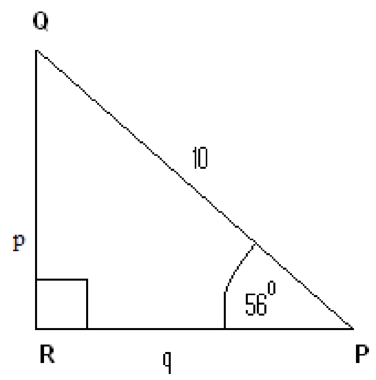


$$\tan 40° = \frac{15}{b}$$
$$b \tan 40° = 15$$
$$b = \frac{15}{\tan 40} = \frac{15}{0.8390} = 17.8$$

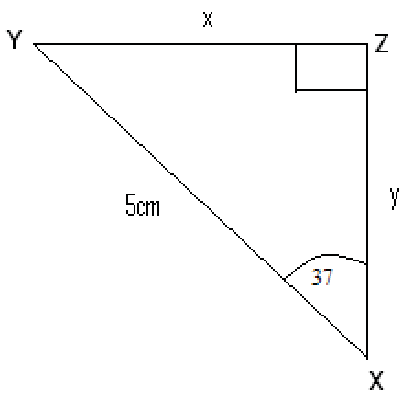
Activity 15.1

In each of the following triangles calculate the lengths and angles that are marked with a letter.

(a) Hypotenuse = 10, one angle = 56°. Calculate p and q.

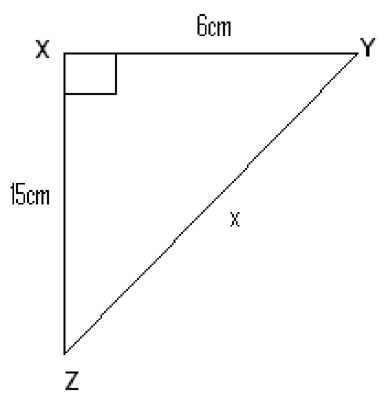


(b) Hypotenuse = 5 cm, one angle = 37°. Calculate x and y.

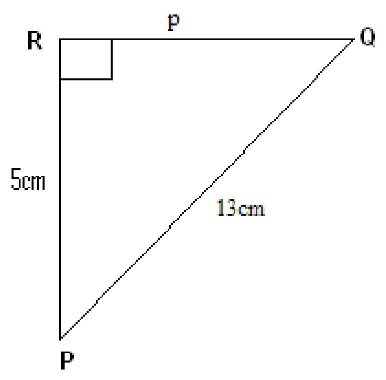


2. Use Pythagoras' theorem to find side 'x' in the following right –angled triangles.

(a)



(b)



Compare your answers with those at the end of the sub-unit. Be sure you remember the trigonometric ratios you learned at your junior certificate level before you continue to the next section. You should also continue only if you answered all questions correctly. If not, try the activity again.

Key Points to Remember

The key points to remember in this sub-unit on trigonometric ratios used in right-angled triangles are:

- **Trigonometric ratios are:**

Note it!

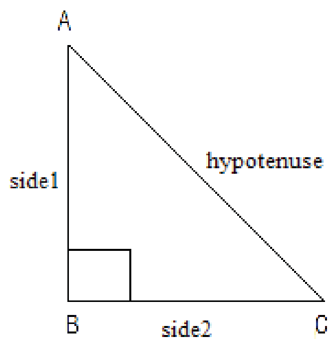
$$\sin A = \frac{side\,opposite\ A}{hypotenuse}$$

$$\cos A = \frac{side\,adjacent\ A}{hypotenuse}$$

$$\tan A = \frac{side, opposite, A}{side, adjacent, A}$$

- **Pythagoras' theorem is:**

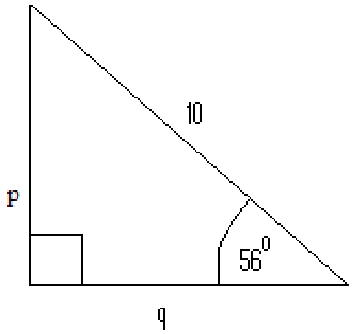
$(hypotenuse)^2 = (side1)^2 + (side2)^2$ using the diagram below



Model Answers

Activity 15.1

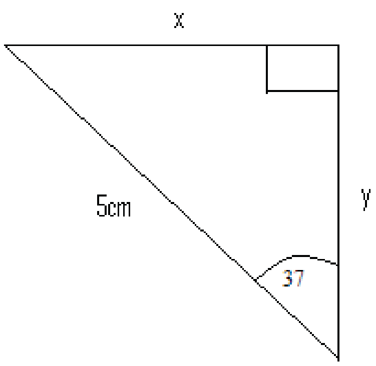
1. (a)



$$\sin 56° = \frac{p}{10}$$
$$10\sin 56° = p$$
$$8.29 = p$$

$$\cos 56° = \frac{q}{10}$$
$$10\cos 56° = q$$
$$5.59 = q$$

(b)

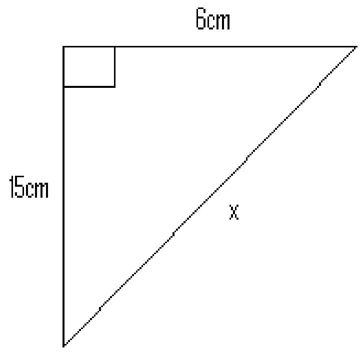


$$\sin 37° = \frac{x}{5}$$
$$5\sin 37 = x$$
$$6.01 = x$$

$$\cos 37° = \frac{y}{5}$$
$$5\cos 37 = y$$
$$7.99 = x$$

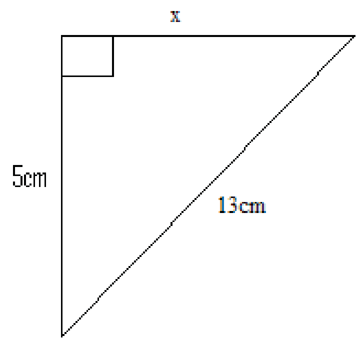
2.

(a)



$$p^2 = \sqrt{15^2 - 6^2}$$
$$p^2 = 15^2 + 6^2$$
$$p = \sqrt{15^2 + 6^2}$$
$$p = \sqrt{261}$$
$$p = 16.16$$

(b)



$$x^2 = 13^2 - 5^2$$
$$\sqrt{x^2} = \sqrt{13^2 - 5^2}$$
$$x = \sqrt{144}$$
$$x = 12$$

Lesson 2 The Sine Rule

Imagine that you are building a house for your family. Unfortunately, the site has a rock too large to move exactly where you wanted to place one corner of the house. The only solution is to move the corner so that it no longer forms a right angle. How can you calculate the lengths of the walls of your house so that you can buy the building materials needed?

Not all structures are built with right angles, nor are right-angled triangles found consistently in other situations where trigonometry is applied in everyday life. For this reason, we need to find ways to make measurements using triangles that do not contain a right angle.

By the end of this sub-unit, you should be able to:

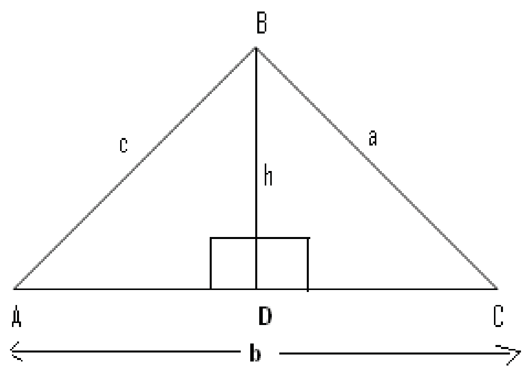
- Define the meaning of the term sine rule;
- Explain how the ratio can be used to calculate different dimensions of triangles;
- Decide where the sine rule can be applied appropriately;
- Calculate sides and angles of triangles that are not right-angled using the sine rule.

Important Note

It is very important for you to remember that the ratios sine, cosine, tangent and Pythagoras' Theorem are used only in right-angled triangles. For triangles that do not have a right-angle, we need to use other formulae; these are called the sine rule and the cosine rule.

Don't forget that the letters a, b and c represent the different sides of a triangle and their lengths. The upper-case or capital letters A, B and C refer to the vertices or points where two sides of a triangle meet, as well as the angles formed.

Lesson 3 Deriving the Sine Formula
You can derive the sine formula by drawing a triangle which has height *h,* as shown below.



$\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$

$c \sin A = h$ and $a \sin C = h$

$c \sin A = a \sin C$

Divide by sin C on both sides

$$\frac{c \sin A}{\sin C} = \frac{a \cancel{\sin C}}{\cancel{\sin C}}$$

$$\frac{c \sin A}{\sin C} = \frac{a}{1}$$

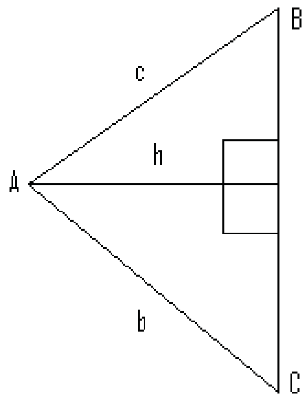
Divide both sides by sin A

$$\frac{c \sin A}{\sin C \sin A} = \frac{a}{\sin A}$$

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

When a perpendicular line is drawn from A to BC the above steps can be repeated to show that $\frac{a}{\sin A} = \frac{b}{\sin B}$ as shown below.



sin B = $\frac{h}{c}$ and sin C = $\frac{h}{b}$

When we multiply both sides in the first equation by *c* we get,

c sin B = h

And when we multiply both sides of the second equation by *b* we get,

b sin C = h

Therefore,

c sin B = b sin C

When we divide both sides by sine B we get,

$$\frac{c\,\sin B}{\sin B} = \frac{b\sin C}{\sin B}$$

$$\frac{c}{1} = \frac{b\sin C}{\sin B}$$

When we divide both sides by sin C we get,

$$\frac{c}{1\times\sin C} = \frac{b\sin C}{\sin B\times\sin C}$$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

We have also shown that $\frac{c}{\sin C} = \frac{a}{\sin A}$

Therefore,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The above formulae is called the sine rule or the sine formulae.

This formula can be broken down into three equations as follows:

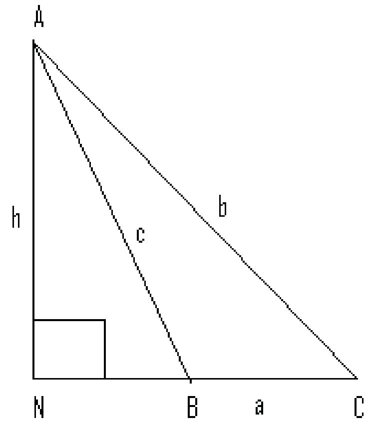
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

The formulae can also be shown to work for the obtuse- angled triangle ABC drawn below. Try it yourself.

Triangle ABC is tilted in such a way that we are unable to draw a perpendicular height inside it. Instead, we need to create new lines outside the original triangle ABC to work out its dimensions. First draw an extension of the line CB. When a perpendicular line (h) is drawn from A it touches the extension of CB at N. Notice that we now have a new triangle ANC with one right angle.



In triangle ANC,

First find the formulae for sine C:

$$\sin C = \frac{h}{b}$$

Then make h the subject of the formulae:

b sinC $= h$

$$\sin(180° - B) = \frac{h}{c}$$
$$c\sin(180° - B) = h$$

NB $\sin(180° - B) = \sin B$ (this is because the sine of any pair of supplementary angles are equal).

To check this, find the sine of 120 and the sine of its supplement 60. They both give us 0.866.

$$c\sin(180° - B) = c\sin B = h$$

$$b\sin C = c\sin B$$

The example above shows that the sine formula holds for both acute and obtuse-angled triangles.

Choosing the Appropriate Equation to Solve Trigonometric Problems.

How do you know which equation to use when solving angles and sides in a given triangle?

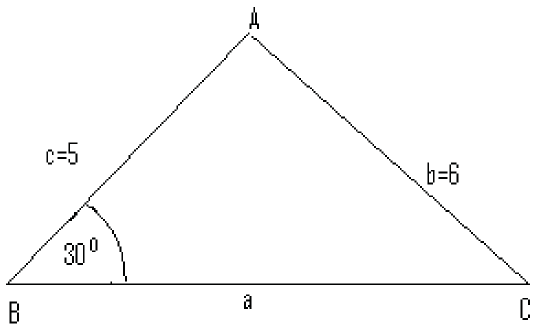
Tip

Tips:

- Choose the formulae where only one unknown is formed when the given information is substituted into the equation.
- The equation should also contain the side or angle to be calculated.

Example 1

Calculate the size of angle C given that $b = 6$ cm, $c = 5$ cm and angle $B = 30°$



In this example we need the equation that contains side b and c and angle B.

Step 1 - Choose the equation.

$$\sin \frac{b}{\sin B} = \frac{c}{\sin C}$$

Step 2 - Substitute the given information into the equation.

$$\sin \frac{6}{\sin 30°} = \frac{5}{\sin C}$$

Remember that angle C is the only unknown.

Step 3 - Make sine C the subject of the formula.

Remove denominators by cancelling $\sin 30°$ and sine C.

Cancel $\sin 30°$

$$\frac{6\sin 30^0}{\sin 30^0} = \frac{5\sin 30^0}{\sin C}$$

$$6 = \frac{5\sin 30^0}{\sin C}$$

Multiply by sine C both sides and cancel sine C

$$\frac{6}{1} * \sin C = \frac{5\sin 30° * \sin C}{\sin C}$$

6 * sine C =5 sine30Divide by 6 on both sides and cancel 6

$$\frac{6\sin C}{6} = \frac{5\sin 30°}{6}$$

$$\frac{6\sin C}{6} = \frac{5\sin 30°}{6}$$

$$\sin C = \frac{5\sin 30^0}{6}$$

Step 4 - Calculate the angle C using the $\frac{5\sin 30°}{6}$ value.

Find sin 30 from your calculator or sine tables from your mathematical tables.

$\sin 30° = 0.5$

$$\sin C = \frac{5 \times 0.5}{6}$$

$\sin C = 0.416666......$

Let us now round off 0.4166..... to approximately four decimal places=0.4167

Therefore sinC = 0.4167

At this stage we have not yet found angle C.

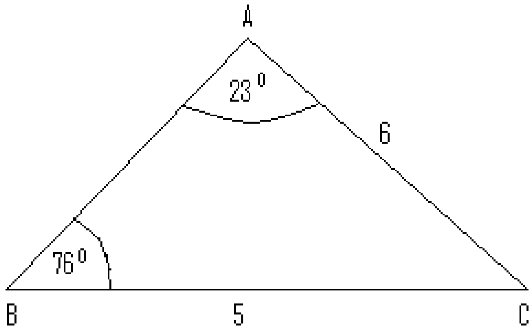
Step 5

After getting 0.4167 we do not remove it from our calculator, we proceed to click the inverse button followed by the sine button on a calculator to get the result $24.63°$. The result is the size of angle C.

$angle\,C = 24.63°$

Example 2

Calculate the unknown quantities in the following triangle:



Angle C $= 180° - (76° + 23°) = 81$ using the angle- sum of a triangle

Angle $C = 81°$

To calculate side c:

Let us now choose a formula that will have c as the only unknown variable in the whole equation.

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 81°} = \frac{6}{\sin 76°}$$

To make c the subject of the formula let us start by multiplying the left hand side and the right hand side of this equation by sin81º to get,

$$\frac{c}{\sin 81°} \times \sin 81° = \frac{6}{\sin 76°} \times \sin 81°$$

The result is $c = \dfrac{6 \sin 81°}{\sin 76°}$

C is now the subject of the formula.

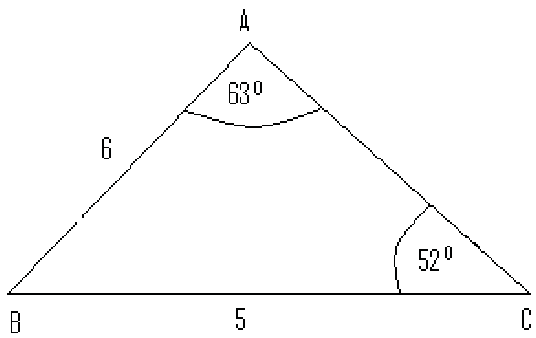
Let us evaluate the right hand side of the equation. Find sin 81º and sin 76º from a calculator or the trigonometric tables.

$$c = \frac{6 \times 0.9877}{0.9703}$$

c = 6.11cm

Example3

Calculate the unknown quantities in the following triangle:



The unknown quantities are side a, side b and angle B.

Like in the first example to get the third angle we use angle – sum of a triangle.

Therefore angle B = 180º – (63 º +52 º) = 65 º.

To find side b we use the following formula which side b as the only unknown. The formulae is:

$$\frac{b}{\sin B} = \frac{6}{\sin 52°}$$

Make b the subject of the formula by multiplying by sin65 on both sides.

$$b = \frac{6 \sin 65°}{\sin 52°}$$

Look up sin 65 º and sin52 º in your calculator or trigonometric tables.

sin 65 º = 0.9063 and

sin 52 º = 0.7880

Therefore,

$$b = \frac{6 \times 0.9063}{0.7880}$$

$$b = 6.9$$

To find side a,

Note it!

$$\frac{a}{\sin A} = \frac{6}{\sin 52°}$$

There are two alternatives because you now have all values except side a. However, it is wise to avoid values you have calculated because it is possible that you made a mistake.

$$\frac{a}{\sin 63°} = \frac{6}{\sin 52°}$$

Find the subject of the formulae by multiplying both sides of the equation by sin 63°.

$$a = \frac{6 \sin 63°}{\sin 52°}$$

$$a = \frac{6 \times 0.8910}{0.7880}$$

$$a = 6.78$$

Activity 15.2

1. Calculate unknown angles and sides of the triangles in the following table:

Activity

A	B	C	a	b	C
	28°	64°	8.52cm		
	86°	48°		7.24cm	

81°	42.5°			28.1cm	
44°	68°				12cm

Compare your answers with the ones at the end of the sub-unit. If you get less than ten items correct, review the examples above. If you get ten or more right, proceed to the next section on the cosine formula.

Key Points to Remember

The key points to remember in this sub-unit on the sine rule are:

- The sine formulae is $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

  Where *a, b* and *c* are sides of a triangle and A, B and C are angles opposite the sides respectively.

- This formula can be broken down into three equations as follows:

  $$\frac{a}{\sin A} = \frac{b}{\sin B}$$

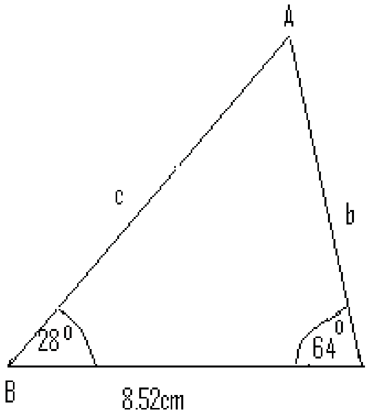
  $$\frac{a}{\sin A} = \frac{c}{\sin C}$$

  $$\frac{b}{\sin B} = \frac{c}{\sin C}$$

- The sine formulae is used when the triangle is not right-angled.
- Choose the formulae where only one unknown is formed when the given information is substituted into the equation.
- The equation should also contain the side or angle to be calculated.
- Sometimes the information given does not allow us to use the sine formula in those triangles that are not right-angled. Another formula that can be used is the cosine formula, which is also called the cosine rule.

Model Answers

Activity 15.2

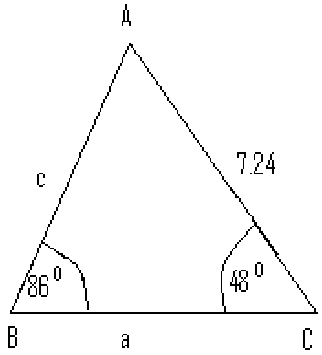


A = {180 º - (28 º + 64 º)}

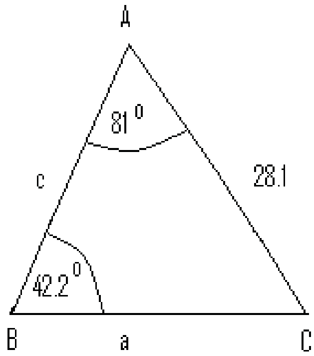
A= 88 º

$$\frac{b}{\sin 28°} = \frac{8.52}{\sin 88°}$$
$$b = \frac{8.52 \times \sin 28°}{\sin 88°}$$
$$b = \frac{8.52 \times 0.4695}{0.9994}$$
$$b = 4.00cm$$

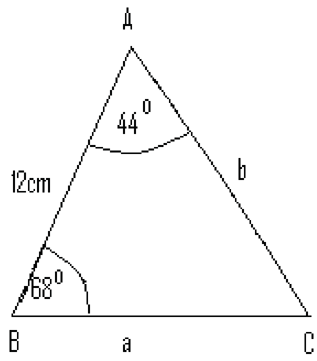
$$\frac{c}{\sin 64°} = \frac{8.52}{\sin 88°}$$
$$c = \frac{8.52 \sin 64°}{\sin 88°}$$
$$c = \frac{8.52 \times 0.8988}{0.9994}$$
$$c = 7.66cm$$



$$A = 180° - (86° + 48°)$$
$$A = 46$$
$$\frac{a}{\sin 46°} = \frac{7.24}{\sin 86°}$$
$$a = \frac{7.24 \times \sin 46°}{\sin 86°}$$
$$a = \frac{7.24 \times 0.7193}{0.9976}$$
$$a = 5.22cm$$



$$\frac{a}{\sin 81°} = \frac{28.1}{\sin 42.5°}$$
$$a = \frac{28.1 \times \sin 81°}{\sin 42.5°}$$
$$a = \frac{28.1 \times 0.9877}{0.6756}$$
$$a = 41.08$$



$$C = 180° - (44° + 68°)$$
$$C = 68°$$

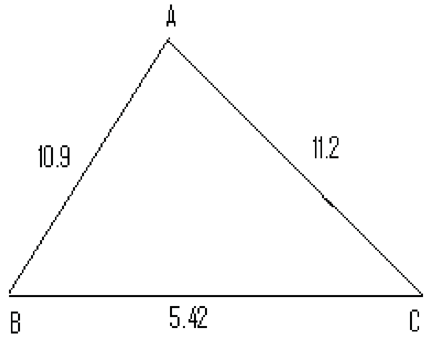
$$\frac{a}{\sin 44°} = \frac{12}{\sin 68°}$$
$$a = \frac{12 \times \sin 44°}{\sin 68°}$$
$$a = \frac{12 \times 0.6947}{0.9272}$$
$$a = 8.99$$

To find side *b*, think about the type of triangle that triangle ABC is. Hint: two of its angles are equal to 68º. What can you say about the sides opposite these two equal angles?

They are equal. Therefore $c = b = 12$ cm.

Lesson 3 The Cosine Formula

Suppose we know the sizes of the three sides of a triangle that is not right-angled. Is it possible to calculate its angles using the sine formulae? Use the following example where only the sides are given to answer this question.



To be able to use the sine formula, at least one angle must be known. There are situations other than this where the sine formula cannot be used. Instead, we can use the cosine formula to find the number of degrees in each angle. The situations where the cosine should be used will become clearer as we derive the formula.

By the end of this sub-unit, you should be able to:

- Define the meaning of the term cosine rule;
- Explain how the cosine rule can be used to calculate different dimensions of triangles;
- Decide where the cosine rule can be applied appropriately;
- Calculate sides and angles of triangles that are not right-angled using the cosine rule.

Finding the Cosine Formulae

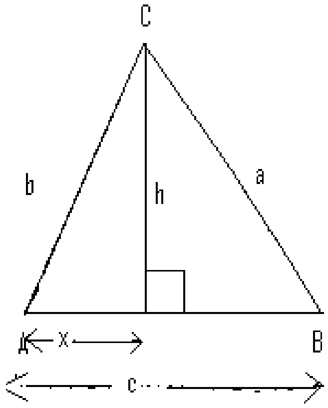
Sometimes the information given does not allow us to use the sine formula. For example, if you were asked to calculate the length of side c, if $a = 4$ $b = 3$ and C = 40°, how would you go about it?

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{4}{\sin A} = \frac{3}{\sin B} = \frac{c}{\sin 40°}$$

In this case no equation will leave one unknown. So we are unable to use the sine formula. Another formula – the cosine formula – is used instead.

Finding the cosine formula.



Using Pythagoras' theorem

$$b^2 = h^2 + x^2$$

$$a^2 = h^2 + (c-x)^2$$

Let us work out the bracket $(c-x)^2$

$$(c-x)^2 = (c-x)(c-x)$$
$$(c-x)^2 = c^2 - 2cx + x^2$$

Let us rewrite $a^2 = h^2 + (c-x)^2$ using the new expression for $(c-x)^2$

then $a^2 = h^2 + c^2 - 2cx + x^2$

By rearranging: $a^2 = c^2 + (h^2 + x^2) - 2cx$

Remember that:

$$b^2 = h^2 + x^2.$$

We then substitute this into our equation to get

$$a^2 = c^2 + b^2 - 2cx$$

$$\cos A = \frac{x}{b}$$
$$x = b\cos A$$

Substituting bcosA for x in $a^2 = c^2 + b^2 - 2cx$ we get

$$a^2 = c^2 + b^2 - 2cb\cos A$$

This is called the cosine formula.

Forms of the Cosine Formula

$$a = b + c - 2bc\cos A$$

$$b = a + c - 2ac\cos B$$

$$c = a + b - 2ab\cos C$$

Tip

Tips

(a) As we did with the sine formula, choose the formula that leaves one unknown when the given information is substituted.

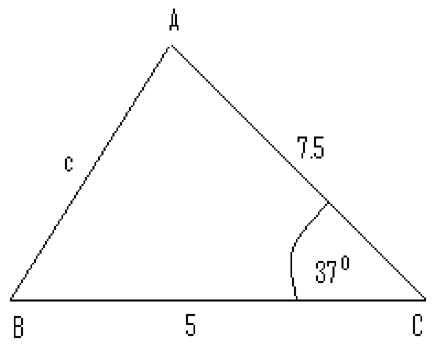
(b) The side or angle that is to be calculated should be in the formula.

(c) To easily get the terms on the right hand side look at the following patterns:

Subject of the formula	First two terms added (i.e. the remaining sides)	The sides that are multiplied by -2 (remaining sides)	Angle in the last term (this is the angle opposite the side that is the subject of the formula)
a	$b^2 + c^2$	bc	A
b	$a^2 + c^2$	ac	B
c	$a^2 + b^2$	ab	C

Example 1

Calculate c given that $a = 5$, $b = 7.5$ and angle C = 37º



If we are looking for *c,* the formula chosen should have *c* as its subject.

The two remaining sides are *a* and *b*, therefore our formula should contain,

$c^2 = a^2 + b^2$ …………..

If we have $a^2 + b^2$, then -2 is multiplied by *ab* so that our formula becomes,

$c^2 = a^2 + b^2 - 2ab$......

The cosine that multiplies -2ab is that of the angle opposite the side that is the subject of the formula as shown below,

$c^2 = a^2 + b^2 - 2ab\cos C$

Substituting the given values into our equation we get,

$c^2 = 5^2 + 7.5^2 - 2\times 5\times 7.5\times \cos 37$

Work out $b^2 + a^2$ and $-2bacosC$ separately

$c^2 = 25 + 56.25 - 75\cos 37$

$c^2 = 81.25 - 75\cos 37$

NOTE: $a^2 + b^2 = 81.25$

$c^2 = 81.25 - 75\times 0.7986$

$c^2 = 81.25 - 59.9$

NOTE: $2ab\cos C = 59.9$

We then subtract 59.9 from 81.25 as the last step.

Tip

DO NOT SUBTRACT BEFORE MULTIPLYING. Remember the order of operations by recalling the mnemonic BODMAS, which stands for Bracket, Of, Division, Multiplication, Addition and Subtraction.

$$c^2 = 21.35$$

Find the square root of both sides.

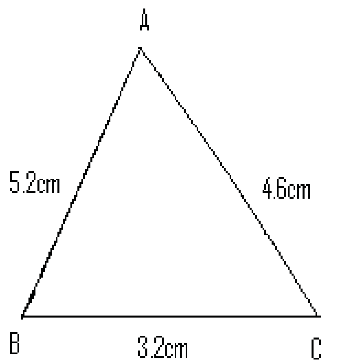
$$\sqrt{c^2} = \sqrt{21.35}$$

$$c = 4.62$$

Example 2

Calculate the size of angle C given that,

a = 3.2cm, b = 4.6cm, c = 5.2cm



What formula should you use?

____

Compare your formula with the one below:

Finding cosC is best since you are asked to calculate C. So, the formula to use is:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Where c^2 is the subject of our formula, while a and b are the remaining sides.

Substitute the given information into the equation:

Compare your answer with the one below:

$$5.2^2 = 3.2^2 + 4.6^2 - 2 \times 3.2 \times 4.6 \times \cos C$$

Simplify by calculating the squares and the product:

Compare your answer with the one below:

$$27.04 = 10.24 + 21.16 - 29.44 \times \cos C$$

Add 10.24 and 21.16 and then subtract that amount from each side of the equation:

Compare your answer with the one below:

$$-4.36 = -29.44 \times \cos C$$

Divide both sides by -29.44 to isolate $\cos C$

Compare your answer with the one below:

$$\frac{-4.36}{-29.44} = \frac{-29.44\cos C}{-29.44}$$

$$\frac{-4.36}{-29.44} = \frac{-\cancel{29.44}\cos C}{-\cancel{29.44}}$$

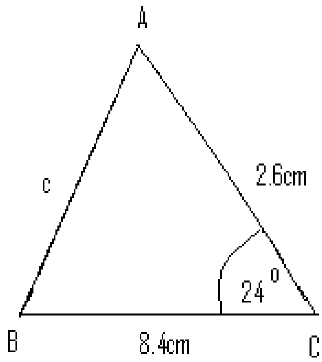
$$0.1481 = \cos C$$

Find the inverse of cos 0.1481 to get angle C

Compare your answer with the one below:

$$81.4° = C$$

Example 3
Calculate c in the following triangle,



c has to be the subject of the formula because it is required.

Which formula should you use?

Compare your formula with the one below:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Substitute the given information into the equation:

Compare your answer with the one below:

$$c^2 = 8.4^2 + 2.6^2 - 2 \times 8.4 \times 2.6 \times \cos 24°$$

Simplify by calculating the squares and the product.

Compare your answer with the one below:

$$\cos 24° = 0.9135 \text{ So,}$$

$$c^2 = 70.56 + 6.76 - 43.89 \times 0.9135$$

Multiply 0.9135 by 43.89 and simplify the right hand side to solve for c^2

Compare your answer with the one below:

$$c^2 = 77.32 - 40.09$$

$$c^2 = 37.23$$

Find the square root of both sides.

Compare your answer with the one below:

$c = 6.1\,\text{cm}$

Activity

Activity 15.3

Use a separate sheet of paper to calculate the lengths and angles left out in the following table:

	A	B	C	a	b	c
Question 1	______	______	110°	3.7	4.2	_____
Question 2	60.4°	______	______	______	21.2	14.3
Question 3	_______	95.5°	______	4.32	_____	2.81
Question 4	______	______	_____	2.43	7.12	6.43

Compare your answers with the ones worked out below. If you get less than ten of the twelve items correct, review the examples above. If you get ten or more right, proceed to the next section on the area of triangles.

Note it!

Key Points to Remember

The key points to remember in this sub-unit on sine rule are:

- Cosine formula is:

  $a^2 = c^2 + b^2 - 2cb\cos A,$

  Where a, b and c are sides of a triangle and A, B and C are angles opposite the sides respectively.

- Forms of the cosine formula are:

  $$a = b + c - 2bc\cos A$$

  $$b = a + c - 2ac\cos B$$

  $$c = a + b - 2ab\cos C$$

- These cosine formulae can be used when you are dealing with a triangle that does not include a right or 90° angle.
- Choose the formula where only one unknown is formed when the given information is substituted into the equation.

- The equation should also contain the side or angle to be calculated.

Model Answers

Activity 15.3
Question 1

To find side *c*

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$c^2 = 3.7^2 + 4.2^2 - 2 \times 3.7 \times 4.2 \times \cos 110°$$

$$c^2 = 13.69 + 17.64 - 31.08 \times \cos 110°$$

$$c^2 = 31.33 - 31.08 \times -0.8420$$

$$c^2 = 31.33 + 10.63$$

$$c^2 = 41.96$$

$$c = \sqrt{41.96}$$

$$c = 6.48$$

Question 1

To find angle B

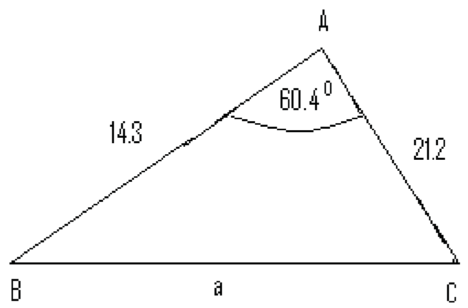
$$B^2 = a^2 + c^2 - 2ac\cos B$$
$$4.2^2 = 3.7^2 + 6.48^2 - 2\times 3.7\times 6.48\times \cos B$$
$$17.64 = 13.69 + 41.99 - 47.95\times \cos B$$
$$17.64 - 55.69 = -47\cos B$$
$$\frac{-38.04}{-47.95} = \cos B$$
$$0.7933 = \cos B$$
$$37.5 = B$$

Question 1

To find angle A

A=180° – (37.5°+110°) = 33°

Question 2



To find side *a*

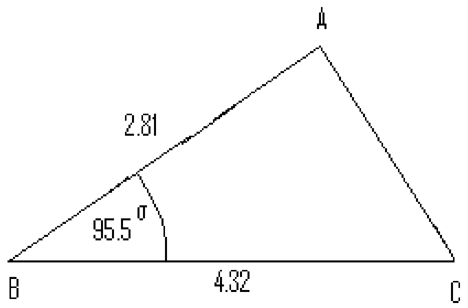
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$a^2 = 21.2^2 + 14.3^2 - 2\times 21.2\times 14.3\times \cos 60.4°$$
$$a^2 = 449.44 + 204.49 - 606.32\times 0.4939$$
$$a^2 = 653.93 - 299.46$$
$$a^2 = 354.33$$
$$\sqrt{a^2} = \sqrt{354.33}$$
$$a = 18.8$$

Question 2

To find angle B

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$21.2^2 = 18.8^2 + 14.2^2 - 218.8 \times 14.2 \times \cos B$$
$$449.44^2 = 353.44 + 201.64 - 3106.93\cos B$$
$$449.44 = 555.08 - 3106.93\cos B$$
$$449.44 - 555.08 = -3106.93\cos B$$
$$-105.64 = -3106\cos B$$
$$0.034001 = \cos B$$
$$88.05° = B$$
$$88.1° = B$$

Question 3



To find side *b*

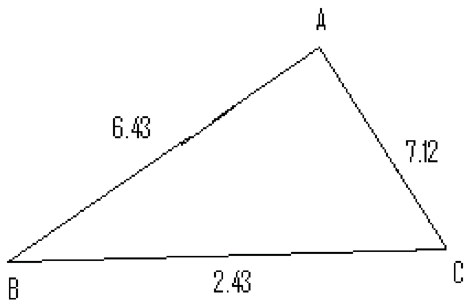
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$b^2 = 4.32^2 + 2.81^2 - 2 \times 4.32 \times 2.81 \times \cos 95.5$$
$$b^2 = 4.32^2 + 2.81^2 - 2 \times 4.32 \times 2.81 \times \cos 95.5$$
$$b^2 = 18.66 + 7.8961 - 24.2784 \times -0.09585$$
$$b^2 = 26.5561 + 2.32698$$
$$b^2 = 28.88$$
$$\sqrt{b^2} = \sqrt{28.88}$$
$$b = 5.37$$

Question 3

To find angle A

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$18.66 = 28.84 + 7.9 - 30.18\cos A$$
$$18.66 - 36.74 = -30.1\cos A$$
$$\frac{-18.08}{-39.18} = \cos A$$
$$0.5991 = \cos A$$
$$53.19° = A$$

Question 4



To find angle A

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$2.43^2 = 7.12^2 + 6.43^2 - 2 \times 7.12 \times 6.4\cos A$$
$$5.9 = 50.69 + 41.34 - 92.59\cos A$$
$$-86.13 = \cos A$$
$$\frac{-86.13}{-92.59} = \cos A$$
$$21.5° = A$$

Question 4

To find angle B

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$7.12^2 = 2.43^2 + 6.43^2 - 2 \times 2.43 \times 6.43\cos A$$

$$50.69 = 5.9 + 41.34 - 31.25\cos B$$

$$50.69 - 47.24 = -31.25\cos B$$

$$\frac{3.45}{-31.25} = \cos B$$

$$83.7° = B$$

Question 4

To find angle C

180° – (21.5°+83.7°)

= 74.8°

Summary of Answers

A	B	C	a	b	c
32.5°	37.5°	110°	3.7	4.2	6.48
60.4°	88.1°	31.5°	18.8	21.2	14.3
53.2°	95.5°	31.1°	4.32	5.37	2.81
21.5°	83.7°	74.8°	2.43	7.12	6.43

Tips

Tip

- To be able to use the sine formulae, at least one angle and two sides need to be known. That means that one of the known sides will opposite the known angle.
- One of the cosine formulae can be used when three sides are given or when an angle and two sides are given.
- Remember the negative for cosine of obtuse angles, that is a negative cosine implies an angle more than 90° or an obtuse angle.

Lesson 4 Area of a Triangle

One of the important uses of trigonometry is its use in finding the area of a triangular surface. Area is a very important aspect in construction. For example, in order to calculate how many tiles you need to cover a triangular section of a floor, you need to know how to find the area of that triangle.

By the end of this sub-unit, you should be able to:

- calculate the area of a triangle that is not right-angled using trigonometry.

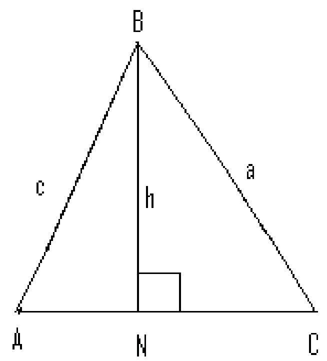
Deriving the Trigonometric Formula for the Area of a Triangle

Think back to the Mathematics course you studied at Junior Secondary Certificate level, can you remember how to find the area of a triangle?

I hope you remembered the following formula for finding the area of a triangle:

$$\text{Area of a Triangle } = \frac{1}{2} base \times height \quad \text{or} \quad \frac{1}{2} bh$$

Consider the following triangle ABC, with a perpendicular line drawn from B to touch the base AC at the point N. In the right-angled triangle ABN.



$$\sin A = \frac{h}{c}$$

$$h = c \sin A$$

When we substitute csinA for h in the formulae of an area of a triangle we get, Area of a triangle $= \frac{1}{2} b \times c \sin A$

$= \frac{1}{2} \times bc \sin A$

Forms of the formulae for the area of a triangle are:

Area $= \frac{1}{2} ab \sin C$

Area $= \frac{1}{2} bc \sin A$

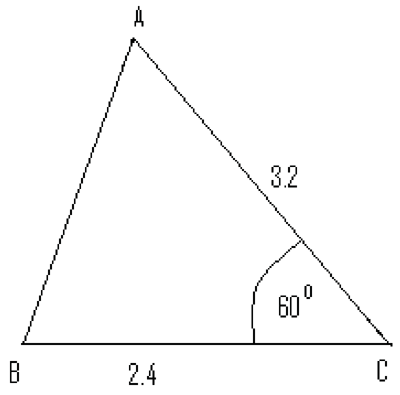
Area $== \frac{1}{2} ac \sin B$

Note that in these formulae, the angle used is the one between the two sides in the formula, as shown below:

Formula	Angle between	Sides
area $= \frac{1}{2} ab \sin C$	C	a and b
area $= \frac{1}{2} bc \sin A$	A	b and c
area$== \frac{1}{2} ac \sin B$	B	a and c

This means that, if the angle between the two sides is not given, it has to be calculated so that it can be used in the formula.

Example 1



Calculate the area of a triangle with the following dimensions:

$$a = 2.4, \qquad b = 3.2 \qquad and\, C = 60°$$

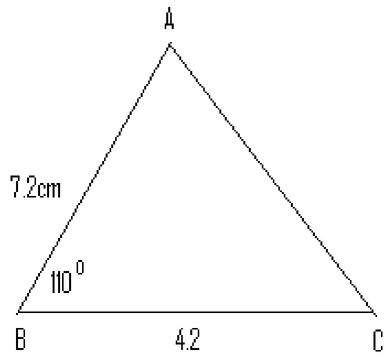
$$= \frac{1}{2} ab \sin 60°$$

$$\text{area of a triangle} = \frac{1}{2} \times 2.4 \times 3.2 \times 0.886$$

$$= 3.4 cm^2$$

Example 2

Find the area of a triangle below where a=4.2, c=7.1, B=110º



What formula should you use?

____

Compare your formulae with the one below:

We are given two sides and the angle between them. So the three values can be used together in the formula:

$$\text{Area of a triangle} = \frac{1}{2} \times ac \sin 110°$$

Substitute the known values and the value for sin110° in the formula and then multiply the numbers.

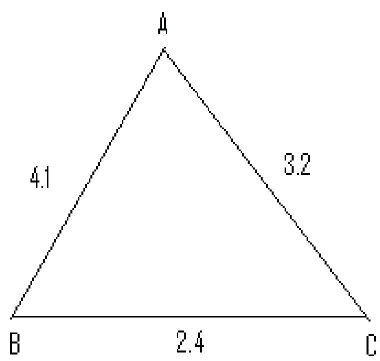
Compare your solution with the one below:

$$= \frac{1}{2} \times 4.2 \times 7.2 \times \sin 110°$$
$$= \frac{1}{2} \times 4.2 \times 7.2 \times 0.9397$$
$$= 14.2$$

Example 3

Calculate the area of a triangle where

$a = 2.4, b = 3.2\ c = 4.1$



For us to use the formula for finding the area of a triangle, we need an angle between any of the sides given. So our first step is to find that angle. To find angle A, what formula should you use?

____

Compare your formula with the one below:

To find A,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Substitute the known values in the formula:

Compare your answer with the one below:

$$2.4^2 = 3.2^2 + 4.1^2 - 2 \times 3.2 \times 4.1 \times \cos A$$

Reduce to find a value for cosA:

Compare your answer with the one below:

$$\frac{5.76 - 27.05}{26.24} = \sin A$$

$$0.8114 = \sin A$$

Use your calculator or sine table to find the angle:

Compare your answer with the one below:

A = 144

What formula should you use to calculate the area of the triangle?

Compare your formula with the one below:

$$\text{Area of a } \Delta = \frac{1}{2} \times bc \sin A$$

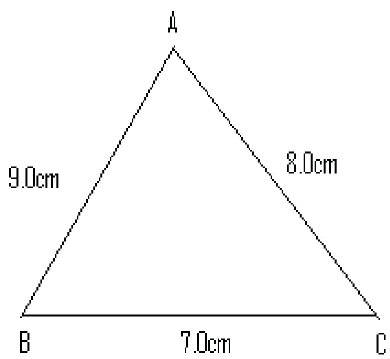
Using the formula, the values of the sides and sine144°, calculate the area of the triangle.

Compare your answer with the one below:

$$= \frac{1}{2} \times 3.2 \times 4.1 \times \sin 144°$$
$$= 3.86$$

Example 4

A=7.0, b = 8.0, c = 9.0



To calculate the area we need two sides and the angle between them. So we calculate angle B. To find angle B, what formula should you use?

Compare your formula with the one below:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

Substitute the known values into the formula:

Compare your answer with the one below:

$$8^2 = 7^2 + 9^2 - 2 \times 7 \times 9 \times \cos B$$

Reduce to solve for cosB:

Compare your answer with the one below:

$$64 = 49 + 81 - 126\cos B$$

$$64 - 49 = -126\cos B$$

$$15 = -126\cos B$$

$$\frac{15}{-126} = \frac{-126\cos B}{-126}$$

$$-1190 = \cos B$$

Reduce to solve for B:

Compare your answer with the one below:

$$B = 96.8°$$

So, for the area of a triangle we need to use two sides and the angle between them. If the angle is B, the two sides will be a and c. What formula should you use?

Compare your formula with the one below:

$$\text{Area of triangle} = \frac{1}{2} \times ac \sin B$$

Substitute the values of angle B, side a and c in the equation.

Compare your answer with the one below:

$$= \frac{1}{2} \times 7 \times 9 \times \sin 96.8°$$

Look up $\sin 96.8°$ in your mathematical tables or use your calculator.

$$= \frac{1}{2} \times 63 \times 0.9930$$

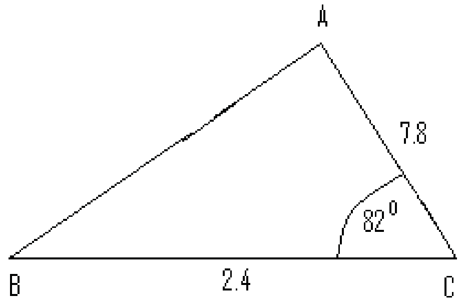
Multiply out the values to get the area as,

$$= 31.28cm^2$$

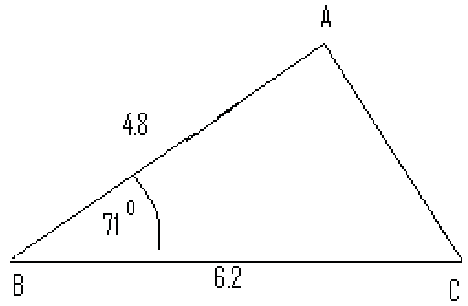
Activity

Activity 15.4
Find the areas of the following triangles

1. a = 2.4, b = 7.8, C = 82°



2. a = 6.2, c = 4.8, B = 71°



3. a = 7.8, b = 9.2, c = 11.3

Note it!

Key Points to Remember

The key points to remember in this sub-unit are:

- We can use trigonometry to find the area of a triangle.
- The formula for the area of a triangle is

$$\text{Area of a Triangle} = \frac{1}{2} \times ab\sin C$$

Where a, b and c are sides of a triangle and A, B and C are angles opposite the sides respectively.

- Forms of the formulae for the area of a triangle include:

$$\frac{1}{2} \times ab\sin c$$

$$\frac{1}{2} \times bc\sin A$$

and

$$\frac{1}{2} \times ac\sin B$$

- The formula can be expressed in words as:

half the product of two sides of a triangle multiplied by the sine of the angle between those two sides.

Model Answers

Activity 15.4

1. **Area of the triangle =** $\frac{1}{2} \times ab\sin C$

$$\frac{1}{2} \times 2.4 \times 7.8 \times \sin 82$$

$$9.27$$

2. **Area of a triangle=** $\frac{1}{2} \times ac\sin B$

$$\frac{1}{2} \times 6.2 \times 4.8 \times \sin 71$$

$$14.07$$

3. **We need to find one of the angles. Let us find angle A, using the cosine formula.**

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$7.8^2 = 9.2^2 + 11.3^2 - 2 \times 9.2 \times 11.3 \times \cos A$$
$$60.84 = 84.64 + 127.69 - 207.92\cos A$$
$$60.84 = 212.33 - 207.92\cos A$$
$$207.92\cos A = 212.33 - 60.84$$
$$\cos A = \frac{151.49}{207.92}$$
$$angle A = 43.23°$$

$$\frac{1}{2} \times c \times b \times \sin A$$

Area of triangle ABC= $\frac{1}{2} \times 11.3 \times 9.2 \times \sin 43.23$

35.6

Lesson 5 Angles of Elevation and Angle of Depression

So far, we have been talking about measurements in two dimensions, using triangles that lie on a flat surface. Sometimes, however, we need to measure things in three dimensions. For example, imagine you are sailing in a boat on a lake, headed toward the rocks at the base of a cliff. From your map, you know the height of the cliff. How can you find out how far the boat is from the rocks?

If you had the equipment to measure the angle from your position to the top of the cliff, you could use trigonometry to calculate how far you have to go before hitting the rocks. This type of problem requires the use of angles of elevation and depression.

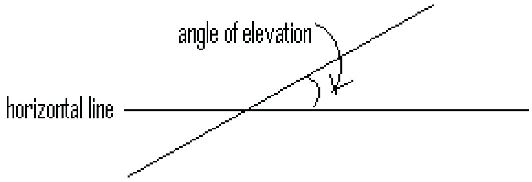
By the end of this sub-unit, you should be able to:

- explain what is meant by the angle of elevation and the angle of depression;
- identify these angles in examples;
- calculate the angle of elevation and the angle of depression;
- apply these skills to calculate measurements in practical problems drawn from everyday life.

This sub-unit is about 12 pages in length.

Identifying the Angle of Elevation and Depression

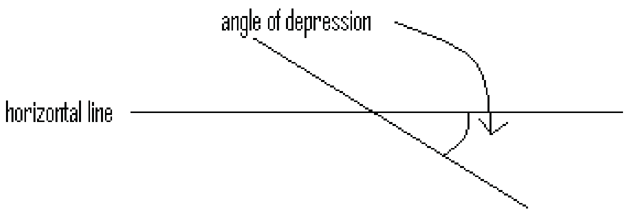
An angle of elevation is an angle measured from the horizontal plane upward as shown in the diagram.



The angle of elevation is the angle rising up from the horizon.

An angle of elevation is very important when we want to measure tall objects such as buildings, trees, etc., when the process can't be done physically. For example, it would be dangerous and impractical to climb to the top of a tree to use a measuring tape to find out how tall it is.

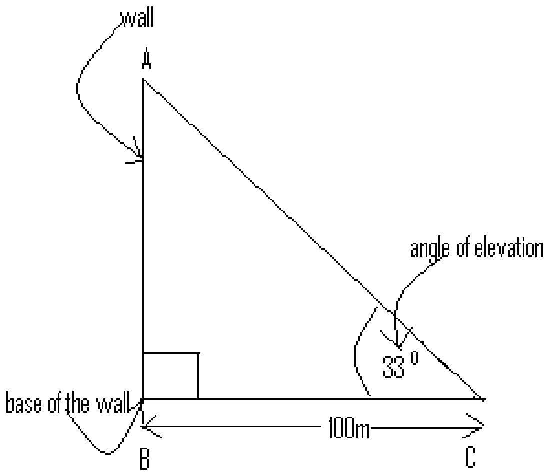
An angle of depression is measured from the horizontal plane downward.

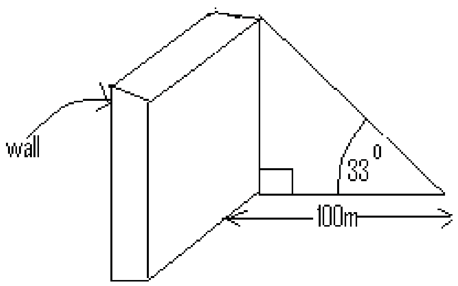


The angle of depression is the angle dropping down from the horizon.

Example 1

A builder wants to measure the height of a tall building, but he doesn't have a ladder tall enough to reach the top. He walks away from the base of the wall and uses a measuring tape to mark off a distance of 100 metres. Next, he measures the angle of elevation from where he stands to the top of the building. The angle of elevation is 33° degrees. How tall is the building?





The point C represents the position where the builder stands.

Write a formula that can be used to find the height (*h*)of the building.

Compare your formula with the one below:

$$\tan 33 = \frac{opposite}{adjacent} = \frac{h}{100}$$

Making AB the subject of the formula.

Compare your answer with the one below:

$$100 \times \tan 33 = h$$
$$h = 64.9\,m$$

The height of side AB of our triangle is 64.9 meters. Is that the height of the building or did we forget something?

When the builder measured the angle to the top of the building, he was standing up straight on the same level as the base of the building. That means his eyes were about 1.6 meters above the ground level. As a result, the imaginary triangle we drew to solve this problem was also 1.6 meters above ground level at the base of the wall. We need to add 1.6 meters to the answer above to get the height of the wall.

Example 2

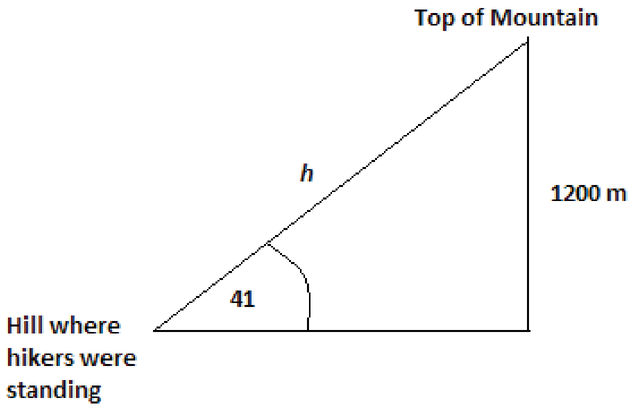
Two people are hiking in the mountains and lose their way. They climb to the top of a hill and catch sight of a mountain they recognize. According to their map, the elevation (height above sea level) of the mountain they see is 1500 m and the elevation of the hill they are standing on is 300 m. Using a clinometer, they measure the angle from their position to the top of the mountain at 41^{o} degrees. As the crow flies (that is, in a straight line) how far away is the top of the mountain?

Let *h* be the distance from the hikers to the top of the mountain.

Compare your formula with the one below:

$$\sin 41° = \frac{opposite}{hypotenuse} = \frac{1200m}{h}$$

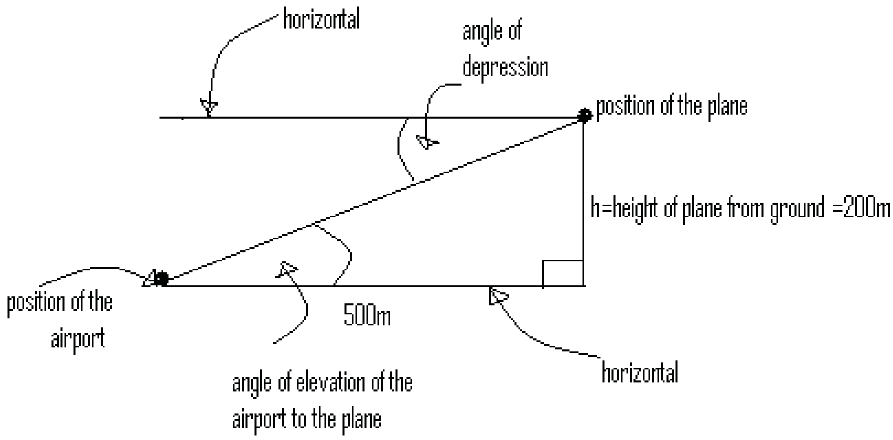
Compare your answer with the one below:



$$\sin 41° = \frac{1200}{h}$$
$$h \sin 41° = 1200$$
$$h = \frac{1200}{\sin 41°}$$
$$h = \frac{1200}{0.6561}$$
$$h = 1829\,m$$

Example 3

While flying a small plane, the pilot has some problems with his instruments. She needs to land at an airfield 500 m away, but there is fog and she can't see the ground. The altimeter tells her that the plane is flying 200 m above the ground level at the airstrip, but she needs to find the right angle for her descent so that the plane won't crash. What is the angle of depression from the airplane to the airport?



What is the relationship between the angle of elevation and the angle of depression in this diagram?

Compare your answer with the one below:

They are alternate angles and are therefore equal (**refer to the section on properties of angle in parallel lines cut by a transversal**).

We can use trigonometry to calculate an angle of elevation in the diagram.

Let ɵ be the angle of elevation.

$$\text{Ta } \theta = \frac{oppposite}{adjacent} = \frac{200}{500}$$

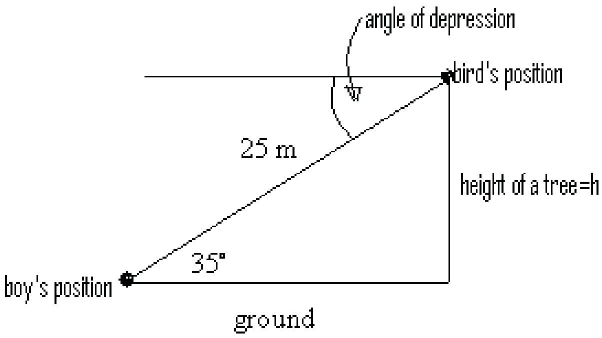
$$= 0.4$$

$$\theta = 21.8°$$

The angle of elevation =21.8°=the angle of depression.

Example 4

A boy is sitting on a rock and watching a bird in a tree some distance away. The ground between the boy and the base of the tree is level, while the angle of elevation from the boy to the bird is 35°. The boy knows from past experience that he can throw a rock 25 meters and hit a target. So he throws a rock at the bird, but it flies away just before the rock arrives. How tall is the tree?



Compare your formulae with the one below:

$$\sin 35° = \frac{oppposite}{hypotenuse} = \frac{h}{25}$$

Compare your answer with the one below:

$$25\sin 35° = h$$
$$14.34m$$

The height of the tree is 14.34 m.

Activity

Activity 15.5

1. Calculate the angle of elevation from the eyes of a man 1.5 m tall looking at the top of a cliff 15 m high. The man is standing 5 m away from the cliff.

2. Calculate the height of a building if the angle of elevation to the top of the building is 82° at a distance of 9 m from the foot of the building.

----

3. The height of a lighthouse is 30 m. The angle of depression [FROM THE TOP OF THE LIGHT HOUSE TO THE]ship is 7°. Calculate the distance of the ship from the lighthouse.

----

4. The angle of depression from the top of a mobile phone mast to a rock is 41°. The rock is 18.7 m from the foot of the mast on level ground. Determine the height of the mast.

----

5. Calculate the length of a vertical pole if the shadow of the pole is 4 m long when the angle of elevation of the sun is 43°.

----

6. The length of a rope from the top of a mast to a point 20 m from the foot of the mast is 60 m. Calculate the height of the mast.

Note it!

Key Points to Remember

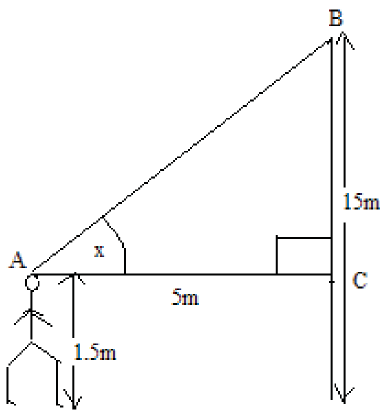
The key points to remember in this sub-unit on are:

- **An angle of elevation** is an angle measured from the horizontal plane upward.
- **An angle of depression** is an angle measured from the horizontal plane downward.
- Both types of angles are useful in real life to calculate the dimensions of things that cannot be done physically, e.g. a tall object or the distance of an object whose height is known from a given point, etc.

Model Answers

Activity 15.5

1.



$$BC = BD - DC$$

$$BC = 15m - 1.5 = 13.5m$$

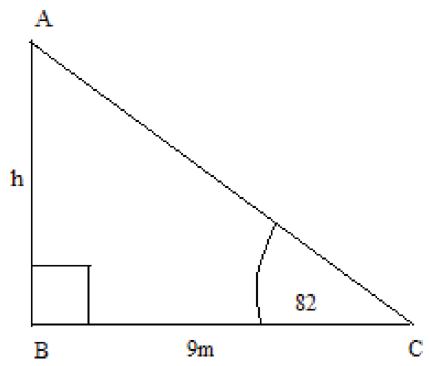
$$\tan x = \frac{13.5m}{5}$$

$$\tan x = 2.7$$

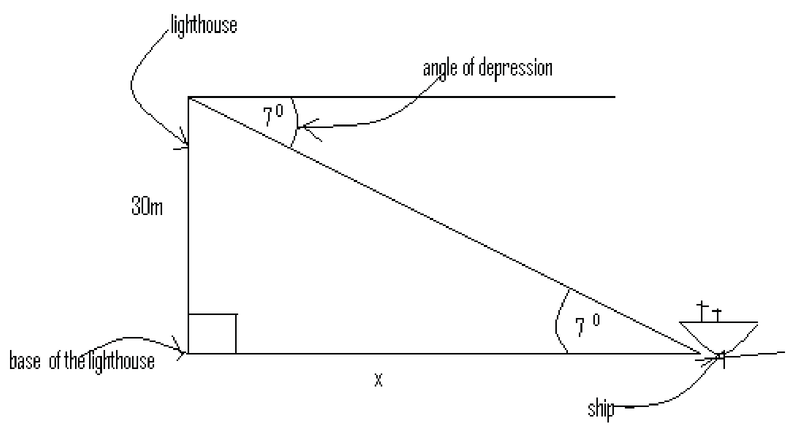
$$x = 69.68$$

2.

Let' h' be the height of the building AB



$$\tan 82° = \frac{h}{9m}$$
$$9\tan 82° = h$$
$$64.03 = h$$



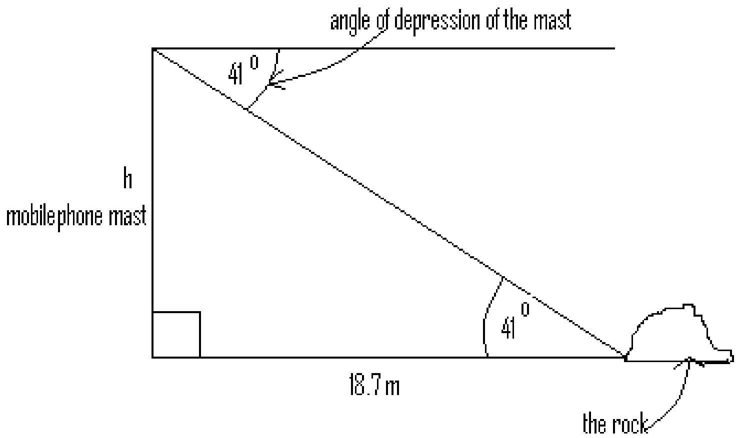
3.

The distance of the ship from the lighthouse = x in the diagram.

The angle of elevation=angle of depression (they are alternate angles is parallel lines)

$$\tan 7° = \frac{30m}{x}$$
$$x \tan 7° = 30m$$
$$x = \frac{30}{\tan 7°} = \frac{30}{0.1228} = 252m$$

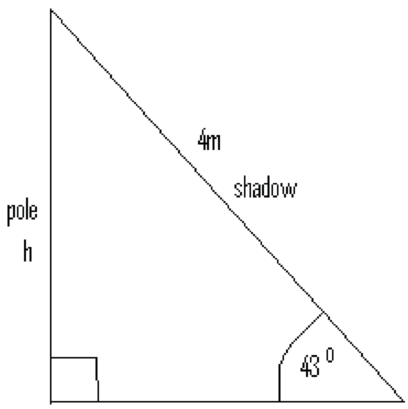
4.



Angle of depression=angle of elevation,

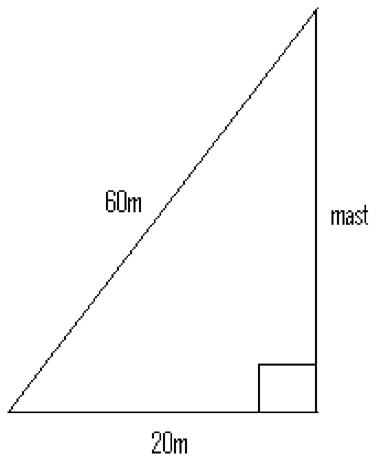
$$\tan 41° = \frac{h}{18.7}$$
$$18.7 \times \tan 41° = h$$
$$16.3 = h$$
$$h = 16.3$$

5.



$$\sin 43° = \frac{h}{4}$$
$$4 \times \tan 43° = h$$
$$3.7 = h$$
$$h = 3.7$$

6.



$$60^2 - 20^2 = h^2$$
$$3600 - 400 = h^2$$
$$\sqrt{3200} = \sqrt{h^2}$$
$$56.6 = h$$

\

Summary

Unit Summary

$(a)\,sine\ rule$

$$\frac{a}{sineA} = \frac{b}{sineB} = \frac{c}{sineC}$$

Choosing the Appropriate Sine Rule to Solve Problems.

Tips:

- Choose the formulae where only one unknown is formed when the given information is substituted into the equation.
- The equation should also contain the side or angle to be calculated.

$(b)\,cosine\ rule$

$$a = b + c - 2bccosA$$

$$b = a \mid c \quad 2accos$$

$$c = a + b - 2abcosC$$

Tips

- To be able to use sine formulae at least one angle and two sides should be given. There should also be a side and its opposite angle. These could be calculated, if possible.
- The cosine formula can be used when three sides are given or when an angle and two sides are given.

Remember the negative for cosine of obtuse angles,

$$(c)\,area\ of\ a\ triangle = \frac{1}{2}base \times height$$

Forms of the formulae for the area of a triangle,

$$area = \frac{1}{2} \times absineC$$

$$area = \frac{1}{2} \times bcsineA$$

$$area = \frac{1}{2} \times acsineB$$

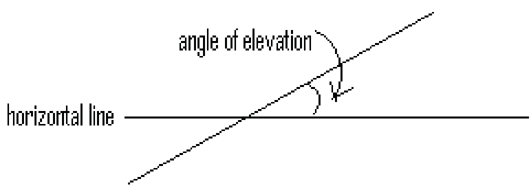
Note that in these formulas, the angle used is the one between the two sides in the formula, as shown below:

Formula	Sides	Angle between
$area = \frac{1}{2} \times absineC$	a and b	C
$area = \frac{1}{2} \times bcsineA$	b and c	A
$area = \frac{1}{2} \times bcsineA$	a and c	B

This means that if the angle between the two sides is not given it has to be calculated so that it can be used in the formula.

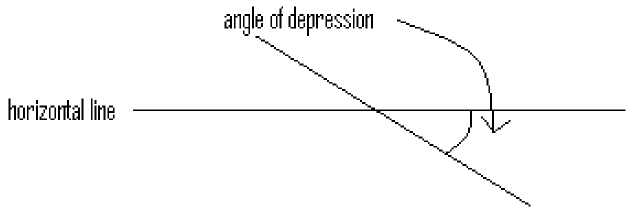
(d) Angles of elevation

An angle of elevation is measured from the horizontal plane upward, as shown in the diagram.



(e) Angle of depression

An angle of depression is measured from the horizontal plane downward.



You have completed the material for this unit on Trigonometry. You should now spend some time reviewing the content in detail.
Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have.
Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

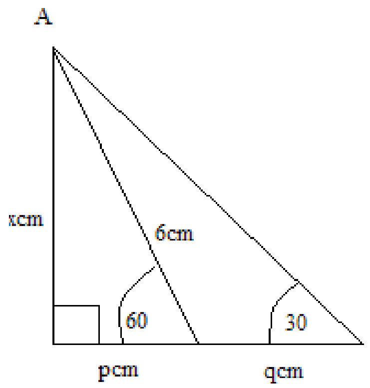
Assignment

Time: 1hr.
Total Marks: 50

Marks for each question or part of a question are shown in [brackets].

Answer all questions.

1. Find the values of p and q in the following diagram.

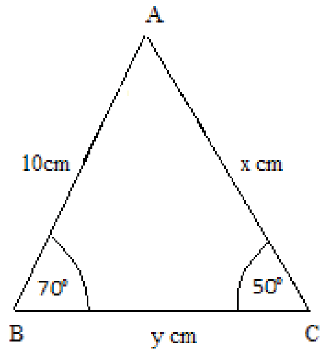


$p[4]$ & $q[9]$

2. Find the angle B of a triangle in which the angle A is 57°, the length of AC is 10 cm, and the length of BC is 9 cm.

[6]

3. Find the sizes of the two remaining sides of a triangle given one side=10 cm one angle=50° and another=70°.



$x[5]$ & $y[6]$

4. Find the largest angle of a triangle, where the three sides are 6 cm, 7 cm, and 8 cm.

[5]

5. Calculate the length of side BC for a triangle, given that AB=7 cm, AC=11 cm, and the angle A =143°.

[5]

6. Calculate the angle B of a triangle in which the angle A=57°, the length of AC is 10 cm, and the length of BC is 11 cm.

[5]

7. The angle of depression of an object on the ground from the top of a building is 34°. The horizontal distance from the object on level ground to the base of the building is 76 metres. What is the height of the building?

[3]

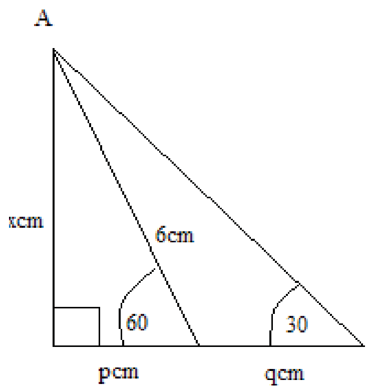
8. A building 26 m high casts a shadow 30 m long. What is the angle of elevation of the sun?

[3]

Compare your answers to those provided below. Pay particular attention to any mistakes you made and clarify those misunderstandings.

Model Answers to the Assignment

1.

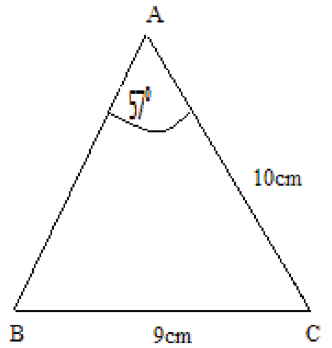


$$\cos 60° = \frac{p}{6}$$
$$6\cos 60° = p$$
$$5 \times 0.5 = p$$
$$2.5 = p$$

$$\sin 60° = \frac{x}{2.5}$$
$$2.5\sin 60° = x$$
$$2.5 \times 0.866 = x$$
$$2.17 = x$$

$$\tan 30° = \frac{2.17}{p+q} = \frac{2.17}{2.5+q}$$
$$(2.5+q)\tan 30 = 2.17$$
$$(2.5+q) = \frac{2.17}{\tan 30} = \frac{2.17}{5774}$$
$$(2.5+q) = 3.76$$
$$q = 3.76 - 2.5 = 1.26cm$$

2.



$$\frac{9}{\sin 57°} = \frac{10}{\sin B}$$

$$\frac{9\sin B}{\sin 57°} = 10$$

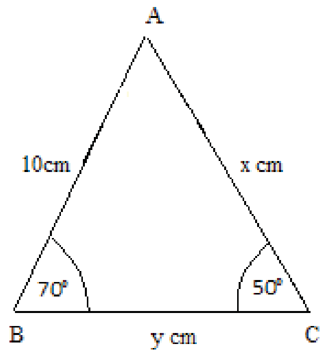
$$9\sin B = 10\sin 57°$$

$$\sin B = \frac{10\sin 57°}{9}$$

$$\sin B = 0.93186$$

$$B = 68.7°$$

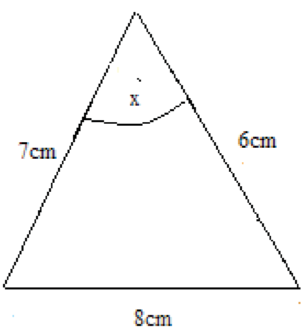
3.



$$\frac{10}{\sin 50} = \frac{x}{\sin 70}$$
$$10\sin 70 = x\sin 50$$
$$\frac{10\sin 70}{\sin 50} = x$$
$$\frac{9.3969}{0.7660} = x$$
$$12.27 = x$$

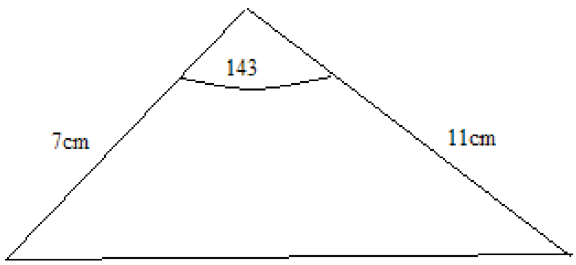
$$\frac{10}{\sin 50} = \frac{y}{\sin 180 - (70+50)}$$
$$\frac{10}{\sin 50} = \frac{y}{\sin 60}$$
$$10\sin 60 = y\sin 50$$
$$\frac{10\sin 60}{\sin 50} = y$$
$$\frac{8.66}{0.766} = y$$
$$11.31cm$$

4.



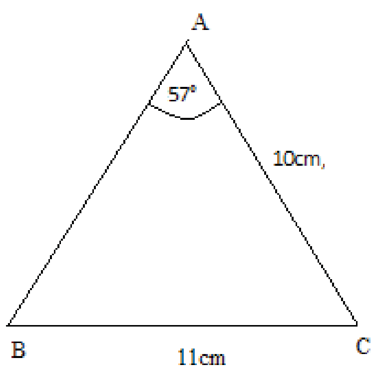
$$8^2 = 6^2 + 7^2 - 2 \times 6 \times 7 \times \cos x$$
$$64 = 36 + 49 - 84\cos x$$
$$\frac{64 - 34 - 49}{-84} = \cos x$$
$$\frac{-21}{-84} = \cos x$$
$$0.25 = \cos x$$
$$75.5 = x$$

5.



$$BC^2 = 7^2 + 11^2 - 2 \times 7 \times \cos 143$$
$$BC^2 = 49 + 121 - 154\cos 143°$$
$$BC^2 = 170 + 154\cos 143°$$
$$BC = \sqrt{170 + 122}$$
$$BC = 17.12$$

6.



$$\frac{10}{\sin b} = \frac{11}{\sin 57°}$$

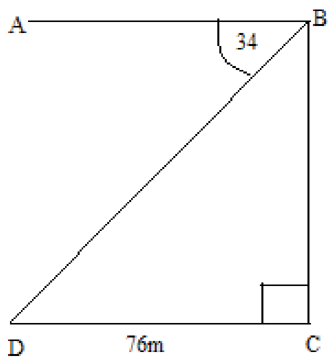
$$10 \sin 57 = 11 \sin B$$

$$\frac{10 \sin 57}{11} = \sin B$$

$$0.7624 = \sin B$$

$$49.7 = B$$

7.



Let BC be the height of the building and the D the position of the object on the ground.

ABD the angle of depression = BDC the angle of elevation.

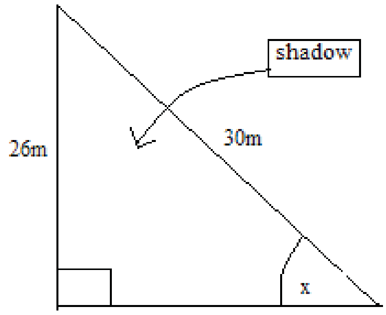
$$\tan BDC = \frac{BC}{76cm}$$

$$\tan 34° = \frac{BC}{76cm}$$

$$76 \tan 34 = BC$$

$$51.26 = BC$$

8.



Let x be the angle of elevation.

$$\sin x = \frac{26}{30}$$
$$x = 60.07°$$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

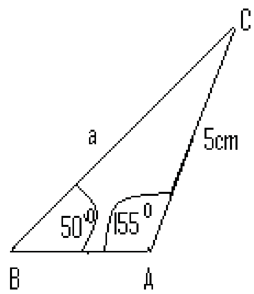
Time: 2hrs

Total Marks: 103

Marks for each question or part of a question are shown in [brackets].

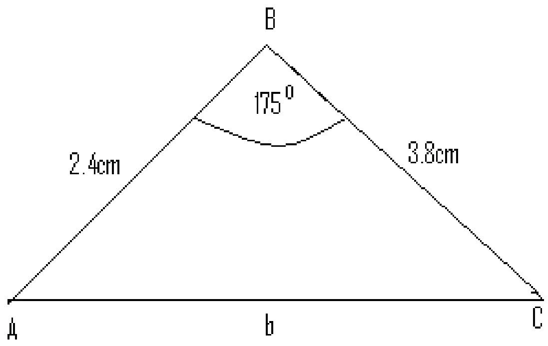
Answer all questions:

1.a Calculate the length of a in the triangle below:



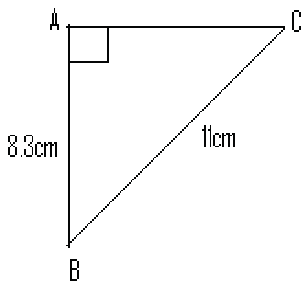
[4]

1.b Calculate the length of b in the triangle below:



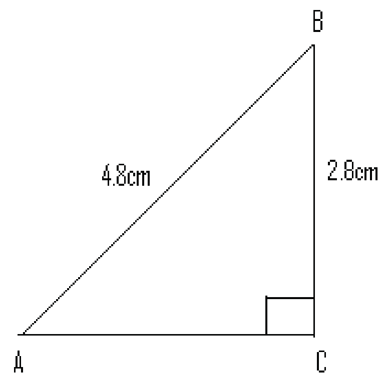
[6]

2.a Calculate angle B in the triangle below:



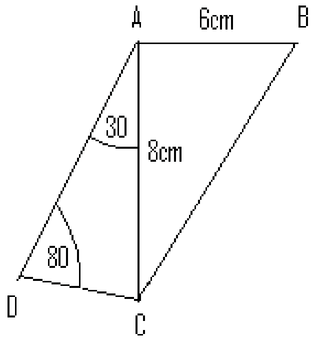
[3]

2.b Calculate angle A in the triangle below:



[3]

3. The points A, B, C and D in the figure below are on level ground. A surveyor knows that AB = 3km and angle CBA= 65.5°, and needs to know the distance AC and AD.



(a) The distance AC cannot be measured directly because of large lake between A and C. By measurement, he finds that angle CAB=84.5°.

Calculate the angle ACB and the length AC.

[6]

(b) He cannot use the same method to find AD because a wood obstructs his view of D from A. By measurement he finds that BD = 4 km and angle ABD =60°. Calculate the length of AD.

[6]

4. ABCD is a quadrilateral such that AB = 6.0 cm, AC=8.0 cm, angle DAC = 30°, ADC=80° and angle BAC is a right angle.

Calculate:

(a) the length of BC

[4]

(b) the length of AD

[5]

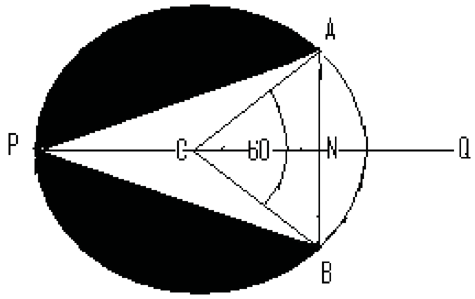
(c) the area of triangle ADC

[3]

(d) the area of the quadrilateral ABCD

[4]

5. The diagram below shows a circle of radius 10 cm with centre C and a chord AB such that angle ACB=60°.



Given that the diameter PCNQ is an axis of symmetry of the diagram, calculate:

(a) the length of AB

[5]

(b) the length of AN

[3]

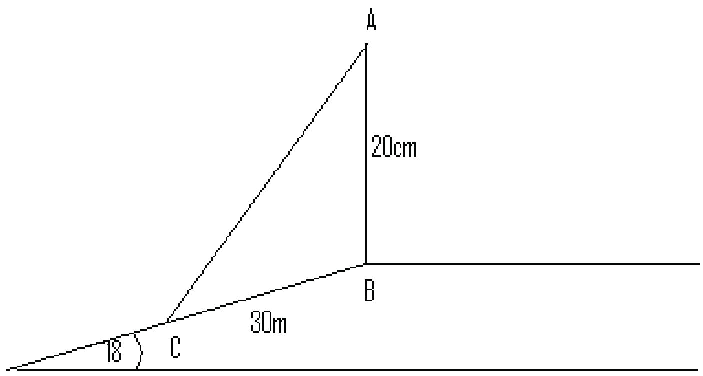
(c) the area of triangle ACP

[4]

(d) the area shaded in the diagram, taking π to be 3.14

[7]

6. A radio mast AB of height 20 m stands at the top of a slope which is inclined at 18° to the horizontal. The mast is supported by a wire AC attached to a point C on the slope. BC=30 m.



(a) Calculate:

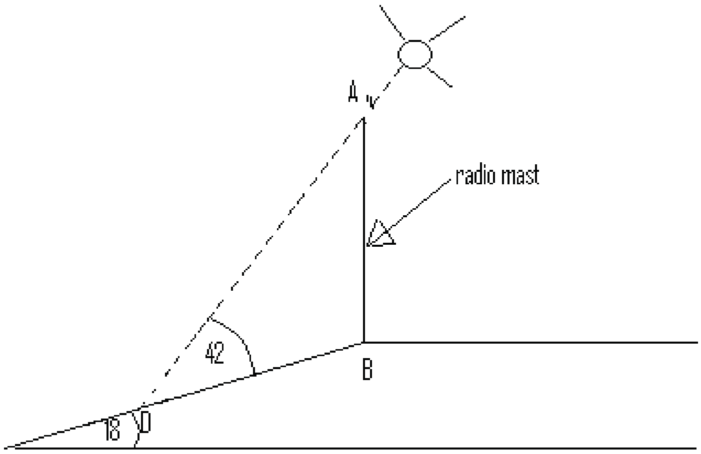
(i) angle ABC

[2]

(ii) the length of the wire AC.

[7]

(b) When the sun is in a certain position, the shadow cast by the mast lies down the slope, shown in the diagram by line BD. Given that ADB = 42°, calculate:



(i) the angle of elevation of the sun

[2]

(ii) Angle DAB

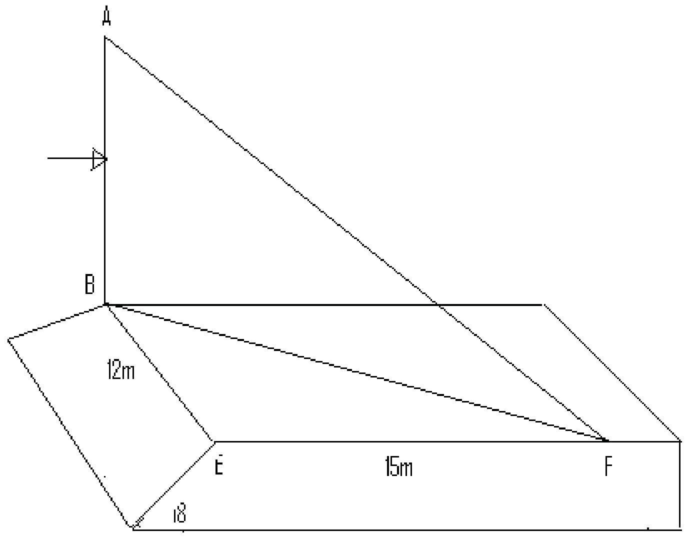
[3]

(iii) the length of shadow BD

[4]

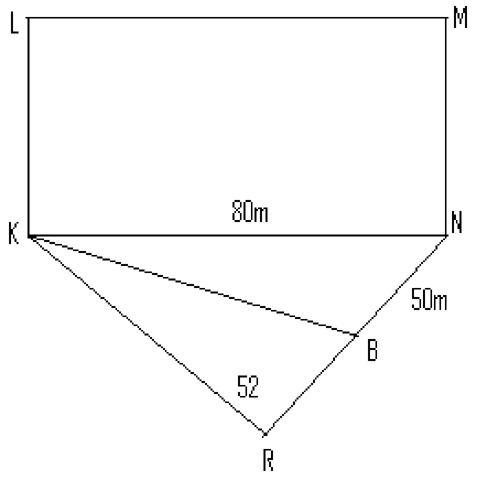
(c)

The mast is supported by another wire AF. The points B, E, and F lie on horizontal ground. Given that angel BEF = 90°, BE = 12 m and EF = 15 m, calculate the length of wire needed to span AF.



[5]

7. In the diagram, the rectangle KLMN represents a vertical cliff and KNBR lies on the horizontal surface of the sea. The sea meets the cliff along the horizontal line KN. LK =56 m, KN = 80 m and NB =50 m angle KRN = 52°.



(a) A boat is at point B, a rock is at point R and RBN is a horizontal straight line.

Calculate:

(i) The distance KR [3]

(ii) The distance KB [7]

(b) A man stands at L on top of a cliff. Calculate his angle of elevation from B (ignore his height).

[3]

(c) The man walks along the top of the cliff from L to M. Calculate the maximum angle of elevation from B that he passes through during his walk.

[4]

Answers

1(a)

$$\frac{a}{\sin 155} = \frac{5}{\sin 50}$$
$$a = \frac{5\sin 155}{\sin 50}$$
$$a = \frac{5 \times 0.4226}{0.7660}$$
$$a = 2.71cm$$

(b)

$$b^2 = a^2 + c^2 - 2ac\cos 175$$
$$b^2 = 3.8^2 + 2.4^2 - 2 \times 3.8 \times 2.4 \times -0.9962$$
$$b^2 = 14.44 + 5.76 - 18.24 \times -0.9962$$
$$b^2 = 14.44 + 5.76 + 18.17$$
$$b^2 = 38.37$$
$$\sqrt{b^2} = \sqrt{38.37}$$
$$b = 6.19$$

2.

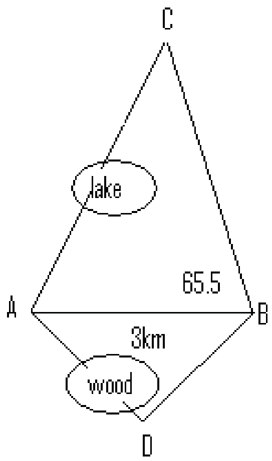
(a)

$$\cos x = \frac{8.3}{11} = 0.7545$$
$$x = 41.01$$

(b)

$$\sin y = \frac{2.8}{4.8} = 0.5833$$
$$y = 35.69$$

3.



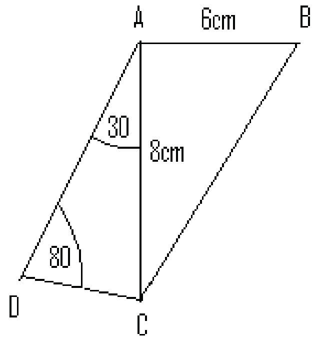
(a) $\angle ACB = 180 - (84.5 + 65.5) = 30$

$$\frac{AC}{\sin \angle ABC} = \frac{AB}{\sin \angle ACB}$$
$$\frac{AC}{\sin 65.5} = \frac{3km}{\sin \angle 30}$$
$$AC = \frac{3 \times \sin 65.5}{\sin \angle 30}$$
$$AC = \frac{3 \times 0.90996}{0.5}$$
$$AC = 8.19km$$

(b)

$$AD^2 = AB^2 + BD^2 - 2AB \times BD \cos \angle ABD$$
$$AD^2 = 3^2 + 4^2 - 2 \times 3 \times 4 \times \cos 60$$
$$AD^2 = 9 + 16 - 24 \cos 60a$$
$$AD^2 = 25 - 12$$
$$AD^2 = 13$$
$$\sqrt{AD^2} = \sqrt{13}$$
$$AD = 3.6km$$

4.



(a)

$$BC^2 = AC^2 + AB^2 (Pythagoras)$$
$$BC^2 = 8^2 + 6^2$$
$$BC^2 = 64 + 36 = 100$$
$$\sqrt{BC^2} = \sqrt{100}$$
$$BC = 10km$$

$$\angle ACD = 180 - (80 + 30) = 70$$

$$\frac{AD}{\sin \angle ACD} = \frac{AC}{\sin \angle ADC}$$

$$AD = \frac{8 \times \sin 70}{\sin 80}$$

$$AD = \frac{8 \times 0.9397}{0.9848}$$

$$AD = 7.63$$

$$area\Delta ADC = \frac{1}{2} \times AD \times AC \times \sin \angle DAC$$

$$area = \frac{1}{2} \times 7.63 \times 8 \times \sin 60$$

$$= 15.26$$

Area of quadrilateral ABCD = area of triangle ADC +Area of triangle ABC.

$$= 15.26 + \left(\frac{1}{2} \times 6 \times 8\right)$$

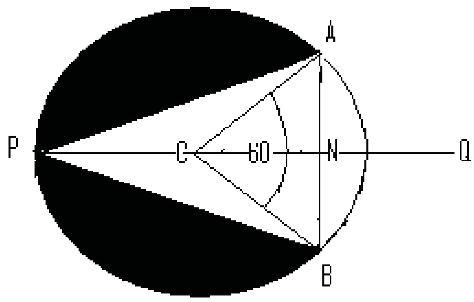
$$= 15.26 + 24$$

$$= 39.26$$

$$=$$

5.

(a)



$$\angle CAB = \frac{180-60}{2}$$

$$\angle CAB = \frac{120}{2} = 60$$

$$\cos CAB = \frac{AN}{10}$$

$$10\cos CAB = AN$$

$$10 \times \cos 30 = AN$$

$$5 = AN$$

$$AB = 2AN = 10$$

(b)

$$areaAPC = \frac{1}{2} \times PC \times CA \times \sin \angle PCA$$

$$= \frac{1}{2} \times 10 \times 10 \times \sin 120$$

$$= 50\sin 120$$

$$= 17.1cm^2$$

(d)

Area of a circle

$$= r^2\pi$$

$$= 10 \times 10 \times 3.14$$

$$= 314cm^2$$

Area of unshaded sector ACB

$$= \frac{60}{360} \times 314$$

$$= 52.33cm^2$$

Total area of the unshaded part of the circle=twice area of APC+ area of unshaded sector.

Area of APC=17.1

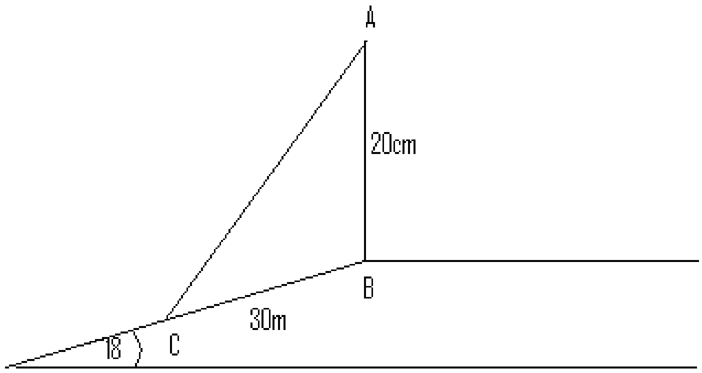
Total area of the unshaded part.

$= 2(17.1) + 52.33$
$= 86.53$

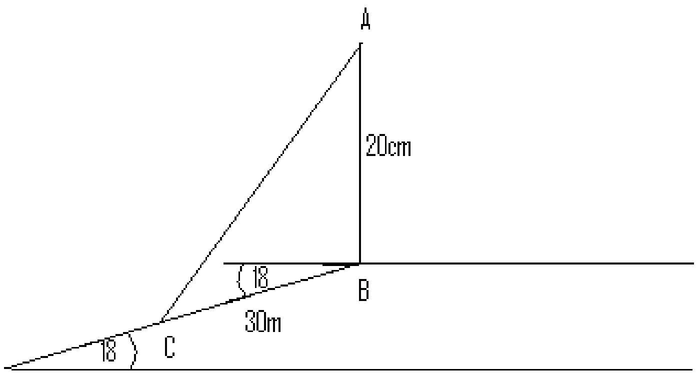
The shaded area = total area of the circle – total area of the unshaded part.

$= 314 - 86.53$
$= 227.47cm^2$

6.



When we extend the horizontal at B we get an angle which is equal to 18 using the property of alternate angles in parallel lines. NB The horizontal lines are parallel, as shown in the diagram below.



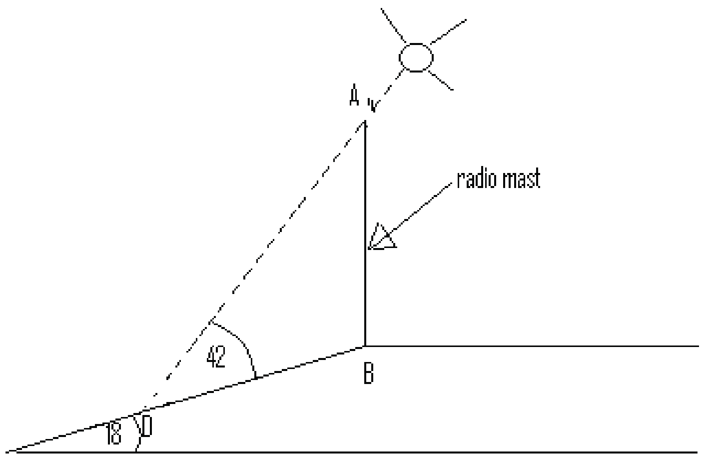
(a) (i)

$\angle ABC = 90 + 18 = 108$

(ii)

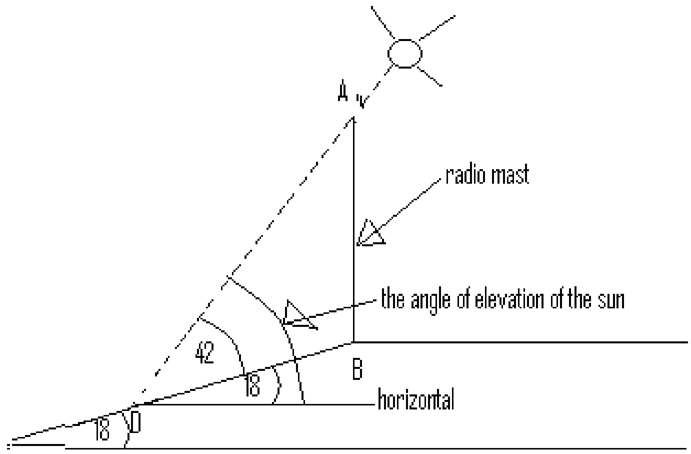
$$AC^2 = AB^2 + BC^2 - 2AB \times BC \cos \angle abc$$
$$AC^2 = 20^2 + 30^2 - 2 \times 20 \times 30 \times \cos 108$$
$$AC^2 = 200 + 900 - 1200 \cos 108$$
$$AC^2 = 1300 - -370.82$$
$$AC^2 = 1300 + 370.82$$
$$AC^2 = 1670.8$$
$$\sqrt{AC^2} = \sqrt{1670.8}$$
$$AC = 40.89$$

(b)



(i)

Here we draw a horizontal at D as shown below. This gives us an angle which is equal to 18 using the property of corresponding angles in parallel lines.



So the angle of elevation of the sun =18+42=60

(ii) To find angle DAB we use angle sum of a triangle.

$\angle DAB = 180 - (42 + 108)$
$= 30$

(iii)

$$\frac{BD}{\sin 30} = \frac{AB}{\sin 42}$$
$$BD = \frac{AB \sin 30}{\sin 42}$$
$$BD = \frac{20 \times 0.5}{0.6691}$$
$$BD = 14.95$$

(c)

$$BF^2 = BE^2 + EF^2$$
$$BF^2 = 12^2 + 15^2$$
$$BF^2 = 144 + 225$$
$$BF^2 = 369$$
$$\sqrt{BF^2} = \sqrt{369}$$
$$BF = 19.2$$

$AF^2 = BF^2 + AB^2$

$AF^2 = 19.2^2 + 20^2$

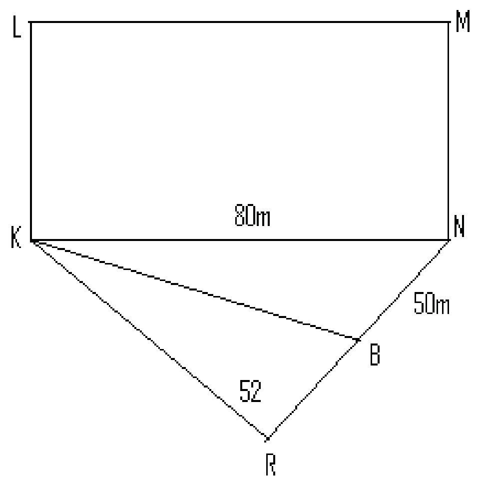
$AF^2 = 368.64 + 400$

$AF^2 = 768.64$

$\sqrt{AF^2} = \sqrt{768.64}$

$AF = 27.7$

7.



(a)

(i)

$$KR = \frac{80 \times \sin 60}{\sin 52}$$

$$KR = \frac{80 \times 0.866}{0.7880}$$

$$KR = 87.9$$

(ii)

$$KB^2 = KN^2 + NB^2 - 2KN \times NB \cos\angle KNB$$
$$KB^2 = 80^2 + 50^2 - 2\times 80\times 50\times \cos 60$$
$$KB^2 = 6400 + 2500 - 8000\times 0.5$$
$$KB^2 = 8900 - 4000$$
$$KB^2 = 4900$$
$$\sqrt{KB^2} = \sqrt{4900}$$
$$KB = 70m$$

(b)

(N.B. the angle of elevation =angle LBK)

$$\tan\angle LBK = \frac{56}{70}$$
$$\tan\angle LBK = 0.8$$
$$\angle LBK = 38.7$$

(c)

We need to first draw a perpendicular from B to KN to meet KN at a point. Name the point X. We should then calculate BX.

$$\sin 60 = \frac{BX}{50}$$
$$50\sin 60 = BX$$
$$50\times 0.866 = BX$$
$$43.3 = BX$$

Angle of elevation=y

$$\tan\angle y = \frac{56}{43.3}$$
$$\tan\angle y = 1.29$$
$$\angle y = 52.3°$$

Unit Contents

Unit 16

Unit 16

Vectors

Introduction

Welcome to the world of Vectors!

Imagine a book falling off a desk. This movement can be represented by a vector. Pushing a car and rotating a door handle can also be represented by a vector. Displacement and force are examples of vector quantities.

You probably have used vectors when classifying quantities into scalars and vectors.

How does a vector differ from a scalar?

____

____

Compare your answer to the following:

A vector differs from a scalar in that it has <u>magnitude</u> and <u>direction</u>. A scalar has magnitude only.

Of course, *the zero or null vector* is the exception.

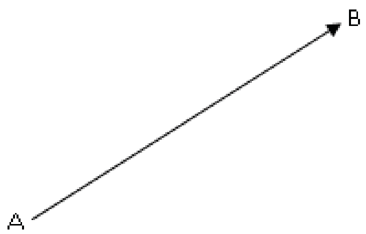
Can you give one example of a vector quantity?

____

Check if your answer is like the following:

A distance from town A to town B is one other example.

You can represent graphically the distance from town A to town B by this vector diagram:



The *length of line AB* is the magnitude and the *arrow* shows the direction.

Algebraically, the vector diagram above can be written as column vector $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$,

where **x** represents horizontal steps(*positive to right, negative to left*) and **y** vertical steps(*positive up*, *negative down*), when moving from A to B.

Now, what are you going to learn in this unit?

In this unit, you are going to learn about determining vectors from other given vectors.

This unit consists of 44 pages. It covers approximately 1% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 15 hours to work on this unit, 10 hours for formal study and 5 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Three Lessons:

Lesson 1 Expressing Parallel Vectors in Terms of Each Other
Lesson 2 Writing any Given Vector in Terms of Base Vectors
Lesson 3 Solving Geometrical Problems Using Vector Methods

Outcomes

Upon completion of this unit you will be able to:

- *express* parallel vectors in terms of each other.
- *write* any given vector in terms of base vectors.
- *solve* geometrical problems using vector methods.

Terminology

Scalar:	A number.
a or <u>a</u>:	Vector notation when letter "a" represents a vector **"a"** not a number.
Magnitude or modulus:	The length of a line which represents a vector.
$\overrightarrow{AB}$:	A vector from A to B.
$\lvert\overrightarrow{AB}\rvert$:	Magnitude or modulus of $\overrightarrow{AB}$.
Equal vectors:	Any vectors which have the same length and direction.
Base vectors:	A pair of non-parallel vectors used to express other vectors.

Position vector: A vector from the origin to a point.

$\begin{pmatrix} x \\ y \end{pmatrix}$: A general position vector of the point (x,y) in an x-y plane; x represents horizontal steps(*positive to right, negative to left*) and y vertical steps(*positive up, negative down*).

$\left|\begin{pmatrix} x \\ y \end{pmatrix}\right|$: magnitude of $\begin{pmatrix} x \\ y \end{pmatrix}$ and is equal to $\sqrt{x^2 + y^2}$

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Expressing Parallel Vectors in Terms of Each Other

Introduction

By the end of this subunit, you should be able to:

- write a vector in terms of another vector, from their graphs, when they are parallel.
- write a vector in terms of another vector, from algebraic calculations, when they are parallel.

This subunit is about 8 pages in length.

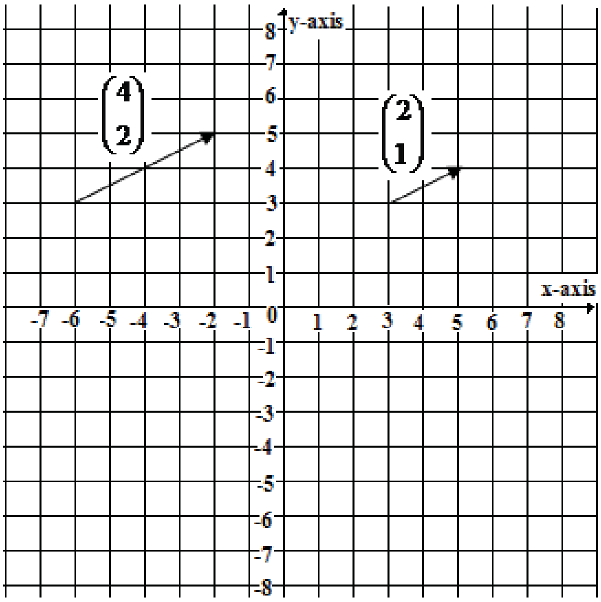
Writing a vector as a scalar multiple of another vector, which is parallel to it.

Any two vectors can be written or expressed as scalar multiples of each other, **if and only if they are parallel.**

Example 1

Look at the column vectors $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}4\\2\end{pmatrix}$:

They can be drawn in the grid below, starting at different points.



Are the vectors parallel or not?

____

Compare your answer with:

The vectors are parallel as they have the same gradient or slope.

What can you say about their directions?

____

Compare your answer with:

Their directions are the same as the arrows point towards the same direction.

What can you say about their magnitudes?

____

Compare your answer with:

The magnitude of $\begin{pmatrix}2\\1\end{pmatrix}$ is half of $\begin{pmatrix}4\\2\end{pmatrix}$ **or** $\begin{pmatrix}4\\2\end{pmatrix}$ is two times the vector $\begin{pmatrix}2\\1\end{pmatrix}$.

How can you write $\begin{pmatrix}4\\2\end{pmatrix}$ in terms of $\begin{pmatrix}2\\1\end{pmatrix}$?

____

Compare your answer with:

$$\begin{pmatrix}4\\2\end{pmatrix} = 2\begin{pmatrix}2\\1\end{pmatrix}$$

How can you write $\begin{pmatrix}2\\1\end{pmatrix}$ in terms of $\begin{pmatrix}4\\2\end{pmatrix}$?

____

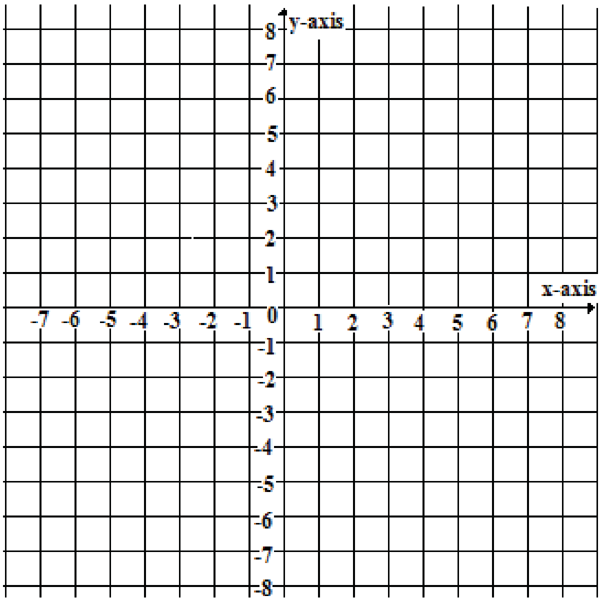
Compare your answer with:

$$\begin{pmatrix}2\\1\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}4\\2\end{pmatrix}$$

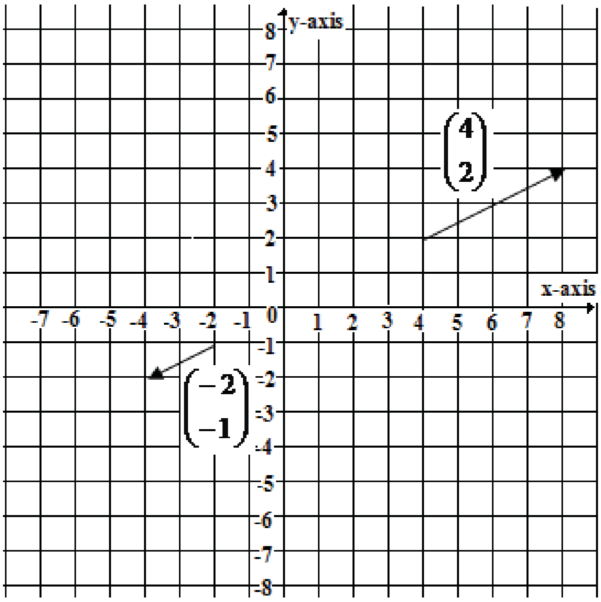
Example 2

Take $\begin{pmatrix}-2\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\2\end{pmatrix}$:

Now, draw $\begin{pmatrix}-2\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\2\end{pmatrix}$ in the grid below, with the tail of $\begin{pmatrix}-2\\-1\end{pmatrix}$ at (-2,-1) and that of $\begin{pmatrix}4\\2\end{pmatrix}$ at (4,2):



Compare your diagram with:



Are the vectors parallel or not?

____

Compare your answer with:

The vectors are parallel as they have the same gradient or slope.

What can you say about their directions?

____

Compare your answer with:

Their directions are not the same as the arrows point towards opposite directions.

Therefore, a negative scalar means direction is reversed.

What can you say about their magnitudes?

____

Compare your answer with:

The magnitude of $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ is half of $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ **or** $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ is two times the vector $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$.

How can you write $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ in terms of $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$?

____

Compare your answer with:

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = -2\begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

How can you write $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ in terms of $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$?

____

Compare your answer with:

$$\begin{pmatrix} -2 \\ -1 \end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

An algebraic method of finding the scalar for a vector that is parallel to another vector.

The method involves forming a vector equation, in which one vector equals the unknown scalar multiplying the other vector.

The equation is then solved for the unknown scalar.

So, if vectors **c** and **d** are parallel,

then **c** = k **d** or **d** = n**c**; k and n are scalars which are reciprocals to each other (k×n = 1).

Are the vectors $\begin{pmatrix} -15 \\ -7.5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ parallel?

____

Compare your answer with:

The vectors $\begin{pmatrix} -15 \\ -7.5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ are parallel.

You cannot always draw the vectors to decide, but it is more accurate to **work it out algebraically.**

How?

If $\begin{pmatrix} -15 \\ -7.5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ are parallel,

$\begin{pmatrix} -15 \\ -7.5 \end{pmatrix} = k \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ OR $n \begin{pmatrix} -15 \\ -7.5 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, where k and n are scalars.

Now, find the scalar k or n:

$\begin{pmatrix} -15 \\ -7.5 \end{pmatrix} = \begin{pmatrix} 4k \\ 2k \end{pmatrix}$ OR $\begin{pmatrix} -15n \\ -7.5n \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Equate corresponding entries to solve for k or n.

What do get when you solve for k?

____

____

____

____

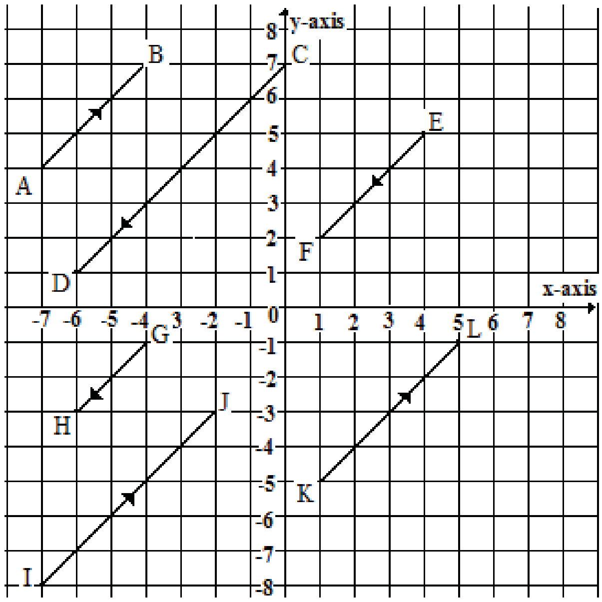
Since 4k = —15 and 2k = —7.5 result in the same value of k, which is —3.25, then you can conclude that $\begin{pmatrix} -15 \\ -7.5 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ are parallel without drawing!

Of course, the minus sign, in —3.25, tells you that their directions are opposite.

Activity 1

Activity 1

1.Use the grid below to answer the questions that follow.



(i) Write the column vectors for the following vectors.

a) $\overrightarrow{\mathbf{AB}}$

b) $\overrightarrow{\mathbf{CD}}$

c) $\overrightarrow{\mathbf{EF}}$

d) $\overrightarrow{\mathbf{GH}}$

e) $\overrightarrow{\mathbf{IJ}}$

f) $\overrightarrow{\mathbf{KL}}$

(ii) Express $\overrightarrow{\mathbf{CD}}$ in terms of $\overrightarrow{\mathbf{AB}}$.

(iii) Express $\overrightarrow{\mathbf{EF}}$ in terms of $\overrightarrow{\mathbf{GH}}$.

(iv) Express $\overrightarrow{\mathbf{KL}}$ in terms of $\overrightarrow{\mathbf{IJ}}$.

Check your performance against the given solutions at the end of this subunit; and if you are satisfied with your performance continue, or otherwise review <u>*expressing parallel vectors in terms of each other.*</u>

Note it!

If you can write one vector in terms of another, then the vectors must be parallel!

Solutions to ACTIVITY 1:

1.(i) Writing the column vectors:

a) $\overrightarrow{\mathbf{AB}} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

b) $\overrightarrow{\mathbf{CD}} = \begin{pmatrix} -6 \\ -6 \end{pmatrix}$

c) $\overrightarrow{\mathbf{EF}} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$

d) $\overrightarrow{\mathbf{GH}} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

e) $\overrightarrow{\mathbf{IJ}} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

f) $\overrightarrow{\mathbf{KL}} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

(ii) Expressing $\overrightarrow{\mathbf{CD}}$ in terms of $\overrightarrow{\mathbf{AB}}$:

$$\overrightarrow{\mathbf{CD}} = \begin{pmatrix} -6 \\ -6 \end{pmatrix} = -2\begin{pmatrix} 3 \\ 3 \end{pmatrix} = -2\,\overrightarrow{\mathbf{AB}}$$

(iii) Expressing $\overrightarrow{\mathbf{EF}}$ in terms of $\overrightarrow{\mathbf{GH}}$:

$$\overrightarrow{\mathbf{EF}} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} -2 \\ -2 \end{pmatrix} = \frac{3}{2}\,\overrightarrow{\mathbf{GH}}$$

(iv) Expressing $\overrightarrow{\mathbf{KL}}$ in terms of $\overrightarrow{\mathbf{IJ}}$:

$$\overrightarrow{\mathbf{KL}} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \frac{4}{5}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{4}{5}\,\overrightarrow{\mathbf{IJ}}$$

Lesson 2 Writing any Given Vector in Terms of Base Vectors

Introduction

By the end of this subunit, you should be able to:

- write a vector in terms of any two non-parallel vectors.

This subunit is about 14 pages in length.

Expressing vectors with base vectors.

As you have seen that vectors can be written or expressed as scalar multiples of each other, this can be extended to writing any given vector in terms of a pair of non-parallel vectors called **base vectors**.

Can any pair of vectors be used as base vectors? ____________________

Compare your answer with:

No! A pair must be a pair of non-parallel vectors.

The simplest base vectors, with both a magnitude of 1unit, are the unit vectors **i** = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and **j** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Determine whether these vectors parallel or not.

____

____

____

Compare your answer with:

Vectors **i** = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and **j** = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are not parallel because you cannot express **i** in terms of **j** , and vice versa.

Example 1

Take, for example, the vector $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$.

$\begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 7 \end{pmatrix}$.

Now, write $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$ in terms of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

____

____

____

Compare your answer with:

Since, $5\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and $7\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$, then $\begin{pmatrix} 5 \\ 7 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 5 \\ 7 \end{pmatrix} = 5$ **i** + 7 **j**.

Example 2

Are the vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ parallel or not?

The vectors $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are not parallel, since they cannot be written or expressed as scalar multiples of each other; therefore, they can be used as base vectors.

Now writing a vector such as $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ in terms of $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$:

$\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \mathrm{k}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \mathrm{l}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, where k and l are scalars.

Write $\mathrm{k}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \mathrm{l}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, as one vector:

$$\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \mathrm{k}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \mathrm{l}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 2k + 3l \\ 3k + 4l \end{pmatrix}$$

Equate corresponding entries to solve the equations simultaneously, for k or l:

$$2 = 2k + 3l \ldots\ldots(i)$$
$$7 = 3k + 4l \ldots\ldots(ii)$$

Multiply equation (i) by –3 and add it to 2 times equation (ii), to eliminate k:

$$-3(2 = 2k + 3l) \ldots\ldots(i)$$
$$2(7 = 3k + 4l) \ldots\ldots(ii)$$

Simplify:

$$-6 = -6k - 9l \ldots\ldots(i)$$
$$14 = 6k + 8l \ldots\ldots(ii)$$

When adding the equations you get:

$$8 = -l$$

$$l = -8$$

Substitute $l = -8$ in equation (i) or equation (ii) to find the value for **k** .

Substituting $l = -8$ in equation (i):

$$2 = 2k + 3(-8)$$

$2 = 2k - 24$

$2 + 24 = 2k - 24 + 24$

$26 = 2k$

$13 = k$

Therefore, k = 13 and l = −8.

Now, $\begin{pmatrix} 2 \\ 7 \end{pmatrix} = 13 \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 8 \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Activity 2

Activity 2

1 Express the following vectors using $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ as base vectors.

a) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

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b) $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$

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c) $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$

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d) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

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Check your performance against the given solutions at the end of this subunit; and if you are satisfied with your performance continue, or otherwise review *writing any given vector in terms of base vectors.*

Note it!

In general, for any vector $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$,

$\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\mathbf{a} = x\,\mathbf{i} + y\,\mathbf{j}$.

When other base vectors are used:

1. Express a given vector as the sum of the scalar multiples of the same base vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ for a given vector $\mathbf{a} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

   $\begin{pmatrix} 0 \\ 2 \end{pmatrix} = k\begin{pmatrix} 1 \\ 2 \end{pmatrix} + l\begin{pmatrix} 3 \\ 2 \end{pmatrix}$,

   Where k and l are scalars, while $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ are base vectors.

2. Simplify the right side by writing $k\begin{pmatrix} 1 \\ 2 \end{pmatrix} + l\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, as one vector.

3. Equate corresponding entries in the vector equation:

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} k+3l \\ 2k+2l \end{pmatrix}.$$

4. Solve the resulting equations simultaneously for the scalars.

Solutions to ACTIVITY 2:

1 Expressing the vectors using $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ as base vectors:

a) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \text{k}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \text{l}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, where k and l are scalars.

Write k $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \text{l}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, as one vector:

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \text{k}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \text{l}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} k+3l \\ 2k+2l \end{pmatrix}$$

Equate corresponding entries to solve the equations simultaneously, for k or l:

$3 = k + 3l \ldots\ldots(i)$

$4 = 2k + 2l \ldots\ldots(ii)$

Multiply equation (i) by -2 and add it to equation (ii), to eliminate k:

$-2(3 = k + 3l)\ldots\ldots(i)$

$4 = 2k + 2l \ldots\ldots(ii)$

Simplify:

$-6 = -2k - 6l \ldots\ldots(i)$

$4 = 2k + 2l \ldots\ldots(ii)$

When adding the equations you get:

$-2 = -4l$

$l = \frac{1}{2}$

Substitute $l = \frac{1}{2}$ in equation (i) or equation (ii) to find the value for **k** .

Substituting $l = \frac{1}{2}$ in equation (i):

$$3 = k + 3\left(\frac{1}{2}\right)$$

$$3 = k + \frac{3}{2}$$

$$3 - \frac{3}{2} = k + \frac{3}{2} - \frac{3}{2}$$

$$\frac{6}{2} - \frac{3}{2} = k$$

$$\frac{3}{2} = k$$

Therefore, $k = \frac{3}{2}$ and $l = \frac{1}{2}$.

Now, $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

b) $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \text{k}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, where k and l are scalars.

Write k $\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, as one vector:

$$\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \text{k}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} k + 3l \\ 2k + 2l \end{pmatrix}$$

Equate corresponding entries to solve the equations simultaneously, for k or l:

$$2 = k + 3l \ldots\ldots(i)$$
$$7 = 2k + 2l \ldots\ldots(ii)$$

Multiply equation (i) by -2 and add it to equation (ii), to eliminate k:

$$-2(2 = k + 3l) \ldots\ldots(i)$$
$$7 = 2k + 2l \ldots\ldots(ii)$$

Simplify:

$-4 = -2k - 6l \ldots\ldots(i)$

$7 = 2k + 2l \ldots\ldots(ii)$

When adding the equations you get:

$3 = -4l$

$l = -\frac{3}{4}$

Substitute $l = -\frac{3}{4}$ in equation (i) or equation (ii) to find the value for **k** .

Substituting $l = -\frac{3}{4}$ in equation (i):

$2 = k + 3\left(-\frac{3}{4}\right)$

$2 = k - \frac{9}{4}$

$2 + \frac{9}{4} = k - \frac{9}{4} + \frac{9}{4}$

$\frac{8}{4} + \frac{9}{4} = k$

$\frac{17}{4} = k$

Therefore, $k = \frac{17}{4}$ and $l = -\frac{3}{4}$.

Now, $\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \frac{17}{4}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

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c) $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 4 \\ 4 \end{pmatrix} = k\begin{pmatrix} 1 \\ 2 \end{pmatrix} + l\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, where k and l are scalars.

Write $k\begin{pmatrix} 1 \\ 2 \end{pmatrix} + l\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, as one vector:

$\begin{pmatrix} 4 \\ 4 \end{pmatrix} = k\begin{pmatrix} 1 \\ 2 \end{pmatrix} + l\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\begin{pmatrix}4\\4\end{pmatrix} = \begin{pmatrix}k+3l\\2k+2l\end{pmatrix}$$

Equate corresponding entries to solve the equations simultaneously, for k or l:

$4 = k + 3l \ldots\ldots(i)$

$4 = 2k + 2l \ldots\ldots(ii)$

Multiply equation (i) by -2 and add it to equation (ii), to eliminate k:

$-2(4 = k + 3l)\ldots\ldots(i)$

$4 = 2k + 2l \ldots\ldots(ii)$

Simplify:

$-8 = -2k - 6l \ldots\ldots(i)$

$4 = 2k + 2l \ldots\ldots(ii)$

When adding the equations you get:

$-4 = -4l$

$l = 1$

Substitute $l = 1$ in equation (i) or equation (ii) to find the value for **k** .

Substituting $l = 1$ in equation (i):

$4 = k + 3(1)$

$4 = k + 3$

$4 - 3 = k + 3 - 3$

$1 = k$

Therefore, $k = 1$ and $l = 1$.

Now, $\begin{pmatrix}4\\4\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix}$.

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d) $\begin{pmatrix}0\\2\end{pmatrix}$

$\begin{pmatrix}0\\2\end{pmatrix} = \text{k}\begin{pmatrix}1\\2\end{pmatrix} + 1\begin{pmatrix}3\\2\end{pmatrix}$, where k and l are scalars.

Write k $\begin{pmatrix}1\\2\end{pmatrix} + 1\begin{pmatrix}3\\2\end{pmatrix}$, as one vector:

$$\begin{pmatrix}0\\2\end{pmatrix} = k\begin{pmatrix}1\\2\end{pmatrix} + l\begin{pmatrix}3\\2\end{pmatrix}$$

$$\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}k+3l\\2k+2l\end{pmatrix}$$

Equate corresponding entries to solve the equations simultaneously, for k or l:

$$0 = k + 3l\ldots\ldots(i)$$
$$2 = 2k + 2l\ldots\ldots(ii)$$

Multiply equation (i) by -2 and add it to equation (ii), to eliminate k:

$$-2(0 = k + 3l)\ldots\ldots(i)$$
$$2 = 2k + 2l\ldots\ldots(ii)$$

Simplify:

$$0 = -2k - 6l\ldots\ldots(i)$$
$$2 = 2k + 2l\ldots\ldots(ii)$$

When adding the equations you get:

$$2 = -4l$$

$$l = -\frac{1}{2}$$

Substitute $l = -\frac{1}{2}$ in equation (i) or equation (ii) to find the value for **k** .

Substituting $l = -\frac{1}{2}$ in equation (i):

$$0 = k + 3\left(-\frac{1}{2}\right)$$

$$0 = k - \frac{3}{2}$$

$$0 + \frac{3}{2} = k - \frac{3}{2} + \frac{3}{2}$$

$$\frac{3}{2} = k$$

Therefore, $k = \frac{3}{2}$ and $l = -\frac{1}{2}$.

Now, $\begin{pmatrix}0\\2\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1\\2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}3\\2\end{pmatrix}$.

Lesson 3 Solving Geometrical Problems Using Vector Methods

Introduction

By the end of this subunit, you should be able to:

- apply vector methods to geometrical problems.

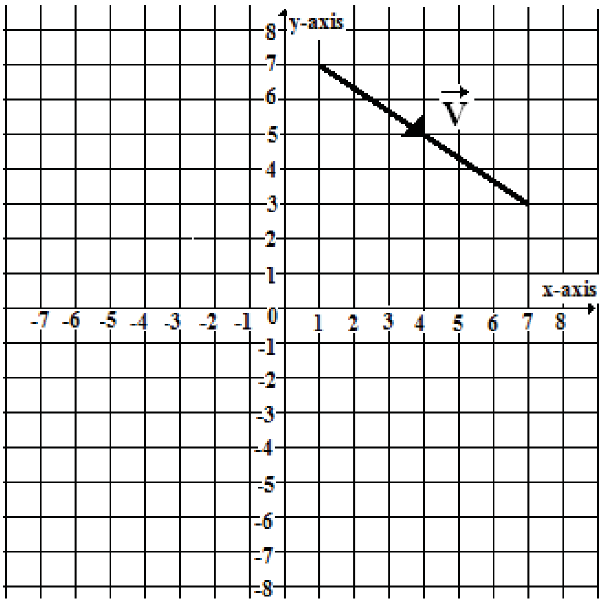
This subunit is about 14 pages in length.

Vectors in geometrical problems.

As some natural phenomena such as force, displacement, velocity and acceleration are vectors, the knowledge of properties vectors is very useful.

Some geometrical problems can be solved using vectors.

Example 1



The vector **v** in the diagram represents the velocity of south-easterly wind blowing in Sahara.

What is the column vector for the velocity of south-easterly wind?

Compare your answer with:

$$\begin{pmatrix} 6 \\ -4 \end{pmatrix}$$

Write down the x-component of the velocity.

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Compare your answer with:

The x-component of the velocity is the top number in the column vector, which is 6.

Which component of the velocity is the bottom number, -4?

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Compare your answer with:

Y-component.

Find the magnitude of the velocity.

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Compare your answer with:

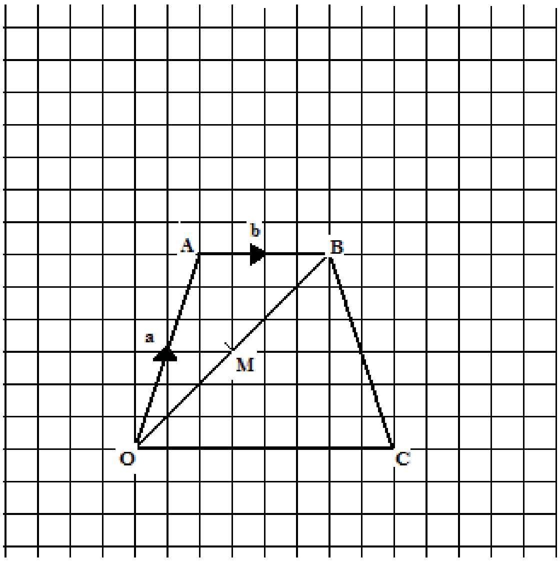
$$\mathbf{v} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$$

$Magnitude\ of\ \mathbf{v}\ is\ |\mathbf{v}|.$

$$|\mathbf{v}| = \left|\begin{pmatrix} 6 \\ -4 \end{pmatrix}\right| = \sqrt{(6^2 + (-4)^2)} = \sqrt{(36 + 16)} = \sqrt{52}\ \ units.$$

Example 2

In the figure below, OABC is a trapezium in which OC = 2AB and M is the midpoint of OB. $\overrightarrow{\mathbf{OA}} = \mathbf{a}$ and $\overrightarrow{\mathbf{AB}} = \mathbf{b}$.



(a) Express the vector $\overrightarrow{\mathbf{OC}}$ in terms of **a** and **b**.

As OC = 2AB, then $\overrightarrow{\mathbf{OC}} = 2\,\mathbf{b}$.

(b) Find the magnitude of vector $\overrightarrow{\mathbf{OC}}$.

Compare your answer with:

$$\overrightarrow{OC} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

Magnitude of $\overrightarrow{OC}$ *is* $\left|\overrightarrow{OC}\right|$.

$$\left|\overrightarrow{OC}\right| = \left|\begin{pmatrix} 8 \\ 0 \end{pmatrix}\right| = \sqrt{(8^2 + 0^2)} = \sqrt{(64 + 0)} = \sqrt{64} = 8 \text{ units.}$$

(c) Express the vector $\overrightarrow{\mathbf{OB}}$ in terms of **a** and **b**.

Compare your answer with:

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \mathbf{a} + \mathbf{b}$$

(d) Find the magnitude of vector $\overrightarrow{\mathbf{OB}}$.

Compare your answer with:

$\overrightarrow{OB} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$

$Magnitude\ of\ \overrightarrow{OB}\ is\ \left|\overrightarrow{OB}\right|.$

$\left|\overrightarrow{OB}\right| = \left|\begin{pmatrix} 6 \\ 6 \end{pmatrix}\right| = \sqrt{(6^2 + 6^2)} = \sqrt{(36 + 36)} = \sqrt{72}\quad units.$

(e) Express the vector $\overrightarrow{\mathbf{CB}}$ in terms of **a** and **b**.

Compare your answer with:

$\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB}.$

$But\ \overrightarrow{CO} = -\overrightarrow{OC}, and\ therefore,\ \overrightarrow{CB} = -\overrightarrow{OC} + \overrightarrow{OB}.$

$\overrightarrow{CB} = -2\mathbf{b} + (\mathbf{a} + \mathbf{b})$

$\overrightarrow{CB} = -2\mathbf{b} + \mathbf{a} + \mathbf{b}$

$\overrightarrow{CB} = \mathbf{a} - \mathbf{b}$

(f) Find the magnitude of vector $\overrightarrow{\mathbf{CB}}$.

Compare your answer with:

$\overrightarrow{CB} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

$Magnitude\ of\ \overrightarrow{CB}\ is\ \left|\overrightarrow{CB}\right|.$

$\left|\overrightarrow{CB}\right| = \left|\begin{pmatrix} -2 \\ 6 \end{pmatrix}\right| = \sqrt{((-2)^2 + 6^2)} = \sqrt{(4 + 36)} = \sqrt{40}\quad units.$

(g) Express the vector $\overrightarrow{\mathbf{OM}}$ in terms of **a** and **b**.

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Compare your answer with:

M is the midpoint of OB; that means $\overrightarrow{\mathbf{OM}}$ is equal to half of $\overrightarrow{\mathbf{OB}}$.

$$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB}$$

$$\text{Since } \overrightarrow{OB} = \mathsf{a} + \mathsf{b},$$

$$\overrightarrow{OM} = \frac{1}{2}(\mathsf{a} + \mathsf{b})$$

$$\overrightarrow{OM} = \frac{1}{2}\mathsf{a} + \frac{1}{2}\mathsf{b}$$

(h) Express the vector $\overrightarrow{\mathbf{MA}}$ in terms of **a** and **b**.

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Compare your answer with:

$$\overrightarrow{MA} = \overrightarrow{MO} + \overrightarrow{OA}.$$

$$\text{But } \overrightarrow{MO} = -\overrightarrow{OM}, \text{and therefore, } \overrightarrow{MA} = -\overrightarrow{OM} + \overrightarrow{OA}.$$

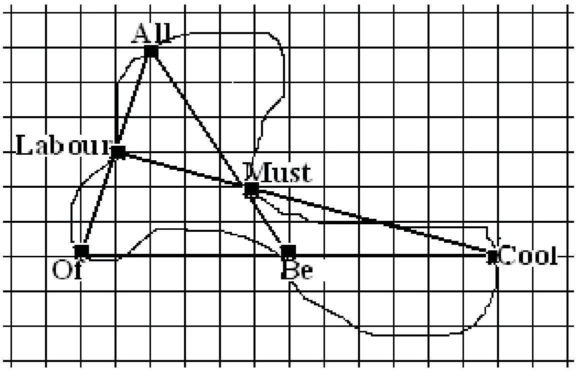
$$\overrightarrow{MA} = -\left(\frac{1}{2}\mathsf{a} + \frac{1}{2}\mathsf{b}\right) + \mathsf{a}$$

$$\overrightarrow{MA} = -\frac{1}{2}\mathsf{a} - \frac{1}{2}\mathsf{b} + \mathsf{a}$$

$$\overrightarrow{MA} = \frac{1}{2}\mathsf{a} - \frac{1}{2}\mathsf{b}$$

Example 3

The diagram shows places in Workplace city. The displacement from Of to All is 3**a**. The displacement from Of to Be is 3**b**. Labour is equidistant from Of and All. Be is equidistant from Of and Cool. The displacement of Must from All is two thirds of the displacement of Be from All.



(i) Express the following displacements in terms of **a** and **b**.

a) Displacement from All to Be.

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Compare your answer with:

Displacement from All to Be is equal to displacement from All to Of plus displacement from Of to Be.

Displacement from All to Of is -3**a**.

Displacement from Of to Be 3**b**.

Therefore, displacement from All to Be is 3**b** + (-3**a**), which is 3**b** – 3**a**.

b) Displacement from All to Must.

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Compare your answer with:

It is given that 'The displacement of Must from All is two thirds of the displacement of Be from All'.

But, the displacement of Be from All is 3**b** – 3**a**, from a).

Therefore, displacement from All to Must is $\frac{2}{3}(3\mathbf{b} - 3\mathbf{a})$, which is $2\mathbf{b} - 2\mathbf{a}$.

c) Displacement from Labour to All.

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Compare your answer with:

It is given that 'Labour is equidistant from Of and All'.

Therefore, the displacement from Labour to All is half of the displacement from Of to All.

This is $\frac{1}{2}(3\mathbf{a})$, which is $\frac{3}{2}\mathbf{a}$.

d) Displacement from Labour to Must.

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Compare your answer with:

Displacement from Labour to Must is equal to the displacement from Labour to All, plus the displacement from All to Must.

Now, use your answers from b) and c).

Displacement from Labour to Must is $(2\mathbf{b} - 2\mathbf{a}) + (\frac{3}{2}\mathbf{a})$.

$$(2\mathbf{b} - 2\mathbf{a}) + \frac{3}{2}\mathbf{a} = 2\mathbf{b} - 2\mathbf{a} + \frac{3}{2}\mathbf{a} = 2\mathbf{b} - \frac{1}{2}\mathbf{a}.$$

e) Displacement from Be to Cool.

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Compare your answer with:

It is given that ‘Be is equidistant from Of and Cool’.

Therefore, the displacement from Be to Cool is equal to the displacement from Of to Be, which is given as 3**b**.

f) Displacement from Must to Be.

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Compare your answer with:

If the displacement of Must from All is two thirds of the displacement of Be from All, the displacement of Be from All is one third of the displacement of Be from All, because AM + MB = AB.

The displacement of Be from All is $3\mathbf{b} - 3\mathbf{a}$, from a).

Therefore, the displacement from Must to Be is $\frac{1}{3}(3\mathbf{b} - 3\mathbf{a})$, which simplifies to $\mathbf{b} - \mathbf{a}$.

g) Displacement from Must to Cool.

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Compare your answer with:

Displacement from Must to Cool is equivalent to the displacement from Must to Be plus the displacement from Be to Cool, from the diagram.

Use your answers from e) and f) to get the required expression in terms of **a** and **b**.

Displacement from Must to Cool is $(3\mathbf{b}) + (\mathbf{b} - \mathbf{a})$.

$3\mathbf{b} + (\mathbf{b} - \mathbf{a})$

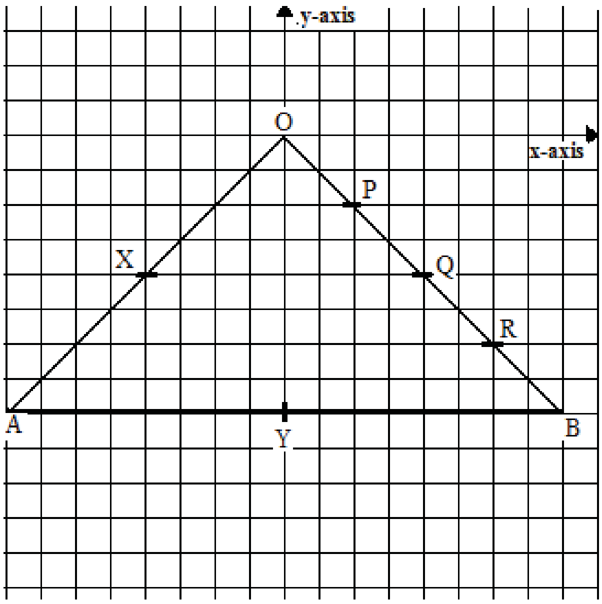
$= 3\mathbf{b} + \mathbf{b} - \mathbf{a}$

$= 4\mathbf{b} - \mathbf{a}$.

Activity 3

Activity 3

1. In the figure below, O is the origin; and the position vectors of A and B are 2**a** and 4**b** respectively. X is the midpoint of OA and Y is the midpoint of AB. OP = PQ = QR = RB.



a) Write down the position vector of X in terms of **a** and **b**.

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b) Write down the position vector of P in terms of **a** and **b**.

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c) Write down the vector $\overrightarrow{\mathbf{XP}}$ in terms of **a** and **b**.

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d) Write down the vector $\overrightarrow{\mathbf{AB}}$ in terms of **a** and **b**.

e) Write down the vector $\overrightarrow{\mathbf{YB}}$ in terms of **a** and **b**.

f) Write down the vector $\overrightarrow{\mathbf{BR}}$ in terms of **a** and **b**.

g) Write down the vector $\overrightarrow{\mathbf{YR}}$ in terms of **a** and **b**.

Check your performance against the given solutions at the end of this subunit; and if you are satisfied with it continue, or otherwise review *solving geometrical problems using vector methods.*

Note it!

It is generally very important to observe the direction when dealing with vectors, as it has been stated that have sense of direction.

Solutions to ACTIVITY 3:

1.

a) Writing down the position vector of X in terms of **a** and **b**.

$$\overrightarrow{OX} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}(2\mathbf{a}) = \mathbf{a}$$

b) Writing down the position vector of P in terms of **a** and **b**.

$$\overrightarrow{OP} = \frac{1}{4}\overrightarrow{OB} = \frac{1}{4}(4\mathbf{b}) = \mathbf{b}$$

c) Writing down the vector $\overrightarrow{\mathbf{XP}}$ in terms of **a** and **b**.

$$\overrightarrow{XP} = \overrightarrow{XO} + \overrightarrow{OP} = -\overrightarrow{OX} + \overrightarrow{OP} = -\mathbf{a} + \mathbf{b}$$

d) Writing down the vector $\overrightarrow{\mathbf{AB}}$ in terms of **a** and **b**.

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -2\mathbf{a} + 4\mathbf{b}$$

e) Writing down the vector $\overrightarrow{\mathbf{YB}}$ in terms of **a** and **b**.

$$\overrightarrow{YB} = \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}(-2\mathbf{a} + 4\mathbf{b}) = -\mathbf{a} + 2\mathbf{b}$$

f) Writing down the vector $\overrightarrow{\mathbf{BR}}$ in terms of **a** and **b**.

$$\overrightarrow{BR} = \frac{1}{4}\overrightarrow{BO} = -\frac{1}{4}\overrightarrow{OB} = -\mathbf{b}$$

g) Writing down the vector $\overrightarrow{\mathbf{YR}}$ in terms of **a** and **b**.

$$\overrightarrow{YR} = \overrightarrow{YB} + \overrightarrow{BR} = (-\mathbf{a} + 2\mathbf{b}) + (-\mathbf{b}) = -\mathbf{a} + \mathbf{b}$$

Unit Summary

Summary

In this unit you learned that any two vectors can be written or expressed as scalar multiples of each other **if and only if they are parallel.**

Any given vector can be written or expressed in terms of a pair of vectors called **base vectors** if the base vectors are not parallel to each other.

The simplest base vectors, with both a magnitude of 1unit, are the unit vectors $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and for any vector $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ in an x –y plane, $\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or **a** = x **i** + y **j**.

When other base vectors are used:

- ✓ Form a vector equation, with a given vector being equal to the sum of the scalar multiples of the base vectors.
- ✓ Simplify the sum of the scalar multiples of the base vectors to one vector.
- ✓ Equate corresponding entries in the vector equation.
- ✓ Solve the resulting simultaneous equations for the scalars.

Some geometrical problems can solved using the fact that vectors can be written or expressed as scalar multiples of other vectors.

You have completed the material for this unit on vectors. You should now spend some time reviewing the content. Once you feel that you can successfully write an exam that covers each of the learning outcomes, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

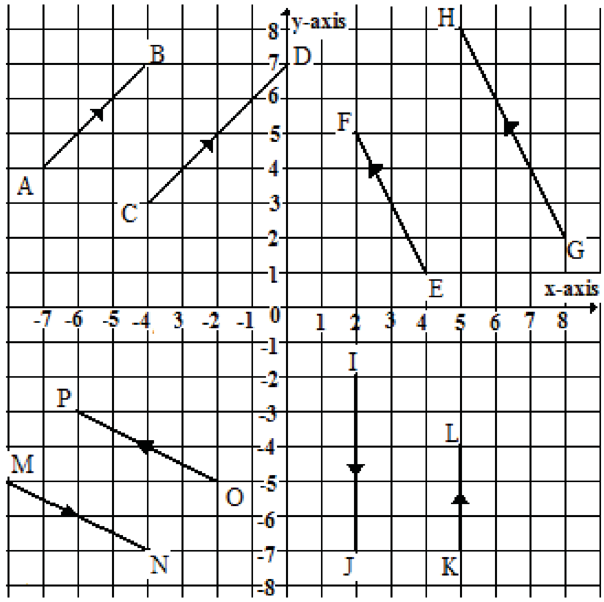
When you work on this assignment, please observe the time allocated and show your working or a reason for your answer.

TOTAL MARKS: 40

TIME: 45 minutes

Assignment

1. Use the grid below to answer the questions that follow.



a) Express $\overrightarrow{\mathbf{CD}}$ in terms of $\overrightarrow{\mathbf{AB}}$. (2 marks)

b) Express $\overrightarrow{\mathbf{EF}}$ in terms of $\overrightarrow{\mathbf{GH}}$. (2 marks)

c) Express $\overrightarrow{\mathbf{IJ}}$ in terms of $\overrightarrow{\mathbf{KL}}$. (2 marks)

d) Express $\overrightarrow{\mathbf{MN}}$ in terms of $\overrightarrow{\mathbf{OP}}$. (2 marks)

2. Express the following vectors using $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\end{pmatrix}$ as base vectors.

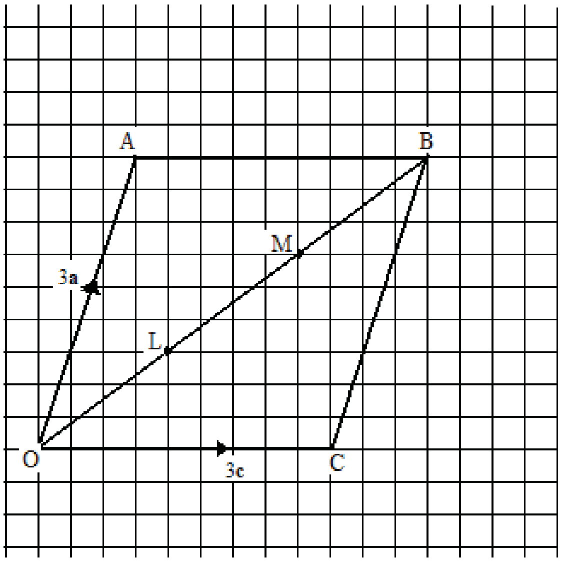
a) $\begin{pmatrix}4\\3\end{pmatrix}$ (4 marks)

b) $\begin{pmatrix}7\\2\end{pmatrix}$ (4 marks)

c) $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ (4 marks)

d) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ (4 marks)

3. OABC is a parallelogram with $\overrightarrow{\mathbf{OA}} = 3\mathbf{a}$, $\overrightarrow{\mathbf{OC}} = 3\mathbf{c}$, OL = $\frac{1}{3}$ OB and 3OM = 2OB.



a) Write down the vector $\overrightarrow{\mathbf{AB}}$ in terms of **a** and **c**. (2 marks)

b) Write down the vector $\overrightarrow{\mathbf{CB}}$ in terms of **a** and **c**. (2 marks)

c) Write down the vector $\overrightarrow{\mathbf{OB}}$ in terms of **a** and **c**. (2 marks)

d) Write down the vector $\overrightarrow{\mathbf{OL}}$ in terms of **a** and **c**. (2 marks)

e) Write down the vector $\overrightarrow{\mathbf{OM}}$ in terms of **a** and **c**. (2 marks)

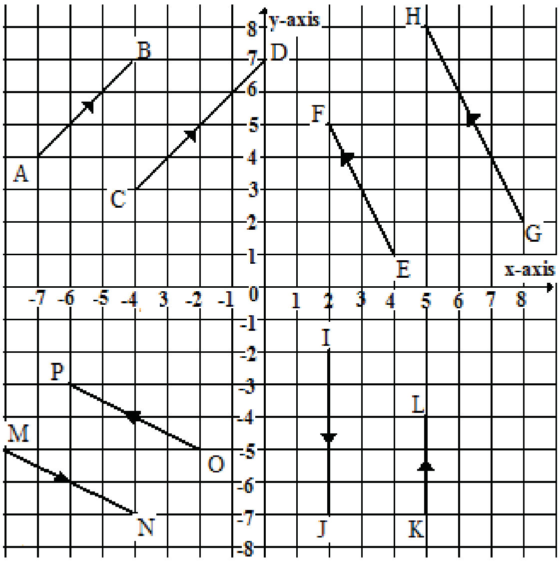
f) Write down the vector $\overrightarrow{\mathbf{AL}}$ in terms of **a** and **c**. (3 marks)

g) Write down the vector $\overrightarrow{\mathbf{MC}}$ in terms of **a** and **c**. (3 marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Solutions to the ASSIGNMENT:

1. Use the grid below to answer the questions that follow.



a) Expressing $\overrightarrow{\mathbf{CD}}$ in terms of $\overrightarrow{\mathbf{AB}}$.

$$\overrightarrow{\mathbf{CD}} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \frac{4}{3}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \frac{4}{3}\overrightarrow{\mathbf{AB}}$$

b) Expressing $\overrightarrow{\mathbf{EF}}$ in terms of $\overrightarrow{\mathbf{GH}}$.

$$\overrightarrow{\mathbf{EF}} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} -3 \\ 6 \end{pmatrix} = \frac{2}{3}\overrightarrow{\mathbf{GH}}$$

c) Expressing $\overrightarrow{\mathbf{IJ}}$ in terms of $\overrightarrow{\mathbf{KL}}$.

$$\overrightarrow{\mathbf{IJ}} = \begin{pmatrix} 0 \\ -5 \end{pmatrix} = -\frac{5}{3}\begin{pmatrix} 0 \\ -3 \end{pmatrix} = -\frac{5}{3}\overrightarrow{\mathbf{KL}}$$

d) Expressing $\overrightarrow{\mathbf{MN}}$ in terms of $\overrightarrow{\mathbf{OP}}$.

$$\overrightarrow{\mathbf{MN}} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} = -\begin{pmatrix} -4 \\ 2 \end{pmatrix} = -\overrightarrow{\mathbf{OP}}$$

2. Express the following vectors using $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as base vectors.

a) $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \end{pmatrix} + l\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, where k and l are scalars.

Write k $\begin{pmatrix}1\\1\end{pmatrix}$ + l $\begin{pmatrix}1\\2\end{pmatrix}$, as one vector:

$\begin{pmatrix}4\\3\end{pmatrix} = k\begin{pmatrix}1\\1\end{pmatrix} + l\begin{pmatrix}1\\2\end{pmatrix}$

$\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}k+l\\k+2l\end{pmatrix}$

Equate corresponding entries to solve the equations simultaneously, for k or l:

$4 = k + l......(i)$

$3 = k + 2l......(ii)$

Multiply equation (i) by -1 and add it to equation (ii), to eliminate k:

$-1(4 = k + l)......(i)$

$3 = k + 2l......(ii)$

Simplify:

$-4 = -k - l......(i)$

$3 = k + 2l......(ii)$

When adding the equations you get:

$-1 = l$

$l = -1$

Substitute $l = -1$ in equation (i) or equation (ii) to find the value for **k** .

Substituting $l = -1$ in equation (i):

$4 = k + (-1)$

$4 = k - 1$

$4 + 1 = k - 1 + 1$

$5 = k$

Therefore, $k = 5$ and $l = -1$.

Now, $\begin{pmatrix}4\\3\end{pmatrix} = 5\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix}$.

b) $\begin{pmatrix}7\\2\end{pmatrix}$

$\begin{pmatrix}7\\2\end{pmatrix} = k\begin{pmatrix}1\\1\end{pmatrix} + l\begin{pmatrix}1\\2\end{pmatrix}$, where k and l are scalars.

Write k $\begin{pmatrix}1\\1\end{pmatrix}+1\begin{pmatrix}1\\2\end{pmatrix}$, as one vector:

$$\begin{pmatrix}7\\2\end{pmatrix}=k\begin{pmatrix}1\\1\end{pmatrix}+l\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}7\\2\end{pmatrix}=\begin{pmatrix}k+l\\k+2l\end{pmatrix}$$

Equate corresponding entries to solve the equations simultaneously, for k or l:

$7=k+l\ldots\ldots(i)$

$2=k+2l\ldots\ldots(ii)$

Multiply equation (i) by -1 and add it to equation (ii), to eliminate k:

$-1(7=k+l)\ldots\ldots(i)$

$2=k+2l\ldots\ldots(ii)$

Simplify:

$-7=-k-l\ldots\ldots(i)$

$2=k+2l\ldots\ldots(ii)$

When adding the equations you get:

$-5=l$

$l=-5$

Substitute $l=-5$ in equation (i) or equation (ii) to find the value for **k** .

Substituting $l=-5$ in equation (i):

$7=k+(-5)$

$7=k-5$

$7+5=k-5+5$

$12=k$

Therefore, $k=12$ and $l=-5$.

Now, $\begin{pmatrix}7\\2\end{pmatrix}=12\begin{pmatrix}1\\1\end{pmatrix}-5\begin{pmatrix}1\\2\end{pmatrix}$.

c) $\begin{pmatrix}3\\3\end{pmatrix}$

$\begin{pmatrix}3\\3\end{pmatrix}=k\begin{pmatrix}1\\1\end{pmatrix}+l\begin{pmatrix}1\\2\end{pmatrix}$, where k and l are scalars.

Write k $\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}1\\2\end{pmatrix}$, as one vector:

$$\begin{pmatrix}3\\3\end{pmatrix} = k\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}k+l\\k+2l\end{pmatrix}$$

Equate corresponding entries to solve the equations simultaneously, for k or l:

$3 = k + l......(i)$

$3 = k + 2l......(ii)$

Multiply equation (i) by -1 and add it to equation (ii), to eliminate k:

$-1(3 = k + l)......(i)$

$3 = k + 2l......(ii)$

Simplify:

$-3 = -k - l......(i)$

$3 = k + 2l......(ii)$

When adding the equations you get:

$0 = l$

$l = 0$

Substitute $l = 0$ in equation (i) or equation (ii) to find the value for **k** .

Substituting $l = 0$ in equation (i):

$3 = k + (0)$

$3 = k$

Therefore, $k = 3$ and $l = 0$.

Now, $\begin{pmatrix}3\\3\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}1\\2\end{pmatrix} = 3\begin{pmatrix}1\\1\end{pmatrix}$.

d) $\begin{pmatrix}0\\2\end{pmatrix}$

$\begin{pmatrix}0\\2\end{pmatrix} = k\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}1\\2\end{pmatrix}$, where k and l are scalars.

Write k $\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}1\\2\end{pmatrix}$, as one vector:

$$\begin{pmatrix}0\\2\end{pmatrix} = k\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}k+l\\k+2l\end{pmatrix}$$

Equate corresponding entries to solve the equations simultaneously, for k or l:

$0 = k + l \ldots\ldots(i)$

$2 = k + 2l \ldots\ldots(ii)$

Multiply equation (i) by -1 and add it to equation (ii), to eliminate k:

$-1(0 = k + l) \ldots\ldots(i)$

$2 = k + 2l \ldots\ldots(ii)$

Simplify:

$0 = -k - l \ldots\ldots(i)$

$2 = k + 2l \ldots\ldots(ii)$

When adding the equations you get:

$2 = l$

$l = 2$

Substitute $l = 2$ in equation (i) or equation (ii) to find the value for **k** .

Substituting $l = 2$ in equation (i):

$0 = k + (2)$

$0 = k + 2$

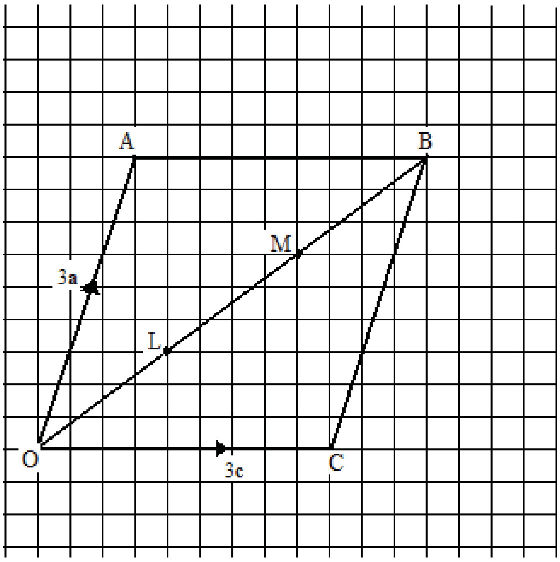
$0 - 2 = k + 2 - 2$

$-2 = k$

Therefore, $k = -2$ and $l = 2$.

Now, $\begin{pmatrix}0\\2\end{pmatrix} = -2\begin{pmatrix}1\\1\end{pmatrix} + 2\begin{pmatrix}1\\2\end{pmatrix}$.

3. OABC is a parallelogram with $\overrightarrow{\mathbf{OA}} = 3\mathbf{a}$, $\overrightarrow{\mathbf{OC}} = 3\mathbf{c}$, OL = $\frac{1}{3}$OB and 3OM = 2OB.



a) Writing down the vector $\overrightarrow{\mathbf{AB}}$ in terms of **a** and **c**.

$\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{OC}} = 3\mathbf{c}$

b) Writing down the vector $\overrightarrow{\mathbf{CB}}$ in terms of **a** and **c**.

$\overrightarrow{\mathbf{CB}} = \overrightarrow{\mathbf{OA}} = 3\mathbf{a}$

c) Writing down the vector $\overrightarrow{\mathbf{OB}}$ in terms of **a** and **c**.

$\overrightarrow{\mathbf{OB}} = \overrightarrow{\mathbf{OA}} + \overrightarrow{\mathbf{AB}} = 3\mathbf{a} + 3\mathbf{c}$

d) Writing down the vector $\overrightarrow{\mathbf{OL}}$ in terms of **a** and **c**

$$\overrightarrow{\mathbf{OL}} = \frac{1}{3}\overrightarrow{\mathbf{OB}} = \frac{1}{3}(3\mathbf{a} + 3\mathbf{c}) = \mathbf{a} + \mathbf{c}$$

e) Writing down the vector $\overrightarrow{\mathbf{OM}}$ in terms of **a** and **c**.

$$\overrightarrow{\mathbf{OM}} = \frac{2}{3}\overrightarrow{\mathbf{OB}} = \frac{2}{3}(3\mathbf{a} + 3\mathbf{c}) = 2\mathbf{a} + 2\mathbf{c}$$

f) Writing down the vector $\overrightarrow{\mathbf{AL}}$ in terms of **a** and **c**.

$\overrightarrow{\mathbf{AL}} = -\overrightarrow{\mathbf{OA}} + \overrightarrow{\mathbf{OL}} = -(3\mathbf{a}) + (\mathbf{a} + \mathbf{c}) = -2\mathbf{a} + \mathbf{c}$

g) Writing down the vector $\overrightarrow{\mathbf{MC}}$ in terms of **a** and **c**.

$\overrightarrow{\mathbf{MC}} = -\overrightarrow{\mathbf{OM}} + \overrightarrow{\mathbf{OC}} = -(2\mathbf{a} + 2\mathbf{c}) + (3\mathbf{c}) = -2\mathbf{a} + \mathbf{c}$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Attempt all the questions. Show your working or a reason for each answer.

TOTAL MARKS: 40

TIME: 45 minutes

1) $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$.

a) Express **a** in terms of **b**. (2 marks)

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____

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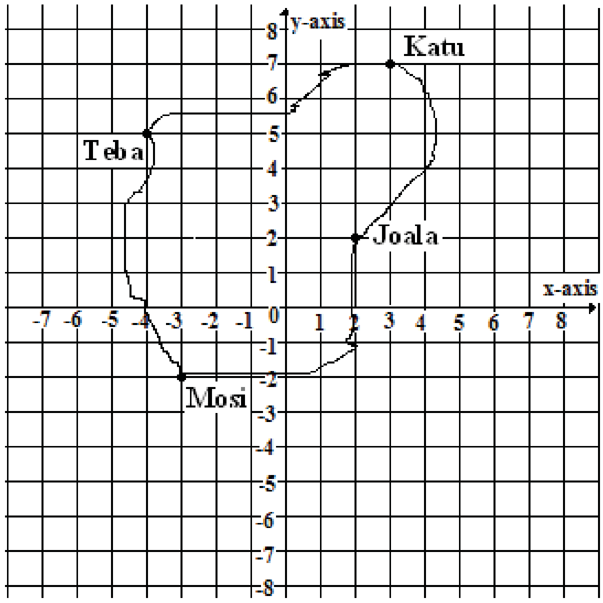
____

b) Find $|\mathbf{a}|$. (2 marks)

c) Express 2**a** – **b** as column vector. (2 marks)

d) Find the magnitude of 2**a** – **b**. (2 marks)

2) The positions of four places in a village are shown in the grid below.



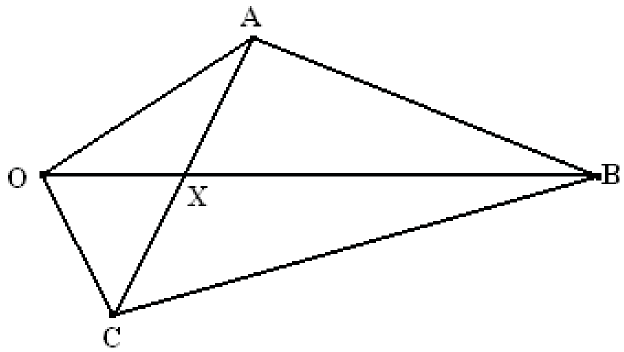
a) Write the position vector of Teba. (2 marks)

b) Write the position vector of Mosi. (2 marks)

c) Write the position vector of Joala. (2 marks)

d) How far is Katu from Joala? (2 marks)

3) The quadrilateral OABC is such that OA = 2**a** and OB = **b**. The line OB intersect with AC at the point X which is the mid-point of AC and 4OX = OB.



Find in terms of **a** and/or **b**, the following vectors.

a) $\overrightarrow{AB}$ (2 marks)

b) $\overrightarrow{OX}$ (2 marks)

c) $\overrightarrow{XA}$ (2 marks)

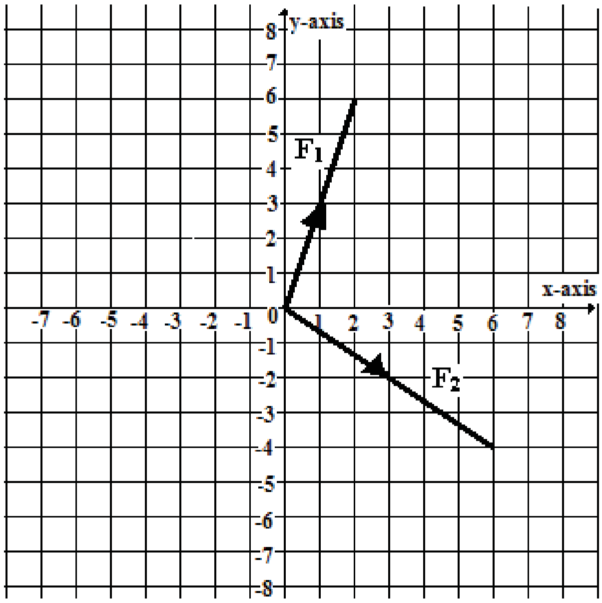
d) $\overrightarrow{XC}$ (2 marks)

e) $\overrightarrow{OC}$ (3 marks)

f) $\overrightarrow{CB}$ (3 marks)

4) The diagram shows two forces $\mathbf{F_1}$ and $\mathbf{F_2}$ applied by **Setŝabelo** and **Tŝoarelo** respectively, when pulling a bag of

rice across the floor.



a) Write down the column vector for $\mathbf{F_1}$. (2 marks)

b) Write down the column vector for $\mathbf{F_2}$. (2 marks)

c) Find the resultant force applied by the two people. (2 marks)

d) Find the magnitude of the resultant force. (2 marks)

Solutions to assessment questions.

1) $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 12 \\ 8 \end{pmatrix}$.

a) **a** in terms of **b**.

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 12 \\ 8 \end{pmatrix}$$

b) Finding $|\mathbf{a}|$.

$$|\mathbf{a}| = \left|\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right| = \sqrt{(3^2 + 2^2)} = \sqrt{(9+4)} = \sqrt{13} \quad \textit{units.}$$

c) Expressing $2\mathbf{a} - \mathbf{b}$ as column vector.

$$2\mathbf{a} - \mathbf{b} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 12 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 12 \\ 8 \end{pmatrix} = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$$

d) Magnitude of $2\mathbf{a} - \mathbf{b}$.

$$|2\mathbf{a} - \mathbf{b}| = \left|\begin{pmatrix} -6 \\ -4 \end{pmatrix}\right| = \sqrt{((-6)^2 + (-4)^2)} = \sqrt{(36+16)} = \sqrt{52} \quad \textit{units.}$$

2)

a) The position vector of Teba.

$$\begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

b) The position vector of Mosi.

$\begin{pmatrix} -3 \\ -2 \end{pmatrix}$

c) The position vector of Joala.

$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

d) Distance of Katu from Joala.

Column vector for displacement from Katu from Joala is $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$

$$Distance = \left|\begin{pmatrix} -1 \\ -5 \end{pmatrix}\right| = \sqrt{((-1)^2 + (-5)^2)} = \sqrt{(1+25)} = \sqrt{26} \quad units.$$

3)

a) $\overrightarrow{AB}$ in terms of **a** and/or **b**.

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -2\mathbf{a} + \mathbf{b}$.

b) $\overrightarrow{OX}$ in terms of **a** and/or **b**.

It is given that 4OX = OB. 4OX = OB means that OX is one quarter of OB.

Therefore, $\overrightarrow{OX} = \frac{1}{4}\overrightarrow{OB} = \frac{1}{4}\mathbf{b}$.

c) $\overrightarrow{XA}$ in terms of **a** and/or **b**.

$\overrightarrow{XA} = \overrightarrow{XO} + \overrightarrow{OA} = -\overrightarrow{OX} + \overrightarrow{OA} = -\frac{1}{4}\mathbf{b} + 2\mathbf{a}$.

d) $\overrightarrow{XC}$ in terms of **a** and/or **b**.

X which is the mid-point of AC, implies that $\overrightarrow{XC}$ is parallel and equal in magnitude to $\overrightarrow{XA}$, but its direction is opposite to that of $\overrightarrow{XA}$. Therefore, $\overrightarrow{XC} = \frac{1}{4}\mathbf{b} - 2\mathbf{a}$.

e) $\overrightarrow{OC}$ in terms of **a** and/or **b**.

$$\overrightarrow{OC} = \overrightarrow{OX} + \overrightarrow{XC} = \frac{1}{4}\mathbf{b} + (\frac{1}{4}\mathbf{b} - 2\mathbf{a}) = \frac{1}{2}\mathbf{b} - 2\mathbf{a}.$$

f) $\overrightarrow{CB}$ in terms of **a** and/or **b**.

$$\overrightarrow{CB} = \overrightarrow{CX} + \overrightarrow{XB} = -\overrightarrow{XC} + \overrightarrow{XB} = -(\frac{1}{4}\mathbf{b} - 2\mathbf{a}) + \overrightarrow{XB}.$$

$\overrightarrow{XB}$ is three quarters of OB.

$$\overrightarrow{CB} = -(\frac{1}{4}\mathbf{b} - 2\mathbf{a}) + \frac{3}{4}\mathbf{b} = \frac{1}{2}\mathbf{b} + 2\mathbf{a}.$$

4)

a) Column vector for $\mathbf{F_1}$ is $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$.

b) Column vector for $\mathbf{F_2}$ is $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$.

c) The resultant force = $\mathbf{F_1} + \mathbf{F_2} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 6 \\ -4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$.

d) The magnitude of the resultant force:

$$\left|\begin{pmatrix} 8 \\ 2 \end{pmatrix}\right| = \sqrt{(8^2 + 2^2)} = \sqrt{(64 + 4)} = \sqrt{68} \quad units.$$

Unit Contents

Unit 17

Unit 17

Quadratic Equations

Introduction

A quadratic function **f(x)** = $ax^2 + bx + c$, where a, b, and c are constants, has a graph which is called a parabola. The constants **b** and **c** can be any numbers, but **a** *must never be zero*.

An equation which results from the quadratic function f(x) =$ax^2 + bx + c$, when f(x) = 0 is called quadratic equation.

That is $ax^2 + bx + c = 0$ is a quadratic equation in x.

Of course, quadratic equations can be written in different forms, but all of which can be reduced to standard form $\mathbf{ax^2 + bx + c = 0}$.

When quadratic equations are written in different forms, they should be reduced to standard form $\mathbf{ax^2 + bx + c = 0}$ so that the constants a, b, and c can be identified.

This unit consists of 115 pages. It covers approximately 5% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 40 hours to work on this unit, 30 hours for formal study and 10 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Six Lessons:

Lesson 1 Solving Quadratic Equations by Factorisation
Lesson 2 Solving Quadratic Equations by Completing the Square
Lesson 3 Solving Quadratic Equations by Quadratic Formula, Using a Calculator
Lesson 4 Solving Equations that can be Reduced to Quadratic Form
Lesson 5 Solving Simultaneous Equations Involving Quadratic Equations
Lesson 6 Solving Problems which give Rise to Quadratic Equations

Take a minute to read the following learning outcomes. They are a guide to what you should focus on while studying this unit.

Upon completion of this unit you will be able to:

Outcomes

- *solve* quadratic equations by factorisation.
- *solve* quadratic equations by completing the square.
- *solve* quadratic equations by quadratic formula, using a calculator.
- *solve* equations that can be reduced to quadratic form.
- *solve* simultaneous equations involving quadratic equations.
- *solve* problems which give rise to quadratic equations.

Terminology

$ax^2 + bx + c = 0$:	Standard form of quadratic equation.
Coefficient:	A number which multiplies a variable.
Equation:	A mathematical statement that uses **equal sign** to connect two expressions.
Term:	What is added in an expression.
Variable:	A letter which represents an unknown in an equation.
$x^2 - y^2 = (x+y)(x-y)$:	Factorisation of the difference of two squares.
Zero factor property:	If the product of any real numbers is zero, then at least one of the numbers is zero.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Solving Quadratic Equations by Factorisation

Introduction

By the end of this subunit, you should be able to solve the equations which can be solved by factorisation.

This subunit is about 30 pages in length.

Solving quadratic equations by zero factor property

Zero factor property is used to solve quadratic equations by factorisation.

Solving by factorisation example 1

Look at the equation: **$(x+2)(2x-3)=0$**.

Is it a quadratic equation?

____

Compare your answer with:

Since $(x+2)(2x-3)=0$ is the same as $2x^2 + x - 6 = 0$, the starting equation is a quadratic equation.

If you do not remember how to multiply through the brackets, refer to the unit on simplifying expressions, under *expansion of double brackets*. Learn it well as it is a foundational skill for this unit.

Now, if $(x+2)(2x-3)=0$, then $(x+2)=0$ or $(2x-3)=0$, using zero factor property.

If $(x+2)=0$, then

$x+2-2=0-2$

$x=-2$.

If $(2x-3)=0$, then

$2x-3+3=0+3$

$2x=3$

$$\frac{2x}{2}=\frac{3}{2}$$

$x=\frac{3}{2}$.

Therefore, the solutions are -2 and $\frac{3}{2}$.

Solving by factorisation example 2

Let us work together on this next example:

$5x(7x+3)=0$.

Now, if $5\mathbf{x}(7\mathbf{x}+3)=0$, what statement can you write about 5x and 7x+3, using the zero factor property?

____

Compare your answer with:

5x = 0 or 7x + 3 = 0

Then, if 5x = 0, what should be x?

____

____

____

Compare your answer with:

5x = 0

$$\frac{5\mathbf{x}}{5}=\frac{0}{5}$$

$$\mathbf{x}=0$$

Similarly, if 7x + 3 = 0, what should be x?

____

____

____

____

Compare your answer with:

7x + 3 = 0

$$7\mathbf{x}+3-3=0-3$$

$$7\mathbf{x}=-3$$

$$\mathbf{x}=\frac{-3}{7}$$

So what are the solutions?

____

Compare your answer with:

The solutions are 0 and $\frac{-3}{7}$.

So, zero factor property can be used to solve quadratic equations when the right side equals zero; and the left is a product of factors with two terms in brackets. Of course, when one of the two factors is one term like in $5\mathbf{x}(7\mathbf{x}+3)$, 5x is not written in brackets.

Solve the equations in *Activity 1* below:

Activity 1

(a) $(4x + 3)(3x - 4) = 0$

(b) $(3x - 4)(3x - 4) = 0$

(c) $(2\mathbf{x} + 1)^2 = 0$

(d) $(2\mathbf{y} + 4\tfrac{1}{2})(\mathbf{y} - 3\tfrac{2}{3}) = 0$

(e) $(2\mathbf{y} + 1.42)(3\mathbf{y} + \sqrt{0.81}) = 0$

Compare your solutions with the answers given at the end of the subunit. If you scored at least 80% continue. If not, review *solving quadratic equations with the product of factors on the left side and zero on the right side* and try this activity again.

Solving quadratic equations when b =0

Solving when b = 0 example 1

Suppose you are to solve $3x^2 = 8$:

Even though, for consistency, we will solve the equation using the zero factor property, a *faster* approach is to divide each side by 3 and then take the square root of each side:

$3\mathbf{x}^2 = 8$

$\frac{3\mathbf{x}^2}{3} = \frac{8}{3}$

$\mathbf{x}^2 = \frac{8}{3}$

$\mathbf{x} = \pm\sqrt{\frac{8}{3}}$

Therefore, the solutions are $\sqrt{\frac{8}{3}}$ and $-\sqrt{\frac{8}{3}}$.

The following shows the zero factor method:

$3\mathbf{x}^2 = 8$

Write it in standard form: $3x^2 - 8 = 0$

Divide by 3 on both sides: $\mathbf{x}^2 - \frac{8}{3} = \mathbf{0}$

Since $\sqrt{\frac{8}{3}} \times \sqrt{\frac{8}{3}} = \frac{8}{3}$, then: $\mathbf{x}^2 - \left(\sqrt{\frac{8}{3}}\right)^2 = \mathbf{0}$

Factorise, using difference of two squares:

$$\left(x-\sqrt{\frac{8}{3}}\right)\left(x+\sqrt{\frac{8}{3}}\right)=0$$

Use zero factor property:

$$\left(x-\sqrt{\frac{8}{3}}\right)=0 \text{ or } \left(x+\sqrt{\frac{8}{3}}\right)=0$$

If $\left(\mathbf{x}-\sqrt{\frac{8}{3}}\right)=0$, then

$$\mathbf{x}-\sqrt{\frac{8}{3}}+\sqrt{\frac{8}{3}}=0+\sqrt{\frac{8}{3}}$$

$$\mathbf{x}=\sqrt{\frac{8}{3}}$$

If $\left(\mathbf{x}+\sqrt{\frac{8}{3}}\right)=0$, then

$$\mathbf{x}+\sqrt{\frac{8}{3}}-\sqrt{\frac{8}{3}}=0-\sqrt{\frac{8}{3}}$$

$$\mathbf{x}=-\sqrt{\frac{8}{3}}$$

Therefore, the solutions are $\sqrt{\frac{8}{3}}$ and $-\sqrt{\frac{8}{3}}$.

Solving when b = 0 example 2

Let us work together on this next example:

$-2x^2 + 3 = 0.$

Divide by -2 on both sides.

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Compare your answer with:

$-2x^2 + 3 = 0$

$$\mathbf{x^2 - \frac{3}{2} = 0}$$

Factorise, using difference of two squares.

____

Compare your answer with:

$$\mathbf{x}^2 - \frac{3}{2} = \mathbf{0}$$

$$\left(\mathbf{x} - \sqrt{\frac{3}{2}}\right)\left(\mathbf{x} + \sqrt{\frac{3}{2}}\right) = 0$$

Use zero factor property to solve.

Compare your answer with:

$$\left(\mathbf{x} - \sqrt{\frac{3}{2}}\right)\left(\mathbf{x} + \sqrt{\frac{3}{2}}\right) = 0$$

$$\left(\mathbf{x} - \sqrt{\frac{3}{2}}\right) = 0 \text{ or } \left(\mathbf{x} + \sqrt{\frac{3}{2}}\right) = 0$$

If $\left(\mathbf{x} - \sqrt{\frac{3}{2}}\right) = 0$, then

$$\mathbf{x} - \sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}} = 0 + \sqrt{\frac{3}{2}}$$

$$\mathbf{x} = \sqrt{\frac{3}{2}}$$

If $\left(\mathbf{x} + \sqrt{\frac{3}{2}}\right) = 0$, then

$$\mathbf{x} + \sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}} = 0 - \sqrt{\frac{3}{2}}$$

$$\mathbf{x} = -\sqrt{\frac{3}{2}}$$

Therefore, the solutions are $\sqrt{\frac{3}{2}}$ and $-\sqrt{\frac{3}{2}}$.

Solving when b = 0 example 3

Take for example $5x^2 = 3x$.

Write it in standard form.

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Compare your answer to the following:

You should get $5\mathbf{x}^2 - 3\mathbf{x} = 0$

Factorise **x** from the two terms.

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Compare your answer to the following:

You should get $\mathbf{x}(5\mathbf{x} - 3) = 0$

Use zero factor property to solve.

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Compare your answer to the following:

If $x(5x – 3) = 0$, then $x = 0$ or $(5x – 3) = 0$, using zero factor property.

If $(5x - 3) = 0$, then

$5x - 3 + 3 = 0 + 3$

$5x = 3$

$$\frac{5\mathbf{x}}{5} = \frac{3}{5}$$

$$\mathbf{x} = \frac{3}{5}$$

Therefore, the solutions are 0 and $\frac{3}{5}$.

Note that when solving quadratic equations in the form $ax^2 + c = 0$:

- b = 0
- $\frac{c}{a}$ must be negative
- when $\frac{c}{a}$ is positive the equation cannot be solved with real numbers.

Solve the equations in *Activity 2* below:

Activity 2

(a) $\mathbf{x}^2 = 100$

(b) $121\mathbf{y}^2 - 0.25 = 0$

(c) $4\mathbf{x}^2 = 16^{\frac{1}{4}}$

(d) $\mathbf{x}(3\mathbf{x} - 4) = -4\mathbf{x} + 81$

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(e) $(3\mathbf{x} - 4)(3\mathbf{x} + 4) = 0$

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Check your performance against the given solutions at the end of the subunit. If you can comfortably answer the questions continue. Otherwise review *solving quadratic equations, not expressed as the product of factors equal to zero, when **b** =0.*

Solving quadratic equations by factorisation when b $\neq$ 0 and c $\neq$ 0

What can be done when b $\neq$ 0 and c $\neq$ 0?

Some of these quadratic equations can be solved by factoring.

Solving by factorisation example 1

Take for example, $\mathbf{x}^2 - 4\mathbf{x} + 3 = 0$.

Work out the product **ac:**

ac = 1 × 3 = 3

List numbers which divide exactly into the product 3**:**

$\pm 1, \pm 3$

Look for a pair(s) of numbers with the sum equal to coefficient of **x** in $\mathbf{x}^2 - 4\mathbf{x} + 3 = 0$:

The pair from **-1** and **-3** has the sum **-4**

Rewrite the equation by replacing the middle term by two terms of x, with **-1** and **-3** as their coefficients (*In this case, it does not matter if the order is -1 and then -3 or -3 and then -1)*:

$\mathbf{x}^2 - 4\mathbf{x} + 3 = 0$

$\mathbf{x}^2 - 1\mathbf{x} - 3\mathbf{x} + 3 = 0$

Factorise the first two terms and the other two separately:

$\mathbf{x}^2 - 1\mathbf{x} - 3\mathbf{x} + 3 = 0$

$(\mathbf{x}^2 - 1\mathbf{x}) - (3\mathbf{x} - 3) = 0$

$\mathbf{x}(\mathbf{x} - 1) - 3(\mathbf{x} - 1) = 0$

Complete the factorisation:

$\mathbf{x}(\mathbf{x} - 1) - 3(\mathbf{x} - 1) = 0$

$(\mathbf{x} - 1)(\mathbf{x} - 3) = 0$

If $(\mathbf{x} - 1)(\mathbf{x} - 3) = 0$, then $(\mathbf{x} - 1) = 0$ or $(\mathbf{x} - 3) = 0$, using zero factor property.

If $(\mathbf{x} - 1) = 0$, then

$\mathbf{x} - 1 + 1 = 0 + 1$

$\mathbf{x} = 1$

If $(\mathbf{x} - 3) = 0$, then

$\mathbf{x} - 3 + 3 = 0 + 3$

$\mathbf{x} = 3$

Therefore, the solutions are a repeated solution 1 and 3 .

Take for example, $\mathbf{x}^2 - 4\mathbf{x} + 3 = 0$.

Factorise **x** from the two terms on the left side:

$3x^2 + 4x = 0$

$x(3x + 4) = 0$

Now, if x (3x + 4) = 0, then x = 0 or (3x + 4) = 0, using zero factor property.

If (3x + 4) = 0, then

$3x + 4 - 4 = 0 - 4$

$3x = -4$

$$\frac{3\mathbf{x}}{3} = \frac{-4}{3}$$

$x = -1\frac{1}{3}$.

Therefore, the solutions are 0 and $-1\frac{1}{3}$

Solving by factorisation example 2

Take the example $\mathbf{x}^2 - 6 = 5\mathbf{x}$

Write it in standard form by subtracting 5x from each side:

$\mathbf{x}^2 - 6 = 5\mathbf{x}$

$\mathbf{x}^2 - 5\mathbf{x} - 6 = 5\mathbf{x} - 5\mathbf{x}$

$\mathbf{x}^2 - 5\mathbf{x} - 6 = 0$

Work out the product **ac:**

ac = 1 × -6 = -6

List numbers which divide exactly into the product -12**:**

$\pm 1, \pm 2, \pm 3, \pm 6$

Look for a pair(s) of numbers with the sum equal to coefficient of **x** in $\mathbf{x}^2 - 5\mathbf{x} - 6 = 0$:

The pair ***1*** *and* ***-6*** or ***-2*** *and* ***-3*** has the sum **-5.**

But, the pair, *1* and *-6*, whose product is equal to ac, that is -6, is correct for

$\mathbf{x}^2 - 5\mathbf{x} - 6 = 0$:

Rewrite the equation by replacing the middle term by two terms of x, with **1** and **-6** as their coefficients *(In this case, it does not matter if the order is -6 and then 1 or 1 and then -6)*:

$\mathbf{x}^2 - 5\mathbf{x} - 6 = 0$

$\mathbf{x}^2 - 6\mathbf{x} + 1\mathbf{x} - 6 = 0$

Factorise the first two terms and the other two separately:

$\mathbf{x}^2 - 6\mathbf{x} + 1\mathbf{x} - 6 = 0$

$(\mathbf{x}^2 - 6\mathbf{x}) + (1\mathbf{x} - 6) = 0$

$\mathbf{x}(\mathbf{x} - 6) + 1(\mathbf{x} - 6) = 0$

Complete the factorisation:

$\mathbf{x}(\mathbf{x} - 6) + 1(\mathbf{x} - 6) = 0$

$(\mathbf{x} - 6)(\mathbf{x} + 1) = 0$

If $(\mathbf{x} - 6)(\mathbf{x} + 1) = 0$, then $(\mathbf{x} - 6) = 0$ or $(\mathbf{x} + 1) = 0$, using zero factor property.

If $(\mathbf{x} - 6) = 0$, then

$\mathbf{x} - 6 + 6 = 0 + 6$

$\mathbf{x} = 6$

If $(\mathbf{x} + 1) = 0$, then

$\mathbf{x} + 1 - 1 = 0 - 1$

$\mathbf{x} = -1$

Therefore, the solutions are a repeated solution 6 and -1

Solving by factorisation example 3

Let us work through the following equation together:

$2\mathbf{x}^2 + 11\mathbf{x} + 3 = 6 - 2\mathbf{x}^2$

Write it in standard form:

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Compare your answer with the following:

Add $2x^2$ to each side:

$2\mathbf{x}^2 + 11\mathbf{x} + 3 = 6 - 2\mathbf{x}^2$

$2\mathbf{x}^2 + 2\mathbf{x}^2 + 11\mathbf{x} + 3 = 6 - 2\mathbf{x}^2 + 2\mathbf{x}^2$

$4\mathbf{x}^2 + 11\mathbf{x} + 3 = 6$

Subtract 6 from each side:

$4\mathbf{x}^2 + 11\mathbf{x} + 3 = 6$

$4\mathbf{x}^2 + 11\mathbf{x} + 3 - 6 = 6 - 6$

$4\mathbf{x}^2 + 11\mathbf{x} - 3 = 0$

Work out the product **ac:**

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Compare your answer with the following:

ac = 4 x (-3) = -12

List numbers which divide exactly into that product **-12:**

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Compare your answer with the following:

$1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12.$

Look for a pair(s) of numbers with the sum equal to coefficient of **x** in

$4x^2 + 11x - 3 = 0$:

____

Compare your answer with the following:

The pair from **-1** and **+12** has the sum **+11**.

Rewrite the equation by replacing the middle term by two terms of x, with **-1** and **+12** as their coefficients *(in this case, it does not matter if the order is -1 and then +12 or +12 and then -1)*:

Compare your answer with the following:

$$4\mathbf{x}^2 + 11\mathbf{x} - 3 = 0$$

$$4\mathbf{x}^2 + 12\mathbf{x} - 1\mathbf{x} - 3 = 0$$

Factorise the first two terms and the other two separately:

Compare your answer with the following:

$$4\mathbf{x}^2 + 12\mathbf{x} - 1\mathbf{x} - 3 = 0$$

$$(4\mathbf{x}^2 + 12\mathbf{x}) - (1\mathbf{x} + 3) = 0$$

$$4\mathbf{x}(\mathbf{x} + 3) - 1(\mathbf{x} + 3) = 0$$

Complete the factorisation:

Compare your answer with the following:

$$4\mathbf{x}(\mathbf{x} + 3) - 1(\mathbf{x} + 3) = 0$$

$$(\mathbf{x} + 3)(4\mathbf{x} - 1) = 0$$

Use zero factor property to solve:

Compare your answer with the following:

If $(\mathbf{x}+3)(4\mathbf{x}-1)=0$, then $(\mathbf{x}+3)=0$ or $(4\mathbf{x}-1)=0$, using zero factor property.

If $(\mathbf{x}+3)=0$, then

$\mathbf{x}+3-3=0-3$

$\mathbf{x}=-3$

If $(4\mathbf{x}-1)=0$, then

$4\mathbf{x}-1+1=0+1$

$4\mathbf{x}=1$

$\frac{4\mathbf{x}}{4}=\frac{1}{4}$

$\mathbf{x}=\frac{1}{4}$

Therefore, the solutions are a repeated solution -3 and $\frac{1}{4}$

If you are comfortable with how to solve quadratic equations by factorisation, solve the equations in activity 3 below. If not, review the examples again.

Solve the equations in *Activity 3* below:

Activity 3

(a) $\mathbf{y}^2+10\mathbf{y}+25=0$

(b) $2\mathbf{m}^2 - 8\mathbf{m} - 10 = 0$

(c) $2\mathbf{x}^2 + 7\mathbf{x} = -6$

(d) $5\mathbf{n}^2 - 27\mathbf{n} + 10 = 0$

(e) $2\mathbf{x}^2 + 4\mathbf{x} = \mathbf{x} + 5$

Check your performance against the given solutions. If you are satisfied with your performance continue. Otherwise, review *solving quadratic equations by factorisation when b ≠ 0 and c ≠ 0:*

Key Points to Remember

The key points to remember in this subunit on solving quadratic equations by factorisation are:

1. Write the equation in standard form, $\mathbf{a\,x^2 + b\,x + c = 0}$ if it is not already written.
2. Factorise the expression, $\mathbf{a\,x^2 + b\,x + c}$, on the left, if it is not in a factorised form.
3. Then, use the zero factor property to get the solutions.

Solutions to the Subunit Activities

Solutions to ACTIVITY 1:

(a) $(4x + 3)(3x - 4) = 0$

If $(4x + 3)(3x - 4) = 0$, then $(4x + 3) = 0$ or $(3x - 4) = 0$, using zero factor property.

If $(4x + 3) = 0$, then

$4x + 3 - 3 = 0 - 3$

$4x = -3$

$$\frac{4\mathbf{x}}{4} = \frac{-3}{4}$$

$$\mathbf{x} = \frac{-3}{4}$$

If $(3x - 4) = 0$, then

$3x - 4 + 4 = 0 + 4$

$3x = 4$

$$\frac{3\mathbf{x}}{3} = \frac{4}{3}$$

$$\mathbf{x} = \frac{4}{3}$$

Therefore, the solutions are $\frac{-3}{4}$ and $\frac{4}{3}$.

(b) $(3x - 4)(3x - 4) = 0$

If $(3x - 4)(3x - 4) = 0$, then $(3x - 4) = 0$ or $(3x - 4) = 0$, using zero factor property.

If $(3x - 4) = 0$, then

$3x - 4 + 4 = 0 + 4$

$3x = 4$

$$\frac{3\mathbf{x}}{3} = \frac{4}{3}$$

$\mathbf{x} = \frac{4}{3}$

If (3x - 4) = 0, then

3x – 4 + 4 = 0 + 4

3x = 4

$\frac{3\mathbf{x}}{3} = \frac{4}{3}$

$\mathbf{x} = \frac{4}{3}$

Therefore, the solutions are a repeated solution $\frac{4}{3}$ and $\frac{4}{3}$.

(c) $(2\mathbf{x}+1)^2 = 0$

If $(2\mathbf{x}+1)^2 = 0$, then (2x + 1) = 0 or (2x + 1) = 0, using zero factor property.

If (2x +1) = 0, then

2x + 1 - 1 = 0 - 1

2x = -1

$\frac{2\mathbf{x}}{2} = \frac{-1}{2}$

$\mathbf{x} = \frac{-1}{2}$

If (2x +1) = 0, then

2x + 1 - 1 = 0 - 1

2x = -1

$\frac{2\mathbf{x}}{2} = \frac{-1}{2}$

$\mathbf{x} = \frac{-1}{2}$

Therefore, the solutions are a repeated solution $\frac{-1}{2}$ and $\frac{-1}{2}$.

(d) $(2\mathbf{y}+4\tfrac{1}{2})(\mathbf{y}-3\tfrac{2}{3}) = 0$

If $(2\mathbf{y}+4\tfrac{1}{2})(\mathbf{y}-3\tfrac{2}{3}) = 0$, then $(2\mathbf{y}+4\tfrac{1}{2}) = 0$ or $(\mathbf{y}-3\tfrac{2}{3}) = 0$, using zero factor property.

If $(2\mathbf{y}+4\tfrac{1}{2})=0$, then

$2\mathbf{y}+4\tfrac{1}{2}-4\tfrac{1}{2}=0-4\tfrac{1}{2}$

$2\mathbf{y}=-4\tfrac{1}{2}$

$\frac{2\mathbf{y}}{2}=\frac{-4\tfrac{1}{2}}{2}$

$\mathbf{x}=-2\tfrac{1}{4}$

If $(\mathbf{y}-3\tfrac{2}{3})=0$, then

$\mathbf{y}-3\tfrac{2}{3}+3\tfrac{2}{3}=0+3\tfrac{2}{3}$

$\mathbf{y}=3\tfrac{2}{3}$

Therefore, the solutions are $-2\tfrac{1}{4}$ and $3\tfrac{2}{3}$.

(e) $(2\mathbf{y}+1.42)(3\mathbf{y}+\sqrt{0.81})=0$

If $(2\mathbf{y}+1.42)(3\mathbf{y}+\sqrt{0.81})=0$, then $(2\mathbf{y}+1.42)=0$ or $(3\mathbf{y}+\sqrt{0.81})=0$, using zero factor property.

If $(2\mathbf{y}+1.42)=0$, then

$2\mathbf{y}+1.42-1.42=0-1.42$

$2\mathbf{y}=-1.42$

$\frac{2\mathbf{y}}{2}=\frac{-1.42}{2}$

$\mathbf{x}=-0.72$

If $(3\mathbf{y}+\sqrt{0.81})=0$, then

$(3\mathbf{y}+\sqrt{0.81})-\sqrt{0.81})=0-\sqrt{0.81})$

$3\mathbf{y}=-\sqrt{0.81})$

$\frac{3\mathbf{y}}{3}=-\frac{\sqrt{0.81}}{3}$

$\mathbf{y}=-\sqrt{\frac{0.81}{9}}$

$\mathbf{y}=-\sqrt{0.09}$

$\mathbf{y}=-0.3$

Therefore, the solutions are -0.72 and -0.3

Solutions to ACTIVITY 2:

(a) $\mathbf{x}^2 = 100$

If $\mathbf{x}^2 = 100$, then take the square root of each side:

$\mathbf{x} = \pm\sqrt{100}$

$\mathbf{x} = \pm 10$

Therefore, the solutions are 10 and -10.

(b) $121\mathbf{y}^2 - 0.25 = 0$

If $121\mathbf{y}^2 - 0.25 = 0$ add 0.25 to both sides:

$121\mathbf{y}^2 - 0.25 + 0.25 = 0 + 0.25$

$121\mathbf{y}^2 = 0.25$

Divide each side by 121 and then take the square root of each side:

$$\frac{121\mathbf{y}^2}{121} = \frac{0.25}{121}$$

$$\mathbf{y}^2 = \frac{0.25}{121}$$

$$\mathbf{y} = \pm\sqrt{\frac{0.25}{121}}$$

$$\mathbf{y} = \pm\frac{1}{22}$$

Therefore, the solutions are $\frac{1}{22}$ and $-\frac{1}{22}$.

(c) $4\mathbf{x}^2 = 16^{\frac{1}{4}}$

If $4\mathbf{x}^2 = 16^{\frac{1}{4}}$, simplify $16^{\frac{1}{4}}$ to 2; divide each side by 4 and then take the square root of each side:

$4\mathbf{x}^2 = 16^{\frac{1}{4}}$

$4\mathbf{x}^2 = 2$

$$\frac{4\mathbf{x}^2}{4} = \frac{2}{4}$$

$$\mathbf{x}^2 = \frac{1}{2}$$

$$\mathbf{x} = \pm\sqrt{\frac{1}{2}}$$

Therefore, the solutions are $\sqrt{\frac{1}{2}}$ and $-\sqrt{\frac{1}{2}}$.

(d) x(3x – 4) = - 4x + 81

If x(3x – 4) = - 4x + 81, remove the brackets by multiplying (3x – 4) by x; :

x(3x – 4) = - 4x + 81

$$3\mathbf{x}^2 - 4\mathbf{x} = -4\mathbf{x} + 81$$

Add 4x to both sides and simplify:

$$3\mathbf{x}^2 - 4\mathbf{x} + 4\mathbf{x} = -4\mathbf{x} + 4\mathbf{x} + 81$$

$$3\mathbf{x}^2 = 81$$

Divide each side by 3 and then take the square root of each side:

$$\frac{3\mathbf{x}^2}{3} = \frac{81}{3}$$

$$\mathbf{x}^2 = 27$$

$$\mathbf{x} = \pm\sqrt{27}$$

Therefore, the solutions are $\sqrt{27}$ and $-\sqrt{27}$

(e) (3x – 4)(3x + 4) = 0

If $(3\mathbf{x} - 4)(3\mathbf{x} + 4) = 0$, then by the *difference of two squares:*

$$(3\mathbf{x} - 4)(3\mathbf{x} + 4) = 0$$

$$9\mathbf{x}^2 - 4^2 = 0$$

$$9\mathbf{x}^2 - 16 = 0$$

$$9\mathbf{x}^2 - 16 + 16 = 0 + 16$$

$$9\mathbf{x}^2 = 16$$

Divide each side by 9 and then take the square root of each side:

$$\frac{9\mathbf{x}^2}{9} = \frac{16}{9}$$

$$\mathbf{x}^2 = \frac{16}{9}$$

$$\mathbf{x} = \pm\sqrt{\frac{16}{9}}$$

$$\mathbf{x} = \pm\frac{4}{3}$$

Therefore, the solutions are $\frac{4}{3}$ and $-\frac{4}{3}$

Solutions to activity 3:

(a) $\mathbf{y}^2 + 10\mathbf{y} + 25 = 0$

Work out the product **ac:**

ac = 1 × 25 = 25

List numbers which divide exactly into the product 3**:**

$\pm 1, \pm 5 \pm 25$

Look for a pair(s) of numbers with the sum equal to coefficient of **y** in $\mathbf{y}^2 + 10\mathbf{x} + 25 = 0$:

The pair from **+5** and **+5** has the sum **+10**

Rewrite the equation by replacing the middle term by two terms of y, with **+5** and **+5** as their coefficients:

$$\mathbf{y}^2 + 10\mathbf{y} + 25 = 0$$

$$\mathbf{y}^2 + 5\mathbf{y} + 5\mathbf{y} + 25 = 0$$

Factorise the first two terms and the other two separately:

$$\mathbf{y}^2 + 5\mathbf{y} + 5\mathbf{y} + 25 = 0$$

$$(\mathbf{y}^2 + 5\mathbf{y}) + (5\mathbf{y} + 25) = 0$$

$$\mathbf{y}(\mathbf{y} + 5) + 5(\mathbf{y} + 5) = 0$$

Complete the factorisation:

$$\mathbf{y}(\mathbf{y} + 5) + 5(\mathbf{y} + 5) = 0$$

$$(\mathbf{y} + 5)(\mathbf{y} + 5) = 0$$

If $(\mathbf{y}+5)(\mathbf{y}+5)=0$, then $(\mathbf{y}+5)=0$ or $(\mathbf{y}+5)=0$, using zero factor property.

If $(\mathbf{y}+5)=0$, then

$\mathbf{y}+5-5=0-5$

$\mathbf{y}=-5$

Again, if $(\mathbf{y}+5)=0$, then

$\mathbf{y}+5-5=0-5$

$\mathbf{y}=-5$

Therefore, the solutions are a repeated solution -5 and -5

(b) $2\mathbf{m}^2-8\mathbf{m}-10=0$

Work out the product **ac:**

ac = 2 × -10 = -20

List numbers which divide exactly into the product 3**:**

$\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20.$

Look for a pair(s) of numbers with the sum equal to coefficient of **m** in $2\mathbf{m}^2-8\mathbf{m}-10=0$:

The pair ***-4*** *and* ***-4*** or *-10 and +2* has the sum **-20.**

But, the pair, *-10* *and* +2, whose product is equal to ac, that is -20, is correct for:

$2\mathbf{m}^2-8\mathbf{m}-10=0$:

Rewrite the equation by replacing the middle term by two terms of m, with **-10** and **+2** as their coefficients (*in this case, it does not matter if the order is -10 and then +2 or +2 and then -10)*:

$2\mathbf{m}^2-8\mathbf{m}-10=0$

$2\mathbf{m}^2+2\mathbf{m}-10\mathbf{m}-10=0$

Factorise the first two terms and the other two separately:

$2\mathbf{m}^2+2\mathbf{m}-10\mathbf{m}-10=0$

$(2\mathbf{m}^2+2\mathbf{m})-(10\mathbf{m}+10)=0$

$2\mathbf{m}(\mathbf{m}+1)-10(\mathbf{m}+1)=0$

Complete the factorisation:

$2\mathbf{m}(\mathbf{m}+1)-10(\mathbf{m}+1)=0$

$(\mathbf{m}+1)(2\mathbf{m}-10)=0$

If $(\mathbf{m}+1)(2\mathbf{m}-10)=0$, then $(\mathbf{m}+1)=0$ or $((2\mathbf{m}-10)=0$, using zero factor property.

If $(\mathbf{m}+1)=0$, then

$\mathbf{m}+1-1=0-1$

$\mathbf{m}=-1$

If $(2\mathbf{m}-10)=0$, then

$2\mathbf{m}-10\mathbf{m}+10=0+10$

$2\mathbf{m}=10$

$\frac{2\mathbf{m}}{2}=\frac{10}{2}$

$\mathbf{m}=5$

Therefore, the solutions are -1 and 5.

(c) $2\mathbf{x}^2+7\mathbf{x}=-6$

Write it in standard form by adding 6 to each side:

$2\mathbf{x}^2+7\mathbf{x}=-6$

$2\mathbf{x}^2+7\mathbf{x}+6=-6+6$

$2\mathbf{x}^2+7\mathbf{x}+6=0$

Work out the product **ac:**

ac = 2 x 6 = 12

List numbers which divide exactly into that product **12:**

$\pm 1,\ \pm 2, \pm 3, \pm 4, \pm 6,\ \pm 12.$

Look for a pair(s) of numbers with the sum equal to coefficient of **x** in $2\mathbf{x}^2+7\mathbf{x}+6=0$:

The pair+ ***4*** *and*+ ***3*** or ***+1*** *and* ***+6*** have the sum **+7**

But, the pair, *+4 and +3*, **whose product is equal to ac, that is +12, is correct for**

$2\mathbf{x}^2+7\mathbf{x}+6=0$:

Rewrite the equation by replacing the middle term by two terms of x, with **+1** and **+6** as their coefficients (*in this case, it does not matter if the order is +1 and then +6 or +6 and then +1)*:

$2\mathbf{x}^2 + 7\mathbf{x} + 6 = 0$

$2\mathbf{x}^2 + 4\mathbf{x} + 3\mathbf{x} + 6 = 0$

Factorise the first two terms and the other two separately:

$2\mathbf{x}^2 + 4\mathbf{x} + 3\mathbf{x} + 6 = 0$

$(2\mathbf{x}^2 + 4\mathbf{x}) + (3\mathbf{x} + 6) = 0$

$2\mathbf{x}(\mathbf{x} + 2) + 3(\mathbf{x} + 2) = 0$

Complete the factorisation:

$2\mathbf{x}(\mathbf{x} + 2) + 3(\mathbf{x} + 2) = 0$

$(\mathbf{x} + 2)(2\mathbf{x} + 3) = 0$

If $(\mathbf{x} + 2)(2\mathbf{x} + 3) = 0$, then $(\mathbf{x} + 2) = 0$ or $(2\mathbf{x} + 3) = 0$, using zero factor property.

If $(\mathbf{x} + 2) = 0$, then

$\mathbf{x} + 2 - 2 = 0 - 2$

$\mathbf{m} = -2$

If $(2\mathbf{x} + 3) = 0$, then

$2\mathbf{x} + 3 - 3 = 0 - 3$

$2\mathbf{x} = -3$

$\frac{2\mathbf{x}}{2} = \frac{-3}{2}$

$\mathbf{m} = \frac{-3}{2}$

Therefore, the solutions are -2 and $\frac{-3}{2}$.

(d) $5\mathbf{n}^2 - 27\mathbf{n} + 10 = 0$

Work out the product **ac:**

ac = 5 × 10 = 50

List numbers which divide exactly into the product 50**:**

$\pm 1, \pm 2, \pm 5, \pm 10, \pm 25, \pm 50.$

Look for a pair(s) of numbers with the sum equal to coefficient of **n** in

$5n^2 - 27n + 10 = 0$:

The pair from **-2** and **-25** has the sum **-27.**

Rewrite the equation by replacing the middle term by two terms of n, with **-2** and **-25** as their coefficients (*in this case, it does not matter if the order is -2 and then -25 or -25 and then -2)*:

$$5n^2 - 27n + 10 = 0$$

$$5n^2 - 25n - 2n + 10 = 0$$

Factorise the first two terms and the other two separately:

$$5n^2 - 25n - 2n + 10 = 0$$

$$(5n^2 - 25n) - (2n - 10) = 0$$

$$5n(n - 5) - 2(n - 5) = 0$$

Complete the factorisation:

$$5n(n - 5) - 2(n - 5) = 0$$

$$(n - 5)(5n - 2) = 0$$

If $(n - 5)(5n - 2) = 0$, then $(n - 5) = 0$ or $(5n - 2) = 0$, using zero factor property.

If $(n - 5) = 0$, then

$$n - 5 + 5 = 0 + 5$$

$$n = 5$$

If $(5n - 2) = 0$, then

$$5n - 2 + 2 = 0 + 2$$

$$5n = 2$$

$$\frac{5n}{5} = \frac{2}{5}$$

$$n = \frac{2}{5}$$

Therefore, the solutions are a repeated solution 5 and $\frac{2}{5}$

(e) $2x^2 + 4x = x + 5$

Write it in standard form:

Subtract 5 from each side:

$$2x^2 + 4x - 5 = x + 5 - 5$$

$2x^2 + 4x - 5 = x$

Subtract x from each side:

$2x^2 + 4x - x - 5 = x - x$

$2x^2 + 3x - 5 = 0$

Work out the product **ac:**

ac = 2 × -5 = -10

List numbers which divide exactly into the product -10**:**

$\pm 1, \pm 2, \pm 5, \pm 10.$

Look for a pair(s) of numbers with the sum equal to coefficient of **x** in

$2x^2 + 3x - 5 = 0$:

The pair+ ***1*** *and*+ ***2*** or **+*5*** *and* ***-2*** have the sum **+3.**

But, the pair +*5 and* **-*2***, **whose product is equal to ac, that is -10, is correct for**

$2x^2 + 3x - 5 = 0$:

Rewrite the equation by replacing the middle term by two terms of x, with **+5** and **-2** as their coefficients (*in this case, it does not matter if the order is -2 and then +5 or +5 and then -2)*:

$2x^2 + 3x - 5 = 0$

$2x^2 - 2x + 5x - 5 = 0$

Factorise the first two terms and the other two separately:

$2x^2 - 2x + 5x - 5 = 0$

$(2x^2 - 2x) + (5x - 5) = 0$

$2x(x - 1) + 5(x - 1) = 0$

Complete the factorisation:

$2x(x - 1) + 5(x - 1) = 0$

$(x - 1)(2x + 5) = 0$

If $(x - 1)(2x + 5) = 0$, then $(x - 1) = 0$ or $(2x + 5) = 0$, using zero factor property.

If $(x - 1) = 0$, then

$x - 1 + 1 = 0 + 1$

$n = 1$

If $(2x + 5) = 0$, then

$2x + 5 - 5 = 0 - 5$

$2\mathbf{x} = -5$

$\frac{2\mathbf{x}}{2} = \frac{-5}{2}$

$\mathbf{x} = \frac{-5}{2}$

Therefore, the solutions are a repeated solution 1 and $\frac{-5}{2}$

Lesson 2 Solving Quadratic Equations by Completing the Square

Introduction

By the end of this subunit, you should be able to complete a square in order to solve a quadratic equation.

This subunit is about 10 pages in length.

Completing the square

Some equations cannot be easily solved by factorisation:

Take, for example, $\mathbf{x}^2 + 4\mathbf{x} + 1 = 0$:

Work out the product **ac:**

ac = 1 x 1 = 1

List numbers which divide exactly into that product **1:**

± 1.

Look for a pair(s) of numbers with the sum equal to coefficient of **x** in

$\mathbf{x}^2 + 4\mathbf{x} + 1 = 0$:

Since there is no pair, the equation is not easily solved by factorisation.

Look at this other example:

$3\mathbf{n}^2 - 4\mathbf{n} - 2 = 0$

Work out the product **ac:**

ac = 3 × -2 = -6

List numbers which divide exactly into the product -6**:**

$\pm 1, \pm 2, \pm 3, \pm 6.$

Look for a pair(s) of numbers with the sum equal to coefficient of **n** in $3\mathbf{n}^2 - 4\mathbf{n} - 2 = 0$:

The pair *-6 and +2* or *-1 and -3* has the sum **-4.**

But, since there is no pair, whose product is equal to ac, that is -6, the equation is not easily solved by factorisation.

Another method of solving quadratic equations is by completing the square.

$(\mathbf{x}+\mathbf{a})^2$ means $(\mathbf{x}+\mathbf{a})(\mathbf{x}+\mathbf{a})$

Simplifying the expression $(\mathbf{x}+\mathbf{a})(\mathbf{x}+\mathbf{a})$:

$(\mathbf{x}+\mathbf{a})(\mathbf{x}+\mathbf{a})$
$= \mathbf{x}(\mathbf{x}+\mathbf{a}) + \mathbf{a}(\mathbf{x}+\mathbf{a})$
$= \mathbf{x}^2 + \mathbf{xa} + \mathbf{ax} + \mathbf{a}^2$
$= \mathbf{x}^2 + 2\mathbf{ax} + \mathbf{a}^2$

Now, first term = the square of x in $(\mathbf{x}+\mathbf{a})^2$

second term = twice the product of x and a in $(\mathbf{x}+\mathbf{a})^2$

third term = the square of a in $(\mathbf{x}+\mathbf{a})^2$

Completing the square example 1:

Look at how to use the above idea in solving quadratic equation $\mathbf{x}^2 + 4\mathbf{x} + 1 = 0$:

$\mathbf{x}^2 + 4\mathbf{x} + 1 = 0$

Subtract 1 from both sides:

$\mathbf{x}^2 + 4\mathbf{x} + 1 - 1 = 0 - 1$

$\mathbf{x}^2 + 4\mathbf{x} = -1$

The first term on the left side of the equation is already a square of x.

Since 4x equals 2(2x), the square of half the coefficient of x added to both sides makes the left side a complete square:

$\mathbf{x}^2 + 4\mathbf{x} = -1$

$\mathbf{x}^2 + 4\mathbf{x} + 2^2 = -1 + 2^2$

$(\mathbf{x}+2)^2 = -1 + 4$

$(\mathbf{x}+2)^2 = 3$

Then take the square root of each side:

$(\mathbf{x}+2)^2 = 3$

$(\mathbf{x} + 2) = \pm\sqrt{3}$

$\mathbf{x} + 2 - 2 = -2 \pm \sqrt{3}$

$\mathbf{x} = -2 \pm \sqrt{3}$

Therefore, the solutions are $-2 + \sqrt{3}$ and $-2 - \sqrt{3}$.

Completing the square example 2:

Let us work through the following equation together:

$3\mathbf{n}^2 - 4\mathbf{n} - 2 = 0$

Add 2 to both sides and simplify:

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Compare your answer with:

$3\mathbf{n}^2 - 4\mathbf{n} - 2 + 2 = 0 + 2$

$3\mathbf{n}^2 - 4\mathbf{n} = 2$

Divide by 3 on both sides to make first term on the left side of the equation a perfect square and simplify:

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Compare your answer with:

$3\mathbf{n}^2 - 4\mathbf{n} = 2$

$\frac{3\mathbf{n}^2 - 4\mathbf{n}}{3} = \frac{2}{3}$

$\mathbf{n}^2 - \frac{4}{3}\mathbf{n} = \frac{2}{3}$

Since $-\frac{4}{3}\mathbf{n}$ equals $2(-\frac{2}{3}\mathbf{n})$, add the square of half the coefficient of n to both sides to make the left side a complete square:

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Compare your answer with:

$$\mathbf{n}^2 - \frac{4}{3}\mathbf{n} = \frac{2}{3}$$

$$\mathbf{n}^2 - \frac{4}{3}\mathbf{n} + \left(-\frac{2}{3}\right)^2 = \frac{2}{3} + \left(-\frac{2}{3}\right)^2$$

$$\left(\mathbf{n} - \frac{2}{3}\right)^2 = \frac{2}{3} + \frac{4}{9}$$

$$\left(\mathbf{n} - \frac{2}{3}\right)^2 = \frac{10}{9}$$

Then take the square root of each side and solve:

Compare your answer with:

$$\left(\mathbf{n} - \frac{2}{3}\right)^2 = \frac{10}{9}$$

$$\left(\mathbf{n} - \frac{2}{3}\right) = \pm\sqrt{\frac{10}{9}}$$

$$\mathbf{n} - \frac{2}{3} + \frac{2}{3} = \frac{2}{3} \pm \sqrt{\frac{10}{9}}$$

$$\mathbf{n} = \frac{4 \pm \sqrt{10}}{3}$$

Therefore, the solutions are $\frac{4+\sqrt{10}}{3}$ and $\frac{4-\sqrt{10}}{3}$.

Completing the square example 3:

Let us work through the following equation together:

$5x^2 + 3x - 1 = 0$

Add 1 to both sides and simplify:

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Compare your answer with:

$5x^2 + 3x - 1 + 1 = 0 + 1$

$5x^2 + 3x = 1$

Divide by 5 on both sides to make first term on the left side of the equation a perfect square and simplify:

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Compare your answer with:

$5x^2 + 3x = 1$

$$\frac{5x^2 + 3x}{5} = \frac{1}{5}$$

$$x^2 + \frac{3x}{5} = \frac{1}{5}$$

Since $+\frac{3x}{5}$ equals $2\left(\frac{3x}{10}\right)$, add the square of half the coefficient of x to both sides to make the left side a complete square:

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Compare your answer with:

$$x^2 + \frac{3x}{5} = \frac{1}{5}$$

$$x^2 + \frac{3x}{5} + \left(\frac{3}{10}\right)^2 = \frac{1}{5} + \left(\frac{3}{10}\right)^2$$

$$\left(x + \frac{3}{10}\right)^2 = \frac{1}{5} + \frac{9}{100}$$

$$\left(x + \frac{3}{10}\right)^2 = \frac{29}{100}$$

Then take the square root of each side and solve:

Compare your answer with:

$$\left(x + \frac{3}{10}\right)^2 = \frac{29}{100}$$

$$\left(x + \frac{3}{10}\right) = \pm\sqrt{\frac{29}{100}}$$

$$x + \frac{3}{10} - \frac{3}{10} = -\frac{3}{10} \pm \sqrt{\frac{29}{100}}$$

$$x = \frac{-3 \pm \sqrt{29}}{10}$$

Therefore, the solutions are $\frac{-3 + \sqrt{29}}{10}$ and $\frac{-3 - \sqrt{29}}{10}$.

If you are comfortable solving quadratic equations by completing the square, solve the equations in activity 4 below. If not, review the examples again.

Solve the equations in *Activity 4* below:

Activity 4

(a) $x^2 + 6x = 2$

(b) $x^2 + 3x + 1 = 0$

(c) $m^2 + 8m - 1 = 0$

(d) $2n^2 - 10n + 1 = 0$

(e) $4\mathbf{x}^2 + 4\mathbf{x} = -8\mathbf{x} - 1$

Compare your answers with the given solutions at the end of this subunit. If you understand each solution continue. If not, review *solving quadratic equations by completing the square.*

Key Points to Remember

The key points to remember in this subunit on *solving quadratic equations by completing the square.*are:

1. Write the equation $\mathbf{a}\,\mathbf{x}^2 + \mathbf{b}\,\mathbf{x} + \mathbf{c} = \mathbf{0}$ as $\mathbf{a}\,\mathbf{x}^2 + \mathbf{b}\,\mathbf{x} = -\mathbf{c}$, if it is not already in that form.
2. Divide both sides by **a** so that the coefficient of $\mathbf{x}^2$ is 1.
3. Add the square of half the coefficient of **x**, $\frac{b}{2a}$ to both sides, in order to get a perfect square, $\left(x + \frac{b}{2a}\right)^2$ on the left.
4. Take square root of each side and simplify to get the two values of **x**.

Solutions to ACTIVITY 4:

(a) $\mathbf{x}^2 + 6\mathbf{x} = 2$

The first term on the left side of the equation is already a square of x.

Since 6x equals 2(3x), the square of half the coefficient of x added to both sides makes the left side a complete square:

$$\mathbf{x}^2 + 6\mathbf{x} = 2$$

$$\mathbf{x}^2 + 6\mathbf{x} + 3^2 = 2 + 3^2$$

$$(\mathbf{x} + 3)^2 = 2 + 9$$

$$(\mathbf{x} + 3)^2 = 11$$

Then take the square root of each side:

$$(\mathbf{x} + 3)^2 = 11$$

$$(\mathbf{x} + 3) = \pm\sqrt{11}$$

$$\mathbf{x} + 3 - 3 = -3 \pm \sqrt{11}$$

$$\mathbf{x} = -3 \pm \sqrt{11}$$

Therefore, the solutions are $-3 + \sqrt{11}$ and $-3 - \sqrt{11}$.

(b) $\mathbf{x}^2 - 3\mathbf{x} - 1 = 0$

Add 1 to both sides and simplify:

$$\mathbf{x}^2 + 3\mathbf{x} + 1 - 1 = 0 - 1$$

$$\mathbf{x}^2 + 3\mathbf{x} = -1$$

The first term on the left side of the equation is already a square of x.

Since 3x equals $2\left(\frac{3\mathbf{x}}{2}\right)$, the square of half the coefficient of x added to both sides makes the left side a complete square:

$$\mathbf{x}^2 + 3\mathbf{x} = -1$$

$$\mathbf{x}^2 + 3\mathbf{x} + \left(\frac{3}{2}\right)^2 = 2 + \left(\frac{3}{2}\right)^2$$

$$\left(\mathbf{x} + \frac{3}{2}\right)^2 = 2 + \frac{9}{4}$$

$$\left(x+\frac{3}{2}\right)^2=\frac{17}{4}$$

Then take the square root of each side:

$$\left(x+\frac{3}{2}\right)^2=\frac{17}{4}$$

$$\left(x+\frac{3}{2}\right)=\pm\sqrt{\frac{17}{4}}$$

$$x+\frac{3}{2}-\frac{3}{2}=-\frac{3}{2}\pm\sqrt{\frac{17}{4}}$$

$$x=\frac{-3\pm\sqrt{17}}{2}$$

Therefore, the solutions are $\frac{-3+\sqrt{17}}{2}$ and $\frac{-3-\sqrt{17}}{2}$.

(c) $m^2+8m-1=0$

Add 1 to both sides and simplify:

$$m^2+8m-1+1=0+1$$

$$m^2+8m=1$$

The first term on the left side of the equation is already a square of m.

Since 8 m equals $2(4m)$, the square of half the coefficient of m added to both sides makes the left side a complete square:

$$m^2+8m=1$$

$$m^2+8m+4^2=1+4^2$$

$$(m+4)^2=1+16$$

$$(m+4)^2=17$$

Then take the square root of each side:

$$(m+4)^2=17$$

$$(m+4)=\pm\sqrt{17}$$

$$m+4-4=-4\pm\sqrt{17}$$

$$m=-4\pm\sqrt{17}$$

Therefore, the solutions are $-4+\sqrt{17}$ and $-4-\sqrt{17}$.

(d) $2\mathbf{n}^2-10\mathbf{n}+1=0$

Subtract 1 from both sides:

$2\mathbf{n}^2-10\mathbf{n}+1-1=0-1$

$2\mathbf{n}^2-10\mathbf{n}=-1$

Divide by 2 on both sides to make first term on the left side of the equation a perfect square and simplify:

$2\mathbf{n}^2-10\mathbf{n}=-1$

$$\frac{2\mathbf{n}^2-10\mathbf{n}}{2}=-\frac{1}{2}$$

$$\mathbf{n}^2-5\mathbf{n}=-\frac{1}{2}$$

Since $-5\mathbf{n}$ equals $2(-\frac{5}{2}\mathbf{n})$, add the square of half the coefficient of n to both sides to make the left side a complete square:

$$\mathbf{n}^2-5\mathbf{n}=-\frac{1}{2}$$

$$\mathbf{n}^2-5\mathbf{n}+\left(-\frac{5}{2}\right)^2=-\frac{1}{2}+\left(-\frac{5}{2}\right)^2$$

$$\left(\mathbf{n}-\frac{5}{2}\right)^2=-\frac{1}{2}+\frac{25}{4}$$

$$\left(\mathbf{n}-\frac{5}{2}\right)^2=\frac{21}{4}$$

Then take the square root of each side and solve:

$$\left(\mathbf{n}-\frac{5}{2}\right)^2=\frac{21}{4}$$

$$\left(\mathbf{n}-\frac{5}{2}\right)=\pm\sqrt{\frac{21}{4}}$$

$$\mathbf{n}-\frac{5}{2}+\frac{5}{2}=\frac{5}{2}\pm\sqrt{\frac{21}{4}}$$

$$\mathbf{n}=\frac{5\pm\sqrt{21}}{2}$$

Therefore, the solutions are $\frac{5+\sqrt{21}}{2}$ and $\frac{5-\sqrt{21}}{2}$.

(e) $4\mathbf{x}^2 + 4\mathbf{x} = -8\mathbf{x} - 1$

Add 8x to both sides and simplify:

$$4\mathbf{x}^2 + 4\mathbf{x} + 8\mathbf{x} = -8\mathbf{x} - 1 + 8\mathbf{x}$$

$$4\mathbf{x}^2 + 12\mathbf{x} = -1$$

Divide by 4 on both sides to make first term on the left side of the equation a perfect square and simplify:

$$4\mathbf{x}^2 + 12\mathbf{x} = -1$$

$$\frac{4\mathbf{x}^2 + 12\mathbf{x}}{4} = -\frac{1}{4}$$

$$\mathbf{x}^2 + 3\mathbf{x} = -\frac{1}{4}$$

Since $3\mathbf{x}$ equals $2(\frac{3}{2}\mathbf{x})$, add the square of half the coefficient of x to both sides to make the left side a complete square:

$$\mathbf{x}^2 + 3\mathbf{x} = -\frac{1}{4}$$

$$\mathbf{x}^2 + 3\mathbf{x} + \left(\frac{3}{2}\right)^2 = -\frac{1}{4} + \left(\frac{3}{2}\right)^2$$

$$\left(\mathbf{x} + \frac{3}{2}\right)^2 = -\frac{1}{4} + \frac{9}{4}$$

$$\left(\mathbf{x} + \frac{3}{2}\right)^2 = 2$$

Then take the square root of each side and solve:

$$\left(\mathbf{x} + \frac{3}{2}\right)^2 = 2$$

$$\left(\mathbf{x} + \frac{3}{2}\right) = \pm\sqrt{2}$$

$$\mathbf{x} + \frac{3}{2} - \frac{3}{2} = -\frac{3}{2} \pm \sqrt{2}$$

$$\mathbf{x} = \frac{3 \pm 2\sqrt{2}}{2}$$

Therefore, the solutions are $\frac{3+2\sqrt{2}}{2}$ and $\frac{3-2\sqrt{2}}{2}$.

Lesson 3 Solving Quadratic Equations by Quadratic Formula, Using a Calculator

Introduction

By the end of this subunit, you should be able to use the formula to solve quadratic equations and the calculator to evaluate explicit expressions from the formula.

This subunit is about 10 pages in length.

Using the quadratic formula

The quadratic formula is derived by completing the square to the standard form of quadratic equation, $\mathbf{a x^2 + b x + c = 0}$.

This is how it is done:

Take constant term **c** to the right:

$a x^2 + b x = -c$

Divide by **a** on both sides:

$$\mathbf{x}^2 + \frac{b}{a}\mathbf{x} = \frac{-c}{a}$$

The square of half of $\frac{b}{a}$ will make $\mathbf{x}^2 + \frac{b}{a}\mathbf{x}$ **,** a perfect square when added to both sides of the equation:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$$

Factorise the perfect square on the left and simplifying $\left(\frac{b}{2a}\right)^2$ on the right:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-c}{a} + \frac{b^2}{4a^2}$$

Express $\frac{-c}{a} + \frac{b^2}{4a^2}$ as a single fraction on the right:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

Take square roots on both sides and remember there are two roots which are opposites

$$\left(x+\frac{b}{2a}\right) = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Remove brackets on the left and take $4a^2$ out of the square root on the right:

$$x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

Isolate x to get the formula:

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

You can also write it as:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

So, $x = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ or $x = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$

This is called the quadratic formula. It comes very handy when the solutions are irrational and are approximated with a calculator.

Now, look at how it used to solve quadratic equations.

Solving using the quadratic equation example 1

Take $2x^2 + 11x + 3 = 6 - 2x^2$.

Write it in standard form: $4x^2 + 11x - 3 = 0$

What are the values for a, b and c in $4x^2 + 11x - 3 = 0$?

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Check your answer with the following:

a = 4, b = 11, and c = -3

Then, you substitute these values for the letters in the formula:

From $\mathbf{x} = \dfrac{-\mathbf{b} \pm \sqrt{\mathbf{b}^2 - 4\mathbf{ac}}}{2\mathbf{a}}$ when, a = 4, b = 11, and c = -3

$$x = \frac{-11 \pm \sqrt{121 + 48}}{8}$$

$$x = \frac{-11 \pm 13}{8}$$

$$x = \frac{2}{8} \text{ or } \frac{-24}{8}$$

$$x = \frac{1}{4} \text{ or } -3$$

Solving using the quadratic equation example 2

Take $x^2 + 2x - 3 = 0$.

What are the values for a, b and c in $x^2 + 2x - 3 = 0$?

Compare your answer with:

a = 1, b = 2, and c = -3

Then, you substitute these values for the letters in the formula:

From $\mathbf{x} = \dfrac{-\mathbf{b} \pm \sqrt{\mathbf{b}^2 - 4\mathbf{ac}}}{2\mathbf{a}}$ when, a = 1, b = 2, and c = -3

$$x = \frac{-2 \pm \sqrt{4 + 12}}{2}$$

$$x = \frac{-2 \pm 4}{2}$$

$$x = \frac{-2 + 4}{2} \text{ or } \frac{-2 - 4}{2}$$

$$x = 1 \text{ or } -3$$

Solving with a calculator

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers.

Write the dividend and divisor in brackets:

$$x = \frac{(-b \pm \sqrt{b^2 - 4ac})}{(2a)}$$

Also, write $b^2 - 4ac$ in brackets so that the calculator can compute it as one:

$$x = \frac{(-b \pm \sqrt{\ }(b^2 - 4ac))}{(2a)}$$

This is equivalent to $(-b + \sqrt{\ }(b^2 - 4ac)) \div (2a) = x$ **or** $(-b - \sqrt{\ }(b^2 - 4ac)) \div (2a) = x$, with the calculator.

When the value of **b** is negative, write it in brackets too.

Then, you cannot easily go wrong.

If you are comfortable solving quadratic equations using the quadratic formula, solve the equations in activity 5 below. If not, review the examples again.

Solve the equations in *Activity 5* below:

Activity 5

(a) $\mathbf{y}^2 + 6\mathbf{y} = 3$

(b) $\mathbf{x}^2 - 3.5\mathbf{x} - 1 = 0$

(c) $\mathbf{m}^2 + 8\mathbf{m} + 1.7 = 0$

(d) $2\mathbf{n}^2 + 10\mathbf{n} - 5 = 0$

(e) $4\mathbf{x}^2 + 4\mathbf{x} = -8\mathbf{x} - 1$

Compare your answers with the given solutions at the end of this subunit. If you scored at least 80% continue. Otherwise review *solving quadratic equations using the quadratic formula* and try this activity again.

Key Points to Remember

The key points to remember in this subunit on solving quadratic equations by quadratic formula are:

1. Write the quadratic equation in standard form, $\mathbf{a\,x^2 + b\,x + c = 0}$.
2. Identify the values **a**, **b** and **c** in the eqaution $\mathbf{a\,x^2 + b\,x + c = 0}$.
3. Substitute the values **a**, **b** and **c** in the formula, $\mathbf{x} = \dfrac{-\mathbf{b} \pm \sqrt{\mathbf{b}^2 - 4\mathbf{ac}}}{2\mathbf{a}}$

   , and evaluate with a calculator.

Solutions to activity 5:

(a) $\mathbf{y}^2 + 6\mathbf{y} = 3$

Write it in standard form by subtracting 3 from each side:

$\mathbf{y}^2 + 6\mathbf{y} - 3 = 3 - 3$

$\mathbf{y}^2 + 6\mathbf{y} - 3 = 0$

Now, a = 1, b = 6, and c = -3

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \frac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = 6, and c = -3:

$$\mathbf{y} = \frac{(-6 \pm \sqrt{(6^2 - 4 \times 1 \times (-3))})}{(2 \times 1)}$$

Therefore, the solutions are $\frac{(-6 + \sqrt{(6^2 - 4 \times 1 \times (-3))})}{(2 \times 1)} \approx 0.464$ and

$\frac{(-6 - \sqrt{(6^2 - 4 \times 1 \times (-3))})}{(2 \times 1)} \approx$ -6.46.

(b) $\mathbf{x}^2 - 3.5\mathbf{x} - 1 = 0$

Now, a = 1, b = 3.5, and c = -1

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \frac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = 3.5, and c = -1:

$$\mathbf{x} = \frac{(-3.5 \pm \sqrt{(3.5^2 - 4 \times 1 \times (-1))})}{(2 \times 1)}$$

Therefore, the solutions are $\frac{(-3.5 + \sqrt{(3.5^2 - 4 \times 1 \times (-1))})}{(2 \times 1)} \approx 0.266$ and

$\frac{(-3.5 - \sqrt{(3.5^2 - 4 \times 1 \times (-1))})}{(2 \times 1)} \approx$ -3.77.

(c) $\mathbf{m}^2 + 8\mathbf{m} + 1.7 = 0$

Now, a = 1, b = 8, and c = 1.7

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \frac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = 8, and c = 1.7:

$$\mathbf{m} = \frac{(-8 \pm \sqrt{(8^2 - 4 \times 1 \times 1.7)})}{(2 \times 1)}$$

Therefore, the solutions are $\frac{(-8+\sqrt{(8^2-4\times1\times1.7)})}{(2\times1)}$ ≈ -0.218 and

$\frac{(-8-\sqrt{(8^2-4\times1\times1.7)})}{(2\times1)}$ ≈ -7.78.

(d) $2\mathbf{n}^2+10\mathbf{n}-5=0$

Now, a = 2, b = 10, and c = -5

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x}=\frac{(-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 2, b = 10, and c = -5:

$$\mathbf{n}=\frac{(-10\pm\sqrt{(10^2-4\times2\times(-5))})}{(2\times2)}$$

Therefore, the solutions are $\frac{(-10+\sqrt{(10^2-4\times2\times(-5))})}{(2\times2)}$ ≈ 0.458 and

$\frac{(-10-\sqrt{(10^2-4\times2\times(-5))})}{(2\times2)}$ ≈ -5.46.

(e) $4\mathbf{x}^2+4\mathbf{x}=-8\mathbf{x}-1$

Write it in standard form:

Add 8x to both sides and simplify:

$$4\mathbf{x}^2+4\mathbf{x}+8\mathbf{x}=-8\mathbf{x}-1+8\mathbf{x}$$

$$4\mathbf{x}^2+12\mathbf{x}=-1$$

Add 1 to both sides and simplify:

$$4\mathbf{x}^2+12\mathbf{x}+1=-1+1$$

$$4\mathbf{x}^2+12\mathbf{x}+1=0$$

Now, a = 4, b = 12, and c = 1

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x}=\frac{(-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 4, b = 12, and c = 1:

$$\mathbf{x}=\frac{(-12\pm\sqrt{(12^2-4\times4\times1)})}{(2\times4)}$$

Therefore, the solutions are $\frac{(-12+\sqrt{(12^2-4\times4\times1)})}{(2\times4)}$ ≈ -0.0858 and $\frac{(-12-\sqrt{(12^2-4\times4\times1)})}{(2\times4)}$ ≈ -2.91.

Lesson 4 Solving Equations that can be Reduced to Quadratic Form

Introduction

By the end of this subunit, you should be able to use your knowledge from the previous subunits to solve some equations which are not quadratic, but which can be reduced to quadratic form.

This subunit is about 10 pages in length.

Reducing to quadratic form

Some equations can also be solved using the methods of solving quadratic equations. In this topic you will learn how to solve non-quadratic equations by reducing them to quadratic form.

Reducing equations to the quadratic form example 1:

Take for example, $\mathbf{x^4 - 11x^2 + 30 = 0}$

How can you solve it, say, by using the formula?

$$x = \frac{(-b\pm\sqrt{(b^2-4ac)})}{(2a)} \quad ?$$

First, reduce this equation to quadratic form by letting a variable, such as "u", be equal to $\mathbf{x}^2$.

Substitute u for $\mathbf{x}^2$ in $x^4 - 11x^2 + 30 = 0$:

$\mathbf{x}^4 - 11\mathbf{x}^2 + 30 = 0$

$\mathbf{u}^2 - 11\mathbf{u} + 30 = 0$

Now, a = 1, b = -11, and c = 30

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \frac{(-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = -11, and c = 30:

$$\mathbf{u} = \frac{(-(-11)\pm\sqrt{((-11)^2-4\times1\times30)})}{(2\times1)}$$

$\mathbf{u} = 6 \ldots \mathbf{or} \ldots 5$

Since u is equal to $\mathbf{x}^2$, this means that:

$\mathbf{x}^2 = 6 \ldots \mathbf{or} \ldots 5$

Take square roots on both sides and remember there are two roots which are opposites:

$x = \pm\sqrt{6} \ldots or \ldots x = \pm\sqrt{5}$

Therefore, the solution set is $\{\sqrt{6}, -\sqrt{6}, \sqrt{5}, -\sqrt{5}\}$

Reducing equations to the quadratic form example 2:

Take for example, $\mathbf{y^4 - 64 = 0}$

First, reduce this equation to quadratic form by letting a variable, such as "u", be equal to $\mathbf{y}^2$.

Substitute u for $\mathbf{y}^2$ in $y^4 - 64 = 0$:

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$\mathbf{y}^4 - 64 = 0$

$\mathbf{u}^2 - 64 = 0$

What are the values for a, b and c in $\mathbf{u}^2 - 64 = 0$?

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Compare your answer with:

a = 1, b = 0, and c = -64

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ calculate the values of u when, a = 1, b = 0, and c = -64:

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Compare your answer with:

$$\mathbf{u} = \frac{(-0 \pm \sqrt{(0^2 - 4 \times 1 \times (-64))})}{(2 \times 1)}$$

$\mathbf{u} = 8 \ldots \mathbf{or} \ldots -8$

Since u is equal to $\mathbf{y}^2$, this means that:

$\mathbf{y}^2 = 8 \ldots \mathbf{or} \ldots -8$

Then take the square root of each side and solve for y:

Compare your answer with:

Now, $\mathbf{y}^2 = 8$ implies that $\mathbf{y} = \pm\sqrt{8}$; **but** $\mathbf{y}^2 = -8$ **is not soluble in real numbers because** $\mathbf{y} = -\sqrt{-8}$ and $\mathbf{y} = \sqrt{-8}$ are not real numbers.

Now by factorisation: $\mathbf{y^4 - 64 = 0}$

Write it as a difference of two squares:

Compare your answer with:

$((y^2)^2 - (8^2) = 0$

Use a difference of two squares to factorise:

Compare your answer with:

$((y^2) - 8)\ ((y^2) + 8) = 0$

Use the zero factor property and solve for y^2 :

Compare your answer with:

$(\mathbf{y}^2) - 8 = 0$ or $(\mathbf{y}^2) + 8 = 0$

$\mathbf{y}^2 = 8$ or $\mathbf{y}^2 = -8$

<u>Then take the square root of each side and solve for y:</u>

Compare your answer with:

Now, $\mathbf{y}^2 = 8$ implies that $\mathbf{y} = \pm\sqrt{8}$; so, y = root 8 or y = 2 root 2 and y = -root 8 or y = -2 root 2.

But $\mathbf{y}^2 = -8$ is not soluble in real numbers because $\mathbf{y} = -\sqrt{-8}$ and $\mathbf{y} = \sqrt{-8}$ are not real numbers.

If you are readily able to solve quadratic equations using the quadratic formula and the factorisation methods, solve the equations in activity 6 below. If not, review the examples again.

Solve the equations in *Activity 6* below:

Activity 6

(a) $\mathbf{y}^4 - 4\mathbf{y}^2 + 3 = 0$

(b) $4\mathbf{x}^4 - 9\mathbf{x}^2 + 2 = 0$

(c) $81\mathbf{m}^4 - 625 = 0$

(d) $\mathbf{n} - 10\sqrt{\mathbf{n}} + 21 = 0$

(e) $2\mathbf{x} + 5 = 7\mathbf{x}^{\frac{1}{2}}$

Check your performance against the given solutions at the end of this subunit. If you can comfortably answer each question continue. Otherwise review *solving equations that can be reduced to quadratic form.*

Key Points to Remember

The key points to remember in this subunit on *solving equations that can be reduced to quadratic form* are:

1. First, reduce the equation to quadratic form by letting a variable, such as "u", be equal to $\mathbf{x}^2$.
2. Substitute u for $\mathbf{x}^2$ in the given equation, to get a new equation which is quadratic in u.
3. Use the quadratic formula to get the values of u.
4. Finally, find the values of x, from the fact that, "u", is equal to $\mathbf{x}^2$.

Solutions to ACTIVITY 6:

(a) $\mathbf{y}^4 - 4\mathbf{y}^2 + 3 = 0$

First, reduce this equation to quadratic form by letting, say u, be equal to $\mathbf{y}^2$.

Substitute u for $\mathbf{y}^2$ in $y^4 - 4y^2 + 3 = 0$:

$\mathbf{y}^4 - 4\mathbf{y}^2 + 3 = 0$

$\mathbf{u}^2 - 4\mathbf{u} + 3 = 0$

Now, a = 1, b = -4, and c = 3

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = -4, and c = 3:

$$\mathbf{u} = \frac{(-(-4) \pm \sqrt{((-4)^2 - 4 \times 1 \times 3)})}{(2 \times 1)}$$

$\mathbf{u} = 3 \ldots \mathbf{or} \ldots 1$

Since u is equal to $\mathbf{y}^2$, this means that:

$\mathbf{y}^2 = 3 \ldots \mathbf{or} \ldots 1$

Take square roots on both sides and remember there are two roots which are opposites:

$\mathbf{y} = \pm\sqrt{3} \ldots \mathbf{or} \ldots \mathbf{x} = \pm\sqrt{1}$

Therefore, the solution set is $\{\sqrt{3}, -\sqrt{3}, 1, -1\}$

(b) $4\mathbf{x}^4 - 9\mathbf{x}^2 + 2 = 0$

First, reduce this equation to quadratic form by letting, say u, be equal to $\mathbf{x}^2$.

Substitute u for $\mathbf{x}^2$ in $4x^4 - 5x^2 + 2 = 0$:

$4\mathbf{x}^4 - 9\mathbf{x}^2 + 2 = 0$

$4\mathbf{u}^2 - 9\mathbf{u} + 2 = 0$

Now, a = 4, b = -9, and c = 2

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 4, b = -9, and c = 2:

$$\mathbf{u} = \frac{(-(-9) \pm \sqrt{((-9)^2 - 4 \times 4 \times 2)})}{(2 \times 4)}$$

$$\mathbf{u} = 2 \ldots \mathbf{or} \ldots \frac{1}{4}$$

Since u is equal to $\mathbf{x}^2$, this means that:

$$\mathbf{x}^2 = 2 \ldots \mathbf{or} \ldots \frac{1}{4}$$

Take square roots on both sides and remember there are two roots which are opposites:

$$\mathbf{x} = \pm\sqrt{2} \ldots \mathbf{or} \ldots \mathbf{x} = \pm\sqrt{\tfrac{1}{4}}$$

Therefore, the solution set is $\left\{\sqrt{2}, -\sqrt{2}, 0.5, -0.5\right\}$

(c) $81\mathbf{m}^4 - 625 = 0$

Write it as a difference of two squares:

$((9\mathbf{m}^2)^2 - (25^2) = 0$

Use a difference of two squares to factorise:

$((9\mathbf{m}^2) - 25)\ ((9\mathbf{m}^2) + 25) = 0$

Use the zero factor property and solve for y^2:

$(9\mathbf{m}^2) - 25 = 0$ or $(9\mathbf{m}^2) + 25 = 0$

$9\mathbf{m}^2 = 25$ or $9\mathbf{m}^2 = -25$

<u>Then take the square root of each side and solve for y:</u>

Now, $9m^2 = 25$ implies that $y = \pm\frac{5}{3}$; **but** $9m^2 = -25$ **is not soluble in real numbers because** $m = -\sqrt{-\frac{25}{9}}$ and $m = \sqrt{-\frac{25}{9}}$ are not real numbers.

(d) $n - 10\sqrt{n} + 21 = 0$

First, reduce this equation to quadratic form by letting, say u, be equal to $\sqrt{n}$.

Substitute u for $\sqrt{n}$ in $n - 10\sqrt{n} + 21 = 0$:

$n - 10\sqrt{n} + 21 = 0$

$u^2 - 10u + 21 = 0$ $\quad\quad$ ($n = \sqrt{n} \times \sqrt{n}$)

Now, a = 1, b = -10, and c = 21

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $x = \frac{(-b \pm \sqrt{(b^2 - 4ac)})}{(2a)}$ when, a = 1, b = -10, and c = 21:

$$u = \frac{(-(-10) \pm \sqrt{((-10)^2 - 4 \times 1 \times 21)})}{(2 \times 1)}$$

$u = 7 \ldots or \ldots 3$

Since u is equal to $\sqrt{n}$, this means that:

$\sqrt{n} = 7 \ldots or \ldots 3$

Square both sides:

$(\sqrt{n})^2 = 7^2 \ldots or \ldots 3^2$

$n = 49 \ldots or \ldots 9$

Therefore, the solutions are 49 and 9.

(e) $2x + 5 = 7x^{\frac{1}{2}}$

Subtract $7x^{\frac{1}{2}}$ from both sides:

$2x + 5 - 7x^{\frac{1}{2}} = 7x^{\frac{1}{2}} - 7x^{\frac{1}{2}}$

$2x + 5 - 7x^{\frac{1}{2}} = 0$

$2x - 7x^{\frac{1}{2}} + 5 = 0$

Reduce this equation to quadratic form by letting, say u, be equal to $x^{\frac{1}{2}}$.

Substitute u for $\mathbf{x}^{\frac{1}{2}}$ in $2\mathbf{x}-7\mathbf{x}^{\frac{1}{2}}+5=0$:

$$2\mathbf{x}-7\mathbf{x}^{\frac{1}{2}}+5=0$$

$$2\mathbf{u}^2-7\mathbf{u}+5=0 \qquad (\ \mathbf{x}=\mathbf{x}^{\frac{1}{2}}\times\mathbf{x}^{\frac{1}{2}}\ )$$

Now, a = 2, b = -7, and c = 5

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x}=\dfrac{(-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 2, b = -7, and c = 5:

$$\mathbf{u}=\frac{(-(-7)\pm\sqrt{((-7)^2-4\times2\times5)})}{(2\times2)}$$

$$\mathbf{u}=\frac{5}{2}\ldots\mathbf{or}\ldots1$$

Since u is equal to $x^{\frac{1}{2}}$, this means that:

$$\mathbf{x}^{\frac{1}{2}}=\frac{5}{2}\ldots\mathbf{or}\ldots1$$

Square both sides:

$$(\mathbf{x}^{\frac{1}{2}})^2=\left(\frac{5}{2}\right)^2\ldots\mathbf{or}\ldots1^2$$

$$\mathbf{x}=\frac{25}{4}\ldots\mathbf{or}\ldots1$$

Therefore, the solutions are $\dfrac{25}{4}$ and 1.

Lesson 5 Solving Simultaneous Equations Involving Quadratic Equations

Introduction

By the end of this subunit, you should be able to solve simultaneous equations which involve quadratic equations.

This subunit is about 20 pages in length.

Solving simultaneous equations

Quadratic equations, like linear equations, can also be solved simultaneously.

Solving simultaneous equations example 1:

Here are simultaneous equations with two variables:

$$y = (2x-1)^2 \ldots\ldots\ldots(i)$$
$$y - 28x - 6 = x^2 \ldots(ii)$$

You, generally, follow the same rules as applied to simultaneous linear equations:

Substitution and elimination methods which you used in ***simultaneous linear equations*** are applicable:

As y is the subject in equation (i), substitute $(2x-1)^2$ for y in equation (ii):

$$(2x-1)^2 - 28x - 6 = x^2 \ldots(ii)$$

Reduce this equation to the standard form $\mathbf{a\,x^2 + b\,x + c = 0}$.

$$(2x-1)^2 - 28x - 6 = x^2 \ldots(ii)$$

$$4x^2 - 4x + 1 - 28x - 6 = x^2 \ldots(ii)$$

$$3x^2 - 32x - 5 = 0 \ldots(ii)$$

Now, $a = 3, b = -32$ and $c = -5$

Therefore, substituting these values into

$$\mathbf{x} = \frac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}:$$

$$x = \frac{(-(-32) \pm \sqrt{((-32)^2 - 4 \times 3 \times -5))}}{(2 \times 3)}$$

$$x = \frac{1024 \pm \sqrt{1084}}{6}$$

$$x = \frac{1024 + \sqrt{1084}}{6} \cdots \text{or} \cdots \frac{1024 - \sqrt{1084}}{6}$$

$$x = \frac{512 + \sqrt{271}}{3} \cdots \text{or} \cdots \frac{512 - \sqrt{271}}{3}$$

Substitute $\frac{512 + \sqrt{271}}{3}$ for x in equation (i) to get corresponding y:

$$y = \left(2\left(\frac{512 + \sqrt{271}}{3}\right) - 1\right)^2 = \left(\frac{1024 + \sqrt{1084}}{3}\right)^2$$

One solution is $\left(\left(\frac{512+\sqrt{271}}{3}\right),\left(\frac{1024+\sqrt{1084}}{3}\right)^2\right) \approx (176{,}123000)\ldots\text{to}\ldots3$ significant figures.

The other solution is

$\left(\left(\frac{512-\sqrt{271}}{3}\right),\left(\frac{1024-\sqrt{1084}}{3}\right)^2\right) \approx (165{,}108000)\ldots\text{to}\ldots3$ significant figures.

Solving simultaneous equations example 2:

Let us work through the following together:

$\mathbf{y} = 2\mathbf{x}^2 - 4\mathbf{x} + 4\ldots\ldots\ldots(\mathbf{i})$

$\mathbf{y} - 2\mathbf{x} + 1 = \mathbf{x}^2\ldots(\mathbf{ii})$

As $\mathbf{y}$ is the subject in equation $(\mathbf{i})$, substitute $2\mathbf{x}^2 - 4\mathbf{x} + 4$ for $\mathbf{y}$ in equation $(\mathbf{ii})$ and simplify the left side:

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Compare your answer with:

$2\mathbf{x}^2 - 4\mathbf{x} + 4 - 2\mathbf{x} + 1 = \mathbf{x}^2\ldots(\mathbf{ii})$

$2\mathbf{x}^2 - 6\mathbf{x} + 5 = \mathbf{x}^2\ldots(\mathbf{ii})$

Reduce this equation to the standard form $\mathbf{a\,x}^2 + \mathbf{b\,x} + \mathbf{c} = \mathbf{0}$.

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Compare your answer with:

$2\mathbf{x}^2 - 6\mathbf{x} + 5 = \mathbf{x}^2\ldots(\mathbf{ii})$

$2\mathbf{x}^2 - 6\mathbf{x} + 5 - \mathbf{x}^2 = \mathbf{x}^2 - \mathbf{x}^2\ldots(\mathbf{ii})$

$\mathbf{x}^2 - 6\mathbf{x} + 5 = 0\ldots(\mathbf{ii})$

What are the values for a, b and c in $\mathbf{x}^2 - 6\mathbf{x} + 5 = 0$?

Compare your answer with:

a = 1, b = -6, and c = 5

From $\mathbf{x} = \frac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ alculate the values of x when, a = 1, b = -6, and c = 5:

Compare your answer with:

$$\mathbf{x} = \frac{(-(-6) \pm \sqrt{((-6)^2 - 4 \times 1 \times 5)})}{(2 \times 1)}$$

$$x = \frac{6 \pm \sqrt{16}}{2}$$

$$x = 5 \cdots \mathbf{or} \cdots 1$$

Substitute 5 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

Compare your answer with:

$y = 2 \times 5^2 - 4 \times 5 - 4$

$\mathbf{y} = 26$

One solution is $(5,26)$

Again, substitute 1 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

Compare your answer with:

$y = 2\times 1^2 - 4\times 1 - 4$

$\mathbf{y} = -6$

Another solution is $(1,-6)$.

If you are readily able to solve simultaneous quadratic equations by substitution or elimination, solve the equations in activity 7 below. If not, review the examples again.

Solve the equations in *Activity 7* below:

Activity 7

(a) $\mathbf{y}^2 = \mathbf{x}\ldots\ldots\ldots(\mathbf{i})$
$\mathbf{y} = \mathbf{x}^2\ldots(\mathbf{ii})$

(b) $\mathbf{y} = 2\mathbf{x} + 3\ldots\ldots\ldots(\mathbf{i})$
$\mathbf{y} = 2\mathbf{x}^2 + 3\ldots(\mathbf{ii})$

(c) $\mathbf{y} = \mathbf{x}^2 - 3 \ldots\ldots\ldots (\mathbf{i})$
$\mathbf{y} = 4\mathbf{x}^2 - 9 \ldots (\mathbf{ii})$

(d) $2\mathbf{y} = 3\mathbf{x}^2 - 4 \ldots\ldots\ldots (\mathbf{i})$
$\mathbf{y} = \mathbf{x}^2 - 2\mathbf{x} + 1 \ldots (\mathbf{ii})$

(e) $y^2 = x^2 ....(i)$
$y^2 - 4x + 3 = 0...(ii)$

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Compare your solutions with the answers at the end of this subunit. If you understand each solution, continue. If not, review *solving simultaneous equations involving quadratic equation* and try this activity again.

Key Points to Remember

The key points to remember in this subunit on *solving simultaneous equations involving quadratic equation* are:

1. Use substitution or elimination method which you used in ***simultaneous linear equations***, to get quadratic equation in x.
2. Evaluate the values of the x with the quadratic formula.
3. Substitute the values in one of the equations to get corresponding values of y.

Solutions to ACTIVITY 7:

(a) $y^2 = x ....(i)$
$y = x^2 ...(ii)$

As y is the subject in equation (ii), substitute x^2 for y in equation (i).

$(x^2)^2 = x \ldots (i)$

$x^4 = x \ldots (i)$

Reduce this equation to the standard form $ax^2 + bx + c = 0$.

$x^4 = x \ldots (i)$

$x^4 - x = x - x \ldots (i)$

$x^4 - x = 0 \ldots (i)$

Factorise the left side:

$x^4 - x = 0 \ldots (i)$

$x(x^3 - 1) = 0 \ldots (i)$

If $x(x^3 - 1) = 0$, then $x = 0$ or $(x^3 - 1) = 0$, using zero factor property.

If $x = 0$, then substitute 0 for x in equation (ii) to get corresponding y:

$y = 0^2$

$y = 0$

One solution is $(0,0)$

If $(x^3 - 1) = 0$, then

$x^3 - 1 + 1 = 0 + 1$

$x^3 = 1$

$x = 1$

Again, substitute 1 for x in equation (ii) to get corresponding y:

$y = 1^2$

$y = 1$

Another solution is $(1,1)$

(b) $y = 2x + 3 \ldots\ldots\ldots (i)$
$y = 2x^2 + 3 \ldots (ii)$

As y is the subject in equation (i), substitute $2x + 3$ for y in equation (ii).

$2x + 3 = 2x^2 + 3 \ldots (ii)$

Reduce this equation to the standard form $\mathbf{a}\,\mathbf{x}^2 + \mathbf{b}\,\mathbf{x} + \mathbf{c} = \mathbf{0}$.

$2\mathbf{x} + 3 = 2\mathbf{x}^2 + 3 \ldots (\mathbf{ii})$

$2\mathbf{x} + 3 - 3 = 2\mathbf{x}^2 + 3 - 3 \ldots (\mathbf{ii})$

$2\mathbf{x} = 2\mathbf{x}^2 \ldots (\mathbf{ii})$

$2\mathbf{x} - 2\mathbf{x} = 2\mathbf{x}^2 - 2\mathbf{x} \ldots (\mathbf{ii})$

$0 = 2\mathbf{x}^2 - 2\mathbf{x} \ldots (\mathbf{ii})$

$2\mathbf{x}^2 - 2\mathbf{x} = 0 \ldots (\mathbf{ii})$

Factorise the left side:

$2\mathbf{x}^2 - 2\mathbf{x} = 0 \ldots (\mathbf{ii})$

$2\mathbf{x}(\mathbf{x} - 1) = 0 \ldots (\mathbf{ii})$

If $2\mathbf{x}(\mathbf{x} - 1) = 0$, then $2\mathbf{x} = 0$ or $(\mathbf{x} - 1) = 0$, using zero factor property.

If $2\mathbf{x} = 0$, x=0; then substitute 0 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding y:

$\mathbf{y} = 2 \times 0 + 3$

$\mathbf{y} = 3$

One solution is $(0,3)$

If $(\mathbf{x} - 1) = 0$, then

$\mathbf{x} - 1 + 1 = 0 + 1$

$\mathbf{x} = 1$

Again, substitute 1 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

$\mathbf{y} = 2 \times 1 + 3$

$\mathbf{y} = 5$

Another solution is $(1,5)$

(c) $\mathbf{y} = \mathbf{x}^2 - 3 \ldots (\mathbf{i})$
$\mathbf{y} = 4\mathbf{x}^2 - 9 \ldots (\mathbf{ii})$

As $\mathbf{y}$ is the subject in equation $(\mathbf{i})$, substitute $\mathbf{x}^2 - 3$ for $\mathbf{y}$ in equation $(\mathbf{ii})$.

$\mathbf{x}^2 - 3 = 4\mathbf{x}^2 - 9 \ldots (\mathbf{ii})$

Reduce this equation to the standard form $\mathbf{a}\,\mathbf{x}^2 + \mathbf{b}\,\mathbf{x} + \mathbf{c} = \mathbf{0}$.

$x^2 - 3 = 4x^2 - 9 \ldots (ii)$

$x^2 - 3 + 3 = 4x^2 - 9 + 3 \ldots (ii)$

$x^2 = 4x^2 - 6 \ldots (ii)$

$x^2 - x^2 = 4x^2 - 6 - x^2 \ldots (ii)$

$0 = 3x^2 - 6 \ldots (ii)$

$3x^2 - 6 = 0 \ldots (ii)$

$3x^2 = 6 \ldots (ii)$

A *faster* approach to find x is to divide each side by 3 and then take the square root of each side:

$3x^2 = 6$

$\frac{3x^2}{3} = \frac{6}{3}$

$x^2 = 2$

$x = \pm\sqrt{2}$

$x = \sqrt{2} \cdots \textbf{or} \cdots -\sqrt{2}$

Substitute $\sqrt{2}$ for x in equation (i) to get corresponding y :

$y = \left(\sqrt{2}\right)^2 - 3$

$y = -1$

One solution is $(\sqrt{2}, -1)$

Again, substitute $-\sqrt{2}$ for x in equation (i) to get corresponding y :

$y = \left(-\sqrt{2}\right)^2 - 3$

$y = -1$

Another solution is $(-\sqrt{2}, -1)$

(d) $2y = 3x^2 - 3 \ldots\ldots\ldots (i)$
$y = x^2 - 2x + 1 \ldots (ii)$

As y is the subject in equation (ii), substitute $x^2 - 2x + 1$ for y in equation (i).

$2\left(\mathbf{x}^2 - 2\mathbf{x} + 1\right) = 3\mathbf{x}^2 - 3\ldots(\mathbf{i})$
$\left(2\mathbf{x}^2 - 4\mathbf{x} + 2\right) = 3\mathbf{x}^2 - 3\ldots(\mathbf{i})$

$2\mathbf{x}^2 - 4\mathbf{x} + 2 = 3\mathbf{x}^2 - 3\ldots(\mathbf{i})$

Reduce this equation to the standard form $\mathbf{a\,x^2 + b\,x + c = 0}$.

$2\mathbf{x}^2 - 4\mathbf{x} + 2 = 3\mathbf{x}^2 - 3\ldots(\mathbf{i})$

$2\mathbf{x}^2 - 4\mathbf{x} + 2 - 2 = 3\mathbf{x}^2 - 3 - 2\ldots(\mathbf{i})$

$2\mathbf{x}^2 - 4\mathbf{x} = 3\mathbf{x}^2 - 5\ldots(\mathbf{i})$

$2\mathbf{x}^2 - 4\mathbf{x} + 4\mathbf{x} = 3\mathbf{x}^2 - 5 + 4\mathbf{x}\ldots(\mathbf{i})$

$2\mathbf{x}^2 = 3\mathbf{x}^2 - 5 + 4\mathbf{x}\ldots(\mathbf{i})$

$2\mathbf{x}^2 - 2\mathbf{x}^2 = 3\mathbf{x}^2 - 5 + 4\mathbf{x} - 2\mathbf{x}^2\ldots(\mathbf{i})$

$0 = \mathbf{x}^2 + 4\mathbf{x} - 5\ldots(\mathbf{i})$

$\mathbf{x}^2 + 4\mathbf{x} - 5 = 0\ldots(\mathbf{i})$

Now, a = 1, b = 4, and c = -5

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = 4, and c = -5:

$$\mathbf{x} = \frac{(-4 \pm \sqrt{(4^2 - 4\times1\times(-5))})}{(2\times1)}$$

$\mathbf{x} = 1\ldots\mathbf{or}\ldots -5$

If $\mathbf{x} = 1$, then substitute 1 for $\mathbf{x}$ in equation $(\mathbf{ii})$ to get corresponding $\mathbf{y}$:

$\mathbf{y} = 1^2 - 2\times1 + 1$

$\mathbf{y} = 0$

One solution is $(1,0)$

Again, substitute -5 for $\mathbf{x}$ in equation $(\mathbf{ii})$ to get corresponding $\mathbf{y}$:

$\mathbf{y} = (-5)^2 - 2\times(-5) + 1$

$\mathbf{y} = 36$

Another solution is $(-5,36)$

(e) $\mathbf{y}^2 = \mathbf{x}^2\ldots\ldots\ldots(\mathbf{i})$
$\mathbf{y}^2 - 4\mathbf{x} + 3 = 0\ldots(\mathbf{ii})$

As $\mathbf{y}^2$ is the subject in equation $(\mathbf{i})$, substitute $\mathbf{x}^2$ for $\mathbf{y}$ in equation $(\mathbf{ii})$.

$\mathbf{x}^2 - 4\mathbf{x} + 3 = 0 \ldots (\mathbf{ii})$

Now, a = 1, b = -4, and c = 3

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = -4, and c = 3:

$$\mathbf{x} = \left(\frac{-(-4) \pm \sqrt{((-4)^2 - 4 \times 1 \times 3)}}{(2 \times 1)} \right)$$

$\mathbf{x} = 3 \ldots \mathbf{or} \ldots 1$

If $\mathbf{x} = 1$, then substitute 1 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

$\mathbf{y}^2 = 1^2$

$\mathbf{y} = 1$

One solution is $(1,1)$

Again, substitute 3 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

$\mathbf{y}^2 = 3^2$

$\mathbf{y} = 3$

Another solution is $(3,3)$

Lesson 6 Solving Problems which give Rise to Quadratic Equations

Introduction

By the end of this subunit, you should be able to apply your knowledge from the earlier subunits in problem solving.

This subunit is about 10 pages in length.

Solving problems

The methods of dealing with quadratic equations can be used in problem solving.

Here is one situation:

One day, Setŝabelo bought two apples from a market. When he arrived home, his sister, Tŝoarelo, asked him to give her one. Setŝabelo said, "You know I like working with numbers. Now, if you can give me the correct answer to my question, you get an apple."

Tŝoarelo asked what the question was. Setŝabelo said, "Four times a real number minus eight times its square root, is negative three. What is the number? "

So, this is how Tŝoarelo, or you can find the number:

Let the number be x.

Form an equation in x from the statement:

"Four times a real number minus eight times its square root, is negative three."

Equation: $4x - 8\sqrt{x} = -3$

Now, solve for the number:

Since, $x = \sqrt{x} \times \sqrt{x}$, write the equation as follows:

$$4\left(\sqrt{x}\right)^2 - 8\left(\sqrt{x}\right) = -3$$

Write it in standard form: $4\left(\sqrt{x}\right)^2 - 8\left(\sqrt{x}\right) + 3 = 0$

Now, $a = 4, b = -8$ and $c = 3$

Therefore,

$$\sqrt{x} = \frac{(-(-8) \pm \sqrt{((-8)^2 - 4 \times 4 \times 3))}}{(2 \times 4)}$$

$$\sqrt{x} = \frac{8 \pm \sqrt{16}}{8}$$

$$\sqrt{x} = \frac{8+4}{8} \cdots \text{or} \cdots \frac{8-4}{8}$$

$$\sqrt{x} = \frac{3}{2} \cdots \text{or} \cdots \frac{1}{2}$$

$$x = \left(\frac{3}{2}\right)^2 \cdots \text{or} \cdots \left(\frac{1}{2}\right)^2$$

$$x = \frac{9}{4} \cdots \text{or} \cdots \frac{1}{4}$$

The number is $\frac{9}{4}$ or $\frac{1}{4}$.

Let us work through the following together:

I think of a number and subtract five times this number from its square. If the result is two, which is this number?

Let the number be x.

Form an equation in x from, "subtract five times this number from its square" and "the result is two":

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Compare your answer with:

$\mathbf{x}^2 - 5\mathbf{x} = 2$

Write it in standard form by subtracting 2 from each side:

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____

Compare your answer with:

$\mathbf{x}^2 - 5\mathbf{x} - 2 = 2 - 2$

$\mathbf{x}^2 - 5\mathbf{x} - 2 = 0$

What are the values for a, b and c in $\mathbf{x}^2 - 5\mathbf{x} - 2 = 0$?

____

Compare your answer with:

a = 1, b = -5, and c = -2

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ calculate the values of x when, a = 1, b = -5, and c = -2:

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$$\mathbf{x} = \frac{(-(-5) \pm \sqrt{((-5)^2 - 4 \times 1 \times (-2))})}{(2 \times 1)}$$

Therefore, the solutions are $\frac{(-(-5) + \sqrt{((-5)^2 - 4 \times 1 \times (-2))})}{(2 \times 1)} \approx 5.37$ and

$$\frac{(-(-5) - \sqrt{((-5)^2 - 4 \times 1 \times (-2))})}{(2 \times 1)} \approx -0.372.$$

If you can comfortably solve problems that give rise to quadratic equations, solve the equations in activity 8 below. If not, review the examples again.

Solve the equations in *Activity 8* below:

Activity 8

(a) The sum of twice the square of a number and three, is seven times the number. Find the number.

(b) Three times the square of a number, plus the number, equals the sum of this number and one. Which number is this?

(c) The length, in metres, of a rectangular garden is equal to the width plus four metres; and the area of the garden is thirty-two square metres.

Find the length of the garden.

(d) A square lawn has the same area as a rectangular lawn, of which, the length is three times the width. The perimeter of the rectangular lawn is forty-eight metres.

Find the side of the square lawn.

(e) The determinant of the matrix $\begin{pmatrix} 2\mathbf{x} & 5 \\ \mathbf{x} & \mathbf{x} \end{pmatrix}$ is four.

Find x.

(Hint: Form an equation using the determinant of the matrix.)

Check your answers with the given solutions at the end of this subunit. If you thoroughly understand each solution, continue. If not, review *solving problems which give rise to quadratic equations.*

Key Points to Remember

The key points to remember in this subunit on *solving problems which give rise to quadratic equations* are:

1. Form mathematical equation(s) from the given information.
2. Solve the equation(s) which have been obtained.
3. Interpret the solution(s) according to the problem.

You have now completed the last subunit of this unit. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Solutions to ACTIVITY 8:

(a) The sum of twice the square of a number and three, is seven times the number. Find the number.

Let the number be $\mathbf{x}$.

Form an equation in x from, "sum of twice the square of a number and three, is seven times the number ":

$$2\mathbf{x}^2 + 3 = 7\mathbf{x}$$

Write it in standard form by subtracting 7x from each side:

$$2\mathbf{x}^2 + 3 - 7\mathbf{x} = 7\mathbf{x} - 7\mathbf{x}$$

$$2\mathbf{x}^2 - 7\mathbf{x} + 3 = 0$$

Now, a = 2, b = -7, and c = 3

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 2, b = -7, and c = 3:

$$\mathbf{x} = \frac{(-(-7) \pm \sqrt{((-7)^2 - 4 \times 2 \times 3)})}{(2 \times 2)}$$

$$\mathbf{x} = 3 \ldots \mathbf{or} \ldots 0.5$$

Therefore, the solutions are 3 and 0.5.

(b) Three times the square of a number, plus the number, equals the sum of this number and one. Which number is this?

Let the number be $\mathbf{x}$.

Form an equation in x from, "three times the square of a number, plus the number, equals the sum of this number and one ":

$$3\mathbf{x}^2 + \mathbf{x} = \mathbf{x} + 1$$

Write it in standard form by subtracting x from each side:

$$3\mathbf{x}^2 + \mathbf{x} - \mathbf{x} = \mathbf{x} - \mathbf{x} + 1$$

$$3\mathbf{x}^2 = 1$$

A *faster* approach is to divide each side by 3 and then take the square root of each side:

$$3\mathbf{x}^2 = 1$$

$$\frac{3\mathbf{x}^2}{3} = \frac{1}{3}$$

$$\mathbf{x}^2 = \frac{1}{3}$$

$$\mathbf{x} = \pm\sqrt{\frac{1}{3}}$$

Therefore, the solutions are $\sqrt{\frac{1}{3}}$ and $-\sqrt{\frac{1}{3}}$

(c) The length, in metres, of a rectangular garden is equal to the width plus four metres; and the area of the garden is thirty-two square metres.

Find the length of the garden.

Let the length of the garden be $\mathbf{x}$.

The width, in terms of x is $\mathbf{x}-4$ from "length, in metres, of a rectangular garden is equal to the width plus four metres".

Since the *area of a rectangle equals length times width*:

Form an equation in $\mathbf{x}$ from, "the area of the garden is thirty-two square metres":

$$\mathbf{x}(\mathbf{x}-4) = 32$$

$$\mathbf{x}^2 - 4\mathbf{x} = 32$$

Write it in standard form by subtracting 32 from each side:

$$\mathbf{x}^2 - 4\mathbf{x} - 32 = 32 - 32$$

$$\mathbf{x}^2 - 4\mathbf{x} - 32 = 0$$

Now, a = 1, b = -4, and c = -32

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = -4, and c = -32:

$$\mathbf{x} = \frac{(-(-4) \pm \sqrt{((-4)^2 - 4\times1\times(-32)))}}{(2\times1)}$$

$$\mathbf{x} = 8\ldots\mathbf{or}\ldots-4$$

Therefore, the solutions are 8 and -4.

Since the length is never negative, 8 metres is the length of the garden.

(d) A square lawn has the same area as a rectangular lawn, of which, the length is three times the width; and the side of square lawn is equal to the length of the rectangular lawn. The perimeter of the rectangular lawn is forty-eight metres.

Find the side of the square lawn.

Since, "square lawn has the same area as a rectangular lawn”:

Area of square = length of rectangle × width of rectangle.

Now, “The perimeter of the rectangular lawn is forty-eight metres” means:

2(length of rectangle + width of rectangle) = 48

(length of rectangle + width of rectangle) = 24

width of rectangle = 24 - length of rectangle

Let the side of the square lawn be $\mathbf{x}$.

So, the length of the rectangular lawn is also $\mathbf{x}$.

Form an equation in $\mathbf{x}$ using *area of square = length of rectangle × width of rectangle*:

Area of square = length of rectangle × width of rectangle.

Side squared = length of rectangle × width of rectangle

$\mathbf{x}^2 = \mathbf{x} \times$ width of rectangle

$\mathbf{x}^2 = \mathbf{x} \times (24 - \mathbf{x})$ (since width of rectangle = 24 - length of rectangle)

$\mathbf{x}^2 = 24\mathbf{x} - \mathbf{x}^2$

Write it in standard form by subtracting 24x from each side and then add $\mathbf{x}^2$ to each side:

$\mathbf{x}^2 - 24\mathbf{x} = 24\mathbf{x} - 24\mathbf{x} - \mathbf{x}^2$

$\mathbf{x}^2 - 24\mathbf{x} = -\mathbf{x}^2$

$\mathbf{x}^2 - 24\mathbf{x} + \mathbf{x}^2 = -\mathbf{x}^2 + \mathbf{x}^2$

$2\mathbf{x}^2 - 24\mathbf{x} = 0$

$2\mathbf{x}(\mathbf{x} - 24) = 0$

If $2\mathbf{x}(\mathbf{x} - 24) = 0$, then $2\mathbf{x} = 0$ or $(\mathbf{x} - 24) = 0$, using zero factor property.

If $2\mathbf{x} = 0$, then

$\mathbf{x} = 0$

If $(\mathbf{x} - 24) = 0$, then

$\mathbf{x} - 24 + 24 = 0 + 24$

$\mathbf{x} = 24$

Therefore, the solutions are 0 and 24.

Since the side of a lawn cannot be zero 24 metres is the side of the square lawn.

(e) The determinant of the matrix $\begin{pmatrix} 2\mathbf{x} & 5 \\ \mathbf{x} & \mathbf{x} \end{pmatrix}$ is four.

Find x.

Determinant = 4 means:

$2\mathbf{x} \times \mathbf{x} - 5 \times \mathbf{x} = 4$

$2\mathbf{x}^2 - 5\mathbf{x} = 4$

Write it in standard form by subtracting 4 from each side:

$2\mathbf{x}^2 - 5\mathbf{x} - 4 = 4 - 4$

$2\mathbf{x}^2 - 5\mathbf{x} - 4 = 0$

Now, a = 2, b = -5, and c = -4

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 2, b = -5, and c = -4:

$$\mathbf{x} = \frac{(-(-5) \pm \sqrt{((-5)^2 - 4 \times 2 \times (-4))})}{(2 \times 2)}$$

Therefore, the solutions are $\dfrac{(-(-5) + \sqrt{((-5)^2 - 4 \times 2 \times (-4))})}{(2 \times 2)} \approx 3.14$ and

$\dfrac{(-(-5) - \sqrt{((-5)^2 - 4 \times 2 \times (-4))})}{(2 \times 2)} \approx -0.637$.

Unit Summary

Summary

In this unit you learned that an equation which results from the quadratic function $f(x) = a x^2 + b x + c$, when $f(x) = 0$ is called quadratic equation.

Quadratic equations can be written in different forms, but all of which can be reduced to standard form **$a x^2 + b x + c = 0$, where a, b, and c are constants.**

Zero factor property is used to solve quadratic equations by factorisation.

Another method of solving quadratic equations is by completing the square.

The quadratic equations can generally be solved by using quadratic formula, which is derived by completing the square to the standard form of quadratic equation, **$a x^2 + b x + c = 0$**.

Some equations can also be solved using the methods of solving quadratic equation when reduced to quadratic form.

Quadratic equations, like linear equations, can also be solved simultaneously.

Quadratic equations can be used in problem solving.

You have completed the material for this unit on quadratic equations. You should now spend some time reviewing the content. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

When you work on this assignment, please observe the time allocated and show your work for each answer.

TOTAL MARKS: 35

TIME: 50 minutes

Work out the following to find how well you have understood this topic:

1. Solve the equations by factorisation.

(a) $2x(x+5)=0$

(4marks)

(b) $(3y-8)(2y-1)=0$

(4marks)

(**c**) $2\mathbf{k}^2 - 72 = 0$

(4marks)

(d) $z^2 = 3z$

(4marks)

(e) $4m^2 - 8m = 8m + 84$

(4marks)

2. Find the solutions of the equations below, by completing the square.

(a) $x^2 - 5x = 6$

(4marks)

(b) $3y^2 - 3y = y^2 + 1$

(4marks)

3. Solve the equations by the quadratic formula, using a calculator:

$$x = \frac{(-b \pm \sqrt{b^2 - 4ac})}{(2a)}$$

(a) $x^2 - 5x = 6$

(4marks)

(b) $3y^2 - 3y = y^2 + 1$

(4marks)

4. Solve these non-quadratic equations by reducing them to quadratic form, and then, solving the quadratic equations which are formed.

(a) $y^3 = 4y$

(5marks)

(b) $y^4 = 6 - 5y^2$

(5marks)

(c) $x + 5\sqrt{x} - 6 = 0$

(6marks)

5. Find the solutions to these simultaneous equations:

$y + 2x - 1 = 0 \ldots (i)$

$y^2 = 3x - 2 \ldots\ldots (ii)$

(6marks)

6. Solve the following problems:

(a)When subtracting four times a number from one, the difference is equal to negative three times the square of the number. Find the number.

(6marks)

(b) A certain number is equal to one minus twice another number; and the square of this number plus two, is equal to three times the other number. Find these two numbers.

(6marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

SOLUTIONS to the ASSIGNMENT QUESTIONS:

1. Solve the equations by factorisation.

   (a) $2\mathbf{x}(\mathbf{x}+5)=0$

If $2\mathbf{x}(\mathbf{x}+5)=0$, then $2\mathbf{x}=0$ or $(\mathbf{x}+5)=0$, using zero factor property.

If $2\mathbf{x}=0$, then

$$\frac{2\mathbf{x}}{2}=\frac{0}{2}$$

$$\mathbf{x}=0$$

If $\mathbf{x}+5=0$, then

$$\mathbf{x}+5-5=0-5$$

$$\mathbf{x}=-5$$

Therefore, the solutions are 0 and -5 .

   (b) $(3\mathbf{y}-8)(2\mathbf{y}-1)=0$

If $(3\mathbf{y}-8)(2\mathbf{y}-1)=0$, then $(3\mathbf{y}-8)=0$ or $(2\mathbf{y}-1)=0$, using zero factor property.

If $3\mathbf{y}-8=0$, then

$$3\mathbf{y}-8+8=0+8$$

$$3\mathbf{y}=8$$

$$\mathbf{y}=\frac{8}{3}$$

$$\frac{3\mathbf{y}}{3}=\frac{8}{3}$$

If $2\mathbf{y}-1=0$, then

$$2\mathbf{y}-1+1=0+1$$

$$2\mathbf{y}=1$$

$$\frac{2\mathbf{y}}{2}=\frac{1}{2}$$

$$\mathbf{y}=\frac{1}{2}$$

Therefore, the solutions are $\frac{8}{3}$ and $\frac{1}{2}$.

(c) $2\mathbf{k}^2 - 72 = 0$

$2\mathbf{k}^2 - 72 = 0$

$2(\mathbf{k}^2 - 36) = 0$

Factorising a difference of two squares in the brackets:

$2(\mathbf{k}^2 - 36) = 0$

$2(\mathbf{k} - 6)(\mathbf{k} + 6) = 0$

If $2(\mathbf{k} - 6)(\mathbf{k} + 6) = 0$, then $(\mathbf{k} - 6) = 0$ or $(\mathbf{k} + 6) = 0$, using zero factor property.

If $\mathbf{k} - 6 = 0$, then

$\mathbf{k} - 6 + 6 = 0 + 6$

$\mathbf{x} = 6$

If $\mathbf{k} + 6 = 0$, then

$\mathbf{k} + 6 - 6 = 0 - 6$

$\mathbf{x} = -6$

Therefore, the solutions are 6 and -6.

(d) $\mathbf{z}^2 = 3\mathbf{z}$

Write it in standard form:

$\mathbf{z}^2 = 3\mathbf{z}$

$\mathbf{z}^2 - 3\mathbf{z} = 3\mathbf{z} - 3\mathbf{z}$

$\mathbf{z}^2 - 3\mathbf{z} = 0$

Factorise **z** from the two terms on the left side:

$\mathbf{z}^2 - 3\mathbf{z} = 0$

$\mathbf{z}(\mathbf{z} - 3) = 0$

Use zero factor property to solve:

If $\mathbf{z}(\mathbf{z} - 3) = 0$, then $\mathbf{z} = 0$ or $(\mathbf{z} - 3) = 0$, using zero factor property.

If $\mathbf{z} - 3 = 0$, then

$\mathbf{z} - 3 + 3 = 0 + 3$

$\mathbf{z} = 3$

Therefore, the solutions are 0 and 3.

(e) $4\mathbf{m}^2 - 8\mathbf{m} = 8\mathbf{m} + 84$

Write it in standard form:

Subtract 84 from each side:

$4\mathbf{m}^2 - 8\mathbf{m} - 84 = 8\mathbf{m} + 84 - 84$

$4\mathbf{m}^2 - 8\mathbf{m} - 84 = 8\mathbf{m}$

Subtract 8 m from each side:

$4\mathbf{m}^2 - 8\mathbf{m} - 8\mathbf{m} - 84 = 8\mathbf{m} - 8\mathbf{m}$

$4\mathbf{m}^2 - 16\mathbf{m} - 84 = 0$

$\mathbf{m}^2 - 4\mathbf{m} - 21 = 0$ (dividing by 4 on both sides)

Work out the product **ac:**

ac = 1 × **-21 = -21**

List numbers which divide exactly into the product -10**:**

$\pm 1, \pm 3, \pm 7, \pm 21.$

Look for a pair(s) of numbers with the sum equal to coefficient of **m** in

$\mathbf{m}^2 - 4\mathbf{m} - 21 = 0$:

The pair+ ***3*** *and* - ***7*** has the sum **-4.**

Rewrite the equation by replacing the middle term by two terms of m , with **+3** and **-7** as their coefficients (*in this case, it does not matter if the order is +3m and then -7m or -7m and then +3m*):

$\mathbf{m}^2 - 4\mathbf{m} - 21 = 0$

$\mathbf{m}^2 - 7\mathbf{m} + 3\mathbf{m} - 21 = 0$

Factorise the first two terms and the other two separately:

$\mathbf{m}^2 - 7\mathbf{m} + 3\mathbf{m} - 21 = 0$

$(\mathbf{m}^2 - 7\mathbf{m}) + (3\mathbf{m} - 21) = 0$

$\mathbf{m}(\mathbf{m} - 7) + 3(\mathbf{m} - 7) = 0$

Complete the factorisation:

$\mathbf{m}(\mathbf{m} - 7) + 3(\mathbf{m} - 7) = 0$

$(\mathbf{m} - 7)(\mathbf{m} + 3) = 0$

If $(\mathbf{m} - 7)(\mathbf{m} + 3) = 0$, then $(\mathbf{m} - 7) = 0$ or $(\mathbf{m} + 3) = 0$, using zero factor property.

If $(\mathbf{m}-7)=0$, then

$\mathbf{m}-7+7=0+7$

$\mathbf{m}=7$

If $(\mathbf{m}+3)=0$, then

$\mathbf{m}+3-3=0-3$

$\mathbf{m}=-3$

Therefore, the solutions are a repeated solution 7 and -3

2. Find the solutions of the equations below, by completing the square.

   (a) $\mathbf{x}^2-5\mathbf{x}=6$

The first term on the left side of the equation is already a square of x.

Since -5x equals $2\left(\frac{-5\mathbf{x}}{2}\right)$, the square of half the coefficient of x added to both sides makes the left side a complete square:

$$\mathbf{x}^2-5\mathbf{x}=6$$

$$\mathbf{x}^2-5\mathbf{x}+\left(\frac{-5}{2}\right)^2=6+\left(\frac{-5}{2}\right)^2$$

$$\left(\mathbf{x}-\frac{5}{2}\right)^2=6+\frac{25}{4}$$

$$\left(\mathbf{x}-\frac{5}{2}\right)^2=\frac{49}{4}$$

<u>Then take the square root of each side:</u>

$$\left(\mathbf{x}-\frac{5}{2}\right)^2=\frac{49}{4}$$

$$\left(\mathbf{x}-\frac{5}{2}\right)=\pm\sqrt{\frac{49}{4}}$$

$$\mathbf{x}-\frac{5}{2}+\frac{5}{2}=\frac{5}{2}\pm\sqrt{\frac{49}{4}}$$

$$\mathbf{x}=\frac{5\pm 7}{2}$$

$$\mathbf{x}=\frac{5+7}{2} \text{ or } \frac{5-7}{2}$$

$\mathbf{x} = 6$ or -1

Therefore, the solutions are 6 and -1.

(b) $3\mathbf{y}^2 - 3\mathbf{y} = \mathbf{y}^2 + 1$

Subtract $\mathbf{y}^2$ from both sides and simplify:

$$3\mathbf{y}^2 - \mathbf{y}^2 - 3\mathbf{y} = \mathbf{y}^2 - \mathbf{y}^2 + 1$$

$$2\mathbf{y}^2 - 3\mathbf{y} = 1$$

Divide by 2 on both sides to make first term on the left side of the equation a perfect square and simplify:

$$2\mathbf{y}^2 - 3\mathbf{y} = 1$$

$$\frac{2\mathbf{y}^2 - 3\mathbf{y}}{2} = \frac{1}{2}$$

$$\mathbf{y}^2 - \frac{3\mathbf{y}}{2} = \frac{1}{2}$$

Since $\frac{-3\mathbf{y}}{2}$ equals $2\left(\frac{-3\mathbf{y}}{4}\right)$, the square of half the coefficient of y added to both sides makes the left side a complete square:

$$\mathbf{y}^2 - \frac{3\mathbf{y}}{2} = \frac{1}{2}$$

$$\mathbf{y}^2 - \frac{3\mathbf{y}}{2} + \left(\frac{-3}{4}\right)^2 = \frac{1}{2} + \left(\frac{-3}{4}\right)^2$$

$$\left(\mathbf{y} - \frac{3}{4}\right)^2 = \frac{1}{2} + \frac{9}{16}$$

$$\left(\mathbf{y} - \frac{3}{4}\right)^2 = \frac{17}{16}$$

Then take the square root of each side:

$$\left(\mathbf{y} - \frac{3}{4}\right)^2 = \frac{17}{16}$$

$$\left(\mathbf{y} - \frac{3}{4}\right) = \pm\sqrt{\frac{17}{16}}$$

$$\mathbf{y} - \frac{3}{4} + \frac{3}{4} = \frac{3}{4} \pm \sqrt{\frac{17}{16}}$$

$$\mathbf{y} = \frac{3 \pm \sqrt{17}}{4}$$

Therefore, the solutions are $\frac{3+\sqrt{17}}{4}$ and $\frac{3-\sqrt{17}}{4}$.

3. Solve the equations by the quadratic formula, using a calculator:

$$x = \frac{(-b \pm \sqrt{b^2 - 4ac})}{(2a)}$$

(a) $\mathbf{x}^2 - 5\mathbf{x} = 6$

Write it in standard form by subtracting 6 from each side:

$\mathbf{x}^2 - 5\mathbf{x} - 6 = 6 - 6$

$\mathbf{x}^2 - 5\mathbf{x} - 6 = 0$

Now, a = 1, b = -5, and c = -6

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \frac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = -5, and c = -6:

$$\mathbf{x} = \frac{(-(-5) \pm \sqrt{((-5)^2 - 4 \times 1 \times (-6))})}{(2 \times 1)}$$

$$\mathbf{x} = \frac{(-(-5) + \sqrt{((-5)^2 - 4 \times 1 \times (-6))})}{(2 \times 1)} \text{ or } \frac{(-(-5) - \sqrt{((-5)^2 - 4 \times 1 \times (-6))})}{(2 \times 1)}$$

$\mathbf{x} = 6$ or -1

Therefore, the solutions are 6 and -1.

(b) $3\mathbf{y}^2 - 3\mathbf{y} = \mathbf{y}^2 + 1$

Write it in standard form:

Subtract $\mathbf{y}^2$ from each side and simplify:

$3\mathbf{y}^2 - \mathbf{y}^2 - 3\mathbf{y} = \mathbf{y}^2 - \mathbf{y}^2 + 1$

$2\mathbf{y}^2 - 3\mathbf{y} = 1$

Subtract 1 from each side and simplify:

$2\mathbf{y}^2 - 3\mathbf{y} - 1 = 1 - 1$

$$2\mathbf{y}^2 - 3\mathbf{y} - 1 = 0$$

Now, a = 2, b = -3, and c = -1

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 2, b = -3, and c = -1:

$$\mathbf{x} = \frac{(-(-3) \pm \sqrt{((-3)^2 - 4 \times 2 \times (-1))})}{(2 \times 2)}$$

$$\mathbf{x} = \frac{(-(-3) + \sqrt{((-3)^2 - 4 \times 2 \times (-1))})}{(2 \times 2)} \text{ or } \frac{(-(-3) - \sqrt{((-3)^2 - 4 \times 2 \times (-1))})}{(2 \times 2)}$$

Therefore, the solutions are $\dfrac{(-(-3) + \sqrt{((-3)^2 - 4 \times 2 \times (-1))})}{(2 \times 2)} \approx 1.78$ and $\dfrac{(-(-3) - \sqrt{((-3)^2 - 4 \times 2 \times (-1))})}{(2 \times 2)} \approx -0.281$.

4. Solve these non-quadratic equations by reducing them to quadratic form, and then, solving the quadratic equations which are formed.

   (a) $\mathbf{y}^3 = 4\mathbf{y}$

Subtract $4\mathbf{y}$ from each side and simplify:

$$\mathbf{y}^3 - 4\mathbf{y} = 4\mathbf{y} - 4\mathbf{y}$$

$$\mathbf{y}^3 - 4\mathbf{y} = 0$$

Factorise **y** from the two terms on the left side:

$$\mathbf{y}^3 - 4\mathbf{y} = 0$$

$$\mathbf{y}(\mathbf{y}^2 - 4) = 0$$

Use zero factor property:

If $\mathbf{y}(\mathbf{y}^2 - 4) = 0$, then $\mathbf{y} = 0$ or $(\mathbf{y}^2 - 4) = 0$, using zero factor property.

$$(\mathbf{y}^2 - 4) = 0$$

$$\mathbf{y}^2 - 2^2 = 0$$

Using the difference of two squares:

$$\mathbf{y}^2 - 2^2 = 0$$

$$(\mathbf{y} - 2)(\mathbf{y} + 2) = 0$$

If $(\mathbf{y}-2)(\mathbf{y}+2)=0$, then $(\mathbf{y}-2)=0$ or $(\mathbf{y}+2)=0$, using zero factor property.

If $(\mathbf{y}-2)=0$, then

$\mathbf{y}-2+2=0+2$

$\mathbf{y}=2$

If $(\mathbf{y}+2)=0$, then

$\mathbf{y}+2-2=0-2$

$\mathbf{y}=-2$

Therefore, the solutions are 0, 2 and -2.

(b) $\mathbf{y}^4=6-5\mathbf{y}^2$

Add $5\mathbf{y}^2$ to each side and simplify:

$\mathbf{y}^4+5\mathbf{y}^2=6-5\mathbf{y}^2+5\mathbf{y}^2$

$\mathbf{y}^4+5\mathbf{y}^2=6$

Subtract 6 from each side and simplify:

$\mathbf{y}^4+5\mathbf{y}^2-6=6-6$

$\mathbf{y}^4+5\mathbf{y}^2-6=0$

Now, reduce this equation to quadratic form by letting a variable, such as “u”, be equal to $\mathbf{y}^2$.

Substitute u for $\mathbf{y}^2$ in $\mathbf{y}^4+5\mathbf{y}^2-6=0$:

$\mathbf{y}^4+5\mathbf{y}^2-6=0$

$\mathbf{u}^2+5\mathbf{u}-6=0$

Now, a = 1, b = 5, and c = -6

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x}=\dfrac{(-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = 5, and c = -6:

$$\mathbf{u}=\frac{(-5\pm\sqrt{(5^2-4\times1\times(-6))})}{(2\times1)}$$

$\mathbf{u}=1\ldots\mathbf{or}\ldots-6$

Since u is equal to $\mathbf{y}^2$, this means that:

$\mathbf{y}^2 = 1 \ldots \mathbf{or} \ldots -6$

Now, $\mathbf{y}^2 = 1$ implies that $\mathbf{y} = \pm 1$; **but** $\mathbf{y}^2 = -6$ **is not solved in real numbers because** $\mathbf{y} = \sqrt{-6}$ and $\mathbf{y} = -\sqrt{-6}$ are not real numbers.

(c) $\mathbf{x} + 5\sqrt{\mathbf{x}} - 6 = 0$

First, reduce this equation to quadratic form by letting a variable, such as "u", be equal to $\sqrt{\mathbf{x}}$.

Substitute u for $\sqrt{\mathbf{x}}$ in $\mathbf{x} + 5\sqrt{\mathbf{x}} - 6 = 0$:

$\mathbf{x} + 5\sqrt{\mathbf{x}} - 6 = 0$

$\mathbf{u}^2 + 5\mathbf{u} - 6 = 0 \ldots\ldots\ldots (\mathbf{x} = \sqrt{\mathbf{x}} \times \sqrt{\mathbf{x}})$

Now, a = 1, b = 5, and c = -6

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = 5, and c = -6:

$$\mathbf{u} = \frac{(-5 \pm \sqrt{(5^2 - 4 \times 1 \times (-6))})}{(2 \times 1)}$$

$\mathbf{u} = 1 \ldots \mathbf{or} \ldots -6$

Since u is equal to $\sqrt{\mathbf{x}}$, this means that:

$\sqrt{\mathbf{x}} = 1 \ldots \mathbf{or} \ldots -6$

Now, $\sqrt{\mathbf{x}} = 1$ implies that $\mathbf{x} = 1$; **but** $\sqrt{\mathbf{x}} = -6$ **is not solved in real numbers because** $\sqrt{\mathbf{x}}$ **is never negative for real numbers**.

5. Find the solutions to these simultaneous equations:

   $\mathbf{y} + 2\mathbf{x} - 1 = 0 \ldots (\mathbf{i})$

   $\mathbf{y}^2 = 3\mathbf{x} - 2 \ldots\ldots (\mathbf{ii})$

Make y the subject in equation $(\mathbf{i})$:

Subtract 2x from each side and simplify:

$\mathbf{y} + 2\mathbf{x} - 2\mathbf{x} - 1 = 0 - 2\mathbf{x} \ldots (\mathbf{i})$

$\mathbf{y} - 1 = -2\mathbf{x} \ldots (\mathbf{i})$

Add 1 to each side and simplify:

$\mathbf{y} - 1 + 1 = -2\mathbf{x} + 1 \ldots (\mathbf{i})$

$\mathbf{y} = -2\mathbf{x} + 1 \ldots (\mathbf{i})$

Now that $\mathbf{y}$ is the subject in equation $(\mathbf{i})$, substitute $-2\mathbf{x}+1$ for $\mathbf{y}$ in equation $(\mathbf{ii})$.

$\mathbf{y}^2 = 3\mathbf{x} - 2 \ldots\ldots (\mathbf{ii})$

$(-2\mathbf{x} + 1)^2 = 3\mathbf{x} - 2 \ldots\ldots (\mathbf{ii})$

Reduce this equation to the standard form $\mathbf{a\,x^2 + b\,x + c = 0}$.

Expand the perfect square on the left:

$(-2\mathbf{x} + 1)^2 = 3\mathbf{x} - 2 \ldots\ldots (\mathbf{ii})$

$(-2\mathbf{x})^2 + 2 \times (-2\mathbf{x}) \times 1 + (1)^2 = 3\mathbf{x} - 2 \ldots\ldots (\mathbf{ii})$

$4\mathbf{x}^2 - 4\mathbf{x} + 1 = 3\mathbf{x} - 2 \ldots\ldots (\mathbf{ii})$

Subtract 3x from each side and simplify:

$4\mathbf{x}^2 - 4\mathbf{x} - 3\mathbf{x} + 1 = 3\mathbf{x} - 3\mathbf{x} - 2 \ldots\ldots (\mathbf{ii})$

$4\mathbf{x}^2 - 7\mathbf{x} + 1 = -2 \ldots\ldots (\mathbf{ii})$

Add 2 to each side and simplify:

$4\mathbf{x}^2 - 7\mathbf{x} + 1 + 2 = -2 + 2 \ldots\ldots (\mathbf{ii})$

$4\mathbf{x}^2 - 7\mathbf{x} + 3 = 0 \ldots\ldots (\mathbf{ii})$

Now, a = 4, b = -7, and c = 3

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 4, b = -7, and c = 3:

$$\mathbf{x} = \frac{(-(-7) \pm \sqrt{((-7)^2 - 4 \times 4 \times 3)})}{(2 \times 4)}$$

$$\mathbf{x} = \frac{(-(-7) + \sqrt{((-7)^2 - 4 \times 4 \times 3)})}{(2 \times 4)} \text{ or } \frac{(-(-7) - \sqrt{((-7)^2 - 4 \times 4 \times 3)})}{(2 \times 4)}$$

$\mathbf{x} = 1 \text{ or } 0.75$

If $\mathbf{x} = 1$, then substitute 1 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

$\mathbf{y} + 2(1) - 1 = 0$

$\mathbf{y} + 2 - 1 = 0$

$\mathbf{y}+1=0$

$\mathbf{y}+1-1=0-1$

$\mathbf{y}=-1$

One solution is $(1,-1)$

Again, substitute 0.75 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

$\mathbf{y}+2(0.75)-1=0$

$\mathbf{y}+1.5-1=0$

$\mathbf{y}+0.5=0$

$\mathbf{y}+0.5-0.5=0-0.5$

$\mathbf{y}=-0.5$

Another solution is $(0.75,-0.5)$

6. Solve the following problems:

   (a) When subtracting four times a number from one, the difference is equal to negative three times the square of the number. Find the number.

Let the number be $\mathbf{x}$.

Form an equation in $\mathbf{x}$ from, "subtracting four times a number from one, the difference is equal to negative three times the square of the number.":

$1-4\mathbf{x}=-3\mathbf{x}^2$

Write it in standard form by adding $3x^2$ to each side:

$1-4\mathbf{x}+3\mathbf{x}^2=-3\mathbf{x}^2+3\mathbf{x}^2$

$1-4\mathbf{x}+3\mathbf{x}^2=0$

$3\mathbf{x}^2-4\mathbf{x}+1=0$

Now, a = 3, b = -4, and c = 1

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x}=\dfrac{(-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 3, b = -4, and c = 1:

$$\mathbf{x}=\frac{(-(-4)\pm\sqrt{((-4)^2-4\times3\times1)})}{(2\times3)}$$

$$\mathbf{x} = \frac{(-(-4) + \sqrt{((-4)^2 - 4\times3\times1)})}{(2\times3)} \text{ or } \frac{(-(-4) - \sqrt{((-4)^2 - 4\times3\times1)})}{(2\times3)}$$

$\mathbf{x} = 1$ or $0.333\ldots$

Therefore, the solutions are 1 and 0.333....

(b) A certain number is equal to one minus twice another number; and the square of this number plus two, is equal to three times the other number. Find these two numbers.

Let the certain number be $\mathbf{x}$.

If another number is $\mathbf{y}$, form one equation from, "A certain number is equal to one minus twice another number.":

$\mathbf{x} = 1 - 2\mathbf{y} \ldots (\mathbf{i})$

Form another equation in $\mathbf{x}$ and $\mathbf{y}$ from, "the square of this number plus two, is equal to three times the other number.":

$\mathbf{x}^2 + 2 = 3\mathbf{y} \ldots (\mathbf{ii})$

Solve these simultaneous equations:

As $\mathbf{x}$ is the subject in equation $(\mathbf{i})$, substitute $1 - 2\mathbf{y}$ for $\mathbf{x}$ in equation $(\mathbf{ii})$.

$\mathbf{x}^2 + 2 = 3\mathbf{y} \ldots (\mathbf{ii})$

$(1 - 2\mathbf{y})^2 + 2 = 3\mathbf{y} \ldots (\mathbf{ii})$

Reduce this equation to the standard form $\mathbf{a\,x^2 + b\,x + c = 0}$.

Expand the perfect square term on the left and simplify:

$(1 - 2\mathbf{y})^2 + 2 = 3\mathbf{y} \ldots (\mathbf{ii})$

$(1)^2 + 2 \times (-2\mathbf{y}) \times 1 + (-2\mathbf{y})^2 + 2 = 3\mathbf{y} \ldots\ldots (\mathbf{ii})$

$1 - 4\mathbf{y} + 4\mathbf{y}^2 + 2 = 3\mathbf{y} \ldots\ldots (\mathbf{ii})$

$3 - 4\mathbf{y} + 4\mathbf{y}^2 = 3\mathbf{y} \ldots\ldots (\mathbf{ii})$

$4\mathbf{y}^2 - 4\mathbf{y} + 3 = 3\mathbf{y} \ldots\ldots (\mathbf{ii})$

Subtract 3y from each side and simplify:

$4\mathbf{y}^2 - 4\mathbf{y} - 3\mathbf{y} + 3 = 3\mathbf{y} - 3\mathbf{y} \ldots\ldots (\mathbf{ii})$

$4\mathbf{y}^2 - 7\mathbf{y} + 3 = 0 \ldots\ldots (\mathbf{ii})$

Now, a = 4, b = -7, and c = 3

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 4, b = -7, and c = 3:

$$\mathbf{x} = \frac{(-(-7) \pm \sqrt{((-7)^2 - 4 \times 4 \times 3)})}{(2 \times 4)}$$

$$\mathbf{x} = \frac{(-(-7) + \sqrt{((-7)^2 - 4 \times 4 \times 3)})}{(2 \times 4)} \text{ or } \frac{(-(-7) - \sqrt{((-7)^2 - 4 \times 4 \times 3)})}{(2 \times 4)}$$

$\mathbf{x} = 1 \text{ or } 0.75$

If $\mathbf{x} = 1$, then substitute 1 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

$\mathbf{y} + 2(1) - 1 = 0$

$\mathbf{y} + 2 - 1 = 0$

$\mathbf{y} + 1 = 0$

$\mathbf{y} + 1 - 1 = 0 - 1$

$\mathbf{y} = -1$

One solution is $(1, -1)$

Again, substitute 0.75 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

$\mathbf{y} + 2(0.75) - 1 = 0$

$\mathbf{y} + 1.5 - 1 = 0$

$\mathbf{y} + 0.5 = 0$

$\mathbf{y} + 0.5 - 0.5 = 0 - 0.5$

$\mathbf{y} = -0.5$

Another solution is $(0.75, -0.5)$

Therefore, the numbers are 1 and -1 **or** 0.75 and -0.5.

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Attempt all the questions. Show your work for each question.

TOTAL MARKS: 60

TIME: 80 minutes

1. Solve the equations by factorisation.

   a) $(\mathbf{y} - 2)(\mathbf{y} - 7) = 0$

(4marks)

b) $(\mathbf{p}+1)(\mathbf{p}+2)=0$

(4marks)

c) $(\mathbf{r}+2)(\mathbf{r}-4)=0$

(4marks)

d) $\mathbf{n}(\mathbf{n}-1)=0$

(4marks)

e) $\mathbf{x}^2 = 4\mathbf{x}$

(4marks)

2. Find the solutions of the equations below, by completing the square.

a) $\mathbf{x}^2 - 4\mathbf{x} = 0$

(4marks)

b) $\mathbf{x}^2 + 6\mathbf{x} = 16$

(4marks)

3. Solve the equations by the quadratic formula, using a calculator:

$$x = \frac{(-b \pm \sqrt{b^2 - 4ac})}{(2a)}$$

a) $\mathbf{y}^2 + 6\mathbf{y} = 0$

(4marks)

b) $3\mathbf{x}^2 + \mathbf{x} = 4$

(4marks)

4. Solve these non-quadratic equations by reducing them to quadratic form, and then, solving the quadratic equations which are formed.

a) $2\mathbf{x}^3 + \mathbf{x}^2 = 3\mathbf{x}$

(5marks)

b) $\mathbf{m}^4 - 6\mathbf{m}^2 + 8 = 0$

(5marks)

5. Find the solutions to these simultaneous equations:

$\mathbf{y} = \mathbf{x}^2 - 2\ldots(\mathbf{i})$

$\mathbf{y} = 2\mathbf{x}^2 - 11\ldots\ldots(\mathbf{ii})$

(6marks)

6. If the sum of a number and two times its reciprocal equals three, which number is this?

(8marks)

SOLUTIONS TO ASSESSMENT:

1. Solve the equations by factorisation.

a) $(\mathbf{y}-2)(\mathbf{y}-7)=0$

If $(\mathbf{y}-2)(\mathbf{y}-7)=0$, then $(\mathbf{y}-2)=0$ or $(\mathbf{y}-7)=0$, using zero factor property.

If $\mathbf{y}-2=0$, then

$\mathbf{y}-2+2=0+2$

$\mathbf{y}=2$

If $\mathbf{y}-7=0$, then

$\mathbf{y}-7+7=0+7$

$\mathbf{y}=7$

Therefore, the solutions are 2 and 7.

b) $(\mathbf{p}+1)(\mathbf{p}+2)=0$

If $(\mathbf{p}+1)(\mathbf{p}+2)=0$, then $(\mathbf{p}+1)=0$ or $(\mathbf{p}+2)=0$, using zero factor property.

If $(\mathbf{p}+1)=0$, then

$\mathbf{p}+1-1=0-1$

$\mathbf{p}=-1$

If $(\mathbf{p}+2)=0$, then

$\mathbf{p}+2-2=0-2$

$\mathbf{p}=-2$

Therefore, the solutions are -1 and -2.

c) $(\mathbf{r}+2)(\mathbf{r}-4)=0$

If $(\mathbf{r}+2)(\mathbf{r}-4)=0$, then $(\mathbf{r}+2)=0$ or $(\mathbf{r}-4)=0$, using zero factor property.

If $(\mathbf{r}+2)=0$, then

$\mathbf{r}+2-2=0-2$

$\mathbf{r}=-2$

If $\mathbf{r}-4=0$, then

$\mathbf{r}-4+4=0+4$

$\mathbf{r}=4$

Therefore, the solutions are -2 and 4.

d) $\mathbf{n}(\mathbf{n}-1)=0$

If $\mathbf{n}(\mathbf{n}-1)=0$, then $\mathbf{n}=0$ or $(\mathbf{n}-1)=0$, using zero factor property.

If $\mathbf{n}-1=0$, then

$n - 1 + 1 = 0 + 1$

$n = 1$

Therefore, the solutions are 0 and 1.

e) $x^2 = 4x$

Write it in standard form:

$x^2 = 4x$

$x^2 - 4x = 4x - 4x$

$x^2 - 4x = 0$

Factorise **z** from the two terms on the left side:

$x^2 - 4x = 0$

$x(x - 4) = 0$

Use zero factor property to solve:

If $x(x - 4) = 0$, then $x = 0$ or $(x - 4) = 0$, using zero factor property.

If $x - 4 = 0$, then

$x - 4 + 4 = 0 + 4$

$x = 4$

Therefore, the solutions are 0 and 4.

2. Find the solutions of the equations below, by completing the square.

(a) $x^2 - 4x = 0$

The first term on the left side of the equation is already a square of x.

Since -4x equals $2(-2x)$, the square of half the coefficient of x added to both sides makes the left side a complete square:

$x^2 - 4x = 0$

$x^2 - 4x + (-2)^2 = 0 + (-2)^2$

$(x - 2)^2 = 0 + 4$

$(x - 2)^2 = 4$

Then take the square root of each side:

$(x - 2)^2 = 4$

$(x - 2) = \pm\sqrt{4}$

$\mathbf{x}-2+2=2\pm\sqrt{4}$

$\mathbf{x}=2\pm 2$

$\mathbf{x}=4$ or 0

Therefore, the solutions are 4 and 0.

(b) $\mathbf{x}^2+6\mathbf{x}=16$

The first term on the left side of the equation is already a square of x.

Since 6x equals $2(3\mathbf{x})$, the square of half the coefficient of x added to both sides makes the left side a complete square:

$\mathbf{x}^2+6\mathbf{x}=16$

$\mathbf{x}^2+6\mathbf{x}+(3)^2=16+(3)^2$

$(\mathbf{x}+3)^2=16+9$

$(\mathbf{x}+3)^2=25$

Then take the square root of each side:

$(\mathbf{x}+3)^2=25$

$(\mathbf{x}+3)=\pm\sqrt{25}$

$\mathbf{x}+3-3=-3\pm\sqrt{25}$

$\mathbf{x}=-3\pm 5$

$\mathbf{x}=2$ or -8

Therefore, the solutions are 2 and -8.

3. Solve the equations by the quadratic formula, using a calculator:

$$x = \frac{(-b\pm\sqrt{b^2-4ac})}{(2a)}$$

a) $\mathbf{y}^2+6\mathbf{y}=0$

Now, a = 1, b = 6, and c = 0

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x}=\dfrac{(-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = 6, and c = 0:

$$\mathbf{x} = \frac{(-6 \pm \sqrt{(6^2 - 4 \times 1 \times 0)}}{(2 \times 1)}$$

$$\mathbf{x} = \frac{(-6 + \sqrt{(6^2 - 4 \times 1 \times 0)}}{(2 \times 1)} \text{ or } \frac{(-6 - \sqrt{(6^2 - 4 \times 1 \times 0)}}{(2 \times 1)}$$

$\mathbf{x} = 0$ or -6

Therefore, the solutions are 0 and -6.

b) $3\mathbf{x}^2 + \mathbf{x} = 4$

Write it in standard form by subtracting 4 from each side:

$3\mathbf{x}^2 + \mathbf{x} - 4 = 4 - 4$

$3\mathbf{x}^2 + \mathbf{x} - 4 = 0$

Now, a = 3, b = 1, and c = -4

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \frac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 3, b = 1, and c = -4:

$$\mathbf{x} = \frac{(-1 \pm \sqrt{(1^2 - 4 \times 3 \times (-4))})}{(2 \times 3)}$$

$$\mathbf{x} = \frac{(-1 + \sqrt{(1^2 - 4 \times 3 \times (-4))})}{(2 \times 3)} \text{ or } \frac{(-1 - \sqrt{(1^2 - 4 \times 3 \times (-4))})}{(2 \times 3)}$$

$\mathbf{x} = 1$ or $\mathbf{x} \approx -1.33$

Therefore, the solutions are 1 and ≈ -1.33.

4. Solve these non-quadratic equations by reducing them to quadratic form, and then, solving the quadratic equations which are formed.

a) $2\mathbf{x}^3 + \mathbf{x}^2 = 3\mathbf{x}$

Subtract $3\mathbf{x}$ from each side and simplify:

$2\mathbf{x}^3 + \mathbf{x}^2 - 3\mathbf{x} = 3\mathbf{x} - 3\mathbf{x}$

$2\mathbf{x}^3 + \mathbf{x}^2 - 3\mathbf{x} = 0$

Factorise **x** from the two terms on the left side:

$2\mathbf{x}^3 + \mathbf{x}^2 - 3\mathbf{x} = 0$

$\mathbf{x}(2\mathbf{x}^2 + \mathbf{x} - 3) = 0$

Use zero factor property:

If $\mathbf{x}(2\mathbf{x}^2+\mathbf{x}-3)=0$, then $\mathbf{x}=0$ or $(2\mathbf{x}^2+\mathbf{x}-3)=0$, using zero factor property.

$2\mathbf{x}^2+\mathbf{x}-3=0$

Now, a = 2, b = 1, and c = -3

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x}=\dfrac{(-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 2, b = 1, and c = -3:

$$\mathbf{x}=\frac{(-1\pm\sqrt{(1^2-4\times2\times(-3))}}{(2\times2)}$$

$$\mathbf{x}=\frac{(-1\pm\sqrt{(1^2-4\times2\times(-3))}}{(2\times2)} \text{ or } \frac{(-1\pm\sqrt{(1^2-4\times2\times(-3))}}{(2\times2)}$$

$\mathbf{x}=1$ or -1.5

Therefore, the solutions are 0,1 and -1,5.

b) $\mathbf{m}^4-6\mathbf{m}^2+8=0$

Now, reduce this equation to quadratic form by letting a variable, such as "u", be equal to $\mathbf{m}^2$.

Substitute u for $\mathbf{m}^2$ in $\mathbf{m}^4-6\mathbf{m}^2+8=0$:

$\mathbf{m}^4-6\mathbf{m}^2+8=0$

$\mathbf{u}^2-6\mathbf{u}+8=0$

Now, a = 1, b = -6, and c = 8

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x}=\dfrac{(-\mathbf{b}\pm\sqrt{(\mathbf{b}^2-4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = -6, and c = 8:

$$\mathbf{u}=\frac{(-(-6)\pm\sqrt{((-6)^2-4\times1\times8)}}{(2\times1)}$$

$\mathbf{u}=4\ldots\mathbf{or}\ldots2$

Since u is equal to $\mathbf{m}^2$, this means that:

$\mathbf{m}^2=4\ldots\mathbf{or}\ldots2$

Take square roots on both sides and remember there are two roots which are opposites:

$\mathbf{m} = \pm\sqrt{4}\ldots\mathbf{or}\ldots\pm\sqrt{2}$

Therefore, the solution set is $\{2,-2,\sqrt{2},-\sqrt{2}\}$

5. Find the solutions to these simultaneous equations:

$\mathbf{y} = \mathbf{x}^2 - 2\ldots(\mathbf{i})$

$\mathbf{y} = 2\mathbf{x}^2 - 11\ldots\ldots(\mathbf{ii})$

As $\mathbf{y}$ is the subject in equation $(\mathbf{i})$, substitute $\mathbf{x}^2 - 2$ for $\mathbf{y}$ in equation $(\mathbf{ii})$.

$\mathbf{x}^2 - 2 = 2\mathbf{x}^2 - 11\ldots(\mathbf{ii})$

Subtract $\mathbf{x}^2$ from each side and simplify:

$\mathbf{x}^2 - \mathbf{x}^2 - 2 = 2\mathbf{x}^2 - \mathbf{x}^2 - 11\ldots(\mathbf{ii})$

$-2 = \mathbf{x}^2 - 11\ldots(\mathbf{ii})$

Add 2 to both sides and simplify:

$-2 + 2 = \mathbf{x}^2 - 11 + 2\ldots(\mathbf{ii})$

$0 = \mathbf{x}^2 - 9\ldots(\mathbf{ii})$

$\mathbf{x}^2 - 9 = 0\ldots(\mathbf{ii})$

Write it as a difference of two squares:

$(\mathbf{x})^2 - (3)^2 = 0$

Use a difference of two squares to factorise:

$(\mathbf{x} + 3)(\mathbf{x} - 3) = 0$

Use the zero factor property and solve for x:

$\mathbf{x} + 3 = 0$ or $\mathbf{x} - 3 = 0$

$\mathbf{x} = -3$ or 3

If $\mathbf{x} = -3$, then substitute -3 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

$\mathbf{y} = (-3)^2 - 2$

$\mathbf{y} = 9 - 2$

$\mathbf{y} = 7$

One solution is $(-3,7)$

Again, substitute 3 for $\mathbf{x}$ in equation $(\mathbf{i})$ to get corresponding $\mathbf{y}$:

$\mathbf{y} = (3)^2 - 2$

$\mathbf{y} = 9 - 2$

$\mathbf{y} = 7$

Another solution is $(3,7)$

6. If the sum of a number and two times its reciprocal equals three, which number is this?

Let the number be $\mathbf{x}$.

Form an equation in $\mathbf{x}$ from, "sum of a number and two times its reciprocal equals three.":

$$\mathbf{x} + 2\left(\frac{1}{\mathbf{x}}\right) = 3$$

Multiply both sides by x and simplify:

$$\mathbf{x}\left(\mathbf{x} + 2\left(\frac{1}{\mathbf{x}}\right)\right) = 3(\mathbf{x})$$

$$\mathbf{x}^2 + 2 = 3\mathbf{x}$$

Write it in standard form by subtracting 3x from each side:

$$\mathbf{x}^2 + 2 - 3\mathbf{x} = 3\mathbf{x} - 3\mathbf{x}$$

$$\mathbf{x}^2 + 2 - 3\mathbf{x} = 0$$

$$\mathbf{x}^2 - 3\mathbf{x} + 2 = 0$$

Now, a = 1, b = -3, and c = 2

When you use a calculator, it is important to introduce brackets in the formula, to avoid getting wrong answers:

From $\mathbf{x} = \dfrac{(-\mathbf{b} \pm \sqrt{(\mathbf{b}^2 - 4\mathbf{ac})})}{(2\mathbf{a})}$ when, a = 1, b = -3, and c = 2:

$$\mathbf{x} = \frac{(-(-3) \pm \sqrt{((-3)^2 - 4 \times 1 \times 2)})}{(2 \times 1)}$$

$$\mathbf{x} = \frac{(-(-3) \pm \sqrt{((-3)^2 - 4 \times 1 \times 2)})}{(2 \times 1)} \text{ or } \frac{(-(-3) \pm \sqrt{((-3)^2 - 4 \times 1 \times 2)})}{(2 \times 1)}$$

$$\mathbf{x} = 2 \text{ or } 1$$

Therefore, the solutions are 2 and 1.

Unit Contents

Unit 18

Unit 18

Perimeters, Areas and Volumes

Introduction

This is another unit on geometrical shapes. The ones we have met so far are flat, or plane shapes. They were in two dimensions. We are going to continue working with them in this unit together with shapes in three dimensions.

This unit is on perimeters, areas and volumes of some geometrical shapes.

This unit consists of 69 pages. It covers approximately 4% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 25 hours to work on this unit, 15 hours for formal study and 10 hours for self-study and completing assessments/assignments.

When reading the following learning outcomes, think about them as a guide to what you should focus on while studying this unit.

This Unit is Comprised of Six Lessons:

Lesson 1 Perimeter and Area of a Triangle, Rectangle and Circle
Lesson 2 Perimeter and Area of a Parallelogram and a Trapezium
Lesson 3 Volume of a Cuboid, Prism and Cylinder
Lesson 4 The Surface Area of a Cuboid and a Cylinder
Lesson 5 Arcs Lenghts and Sector Areas
Lesson 6 The Surface Area and Volume of a Sphere, Pyramid and Cone

Upon completion of this unit you will be able to:

Outcomes

- *calculate*

  the perimeter and area of a rectangle and triangle

  the circumference and area of a circle

  the perimeter and area of a parallelogram and a trapezium

  the volume of a cuboid, prism and cylinder

  the surface area of a cuboid and a cylinder

- *solve* problems involving the arc length and sector area as fractions of the circumference and area of a circle; the surface area and volume of a sphere, pyramid and cone (given formulae for the sphere, pyramid and cone)

Terminology

Perimeter: The perimeter of a shape is the length of its boundary. It is the sum of the lengths of its sides.

Area: Area is the amount of space a flat shape (2 – dimensional shape) occupies.

Surface area: The total area of the exposed sides of an object.

Volume: Amount of space occupied by a 3 – dimensional shape

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Perimeter and Area of a Triangle, Rectangle and Circle

Introduction

By the end of this subunit, you should be able to:

- calculate the perimeter and area of a triangle, rectangle and circle

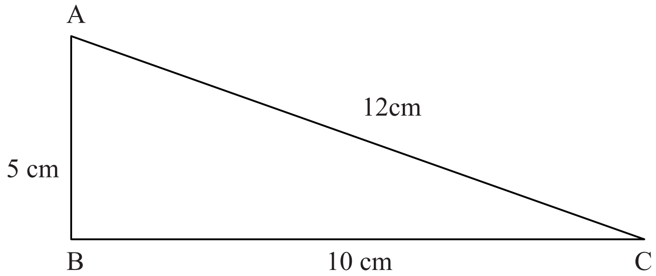
Reflection

In your secondary mathematics course, you did work on the perimeter and area of triangles, rectangles and circles.

We are going to review this work.

The perimeter of a triangle, rectangle and circle

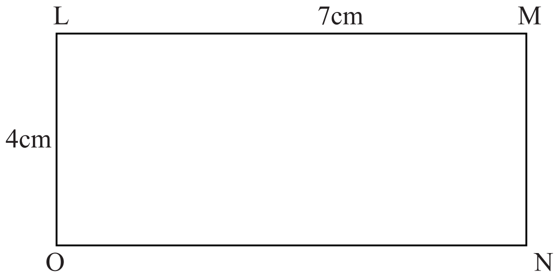
Given triangle ABC below, its perimeter is the sum of the lengths of its sides.



Perimeter of triangle ABC = 5 cm + 12cm + 10cm
= 27 cm

It does not matter what triangle we are using, the perimeter will still be the sum of its sides.

This is rectangle LMNO.



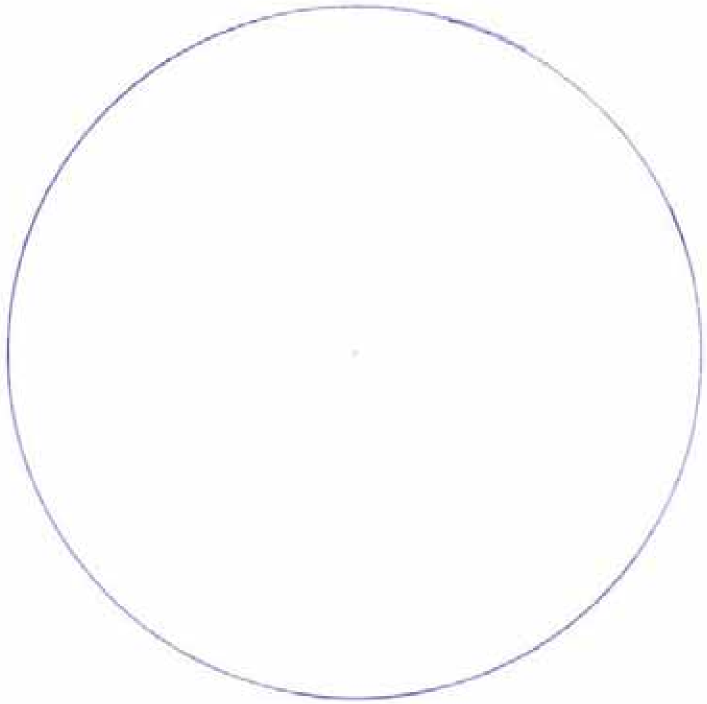
Opposite sides of a rectangle are equal.
This says side LM = side NO = 7cm, LO = MN = 4cm

Therefore the perimeter of rectangle LMNO = LM + MN + NO + OL
= 7cm + 4 cm + 7cm+ 4cm
= 22 cm

OR

perimeter of rectangle LMNO = $(7\text{cm} \times 2) + (4\text{cm} \times 2)$
$= (14\text{ cm}) + (8\text{cm})$
$= 22\text{ cm}$

The circle drawn below has diameter 10 cm.



Perimeter of a circle is calculated using the formula,

Circumference = $\pi \times$ diameter,
= π d, which can be written as

Circumference = $\pi \times 2$radius
= π 2r
= 2 π r

The perimeter of a circle is called the **circumference**.

Note it!

π is not an exact decimal number. A common approximation for π is $\pi \approx 3.14$

The diagrams we are going to use throughout this unit will not be drawn to scale, unless otherwise stated

Therefore circumference = $\pi \times 10\text{cm}$

$= 3.14 \times 10\text{cm}$

$= 31.4 \text{ cm}$

OR

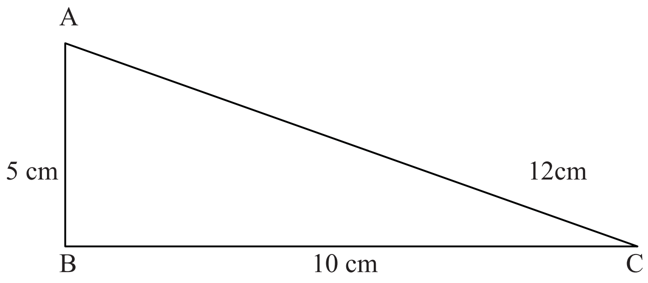
Circumference $= 2\pi r$

$= 2 \times 3.14 \times 5\text{cm}$

$= 31.4 \text{ cm}$

Area of a triangle, rectangle and circle

Using triangle ABC with AB = 5cm, BC = 10cm and AC = 12cm



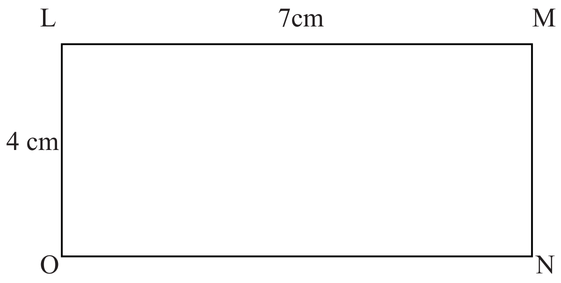
The formula used to calculate area of a triangle is

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

The area of the triangle given above is

$$= \frac{1}{2} \times 10\text{cm} \times 5 \text{ cm}$$

$$= 25 \text{ cm}^2$$



Area of a rectangle is calculated by the formula

$$\text{Area} = \text{length} \times \text{width}$$

The area of the rectangle given above is

$$= 7\text{cm} \times 4\text{cm}$$

$$= 28\text{cm}^2$$

Unlike the circumference, which is the special name for the perimeter of a circle, there is no special name for the area of a circle. The formula we use to calculate the area of a circle is

$$\text{Area} = \pi \times r^2$$

Given a circle with radius 10m. Calculate the area

$$\begin{aligned}\text{Area} &= \pi \times r^2 \\ &= 3.14 \times (10)^2 \\ &= 3.14 \times 100 \\ &= 314\text{m}^2\end{aligned}$$

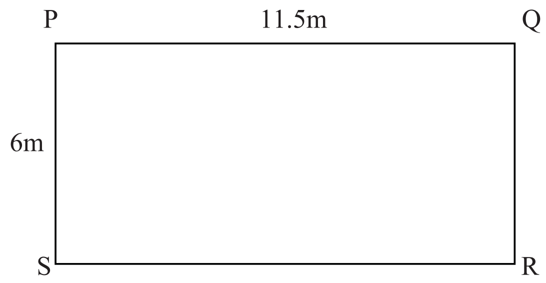
cm^2, mm^2 and m^2 are used to measure relatively small areas. The hectare is commonly used to measure bigger areas; the sizes of farms, towns etc
1 hectare (ha) = 100 m x 100 m

$$= 10\ 000\text{m}^2$$

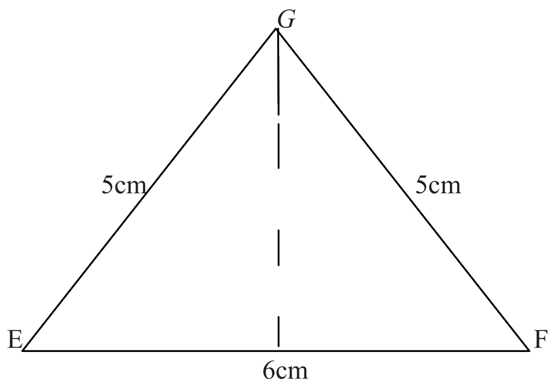
Example 1

Find the perimeter and area of each

(a)



(b)



Triangle EFH has height 4cm

(c) A circle with diameter 45cm.

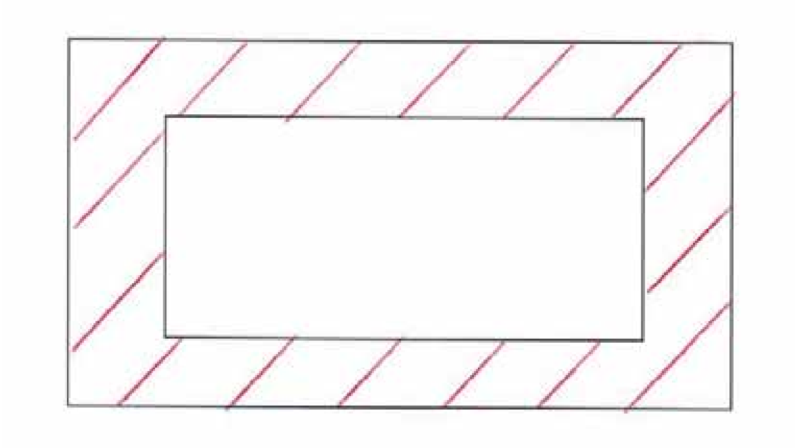
Solutions

	Perimeter	**area**
rectangle PQRS	= 11.5 + 6 + 11.5 + 6 = 35cm	= 11.5cm × 6cm = 69cm^2
triangle EFG	= 5 cm + 5cm + 6cm = 16cm	Area = $\frac{1}{2}$ × base × height = $\frac{1}{2}$ × 6cm × 4cm = 12cm^2
▪ **circle**	= π × diameter = 3.14 × 45	Area = $\pi \times r^2$ = $3.14 \times (22.5)^2$

	= 141.3cm	= 1589.625cm^2

Example 2

The first rectangle has length 7 cm and width 4.5cm; the second one has length 5 cm and width 2 cm.
Calculate the shaded area.



Solution

In this diagram, we have two rectangles, the first one has length 7 cm and width 4.5cm; the second one has length 5 cm and width 2 cm.

Area of the shaded area = area of the 1st rectangle - area of the 2nd rectangle

Area of the 1st rectangle = 7×4.5 cm
= 31.5cm^2

Area of the 2nd rectangle = 5×2 cm
= 10cm^2

= 31.5cm^2 - 10 cm^2 = 21.5cm^2

Example 3

A carpet measuring 4 metres by 6 metres costs M990.

1. Calculate the area of the carpet
2. Calculate the cost per square metre of the carpet

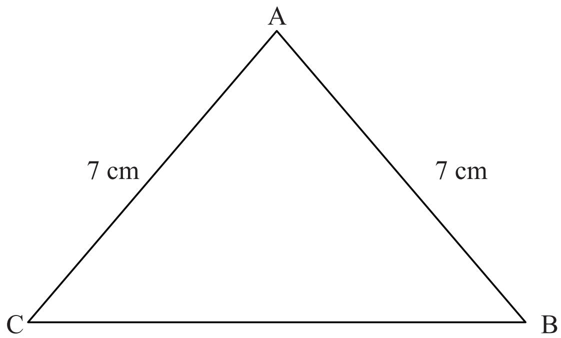
Solution

The carpet is rectangular shaped. Therefore its area $= 4 \times 6$
$= 24\text{m}^2$

If 24m^2 costs M990, 1 m^2 costs M990 $\div 24\text{m}^2$ = M41.25

Finding the Perimeter of a Triangle Given Two Sides and one Angle

There are times when we are given a triangle that has two sides and one angle. This angle is enclosed by the two sides. In the triangle below, angle A = 100°



It is enclosed by side AB and side AC. The formula used to calculate the area of a triangle $= \frac{1}{2} \times AB \times AC \times \sin A$

Side AB can also be named with letter c, for it is opposite angle C. Side AC can also be named with letter b, as it is opposite angle B. The formula can then be written as

$$= \frac{1}{2} \times c \times b \times \sin A$$

$$= \frac{1}{2} \times bc \sin A$$

So the area of triangle ABC $= \frac{1}{2} \times bc \sin A$

$$= \frac{1}{2} \times 7 \times 7 \times \sin 100°$$
$$= \frac{1}{2} \times 49 \times \sin 100°$$
$$= \frac{1}{2} \times 49 \times$$

When the given angle is B, it is enclosed by side AB, which is c, and side BC, which is a. The formula will be

$$= \frac{1}{2} \times AB \times BC \times \sin B$$

$$= \frac{1}{2} \times c \times a \times \sin B$$

$$= \frac{1}{2} \times ac \sin B$$

When the given angle is C, it is enclosed by side AC, which is b, and side BC, which is a. The formula will be

$$= \frac{1}{2} \times AC \times BC \times \sin C$$

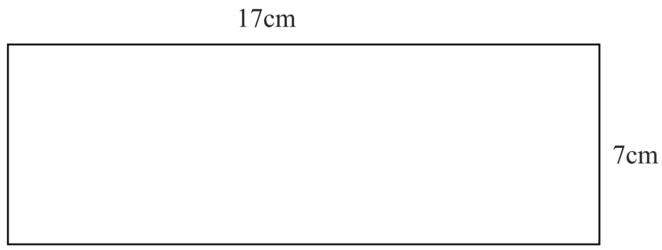
$$= \frac{1}{2} \times b \times a \times \sin C$$

$$= \frac{1}{2} \times ab \sin C$$

Activity 1

Answer all Questions

1. Find the perimeter and area of this rectangle



2. The perimeter of an equilateral triangle is 48cm. How long is each side?

3. Give two formulas for the circumference of a circle.

4. Complete this table. Use $\pi = 3.14$

radius	diameter	circumference	area
7cm		4.396cm	
	16mm		
1.5m			$7.065cm^2$
	18cm	56.52cm	

5. A rectangular garden measures 12 metres by 9.5 metres.
Mrs. Mphoto wants to grass this piece of garden using lawn seed.
If she uses 60 grams of lawn seed per square metre, how many kilograms does she use altogether?

1. 6. Find the areas of the following triangles
   a = 2.4, b = 3.2, C = 100°

   a = 6.18, b = 8.2, B = 120°

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember The key points to remember in this subunit on calculating the perimeter and are of a triangle, rectangle and circle are:

- perimeter is the length of the boundary of the shape

- area is the amount of two dimensional space of a shape

- the formula used to calculate the area of a rectangle = length $\times$ breadth

- the formulae used to calculate the area of a triangle

$$= \frac{1}{2} \times \text{base} \quad \times \text{height},$$

when given the base and the height

OR

$$= \frac{1}{2} \times ab \sin C$$

$$= \frac{1}{2} \times ac \sin B$$

$$= \frac{1}{2} \times bc \sin A$$

when given two sides and the enclosed angle

- the formula used to calculate the area of a circle = π d,
= 2 π r

In the next sub unit, we are going to look at calculating the perimeter and area of a parallelogram and a trapezium

Answers:

(a)

1. Perimeter = 17 + 7 + 17 + 7

   = 48cm

Area = 17 $\times$ 7

= 119 cm^2

(b)

2. An equilateral triangle has all 3 sides equal.

   is 48cm $\div$ 3 sides = 16 cm. Each side is 16cm

(c)

3. Two formulas for the circumference of a circle

$= \pi \times$ diameter
$= \pi$ d,

and

$= \pi\, 2r$
$= 2\pi r$

(d)

4.

radius	diameter	circumference	area
7cm	**14cm**	= 4.396cm	Area = $\pi \times r^2$ **= 153.86cm²**
8mm	16mm	C = $\pi \times$ d **= 50.24cm**	Area = $\pi \times r^2$ **= 200.96mm²**
1.5m	**3m**	C = $\pi \times$ d **= 9.42cm**	7.065cm^2
9cm	18cm	56.52cm	Area = $\pi \times r^2$ **= 254.34cm²**

5. Area of the garden = 12 metres $\times$ 9.5 metres.
   = 114m^2

If 60 grams is used for 1m^2, then for 114m^2 the grams used is
114m^2 $\times$ 60g = 6 840g

- 6.

a = 2.4, b = 3.2, C = 60°

$$= \frac{1}{2} \times ab \sin C$$

$$= \frac{1}{2} \times 2.4 \times 3.2 \times \sin 60°$$

$$= \frac{1}{2} \times 7.68 \times \sin 60°$$

a = 6.18, c = 8.2, B = 64°

$$= \frac{1}{2} \times ac \sin B$$

$$= \frac{1}{2} \times 6.18 \times 8.2 \times \sin 64°$$

$$= \frac{1}{2} \times 50.676 \times \sin 64°$$

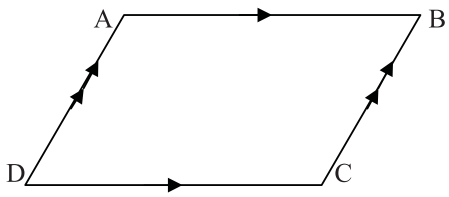
Lesson 2 **Perimeter and Area of a Parallelogram and a Trapezium**

By the end of this subunit, you should be able to

- *calculate* the perimeter of a parallelogram and a trapezium
- *calculate* the area of a parallelogram and a trapezium

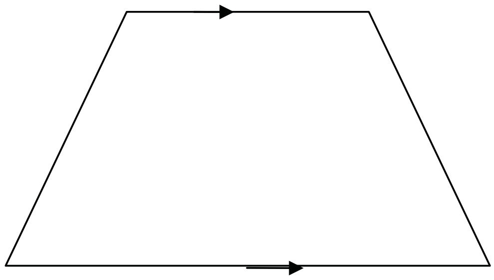
A parallelogram and a trapezium are some of the many quadrilaterals. A quadrilateral is a polygon that has 4 sides.

A parallelogram is a quadrilateral with opposite sides equal and parallel.

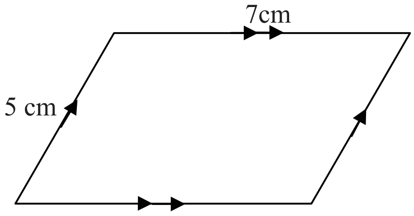


This is parallelogram ABCD

A trapezium is a quadrilateral which has one pair of parallel lines



Even for a parallelogram and a trapezium, perimeter is still the sum of the lengths of its sides.



The perimeter of this parallelogram = 5 + 7 + 5 + 7
= 24 cm

The perimeter of the trapezium given above = 15 + 5 +14 + 9
= 43 cm

Example 1

A parallelogram has a perimeter = 75cm. One of its sides is 6.3 cm. Find the lengths of the other three sides

Solution

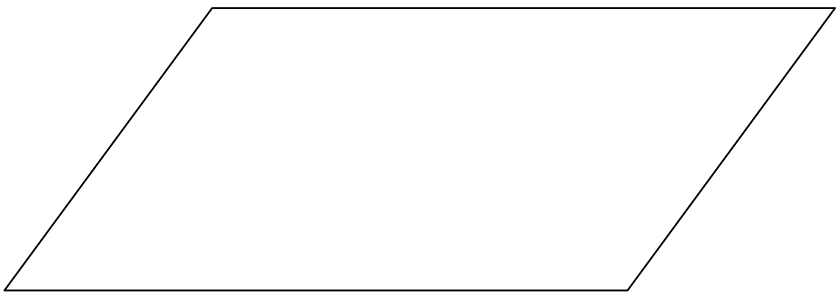
A parallelogram has opposite sides equal and parallel.

$$\text{Perimeter} = (6.3 \times 2) + (2x)$$
$$75 = 12.6 + (2x)$$
$$62.4 = 2x$$
$$31.2 = x$$

The other sides are 6.3 cm, 31.2 cm and 31.2 cm

Area of a Parallelogram

We will start off by finding out how the formula used to calculate the area of a parallelogram is obtained and then we will practice using it.

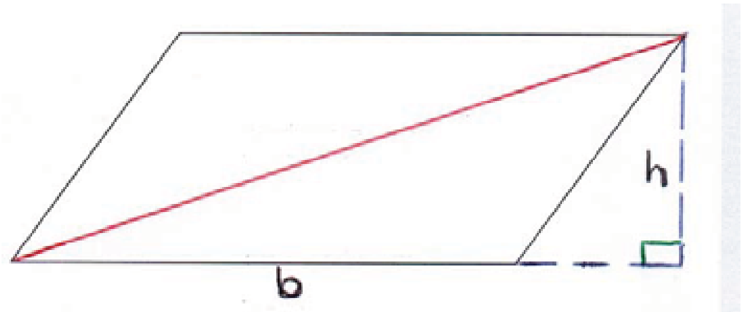


The parallelogram given above can be cut into two equal triangles by its diagonals.

We know that area of a triangle = ½ × base × height
= ½ × b × h

The parallelogram is made up of two triangles.
We therefore can say area of the parallelogram = 2 × area of triangle
= 2(½ × base × height)
= base × height
= b × h

As is the case with triangles, the height must always be measured perpendicularly, that is at right angles to the base.



Example 1

Find the area of the parallelograms given below

Solution

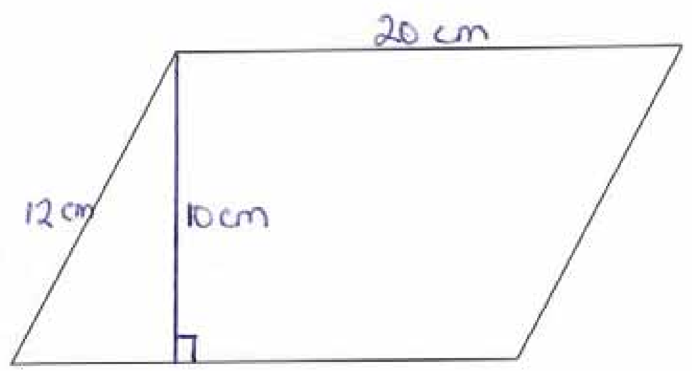
A parallelogram has opposite sides equal and parallel.

Perimeter = (6.3 × 2) + (2x)
75 = 12.6 + (2x)
62.4 = 2x
31.2 = x

The other sides are 6.3 cm, 31.2 cm and 31.2 cm

Example 1

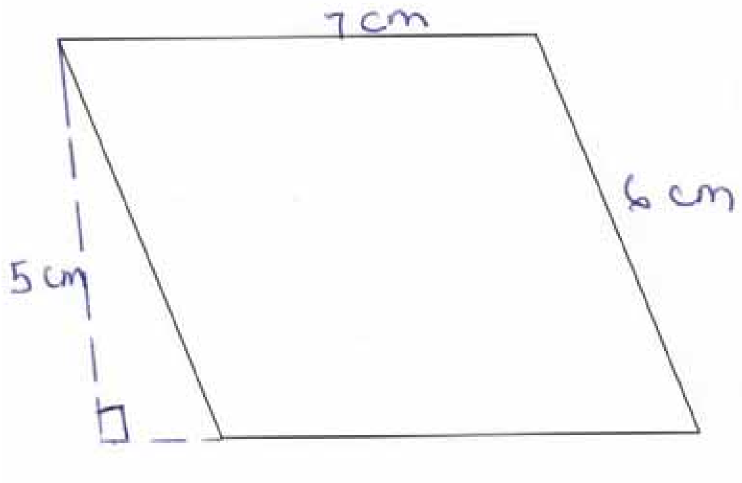
Find the area of the parallelogram given below



Area of the parallelogram = $b \times h$
$= 20 \times 10$
$= 200\ cm^2$

Example 2

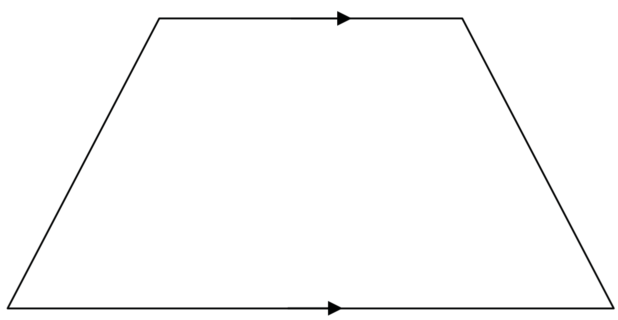
Find the area of the parallelogram given below



Area of the parallelogram = b × h
$= 7 \times 5$
$= 35cm^2$

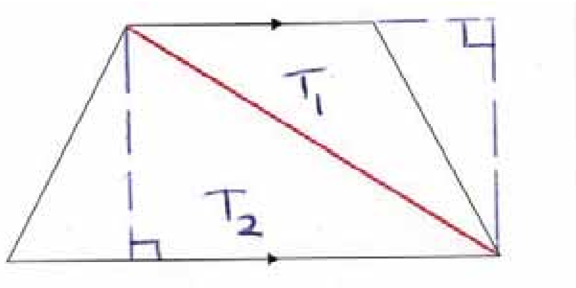
Area of a Trapezium

Again we will start off by finding out how the formula used to calculate the area of a trapezium is obtained and then we will practice using it.



The parallel sides of a trapezium are called its **bases**. The perpendicular distance between the bases is called the **height** of the trapezium.

To find the area of a trapezium, we split it into two triangles.



We have the first triangle, T_1 and the second, T_2

Area of trapezium = area of triangle T_1 + area of triangle T_2

T_1 has base b_1 and height h_1.
T_2 has base b_2 and height h_2.

Area of trapezium = $\frac{1}{2} \times b_1 \times h_1 + \frac{1}{2} \times b_2 \times h_2$

$h_1 = h_2$

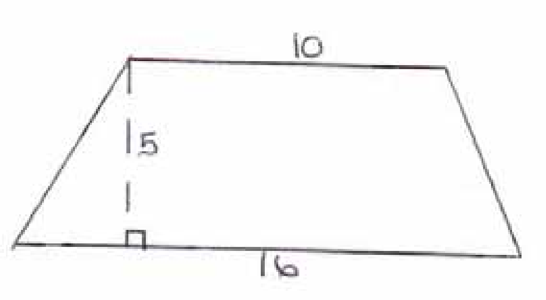
Therefore area of trapezium = $\frac{1}{2} \times b_1 \times h + \frac{1}{2} \times b_2 \times h$

Taking out the common factor of = $\frac{1}{2}$ h

Area of trapezium = $\frac{1}{2}$ h($b_1 + b_2$)

Example 3

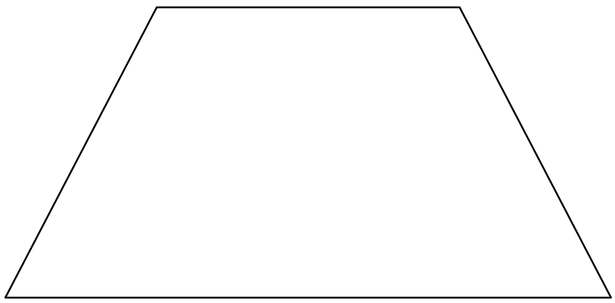
Calculate the area of the given trapezium



Area of trapezium = ½ $h(b_1 + b_2)$
= ½ × 5(10 + 16)
= ½ × 5 (26)
= 65

Example 4

What will be the cost of cementing a floor of this shape that has a height of 75 centimetres, one of the parallel sides is 100 centimetres and the other is 180 centimetres, at M67.50 per square metre?



Solution

75 cm = 0.75m
100 cm = 1 m
180 cm = 1.8m

Area of the floor (which has a shape of a trapezium) = ½ $h(b_1 + b_2)$
= ½ × 0.75(1 + 1.8)
= ½ × 0.75 (2.8)
= 1.05 m^2

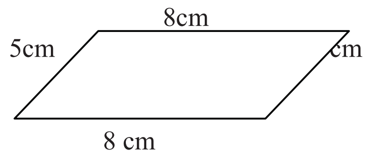
Cost = 1.05 m^2 × M67.50 per square metre?
= M70.88 (to the nearest cent)

Activity 2

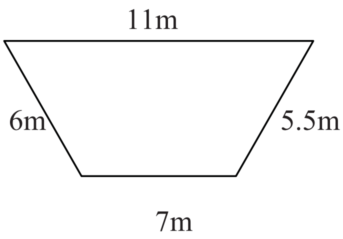
1. A parallelogram is a polygon with _______________ sides.
2. A parallelogram has opposite sides parallel and ____________
3. A trapezium is a polygon with _________________ sides.
4. A trapezium has __________ pair of parallel sides.

5. Find the perimeter of

(a)



(b)



6. Complete these tables

parallelogram	**base**	**height**	**area**
	7cm	3cm	
		8cm	$40cm^2$
	3.6m	7.5m	
	6.6cm		$53.46cm^2$

trapezium	bases		height	area
	4.7cm	6.8cm	8cm	
	18m		4m	50m^2
	82m	57m	175.5m	
	7.3cm	19.7cm	18.6cm	

Compare your answers with those at the end of this subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content and work through the activity again. **Key Point to Remember**

The key points to remember in this subunit on calculating the perimeter and the area of a parallelogram and trapezium are:

- for a parallelogram and a trapezium, perimeter is the sum of the lengths of its sides.
- Area of the parallelogram $= b \times h$

- Area of trapezium = $\frac{1}{2}\ h(b_1 + b_2)$

Answers

1. A parallelogram is a polygon with **four** sides.
2. A parallelogram has opposite sides parallel and **equal**.
3. A trapezium is a polygon with **four** sides.
4. A trapezium has **one** pair of parallel sides.

5.
(a) perimeter = 5cm + 5cm + 8cm + 8cm
= 26 cm
(b) perimeter = 11m + 5.5m + 6cm + 7m
= 29.5m

parallelogram	base	height	area
	7cm	3cm	**21cm^2**
	5cm	8cm	40cm^2
	3.6m	7.5m	**27cm^2**
	6.6cm	**8.1cm**	53.46cm^2

trapezium	bases		height	area
	4.7cm	6.8cm	8cm	**46 cm^2**
	18m	**7m**	4m	50m^2
	82m	57m	175.5m	**12 197.25m^2**
	7.3cm	19.7cm	18.6cm	**251.1 cm^2**

Lesson 3 Volume of a Cuboid, Prism and Cylinder

Introduction

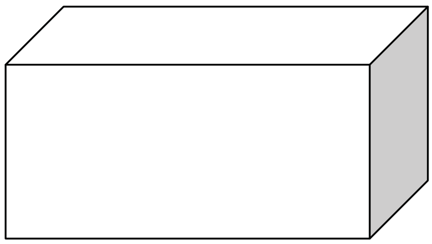
By the end of this subunit, you should be able to

- *calculate* the volume of a cuboid, prism and cylinder

Volume of a solid that is a figure having length, width and height; is a measure of the space it takes up.

Volume brings in another dimension, which is height, for we have so far dealt with length and width.

This is an example of a solid shape.



It is called a **cuboid.**

- A cuboid has six faces, twelve edges and eight vertices
- the opposite faces of a cuboid are parallel

There are a number of items that you come across almost on a daily basis, which are cuboids.

Which ones can you think of?

____

These are some of the few that we could think of; a box of matches, set of mathematical instruments, a dictionary.

All of these have length, width and height.

For us to say we want to calculate the volume of the cuboid, it says we want to calculate the amount of space it can take up.
Let us fill the cuboid with colour.

This is saying how much of the blue space.

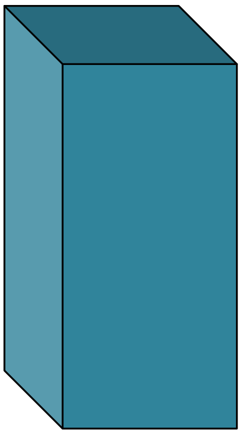
The formula used to calculate volume of a cuboid
$= \text{length} \times \text{width} \times \text{height}$

The length and the width that we have are that of our base, which is a rectangle.

When we were working with rectangles in two dimensions, length $\times$ width gave us the area of the rectangle.

Therefore we can write the formula used to calculate volume of a cuboid as = base area $\times$ height

We may decide to have this cuboid this way.



This means the edge that was our length is no more, so is the width and the height.

Therefore it does not really matter which edge is our length, width or height.

Units of volume are cubic units; cubic centimeters, (cm^3), cubic millimeters, (mm^3), cubic meters, (m^3), and so on.

Example 1

Find the area of a cuboid with the following dimensions; length = 10 cm, width = 4cm and height = 5cm

Solution

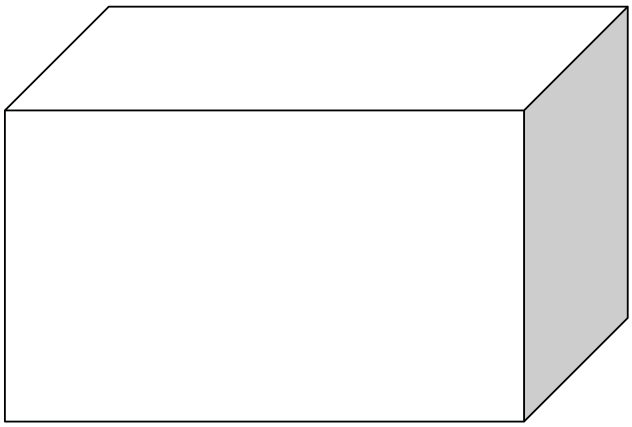
volume of a cuboid = length $\times$ width $\times$ height

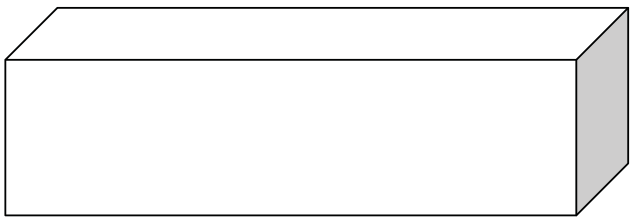
volume of a cuboid = 10cm $\times$ 4cm $\times$ 5cm
= $200cm^3$

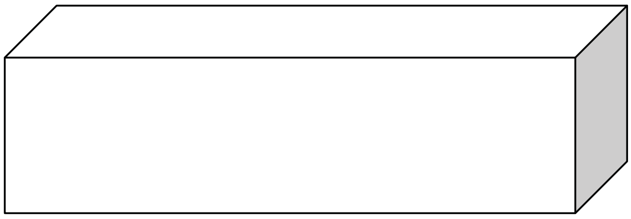
Volume of a Prism

A prism is a solid shape that has the area of cross section when the solid is cut parallel to the base, the same as that of the base.

A cuboid is an example of a prism.
If the cuboid below was to be cut parallel to its base, we would get these shapes

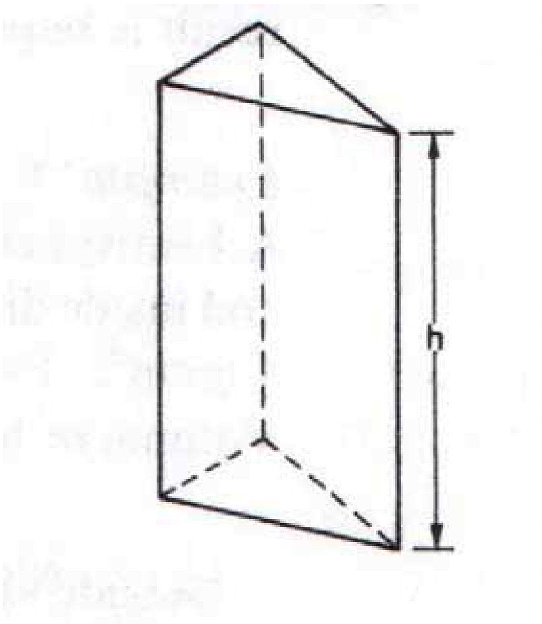






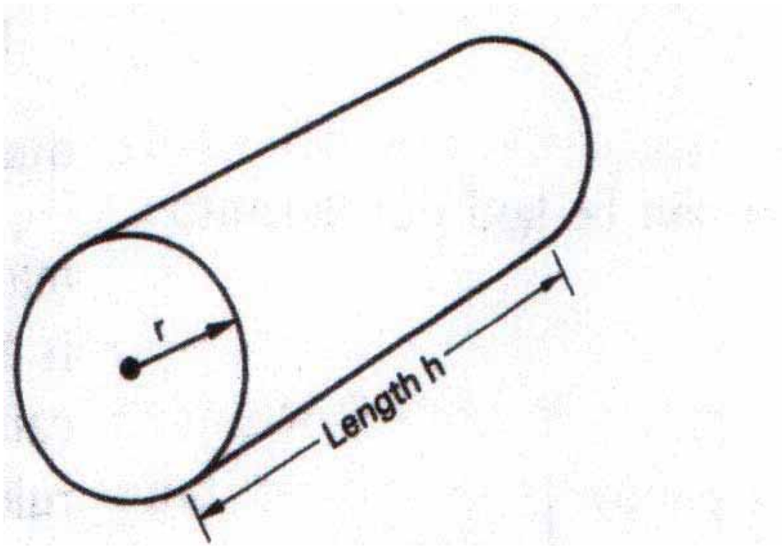
Here are more examples of prisms.

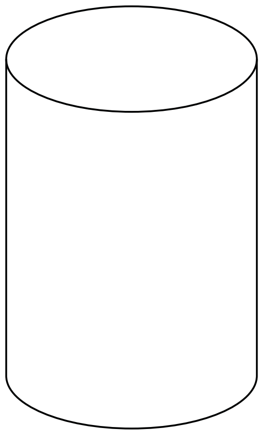
This is a rectangular – based prism.



This is a circular– based prism, usually called cylinder

A cylinder is a prism with a circular base and one curved surface





Which objects that you know, have the shape of a cylinder?

We have a coin, a can of soft drink, and a water pipe. There are many more!

The volume of a cylinder is the product of the base and the height.

Since the bases of a cylinder are circles, the base area, B, is $\pi \times r^2$

volume of a cylinder = base area $\times$ height

$$= \pi \times r^2 \times \text{height}$$

$$= \pi \times r^2 \times h$$

$$= \pi r^2 h$$

The volume of any prism is found by multiplying the base area by height.

Volume of Prism = base area $\times$ height

Example 2

A triangular based prism has base area $20cm^2$ and height 10cm.
What is its volume?

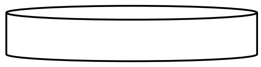
Solution

Volume of Prism = base area $\times$ height
= $20cm^2 \times 10cm$
= $200cm^3$

.

Example 3

Find the volumes of the cylinders below. Use $\pi = 3.14$



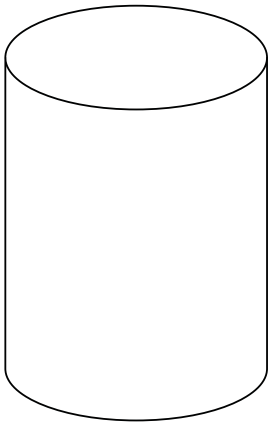
radius = 10 cm

height = 2cm

volume of a cylinder = $\pi r^2 h$

$= 3.14 \times 10^2 \times 2$

$= 628 cm^3$



diameter = 14cm

height = 12cm

diameter = 2 radius

So the radius in this case = 7cm

volume of a cylinder = πr^2h

$= 3.14 \times 7^2 \times 12$

$= 1\ 846.32cm^3$

Activity 3

1. Find the volume of a cuboid 10cm by 5 cm by 12 cm high.

2. A matchbox is 20mm high, 45mm long and 30mm wide. What is its volume?

3. A cylindrical can has base radius 5cm and height 20cm. Taking $\pi = 3.14$ calculate the volume of the can.

4. There is a regulation that there should be at least $18m^3$ of space per person in hospital. What is the greatest number of patients that can be accommodated in a ward which is 20m long, 8 m wide and 3 m high?

5. A certain type of paper is 0.08mm thick. How high is a pile of 150 sheets of this paper? If the paper measures 25cm by 20 cm, what is the volume of the pile?

Check your performance against the given solutions at the end of this subunit. Continue if you are satisfied with your ability to answer the questions. If not, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on calculating the volume of a cuboid, prism and cylinder are:

- a cuboid and a cylinder are some of prisms
- a prism is a solid shape that has the area of cross section when the solid is cut parallel to the base, the same as that of the base
- the volume of any prism is found by multiplying the base area by height.

Answers:

1. volume = $10\text{cm} \times 5\text{cm} \times 12\text{cm} = 600\ \text{cm}^3$

2. volume = $20\text{mm} \times 45\text{mm} \times 30\text{mm} = 27\ 000\ \text{mm}^3 = 27\ \text{cm}^3$

3.
Volume = $\pi r^2 h$

$= 3.14 \times 5^2 \times 20$

$= 1550\ \text{cm}^3$

4.

Number of patients = $\frac{20 \times 8 \times 3}{18}$

$= 26\frac{2}{3}$

So the greatest number of patients that can be accommodated in a ward is 26

5.

Height of the pile = 150 $\times$ 0.08

= 12 mm
= 1.2 cm

Volume = 25cm $\times$ 20 cm $\times$ 1.2cm = 600 cm^3

Lesson 4 The Surface Area of a Cuboid and a Cylinder

Introduction

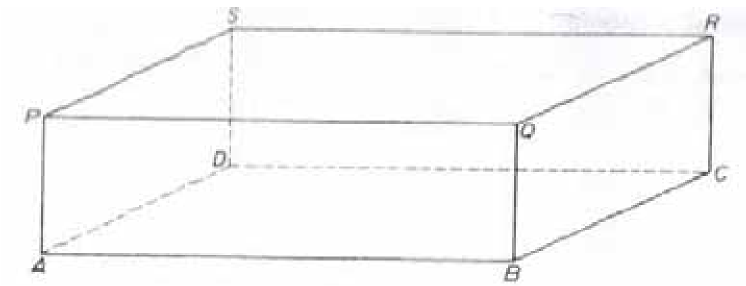
By the end of this subunit, you should be able to

- calculate the surface area of a cuboid and a cylinder.

This subunit is about 4 pages in length.

Surface area of a Cuboid

This is a cuboid ABCDPQRS, with length 8 cm, width 6 cm and height 3 cm.



The total surface area of a cuboid is the total area of all its faces.
A cuboid has six faces. These are all rectangles.

They are ABCD, PQRS, ADSP, BCRQ, ABQP and CDSR

The opposite faces are identical in every respect, i.e they are congruent
ABCD and PQRS are congruent,
ADSP and BCRQ are congruent,
ABQP and CDSR are congruent

So
For ABCD and PQRS
8 cm x 6 cm = 48cm x 2 faces = 96 cm^2

For ADSP and BCRQ
8 cm x 3 cm = 24 cm x 2 faces = 48 cm^2

For ABQP and CDSR
6 cm x 3 cm = 18cm x 2 faces = 36 cm^2

Total surface area = 96 cm^2 + 48 cm^2 + 36 cm^2 = 180cm^2

Another cuboid A'B'C'D'P'Q'R'S' has length 16 cm, width 12 cm and height 6 cm.

Its surface area is

16 cm x 12 cm = 192cm x 2 faces = 384 cm^2

16 cm x 6 cm = 96 cm x 2 faces = 192 cm^2

12 cm x 6 cm = 72cm x 2 faces = 144 cm^2

Total surface area = 384 cm^2 + 192 cm^2 + 144 cm^2 = 720cm^2

The corresponding faces of the two cuboids are similar.

What is the ratio of surface area of the two cuboids?

____

What is the ratio of squares of the lengths of corresponding edges?

____

the ratio of surface area of the two cuboids is 180 : 720 which simplifies to 1 : 4

The ratio of the corresponding edges is

8 : 16 = 1 : 2
6 : 12 = 1 : 2
3 : 6 = 1 : 2

The ratio of squares of the lengths of corresponding edges is the ratio of surface areas of the two cuboids

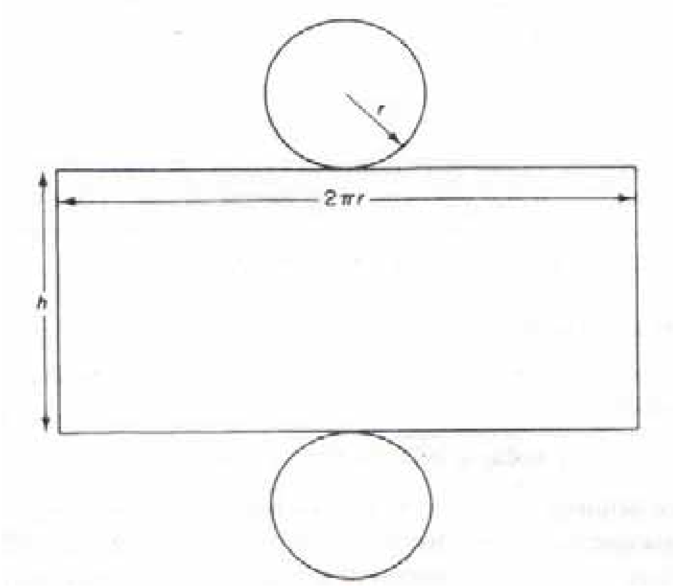
$1^2 : 2^2 = 1 : 4$

Surface Area of a Cylinder

A cylinder has two flat surfaces, the top and the bottom, which are circles, and a curved surface.

Imagine the curved surface of the cylinder opened out and flattened out.

It forms a rectangle whose length is the circumference of the flat surface, and whose width is the height of the cylinder.



The surface area of the curved part = area of the rectangle = $2\pi rh$

The area of the top and the bottom = $2(\pi r^2)$

$= 2\pi r^2$

Total surface area = $2\pi rh + = 2\pi r^2$

$= 2\pi r(h + r)$

Example 1

What is the surface area of a cylinder oh height 10cm and base radius of 7cm?(π = 3.14)

Total surface area $= 2\pi r(h + r)$

$= 2 \times 3.14 \times 7(10 + 7)$

$= 14 \times 3.14(17)$

$= 747.32\ cm^2$

Activity 4

1.

What is the total surface area of a cuboid with length 10 cm, width 8 cm and height 4 cm.

2

Find the surface area of a cylinder of radius 10 cm and height 10 cm. (Take π = 3.14)

3.

Find the surface area of a cylinder of height 15 cm and radius 12cm. (Take π = 3.14)

4.

An enclosed cylindrical petrol tank is to be painted before underground installation. Its dimensions are diameter = 2.3m, length = 3.5m. Use Take π = 3.14 to find the total surface area to be painted.

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on calculating the surface area of a cuboid and cylinder are:

- The total surface area of a cuboid is the total area of all its faces.- Total surface area of a cylinder $= 2\pi r(h + r)$

Answers

1.

10 cm x 8 cm = 80 cm x 2 faces = 160 cm^2

8 cm x 4 cm = 32 cm x 2 faces = 64 cm^2

10 cm x 4 cm = 40cm x 2 faces = 80 cm^2

Total surface area = 160 cm^2 + 64 cm^2 + 80cm^2 = 304cm^2

2.Total surface area $= 2\pi r(h + r)$

$= 2 \times 3.14 \times 10\,(10 + 10)$

$= 62.8\,(20)$

$= 1256\ cm^2$

3.Total surface area $= 2\pi r(h + r)$

$= 2 \times 3.14 \times 15\,(15 + 12)$

$= 94.2\,(27)$

$= 2543.4\ cm^2$

4. Total surface area $= 2\pi r(h + r)$

$= 2\times 3.14 \times 1.15\ (3.5 + 1.15)$

$= 7.222\ (4.65)$

$= 33.5823\text{cm}^2$

$\approx 34\text{cm}^2$

You have now completed work on this subunit on the surface area of a cuboid and a cylinder.. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Lesson 5 Arcs Lenghts and Sector Areas

Introduction

By the end of this subunit, you should be able to:

- solve problems involving the arc length and sector area as fractions of the circumference and area of a circle

In the first sub unit, we were reminded of the formulae used to calculate the perimeter of a circle, which is called circumference, and the area. These are given below:

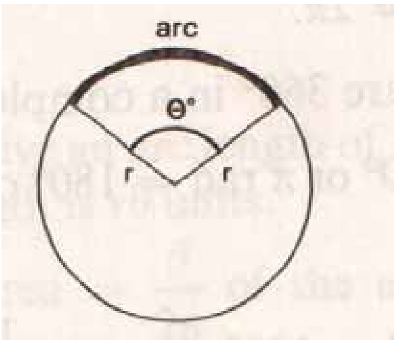
Circumference = $2\pi r$
= πd

Area = $\pi \times r^2$

There are times when we do not work with complete circles. We say we are working with sectors of a circle.

The length of an arc

A **sector** is the region between two radii and the curved part of the circumference. The curved part of the circumference is called an **arc**.



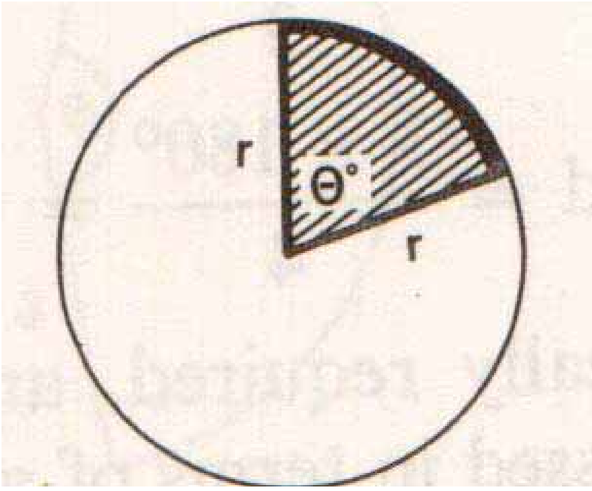
The arc subtends an angle of $\theta°$
Also the length of the arc is a fraction of the length of the circumference.

Therefore the length of an arc subtending an angle of $\theta°$ will be fraction $\frac{\theta}{360}$ of the circumference

$$= \frac{\theta}{360} \times 2\pi r$$

The area of the sector

If the angle of the sector is $\theta°$, the area of the sector is a fraction of the area of the circle. This is the shaded area below.



The area of the circle = $\pi \times r^2$ corresponds to an angle of 360°

Therefore the area of the sector of angle $\theta°$, will be the fraction $\frac{\theta}{360}$ of the area of the circle

$= \frac{\theta}{360} \times \pi \times r^2$

Example 1

A sector has radius 6 cm and an angle 70°
Find the length of the arc and the area of the sector. Use π = 3.14

Solution

Length of arc $= \frac{\theta}{360} \times 2\pi r$

$= \frac{70}{360} \times 2 \times 3.14 \times 6$

$= 7.33\text{cm}$

Area of sector $= \frac{\theta}{360} \times \pi \times r^2$

$= \frac{70}{360} \times 3.14 \times 6^2$

$= 21.9\text{cm}^2$

Example 2

A piece of wire is bent into a complete semicircle of radius 7 m.

What is the total length of the wire?
What area is enclosed?
Use $\pi = 3.14$

Solution

Total length $= 2 \times$ radius + curved arc length

$= 2 \times 7 + \frac{180}{360} \times 2\pi r$

$= 14 + \frac{1}{2} \times 2 \times 3.14 \times 7$

$= 14 + 21.98$

$= 35.98\text{cm}$

Area of sector $= \frac{\theta}{360} \times \pi \times r^2$

$= \frac{180}{360} \times 3.14 \times 7^2$

$= \frac{1}{2} \times 3.14 \times 49$

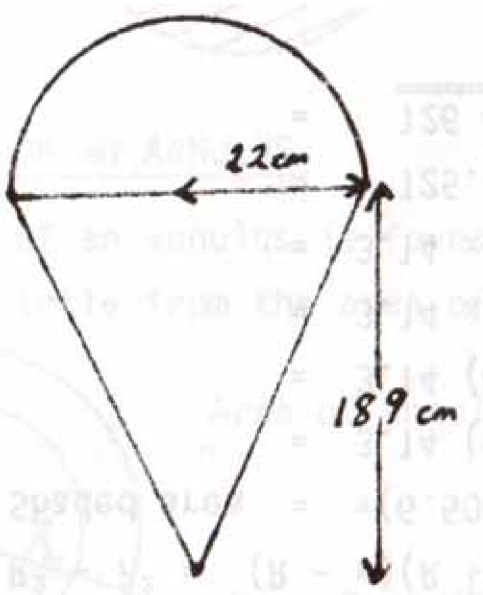
$= 76.93\text{cm}^2$

Activity 5

1. A sector has radius 2 cm and angle 36°. Find the length of its arc and its area.

2. If the length of an arc in a circle of radius 7 cm is 3.6 cm, find the angle subtended. (Take π = 3.14)

3. Calculate the area of the following figure (Take π = 3.14)



Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on calculating the surface area of a cuboid and cylinder are:

- The length of an arc subtending an angle of $\theta° = \frac{\theta}{360} \times 2\pi r$

- The area of the sector of angle $\theta° = \frac{\theta}{360} \times \pi \times r^2$

Answers

1.

$$\text{Length of arc} = \frac{\theta}{360} \times 2\pi r$$

$$= \frac{36}{360} \times 2 \times 3.14 \times 2$$
$$= 1.256 \text{ cm}$$

2.

$$\text{Length of arc} = \frac{\theta}{360} \times 2\pi r$$

$$3.6 = \frac{\theta}{360} \times 2 \times 3.14 \times 7$$
$$\theta = 29.5°$$

3.

Area = area of triangle + area of semicircle

$$= ½ \times 44 \times 189 + \frac{180}{360} \times 3.14 \times 22^2$$

$= ½ \times 44 \times 189 + \frac{1}{2} \times 3.14 \times 22^2$
$= 4158 + 759.88$
$= 5677.76 \text{ cm}^2$

Lesson 6 The Surface Area and Volume of a Sphere, Pyramid and Cone

Introduction

By the end of this subunit, you should be able to:

- solve problems involving the surface area and volume of a sphere, pyramid and cone

This subunit is about 4 pages in length.

Surface area and volume of a sphere

A sphere is a set of all points in 3 – dimensions at a fixed distance from a fixed point.

A good example of a sphere is a soccer ball.

Imagine a sphere enclosed by cylinder with both ends open.
The radius of the sphere is *r*.
Let the radius of the cylinder be *r*.
Then the length of the cylinder will be the diameter of the sphere, which is 2*r*.

If the cylinder is cut out and opened out, it will form a rectangle, that has length equal to the circumference of the circle = $2\pi r$
The height of the cylinder = 2r

Then the area of the sphere = area of the cylinder = length $\times$ height

$= 2\pi r \times 2r$

$= 4\pi r^2$

This practical is used to find the volume of a sphere

Have a cylinder and a marble.

Make sure the marble just fits into the cylinder.

Fill the cylinder with water.

Empty the cylinder and drop the marble into the cylinder.

Add water to this cylinder that now has the marble.

Pour out the water and measure it.

The difference in the two volumes is the volume of the sphere which is two – thirds of the volume of the cylinder.

The volume of the cylinder = $\pi r^2 h$

$= \pi r^2 (2r)$

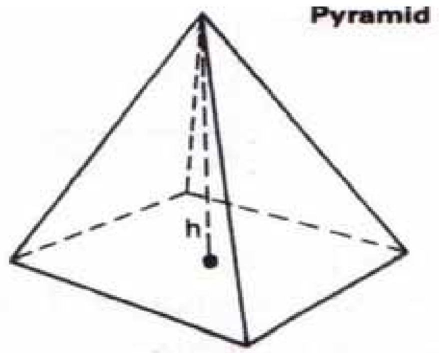
$= 2\pi r^3$

Volume of sphere $= \frac{2}{3}[2\pi r^3]$

$= \frac{4}{3}\pi r^3$

Surface area and volume of a pyramid

Pyramids are some solids with area of cross section that changes. They come to a point at the top.



Pyramids have flat bases, which is a polygon, and it is usually a square. The faces are flat triangles.

The surface area is made up of the area of the base and the areas of the triangles.

In cases where the base is a triangle, the surface area will be 4 times the area of one of the triangular faces.

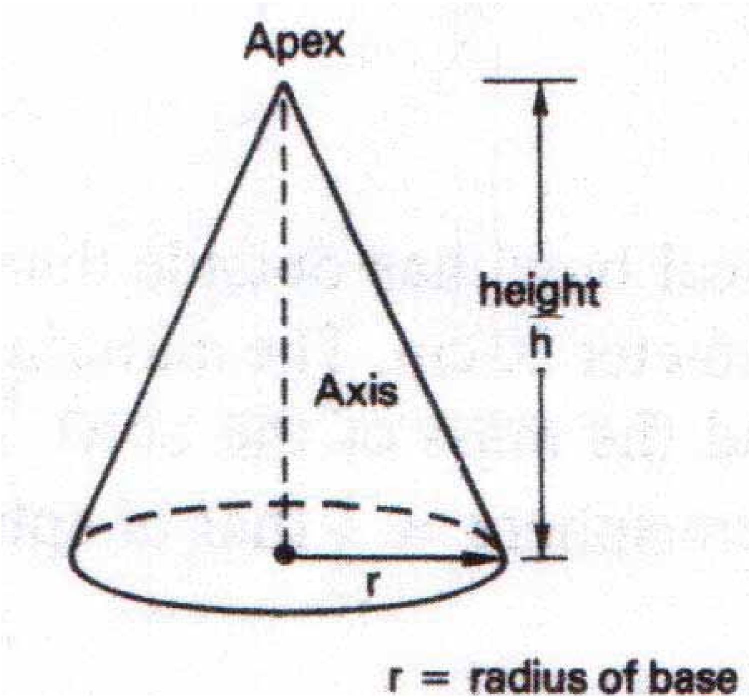
The volume of any pyramid is related to the area, B, of its base and its height, h by the formula

$= \frac{1}{3} \times B \times h$

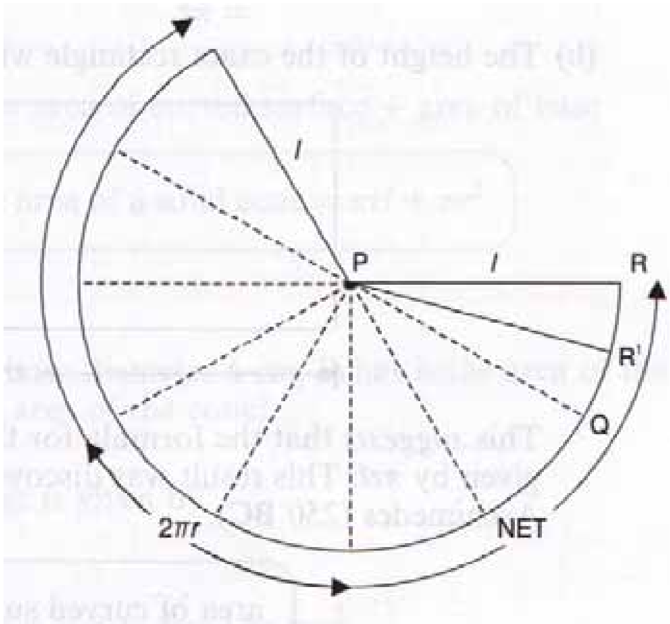
This is for all pyramids, regular or irregular

Surface area and volume of a cone

This is an example of a cone. A cone has a circular base and a curved surface which tapers to a point called the apex.



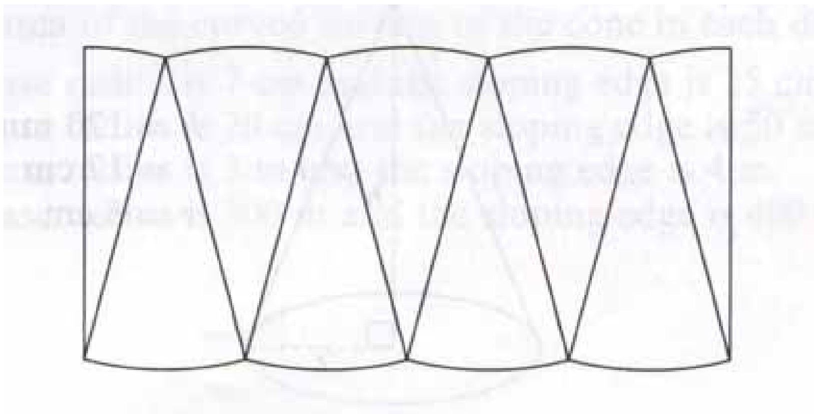
If a cone is opened up, it results in the following shape.



It has been divided into eight equal subsectors. (You could make the subsectors 16, 80!)

Cut out the sectors and glue them to give this shape
Cut the 8th subsector into two equal parts.

The resulting shape is very roughly like a rectangle.



It is also strongly recommended that you also do this practical.

This rectangle has length which is half of the arc length of the net. This is because we arranged half of the sectors along one of the lengths and the other half of the sectors along the other length.

The circumference of the cone is = $2\pi r$

This gives the length of the rectangle as ½ of $2\pi r = \pi r$

The height of the rectangle is *l,* the slant height of the cone.
So the area of the rectangle = $\pi r l$

The total surface area of the cone
= area of the rectangle + area of the base
= $\pi r l + \pi r^2$
= $\pi r (l + r)$

A cone is a pyramid. Therefore the volume of a cone is

= $\frac{1}{3} \times B \times h$

= $\frac{1}{3} \times \pi r^2 \times h$

= $\frac{1}{3} \pi r^2 h$

Example 1

For each of the solids given calculate the surface area and the volume

(a) sphere of radius 3.18 cm (a tennis ball) (π = 3.14)

Surface area of a sphere = $4\pi r^2$

$= 4\pi\, 3.18^2$

$= 127.011744 \text{ cm}^2$

$\approx 127 \text{ cm}^2$

Volume = $\frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi\, 3.18^3$

$= \frac{4}{3}\pi\, 3.18^3$

$= \frac{4}{3}\pi\, 32.157432$

$= 134 cm^3$

(b) a triangular based pyramid of height 10cm if the base is a right – angled triangle with sides 3cm and 4 cm enclosing the right angle?

The surface area is made up of the area of the base and the areas of the triangles.

Area of the base = 4 × area of one triangle

= 4 × ½ × base × height

= 4 × ½ × 4 × 10

= 4 × 20

$= 80\ cm^3$

Volume

= ⅓ × B × h

= ⅓ × ½ × b × l × h

= ⅓ × ½ × 3 × 4 × 10

$= 20cm^3$

(c)

A cone with slant height of 9 cm and the radius of its circular base is 5 cmThe total surface area of the cone $= \pi r (l + r)$

$= \pi 5(9 +5)$

$= 3.14 \times 5(9 +5)$

$= 219.8\ cm^2$

Volume = ⅓ $\pi r^2 h$

$= \frac{1}{3} \times 3.14 \times 5^2 \times 9$
$= 235.5 \text{ cm}^2$

Activity 4

For all questions, take $\pi = 3.14$

1. Assuming the earth to be a sphere of radius 6 000 km, find its circumference and surface area.

2. A spherical lampshade made of glass has a diameter of 18cm. It has a circular hole in the top of it to allow the bulb and the wire to pass through. The diameter of the hole is 7 cm. What is the estimate area of the lampshade?

3. A toy consists of a cone on top of a cylinder. Each has a base radius of 3.5 cm. The height of the cone is 6 cm and that of the cylinder is 5 cm. What is the volume of the toy?

4. Calculate the volume and surface area of a square – based pyramid has length 6 cm, perpendicular height 5cm and a slanting height of 6cm.

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on solving problems involving the surface area and volume of a sphere, pyramid and cone are:

- The surface area of a sphere = $4\pi r^2$
- Volume of sphere $= \frac{4}{3}\pi r^3$
- The surface area of a pyramid is made up of the area of the base and the areas of the triangles.
- The volume of any pyramid, regular or irregular, = $\frac{1}{3} \times B \times h$
- The total surface area of the cone = $\pi r (l + r)$
- Volume of a cone = $\frac{1}{3} \times B \times h$

Answers

1.

Surface area $= 4\pi r^2$

$= 4 \times 3.14 \times 6\ 000^2$

$= 4 \times 3.14 \times 6\ 000^2$
$= 4\ 32\ 000\ 000\ km^2$
$= 4\ .32 \times 10^8\ km^2$

2. area of glass = surface area of sphere – missing area

$= 4\pi r^2 - \pi r^2$

$= 4 \times 3.14 \times 9^2 - 3.14 \times 3.5^2$
$= 4 \times 3.14 \times 9^2 - 3.14 \times 3.5^2$

$= 1017.36 - 38.465$

$= 978.895$

$\approx 980\ cm^2$

3.

volume of toy = volume of cone + volume of cylinder

$= ⅓\ \pi r^2h + \pi r^2h$

$= ⅓ \times 3.14 \times 3.5^2 \times 6 + 3.14 \times 3.5^2 \times 5$

$= 76.93 + 192.325$

$= 269.255 cm^3$

4.

Volume $= ⅓ \times$ base area $\times$ vertical height

$= ⅓ \times 6 \times 6 \times 5$

$= ⅓ \times 6 \times 6 \times 5$

$= 60\ cm^3$

Surface area = area of base + 4 $\times$ area of triangular faces

$= 6 \times 6 + 4 \times ½ \times 6 \times 6$

$= 36 + 72$

$= 108\ cm^2$

You have now completed work on this unit on the surface area and volume of a sphere, pyramid and cone. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Unit Summary

Summary

In this unit you learned that:

- perimeter is the length of the boundary of the shape
- area is the amount of two dimensional space of a shape
- the formula used to calculate the area of a rectangle = length $\times$ breadth
- the formulae used to calculate the area of a triangle

$$= \frac{1}{2} \times \text{base} \times \text{height},$$

when given the base and the height

OR

$$= \frac{1}{2} \times ab \sin C$$

$$= \frac{1}{2} \times ac \sin B$$

$$= \frac{1}{2} \times bc \sin A$$

when given two sides and the enclosed angle

- Circumference $= 2\pi r = \pi d$
  - the formula used to calculate the area of a circle = $\pi \pi r^2$
  - Area of the parallelogram $= b \times h$
  - Area of trapezium = ½ $h(b_1 + b_2)$
  - the volume of any prism is found by multiplying the base area by height.
  - The total surface area of a cuboid is the total area of all its faces.
  - Total surface area of a cylinder $= 2\pi r (h + r)$
  - The surface area of a sphere = $4\pi r^2$
  - Volume of sphere $= \frac{4}{3}\pi r^3$

- The surface area of a pyramid is made up of the area of the base and the areas of the triangles.

- The volume of any pyramid, regular or irregular, = ⅓ × B × h
- The total surface area of the cone = $\pi r (l + r)$
- Volume of a cone = ⅓ × B × h
- The length of an arc subtending an angle of $\theta°$ $= \frac{\theta}{360} \times \pi d$
- The area of the sector of angle $\theta° = \frac{\theta}{360} \times \pi \times r^2$

You have completed the material for this unit on perimeters, areas and volumes. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

Answer All Questions. Show all the necessary working.

Total marks = 28 **Time**: 30 mins

A rectangular warehouse is 100 metres long and 60 metres wide. Calculate its

(a) perimeter [2]

(b) area [2]

Find the circumference of a circle with diameter 125 cm [2]

Find the area of a triangle with $a = 4.2$, $c = 7.1$, $B = 110°$ [3]

Find the perimeter of a quadrant of a circle with radius 6 cm

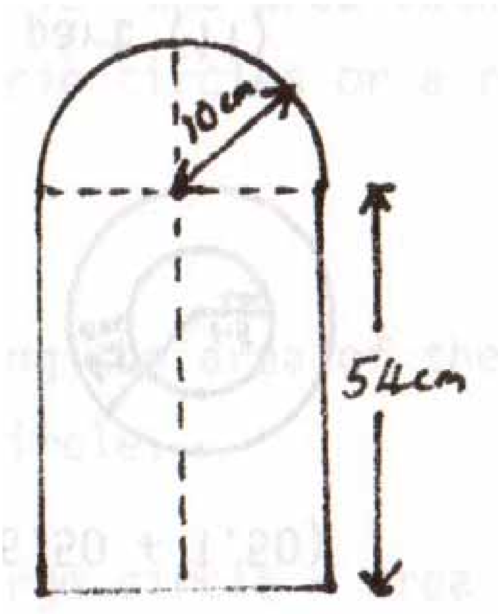
$a = 4.7$, $c = 8.2$, $C = 74°$ [3]

Cylindrical jar A of jam has height 16cm and radius 4cm.

Cylindrical jar B of jam has height 8 cm and radius 8cm.

Which jar contains more jam? How much more? [8]

Calculate the area of the following figure (Take π = 3.14) [5]



What is the surface area of a cylinder of height 25cm and base radius of

17cm?(π = 3.14) [3]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1.

(a) perimeter = 100m + 60m + 100m + 60 m = 320m

(b) area = 100m $\times$ 60m = 6000m^2

2.
Circumference = π d

= 3.14 $\times$ 125

= 392.5 cm

3.

$$\text{Area} = \frac{1}{2} \times \text{ac sin B}$$

$$= \frac{1}{2} \times 4.2 \times 7.1 \times \sin 110°$$

$$= 14.91 \times \sin 110°$$

4. the curved area = $\frac{\theta}{360} \times \pi d = \frac{90}{360} \times 3.14 \times 12$
= 9.42 cm

Two straight lines = 6 + 6
= 12cm

Total area = 9.42 + 12 = 21.42cm

5.

volume of a cylinder = $\pi r^2 h$

$= 3.14 \times 7^2 \times 12$

$= 1\ 846.32 cm^3$

volume of a cylinder = $\pi r^2 h$

$= 3.14 \times 4^2 \times 16$

$= 803.84 cm^3$

volume of a cylinder = $\pi r^2 h$

$= 3.14 \times 8^2 \times 8$

$= 1607.68 cm^3$

$1607.68 cm^3 - 803.84 cm^3 = 803.84\ cm^3$

Jar B has more, 803.84 cm^3 more

6.

Area = area of rectangle + area of semicircle

$= 20 \times 54 + \frac{180}{360} \times 3.14 \times 10^2$

$= 1080 + \frac{1}{2} \times 3.14 \times 10^2$

$= 1080 + 157$

$= 1237 cm^2$

7.

Total surface area $= 2\pi r\,(h + r)$

$= 2 \times 3.14 \times 17\,(25 + 17)$

$= 106.76\,(42)$

$= 4483.92\ cm^2$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Answer All Questions.

Show all the necessary working.

Total marks = 44 **Time**: 1 hour

1.

A piece of string 24 cm long can form the boundary of many different rectangles.

If the width of the rectangle is 1cm, then the length will be 11 cm. What is the perimeter and area?

Complete this table below if the width continues to change as indicated below.

Width (w)	2	3	4	5	6	7	8	9	10	11
Length (l)										
perimeter										
area										

[32]

2.

Find the area of a triangle with $a = 4.7$, $c = 8.2$, $C = 74°$ [3]

3.

What is the height of a cylinder that has surface area = 1570 cm^2 and base radius of 10cm?(π = 3.14) [3]

4.

What is the surface area and volume of a sphere that has radius 18?

(π = 3.14) [3]

5.

A cone of base radius 7cm and height 15 cm is held upside down and filled with water. How much water is in the cone? [3]

Answers

1.

perimeter = 24cm

Area = 11cm^2

Width (w)	2	3	4	5	6	7	8	9	10	11
Length (l)	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**
perimeter	**24**	**24**	**24**	**24**	**24**	**24**	**24**	**24**	**24**	**24**
area	**20**	**27**	**32**	**35**	**36**	**35**	**32**	**27**	**20**	**11**

2.

Area $= \frac{1}{2} \times ac \sin B$

$= \frac{1}{2} \times 4.7 \times 8.2 \times \sin 74°$

$= 19.27 \times \sin 74°$

3.

Height = 15cm

4.

Surface area of a sphere = $4\pi r^2$

$= 4 \times 3.14 \times 18^2$

$= 4069.44 \text{ cm}^2$

Volume of a sphere = $\frac{4}{3}\pi r^3$

$= \frac{4}{3} \times \pi \times 18^3$

$= 24\,416.64\text{cm}^3$

5.

Volume = $\frac{1}{3}\pi r^2 h$

$= \frac{1}{3} \times 3.14 \times 7^2 \times 15$

$= 769.3\text{cm}^2$

Unit Contents

Unit 19

Unit 19

Transformation

Introduction

To transform anything is to change it somehow. An example of a simple transformation could be removing an object from one point to the other. A book may be on the desk, it can be transformed so that is no longer on the desk but on the floor. So it has changed position. Or it could be transformed in that it is still on the desk but it has been rotated. In this topic we are going to learn different kinds of transformations.

This unit consists of 133 pages. It looks huge but the pages have been increased greatly by diagrams. This is approximately 6% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 30 hours to work on this unit, 20 hours for formal study and 10 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Five Lessons:

Lesson 1 Transformations
Lesson 2 The Stretch
Lesson 3 The Shear
Lesson 4 Matrices and Transformations
Lesson 5 Combined Transformation

Outcomes

- *transform* an object by translation
- *transform* an object by reflection
- *transform* an object by rotation
- *transform* an object by enlargement
- *transform* an object by shearing
- *transform* an object by stretching
- *identify and describe* the transformations of shear and stretch connecting given figures
- *find matrices representing transformations*
- *transform* shapes by combining two or three of the following transformations; reflection, rotation, translation, enlargement, shear and stretch.
- *identify and describe* a single transformation that maps the original object to the

final object in combined transformations

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Transformations

In secondary school you have learned the first four transformations which are: translations, reflections, rotations and enlargements. If you have not met any of these four transformations before or you need to remind yourself, please consult you junior mathematics text books.

At the end of this subunit you should be able to:

Identify by looking at the object and its image each of the following transformations:

- Translation
- Reflection
- Rotation
- Enlargement

Describe fully each of the above mentioned transformations.

Transform an object by enlargement, rotation, reflection and translation.

Activity

At this level we are just going to remind ourselves of these first four transformations and move to the other types of transformations.

Activity 1 is meant to help you remember what you did in secondary on transformations.

Activity 1

1.

What kind of transformation does each of the following diagrams show?

a)

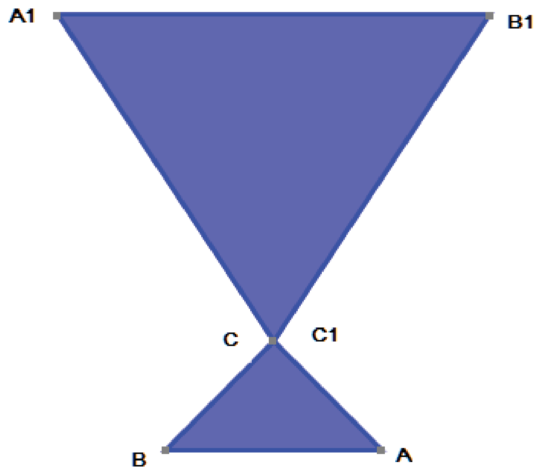


Figure 1

b)

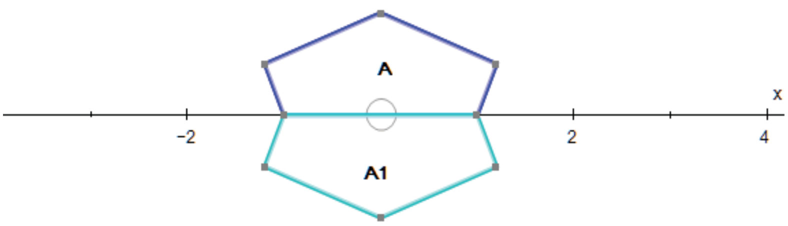


Figure 2

c)

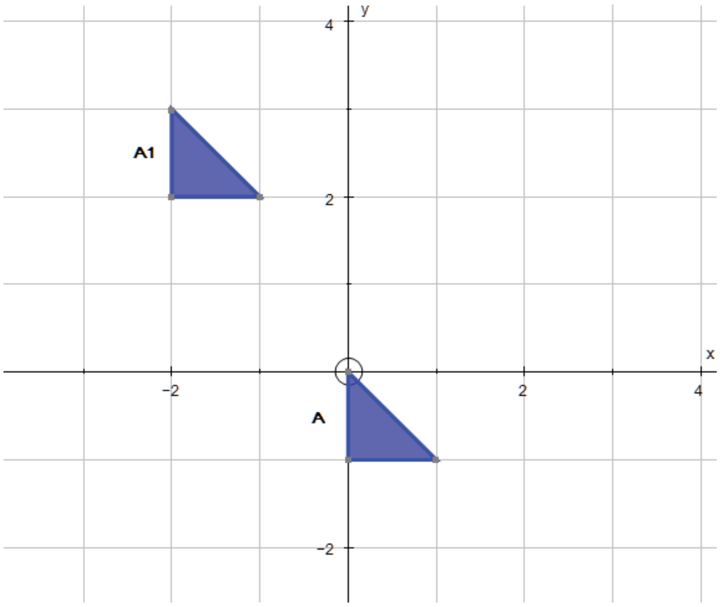


Figure 3

d)

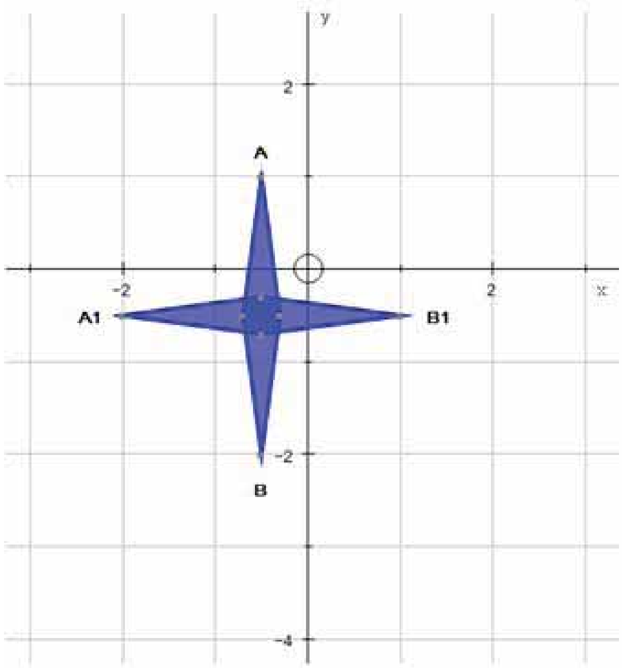


Figure 4

2. Identify and describe fully the transformations in each of the following diagrams that maps shape A onto A1.

a)

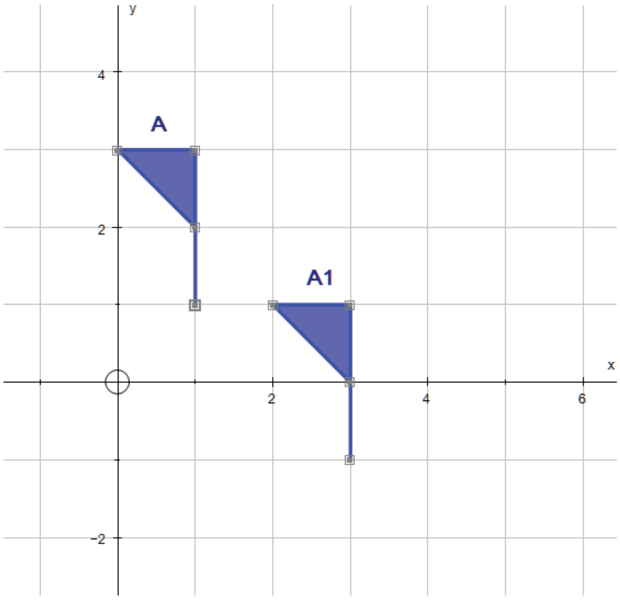


Figure 5

b)

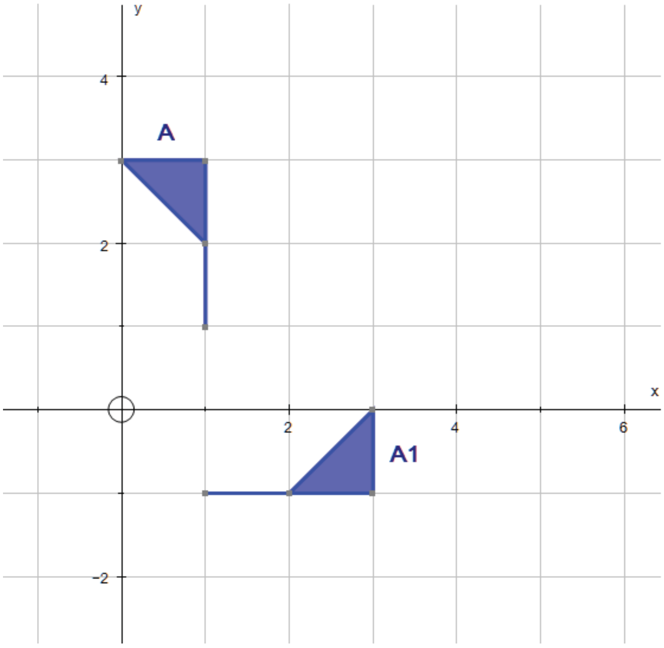


Figure 6

c)

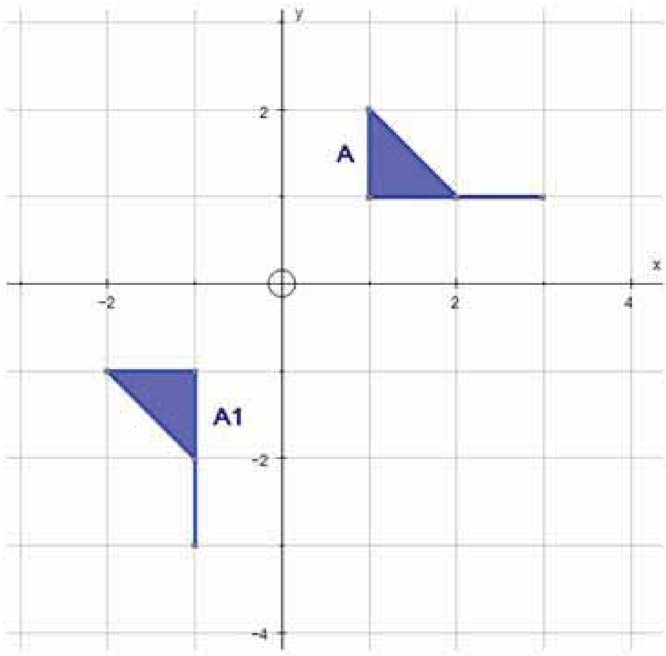


Figure 7

d)

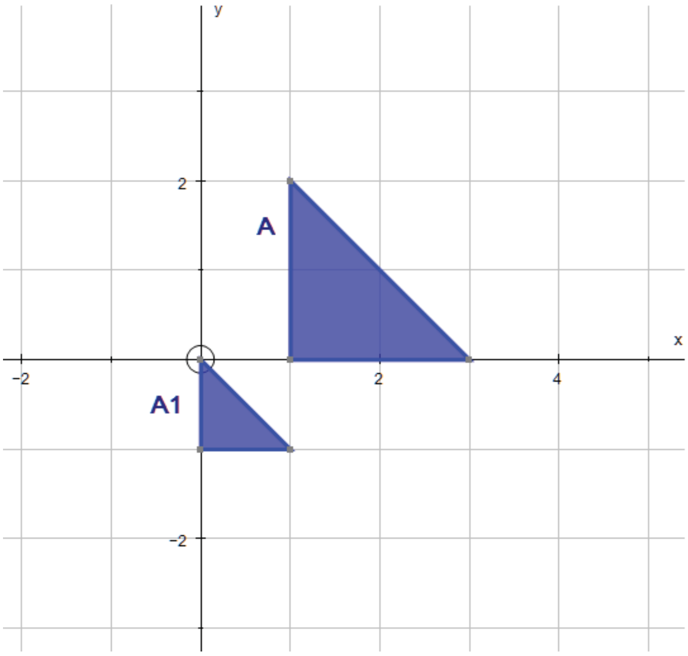


Figure 8

No. 3

a) The single Transformation P maps triangle A onto triangle B as shown below. Describe fully the transformation P.

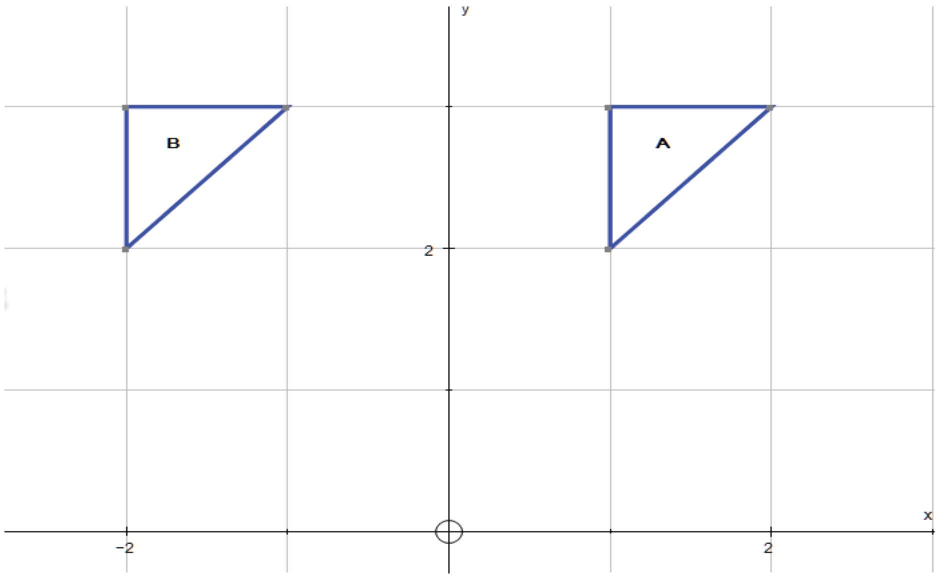


Figure 9

b) The single transformation Q maps triangle A onto triangle C. Describe fully the single transformation Q.

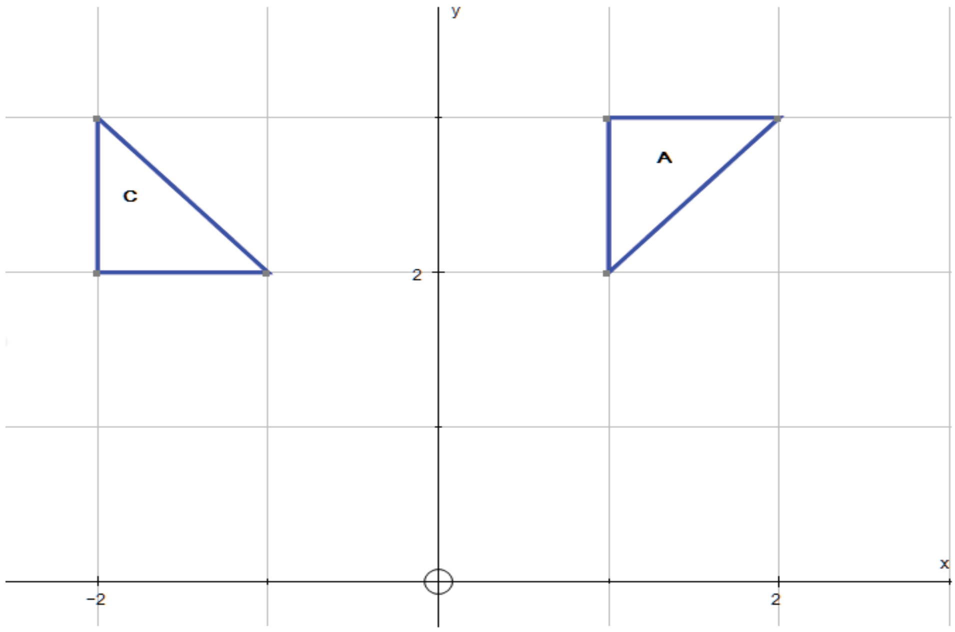


Figure 10

c) The single transformation R maps triangle A onto triangle D. Describe fully the single transformation R.

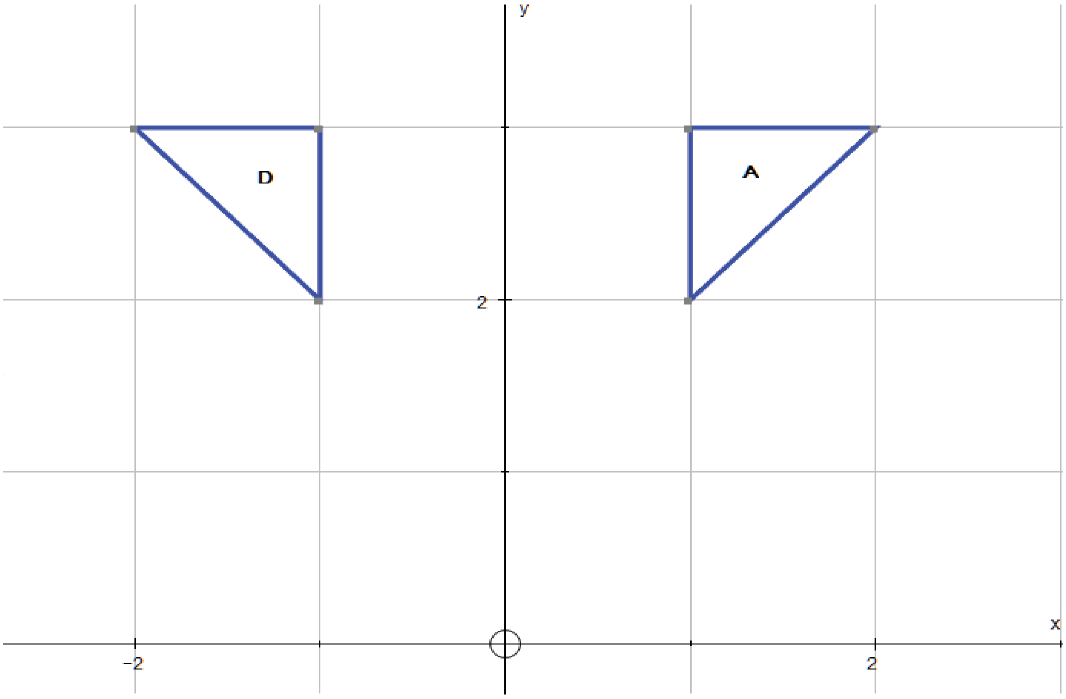


Figure 11

d) The single transformation S maps triangle X onto triangle Z. Describe fully the single transformation S.

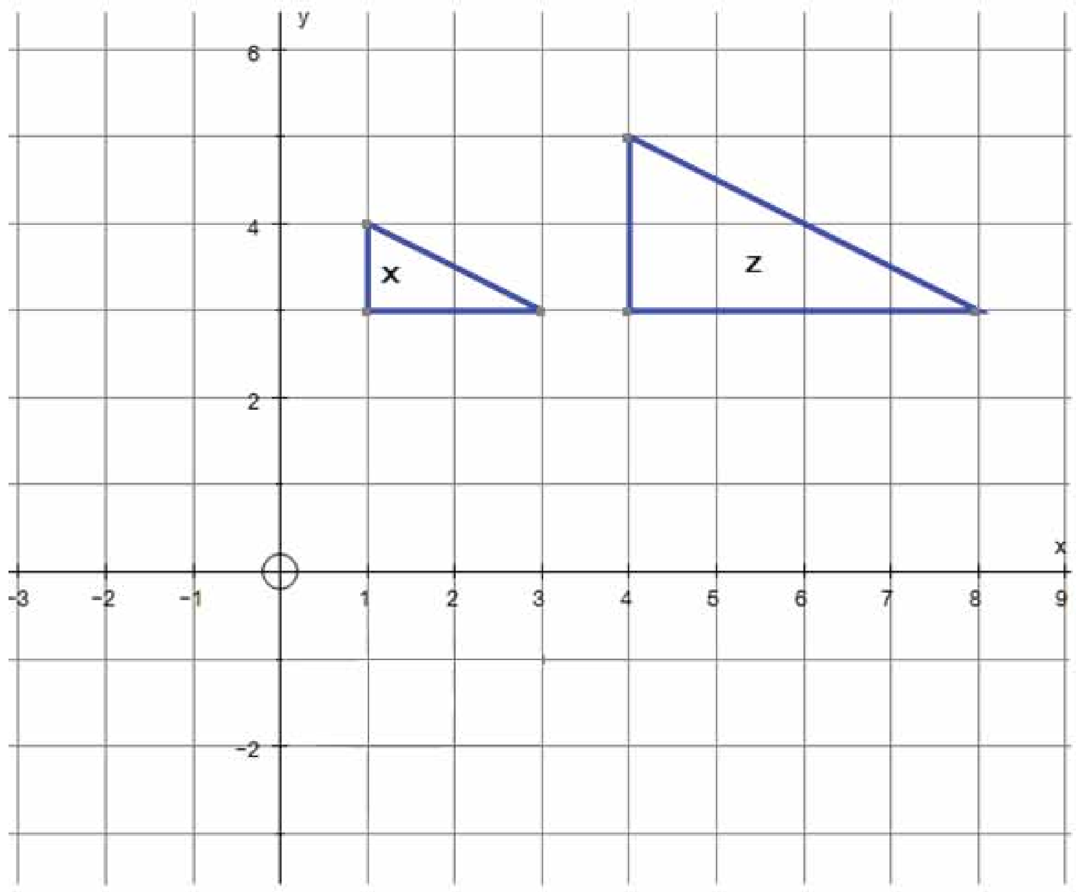
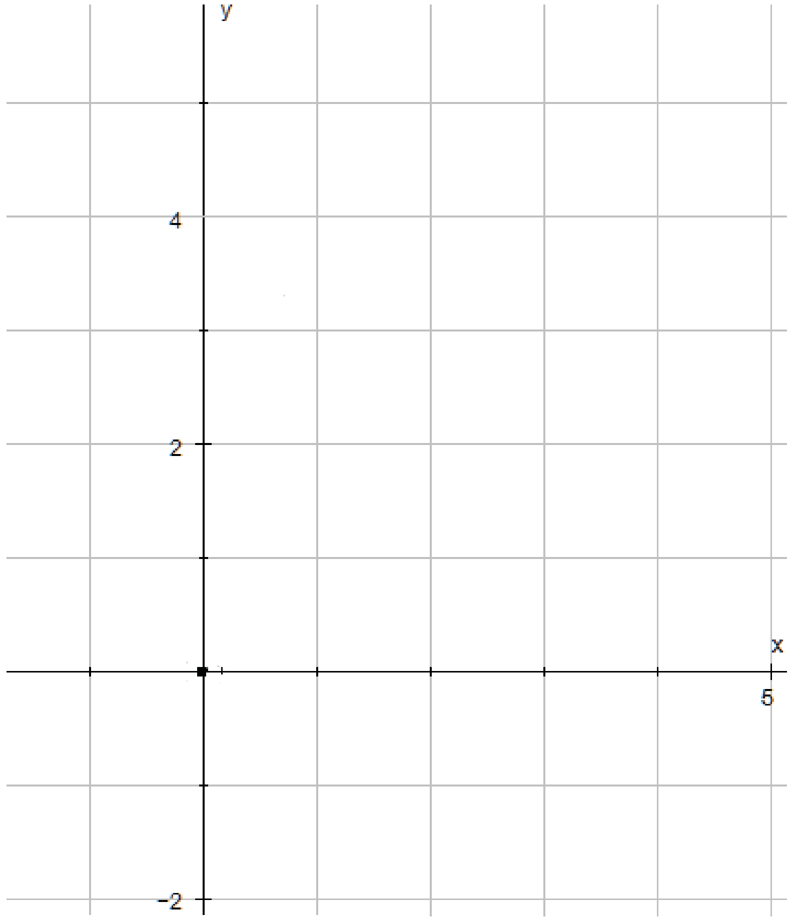


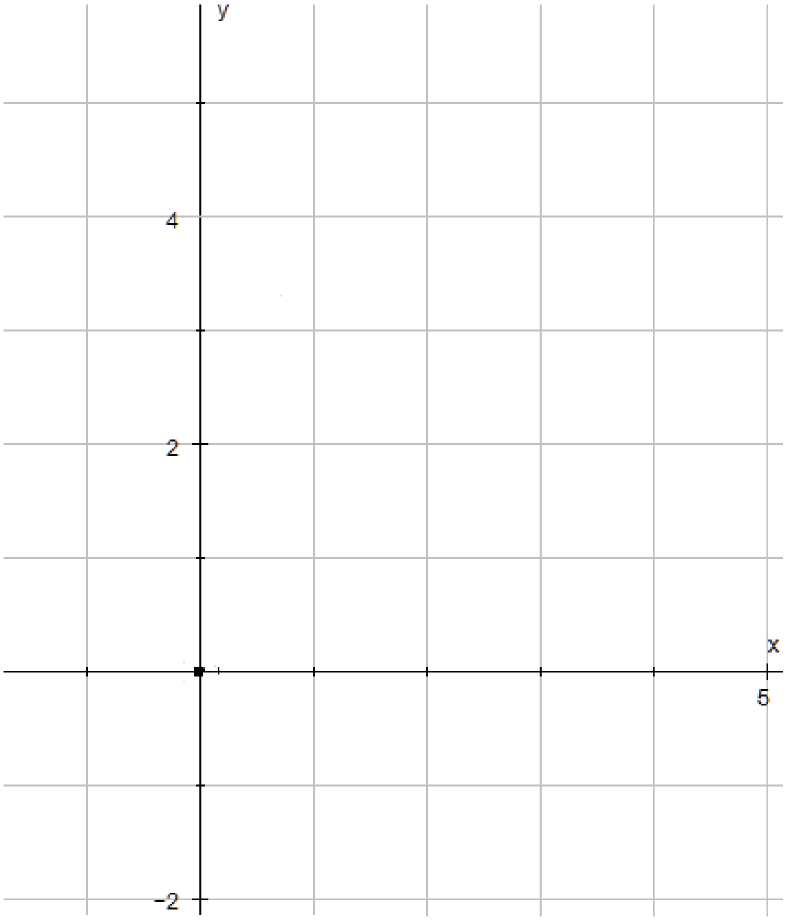
Figure 12

No.4

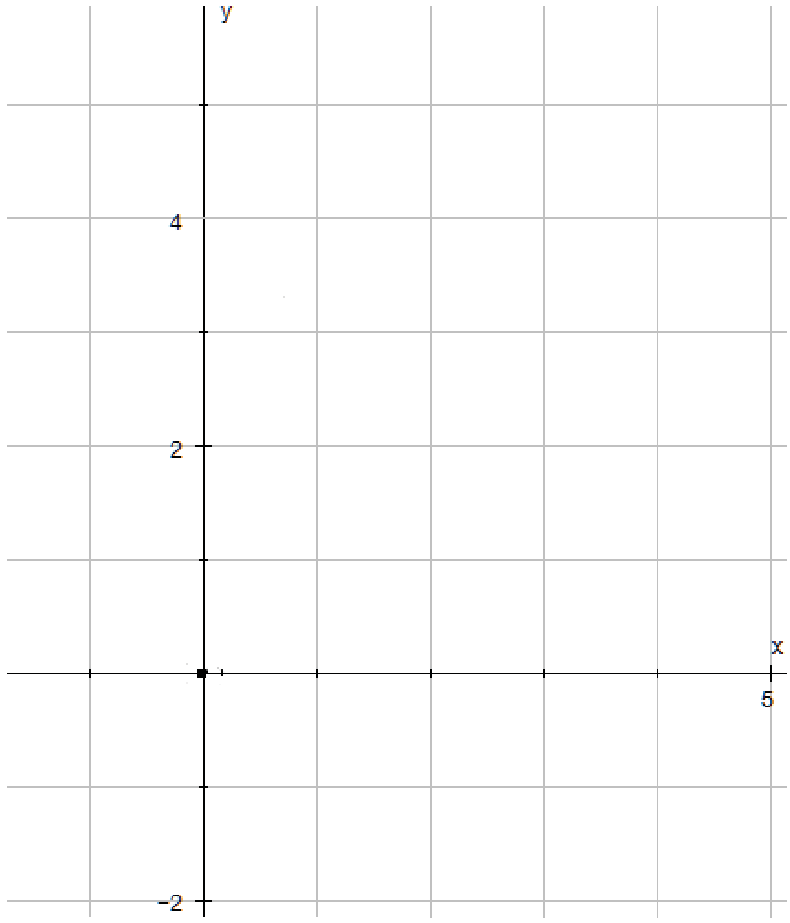
a) Shape A is a triangle such that its coordinates are (0, -2), (-4, 0), (-4, -2). Translate shape A to B with a translation vector $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$. Draw both shape A and its image B.



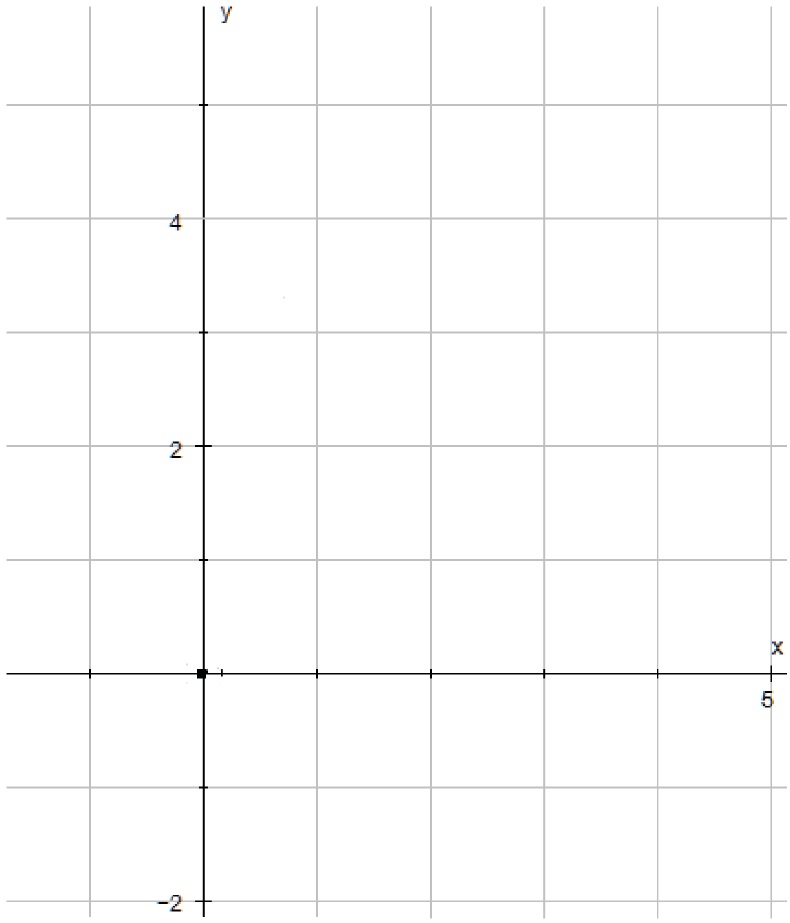
b) Shape C is a triangle with the following coordinates: (0, 1), (2, 0), (2, 1). It is transformed to shape D by a rotation of the centre of rotation (-1, 0) and angle -270^0. Draw both shape C and its image D.



c) Triangle ABC has the following coordinates: A (1, 1), B (3, 1), C (1, 2). It is reflected on the line y=-1. Draw both triangle ABC and its image DEF.



d) Triangle B has the following coordinates: (-8, 8) (-2, 5) (-2, 8). It has been transformed to shape C by an enlargement, scale factor $\frac{1}{3}$ at the centre of enlargement (-8,-1). Draw both shape B and its image C.



Compare your answers with those given at the end of the subunit. I hope you were able to remember all the important features needed in each of the transformations we have just done. If you think you need more you can consult with you junior certificate mathematics text book which discusses all the four transformations in details.

In this Subunit You Have Learned That:

-In a translation the object and the image are the shape and size and are in the same sense. The only difference is the position.

-A translation is defined by a translation vector.

-In a rotation the object and the image have the same shape and size but all the points move so that they are the same distance from a fixed point, the centre of rotation. All the points are turned in arcs with the same angle, the angle of rotation.

-A rotation is defined by the centre of rotation, the angle and the direction (clockwise or anticlockwise).

-In a reflection the object and the image have the same shape and size but in the opposite sense. The two are same distance from the from the mirror line but in opposite directions.

-A reflection is defined by the mirror line.

-In an enlargement the size of the object changes but the shape is still the same.

-The enlargement is defined by the centre of enlargement and the scale factor.

Solutions To activity 1:

1.

a) It is an enlargement.

b) It is a reflection.

c) It is reflection.

d) It is a rotation.

2.

a) It is a translation because the shapes and sizes of the object and the image are the same and have the same sense. The only difference is the position.

The translation vector is found by:

$$\begin{pmatrix} \text{direction and the number of steps moved by each point on the } x-\text{axis} \\ \text{direction and the number of steps moved by each point on the } y-\text{axis} \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

b) It is a rotation because the object and the image have the same shape and size but all the points moved so that they are the same distance from a fixed point, the centre of rotation. All the points turned in arcs with the same angle of rotation.

To find the centre of rotation:

i) Mark a point and its image and draw the perpendicular bisector of the line joining them.

ii) Mark another point and its image and draw the perpendicular bisector of the line joining them.

iii) The point of intersection of the two perpendicular bisectors is the centre of the rotation.

So the centre of rotation is the point (0, 0).

To find the angle of rotation:

i) Mark a point and its image.

ii) Draw a line that joins the point on the object to the centre of rotation.

iii) Draw another line that joins the point on the image to the centre of rotation.

iv)Then the angle between the two lines or its reflex is the angle of rotation. The direction is either clockwise on anticlockwise.

So the angle of rotation in this case is 90^0 clockwise or -90^0.

c) It is a reflection because the object and the image have the same shape and size but in the opposite sense. The two are same distance from the mirror line but in opposite directions.

The mirror line is the line y=-x. The mirror line is found by finding the line equidistant from the object and the image.

d) It is an enlargement because the size of object has changed but the shape is still the same.

The enlargement is also defined by the centre and the scale factor.

We find the scale factor by dividing one side of the image with the corresponding side of the object, or dividing the area of the image with that of the object.

In this case the scale factor $=\frac{4}{2}=2$.

The centre of enlargement is found by joining corresponding points of the object and the image with straight lines. At the point where all these lines meet is the centre of enlargement.

In our case the centre of enlargement is the point (-1,-2).

3. To find how each of the transformations below is found refer to the answers of question 2 above.

a) P is a translation. Translation vector $\begin{pmatrix}3\\0\end{pmatrix}$.

b) Q is a rotation. Centre of rotation is (0, 0). Angle of rotation is 45 anticlockwise.

c) R is a reflection with the mirror line x=0 or the y-axis.

d) S is an enlargement, centre of enlargement (-2,3) and scale factor = $\frac{4}{2}=2$.

4.

a)

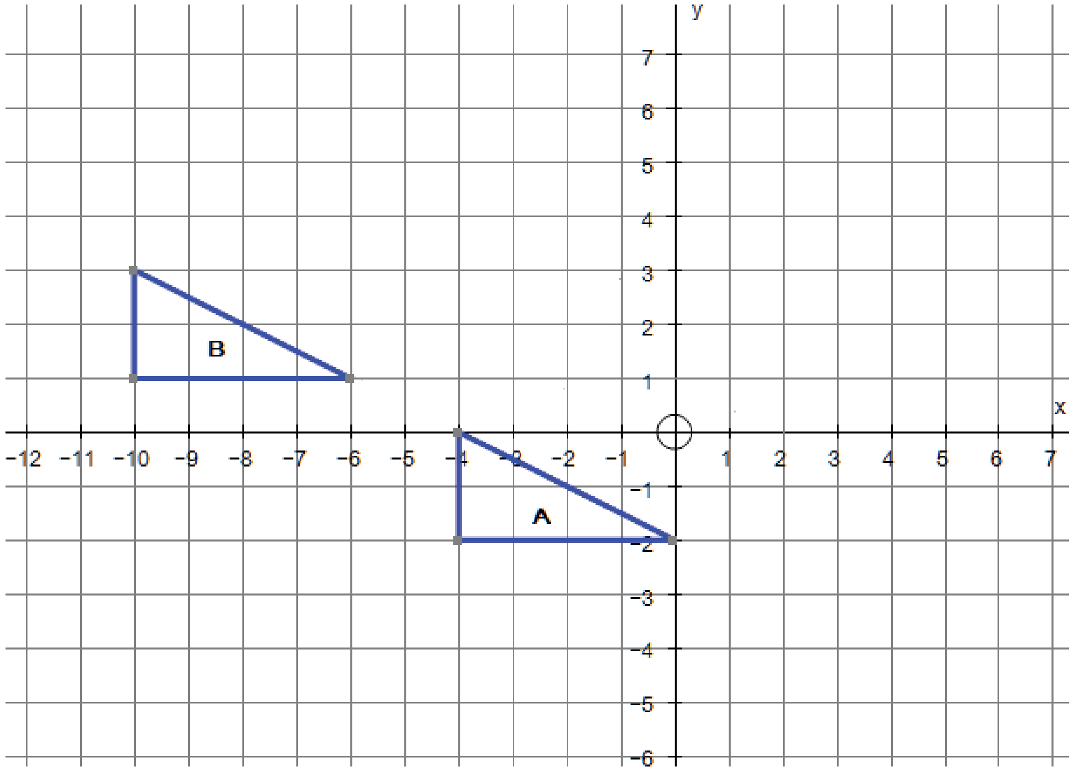


Figure 13

b) -270^0 is the same as 90^0 anticlockwise.

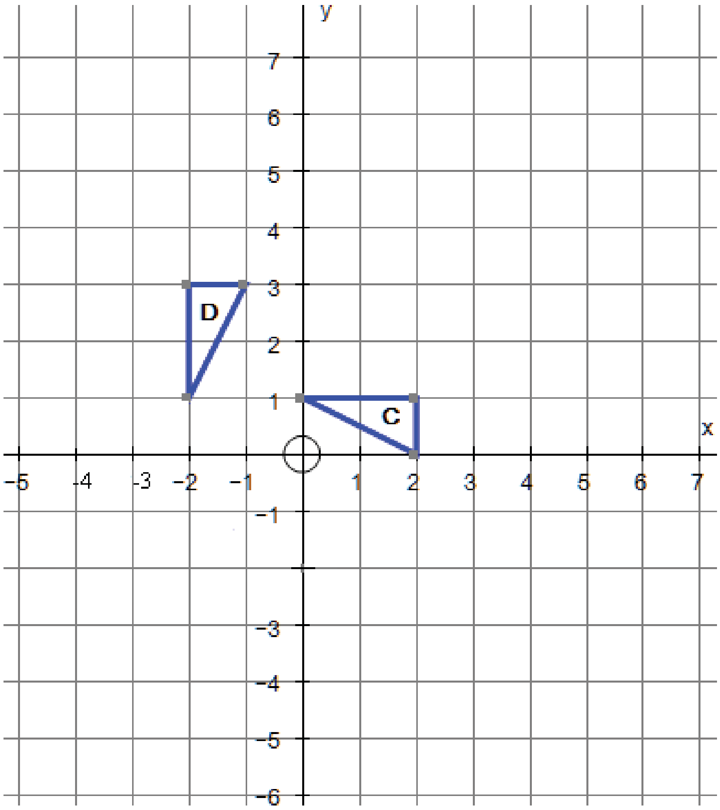


Figure 14

c)

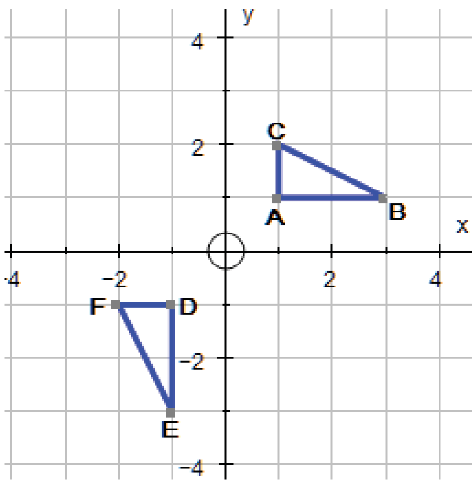


Figure 15

d) To draw the image, first draw the object B and then draw lines that join the points of triangle B to the centre. The points at $\frac{1}{3}$ of each line from the centre are the points of the image C. Join the points to obtain triangle C.

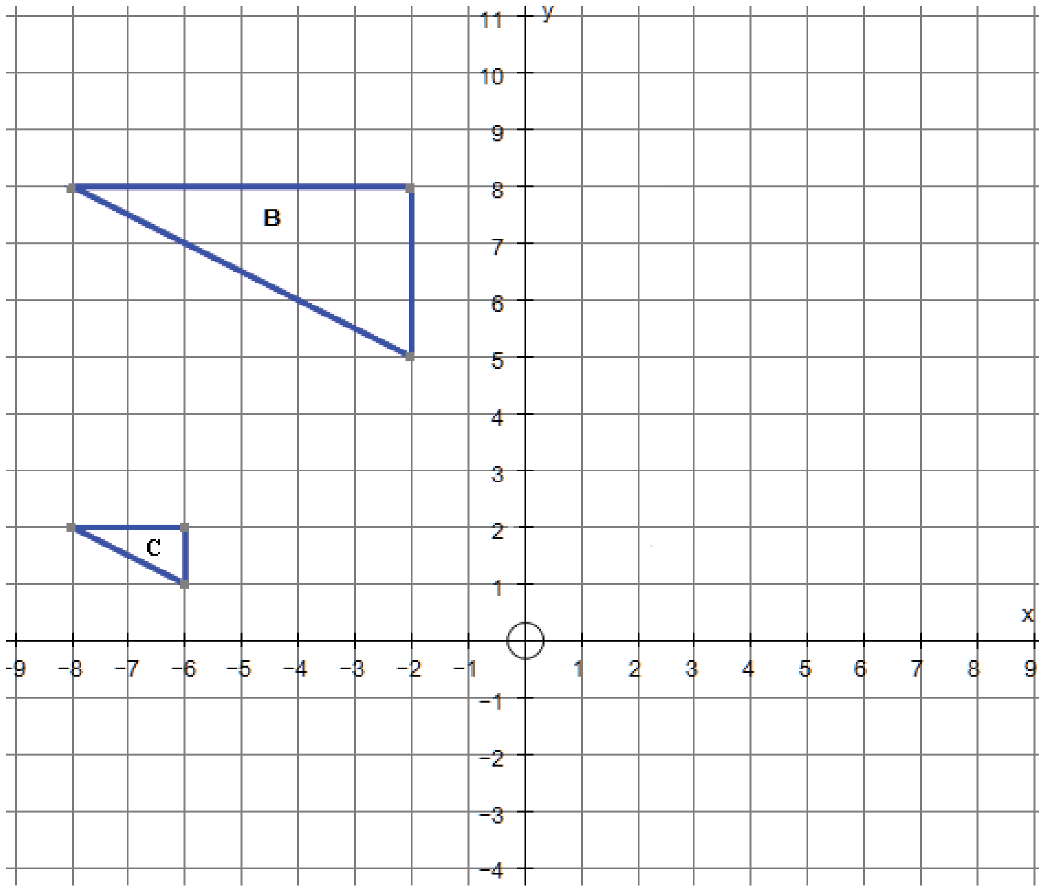


Figure 16

Up to this far we have reviewed the four types of transformation being translation, rotation enlargement and reflection. In the next two subtopics we are going to deal with stretch and shear which are the other two types of transformation.

Lesson 2 The Stretch

At the end of the subunit you should be able to:

-Identify and describe fully a stretch.

-Find the invariant line.

-Find the stretch factor.

-Stretch a shape given either a positive or negative stretch factor.

Apart from the other types of transformation that we now know, a shape can also be transformed by stretch. A stretch is a transformation of the plane in which all points move at right angles to the fixed line, a distance proportional to their distance from the line to start with.

Consider the following example.

Example 1

Consider figure 17 below.

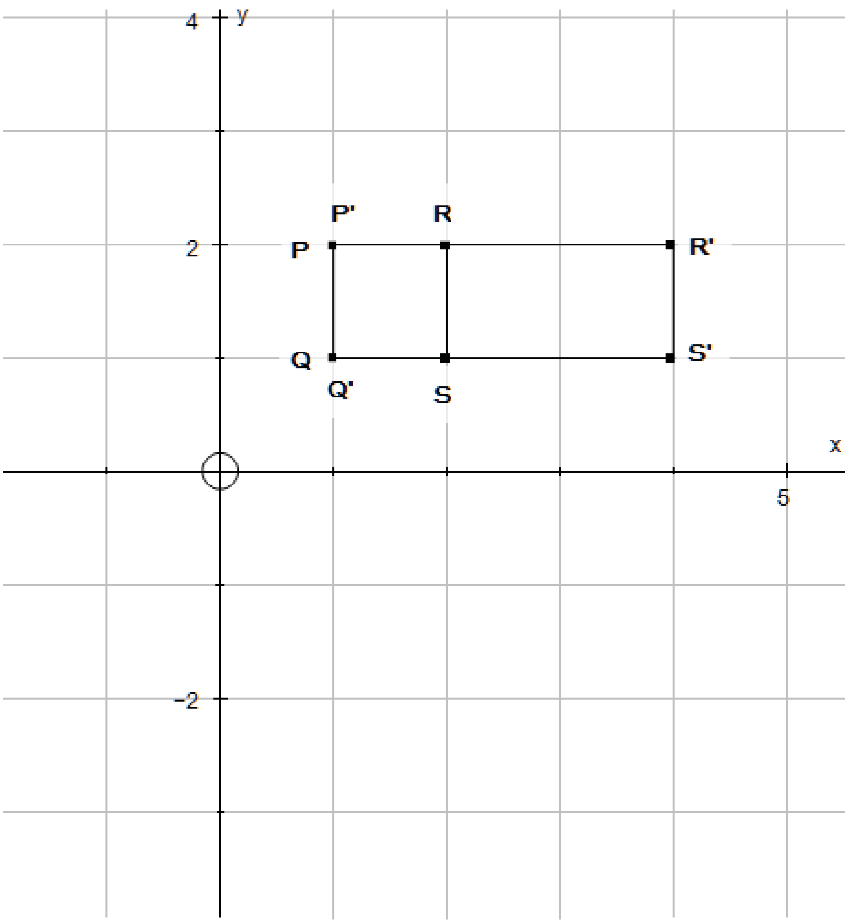


Figure 17

In figure 17 I believe you can see that the general difference between the shapes PQRS and P'Q'R'S' is towards the right. PQRS is a square and P'Q'R'S' is a rectangle. To get P'Q'R'S' from PQRS, the points R and S have moved towards the right, but the points P and Q have not moved so they are said to be invariant. Now the line PQ is an invariant line.

Measure the angles QSR and Q'S'R' and see that QSR= Q'S'R'=90^0.

So the points R and S have moved perpendicular to the invariant line.

Any transformation of this kind is a stretch.

Example 2

Let us work together in this example.

Consider figure 18 below.

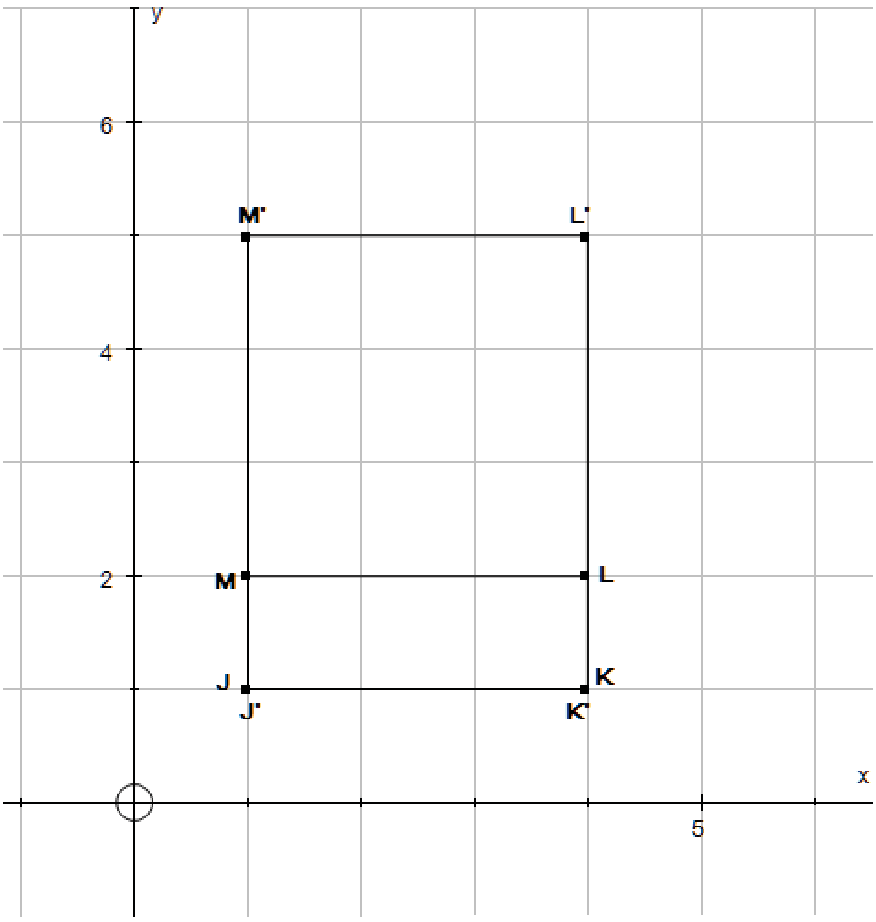


Figure 18

In figure 18 above which are the points that have moved?

___

___

___

Compare your answer with the following.

M and L are the points that have moved.

Measure angles KLM and K'L'M'.

Compare your answer with the following.

KLM = K'L'M' = 90^0.

How have the points moved in relation to JK?

Compare your answer with the following.

The points have moved at right angles or perpendicular to JK.

Which is the invariant line?

Compare your answer with the following.

JK is the invariant line.

What kind of a transformation is this?

Compare your answer with the following.

It is a stretch.

Find the distance JM, The shortest distance from the invariant line JK to the object M.

Compare your answer with the following.

JM= 2 cm

Find the distance JM', the shortest distance from the invariant line to the object M'.

Compare your answer with the following.

JM'=J'M'= 8 cm

Find the area of JKLM and of J'K'L'M'.

____

____

____

Compare your answers with the following.

Area of JKLM= 2cm×6cm=12cm^2 and area of J'K'L'M'=8cm×6cm= 48cm^2.

Find the value of :

a) $\frac{JM'}{JM}$ or $\frac{J'M'}{JM}$

____

____

____

Compare your answer with the following.

$$\frac{JM'}{JM} = \frac{J'M'}{JM} = \frac{8cm}{2cm} = 4$$

b) $\frac{K'L'}{KL}$

____

____

____

Compare your answer with the following.

$$\frac{K'L'}{KL} = \frac{8cm}{2cm} = 4$$

c)

$\frac{area\ of\ J'K'L'M'}{area\ of\ JKLM}$

____

____

____

Compare your answer with the following.

$$\frac{area\ of\ J'K'L'M'}{area\ of\ JKLM} = \frac{48cm^2}{12cm^2} = 4$$

The value that you have obtained in each case in i) above which is 4 is called the *stretch factor*.

Stretch factor which can be either negative or positive is a ratio and is given by:

$$\text{Stretch factor} = \frac{\text{distance of the image from the invariant line}}{\text{distance of the object from the invariant line}} = \frac{\text{area of the image}}{\text{area of the object}}$$

Note:

A stretch is described by

i) An invariant line.

ii) A stretch factor.

Now try the following activity on stretch.

Activity

Activity 2

1. Name the transformation shown in figure 19 below. What are the two factors that qualify it as the transformation you mentioned?

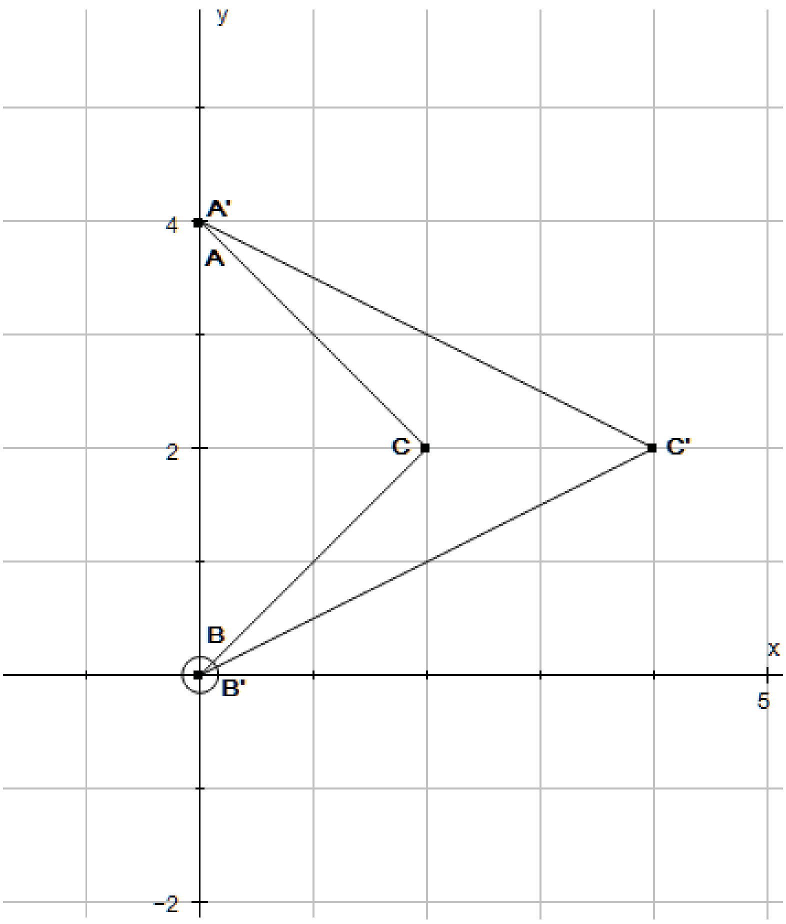


Figure 19

2. In each of the figures below, Shape ABCD has been transformed by a stretch into shape A'B'C'D'. Find the invariant line and the stretch factor.

a)

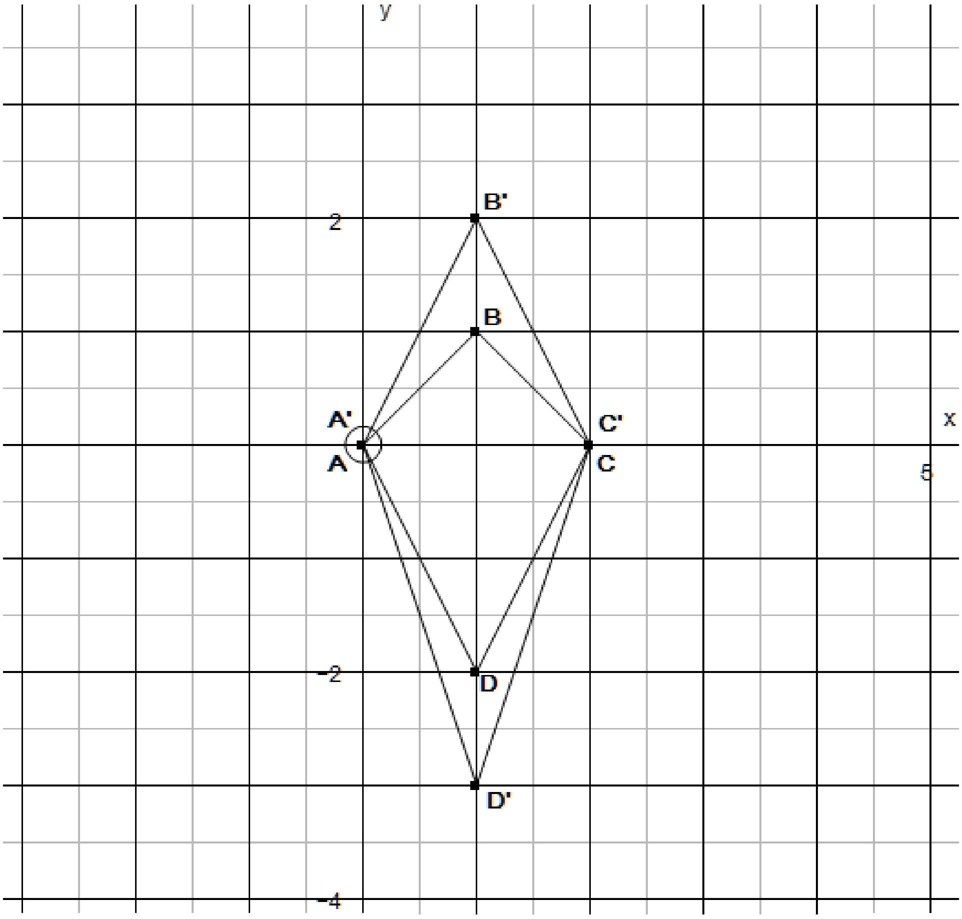


Figure 20

b)

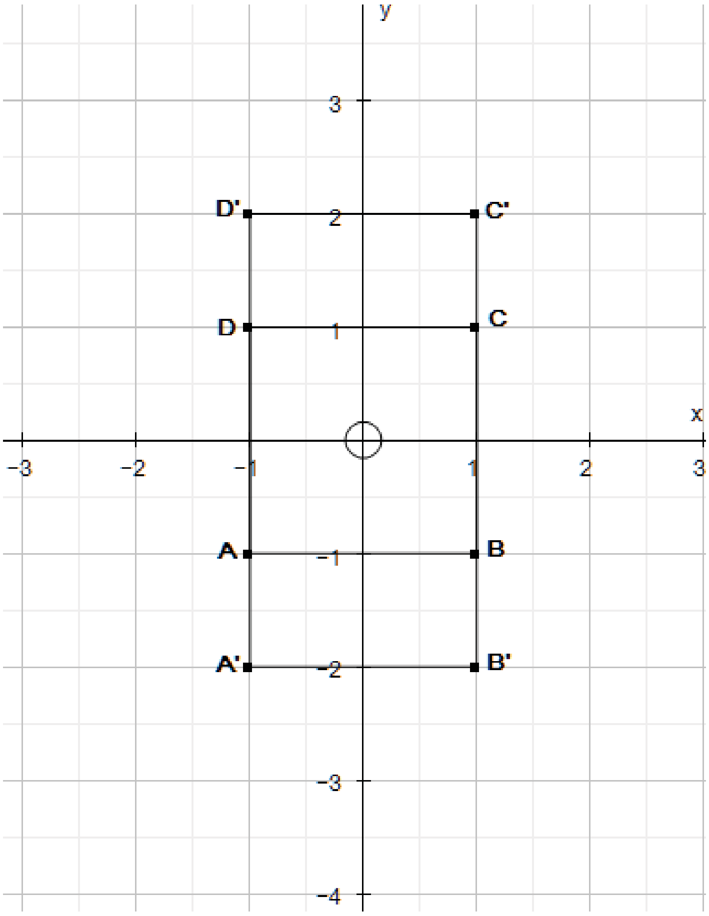


Figure 21

c)

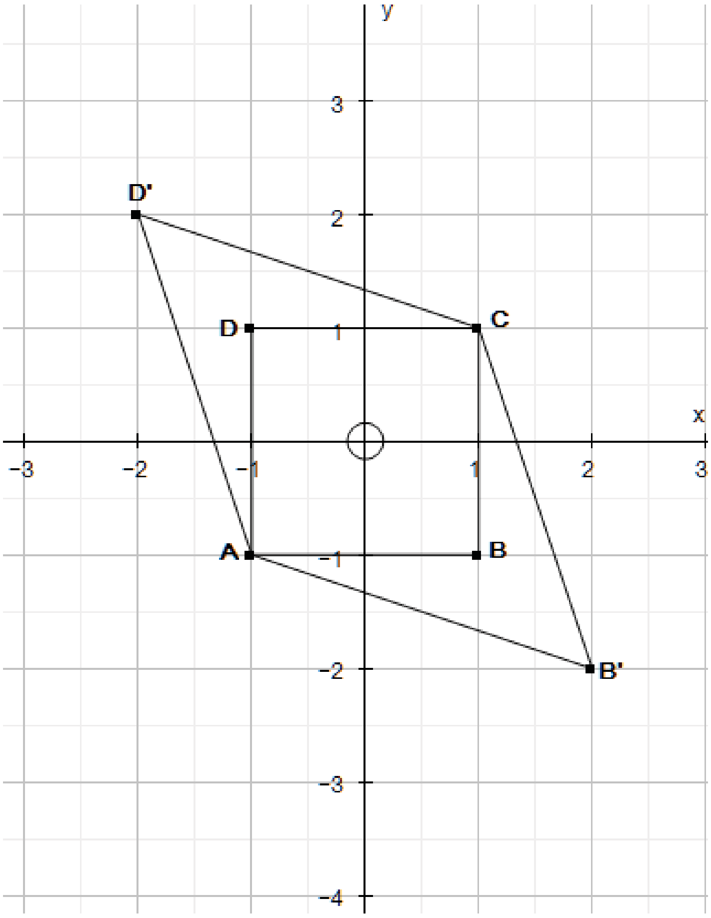
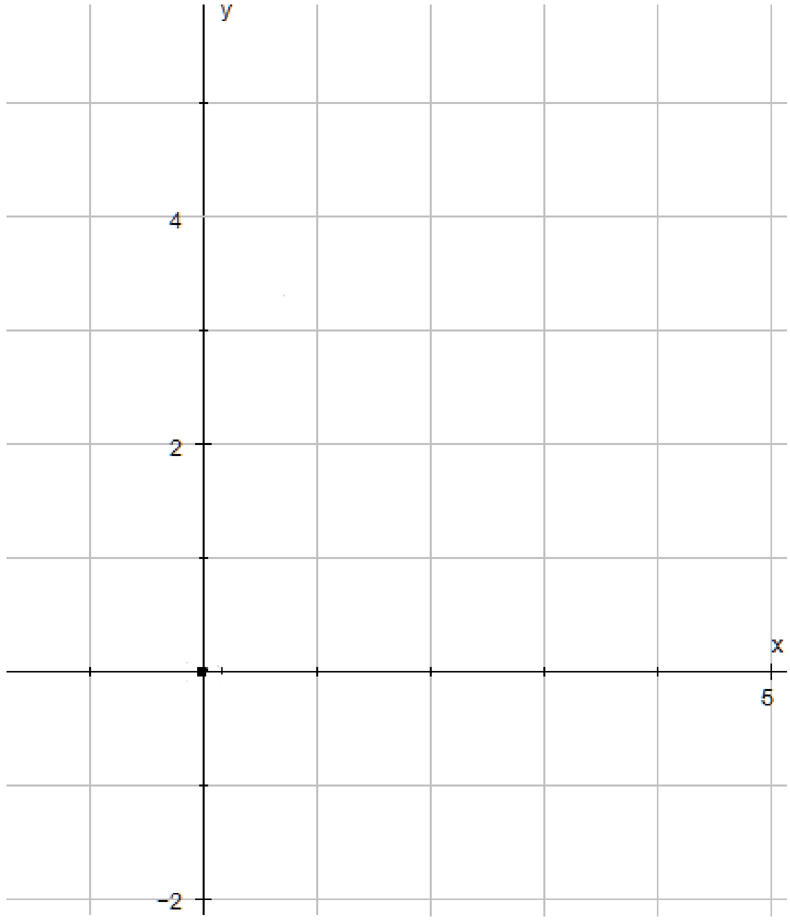


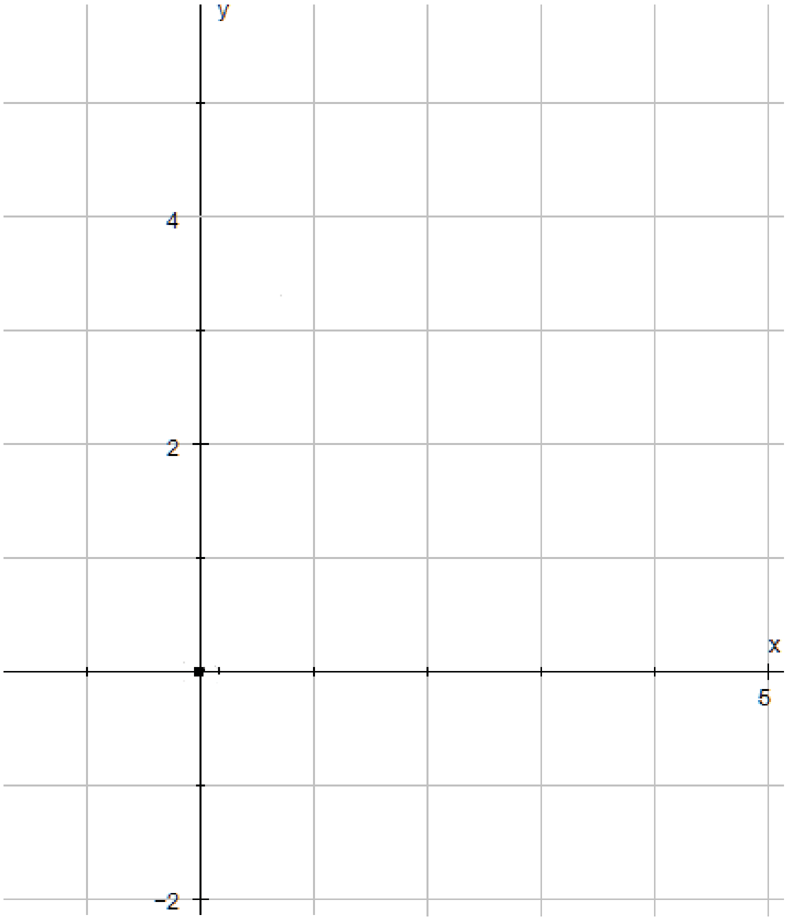
Figure 22

3.A trapezium ABCD has vertices A(1,0), B(1,2), C(3,2) and D(2,0), Trapezium ABCD is mapped onto trapezium A'B'C'D' by a stretch, factor 2 and invariant line y=1. Draw and label trapezium ABCD and A'B'C'D'.



4. Triangle XYZ has vertices X (-1, 1), Y (1, 4) and Z (2, 2). Triangle XYZ is mapped onto triangle X'Y'Z' by a stretch, factor 2 and invariant line x=1.

Draw and label XYZ and X'Y'Z'



Compare your answers with those given at the end of the subunit. I believe you were able to get all the questions right. If you have some wrong, feel free to review the examples given in the subunit on the stretch.

A Negative Stretch Factor

Up to this far we have been dealing with stretches where by the invariant line has always been part of the object, in that in each case some points of the object always remained invariant and did not move.

Example 1

Consider figure 23 below.

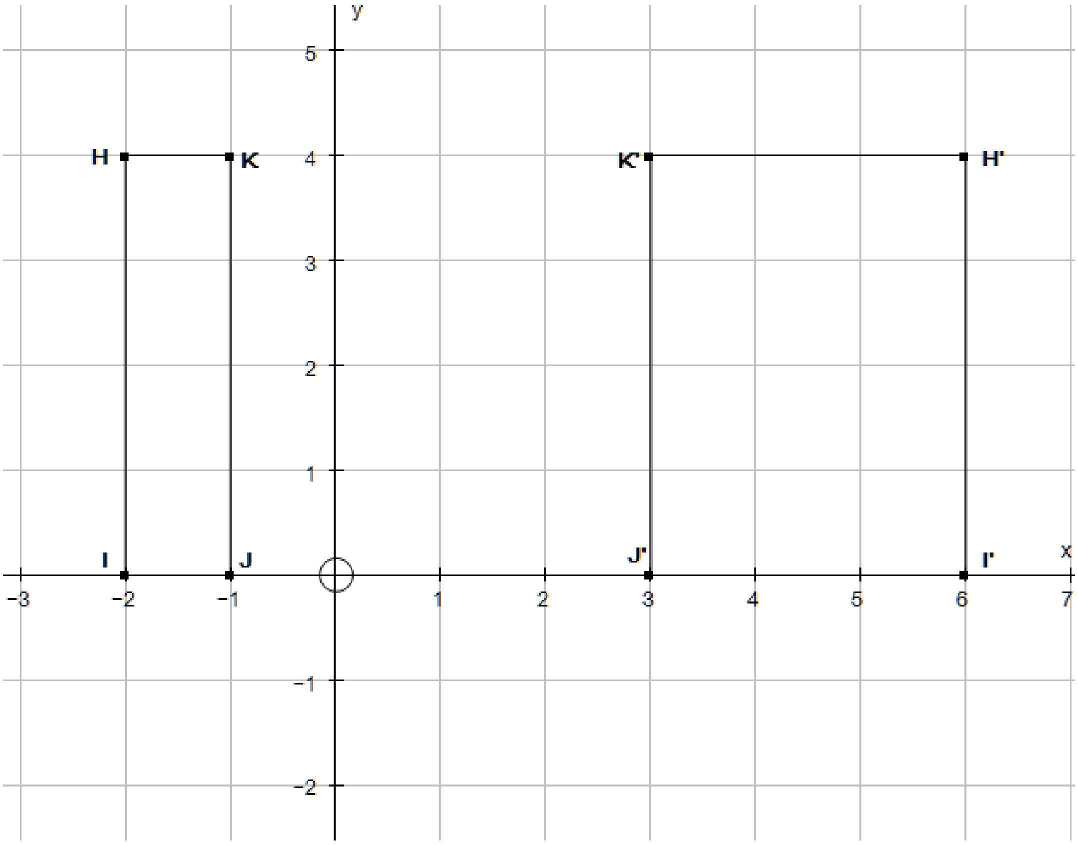


Figure 23

In figure 23 above HIJK is mapped onto H'I'J'K' by a stretch. As you can see the image is totally separated from the object which means that all the points of the object moved. None was invariant. So this suggests that the invariant line is outside the object. I believe we can again see that H'I'J'K' is 3 times bigger than HIJK which means that the distance from the invariant line to any point of H'I'J'K' is 3 times bigger than the distance from the invariant line to any corresponding point of HIJK. This make the y-axis or the line x=0 the invariant line.

Now that we know the invariant line, let us try to find the stretch factor.

Let us label the invariant line ‘Y’.

In figure 23:

YH=-2(because we are counting towards the left hand side which is to the negative side)

YH'=6

YI=-2	YI'=6
YJ=-1	YJ'=3
YK=-1	YK'=3

So the stretch factor $= \frac{YH'}{YH} = \frac{YJ'}{YJ} = \frac{6}{-3} = \frac{YJ'}{YJ} = \frac{YK'}{YK} = \frac{3}{-1} = -3$.

The stretch factor =-3.This is an example of a negative stretch factor.

Note:

With a negative stretch factor:

-The invariant line lies between the object and its image.

-The figure is stretched and reflected about the invariant line.

Example 2

Consider figure 24 below.

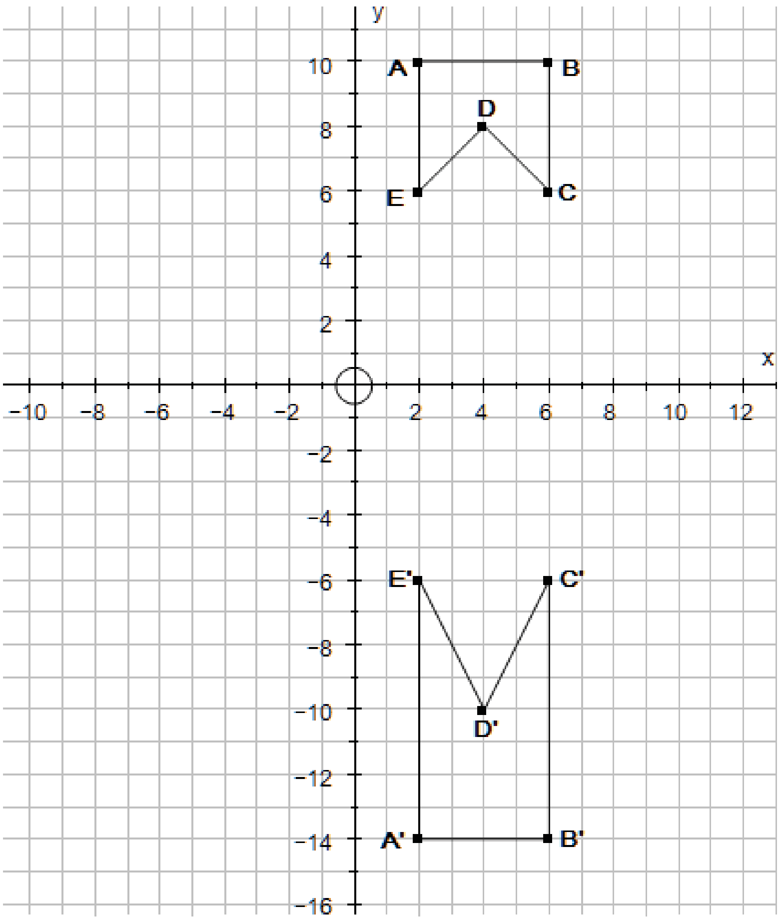


Figure 24

In figure 24 above, object ABCDE has been mapped onto object A'B'C'D'E' by a stretch.

a) Identify the invariant line.

Compare your answer with the following.

The line y = 2 is the invariant line.

b) Naming the invariant line ‘I’, Find:

i) $\frac{IA'}{IA}$

ii) $\frac{IB'}{IB}$

iii) $\frac{IC'}{IC}$

iv) $\frac{ID'}{ID}$

v) $\frac{IE'}{IE}$

Compare your answers with the following.

i) $\frac{IA'}{IA} = \frac{-16}{8} = -2$

ii) $\frac{IB'}{IB} = \frac{-16}{8} = -2$

iii) $\frac{IC'}{IC} = \frac{-8}{4} = -2$

iv) $\frac{ID'}{ID} = \frac{-12}{6} = -2$

v) $\frac{IE'}{IE} = \frac{-8}{4} = -2$

c) Now what is the stretch factor?

Activity

Compare your answer with the following.

The stretch factor is -2.

Now that is another example of a negative stretch factor. I hope you understood. Now try the following activity and see how far you can go.

Activity 3

1)

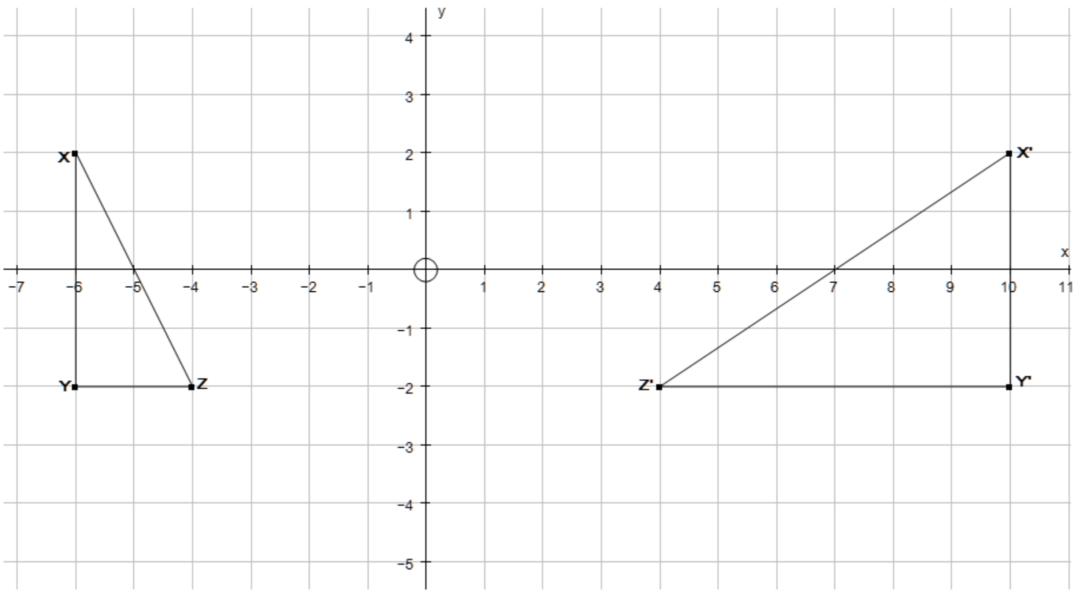


Figure 25

In figure 25 above:

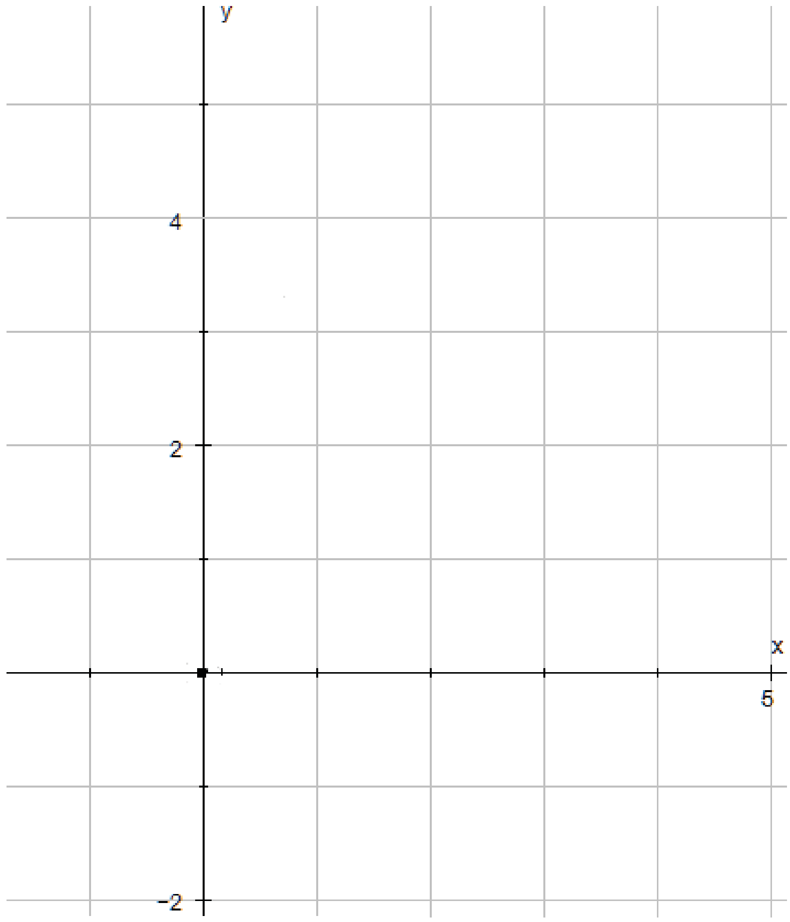
a) Find the invariant line.

____
____
____

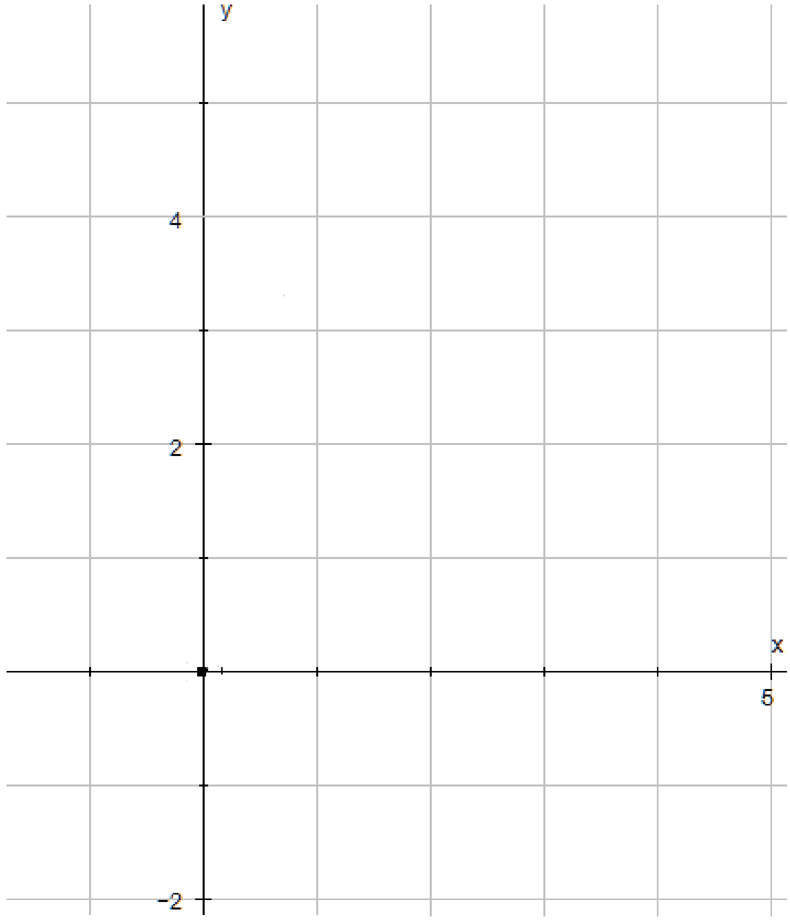
b) Find the stretch factor.

____
____
____

2) A shape DEFGHI has vertices D(-6,2), E(-4,2), F(-4,4), G(-2,4),H(-2,5) and I(-6,5). The shape DEFGHI has been mapped onto D'E'F'G'H'I' by a stretch, stretch factor -2 and invariant line x=-1. Draw and label the shaped DEFGHI and D'E'F'G'H'I'.



3) A quadrilateral ABCD has vertices A(-8,8), B(-4,8),C(-4,4) and D(-8,4). The quadrilateral ABCD has been mapped onto A'B'C'D' by a stretch, factor -2 and invariant line y = x. Draw and label quadrilaterals ABCD and A'B'C'D'.



Compare your answers with those given at the end of the subunit.

We have now come to the end of our discussion on the stretch. If you think you have missed anything go back to this subunit on stretch and try reviewing it.

In this subunit you have learned that:

-A stretch is a transformation of the plane in which all points move at right angles to the fixed line, a distance proportional to their distance from the line to start with.

-A stretch is described by:

i) An invariant line.

ii) A stretch factor.

-

$$Stretch\ factor = \frac{distance\ of\ the\ image\ from\ the\ invariant\ line}{distance\ of\ the\ object\ from\ the\ invariant\ line} = \frac{area\ of\ the\ image}{area\ of\ the\ object}$$

-With a negative scale factor:

The invariant line lies between the object and its image.

The figure is stretched and reflected about the invariant line.

Solutions to Activities:

Solutions To activity 2

1. The transformation shown in figure 19 is a stretch.

The factors that qualify it as a stretch are:

-It has an invariant line, the line AB.

-The points have moved perpendicular to the invariant line.

2.

a) In figure 20 the points A and C have remained invariant, so the invariant line (I) is the line AC or the x-axis.

The stretch factor $= \frac{IB'}{IB} = \frac{2}{1} = 2$.

b) In figure 21 the line y = 0 is the invariant line (I).

The stretch factor $= \frac{ID'}{ID} = \frac{2}{1} = 2$.

c) In figure 22 the line y = 0 is the invariant line (I).

The stretch factor $= \frac{IB'}{IB} = 2$.

3.

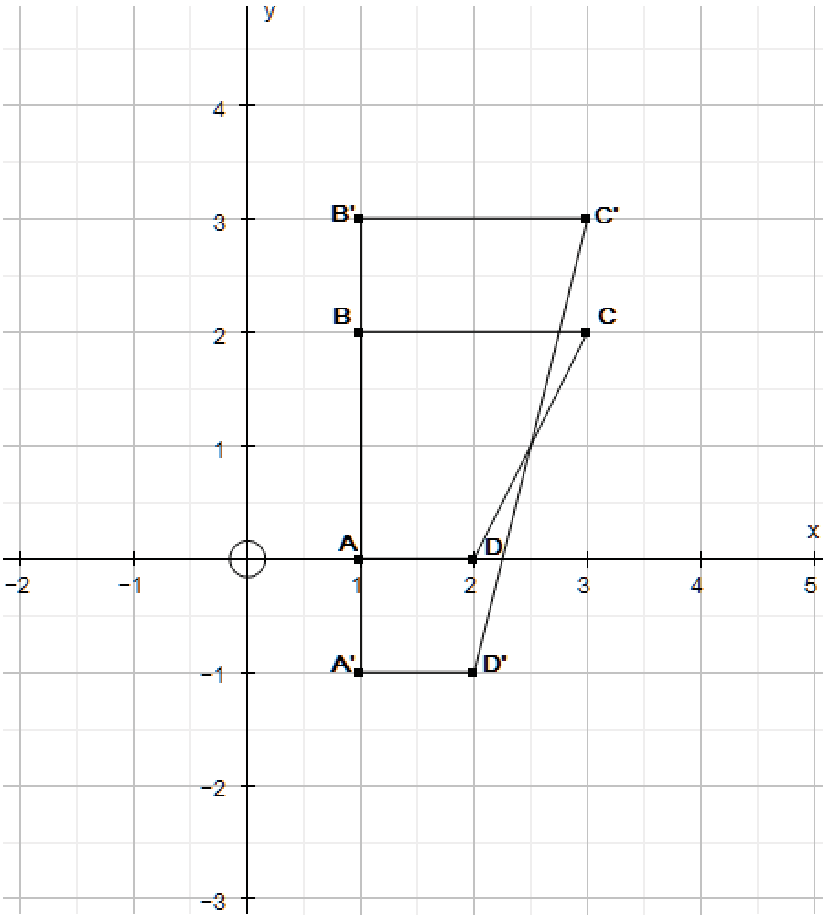


Figure 26

4.

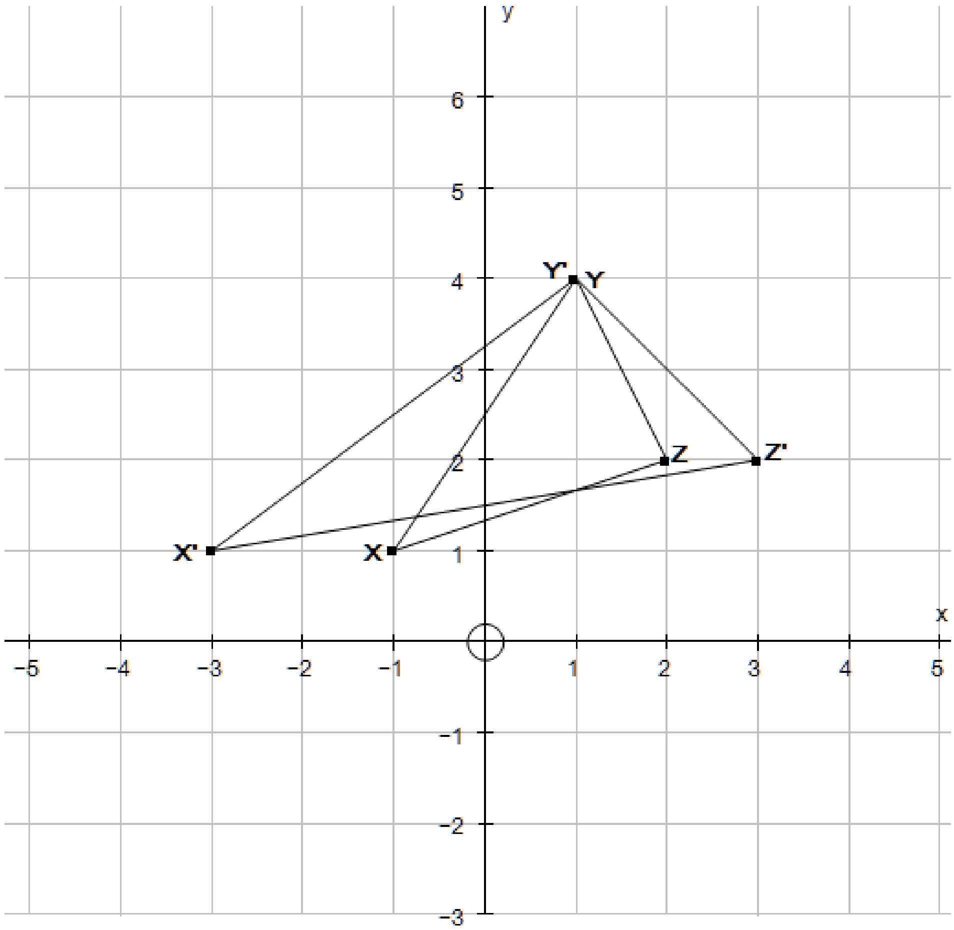


Figure 27

Solutions To activity 2

1. a) The invariant line (I) is the line y = -2.

b) The stretch factor = $\frac{IY'}{IY} = \frac{IX'}{IX} = \frac{12}{-4} = -3$.

2.

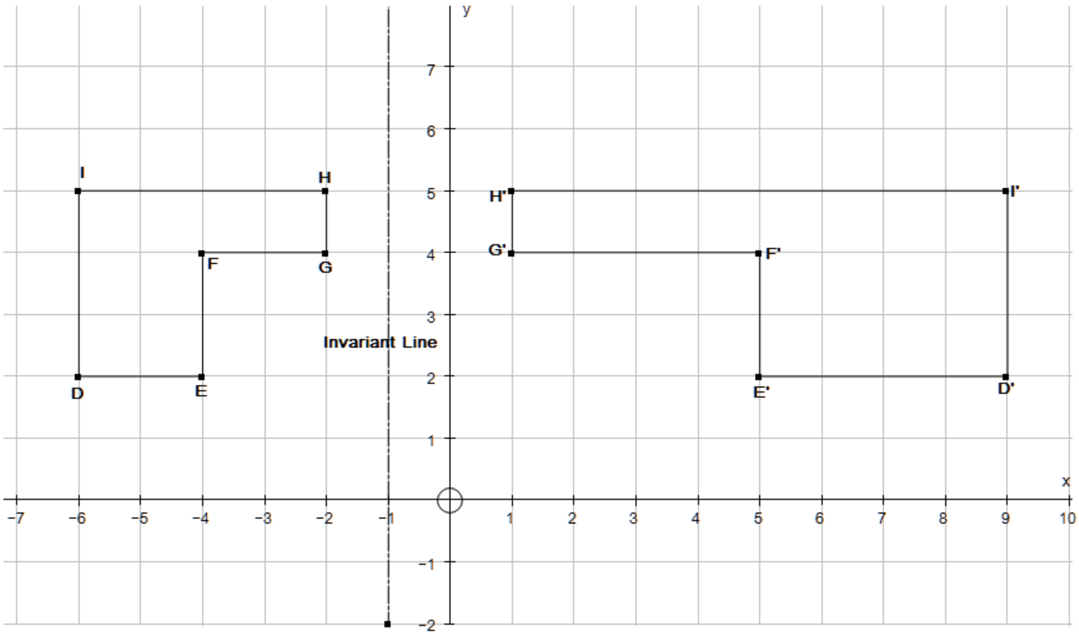

y
x
I
H
F
G
D
E
H'
I'
G'
F'
E'
D'
Invariant Line
7
6
5
4
3
2
1
−1
−2
−7
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6
7
8
9
10

Figure 28

3.

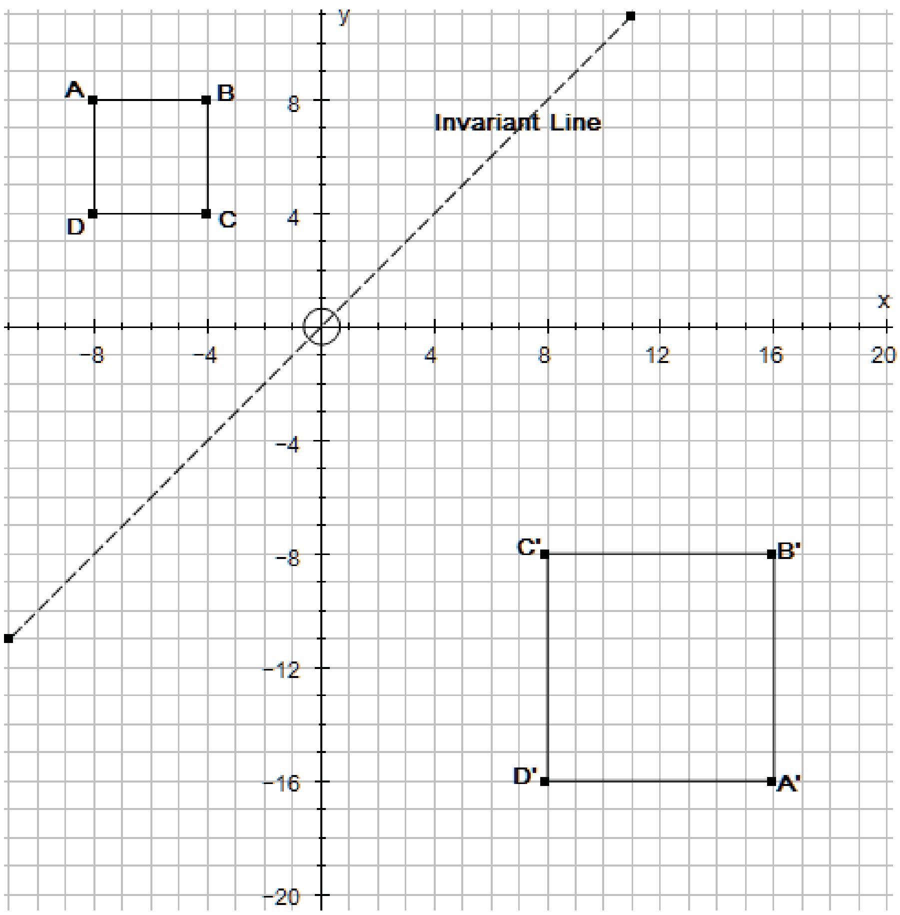


Figure 29

Lesson 3 The Shear

At the end of the subunit you should be able to:

-Identify and describe fully a shear.

-Find the invariant line in a shear.

-Find the shear factor.

-Shear a shape given either a positive or negative shear factor.

A transformation in which all points on the invariant line remain fixed while other points are shifted parallel to the invariant line by a distance proportional to their perpendicular distance from the invariant line is a shear.

Shearing a plane figure does not change its area.

Consider figure 30 below and see how shear transformation works.

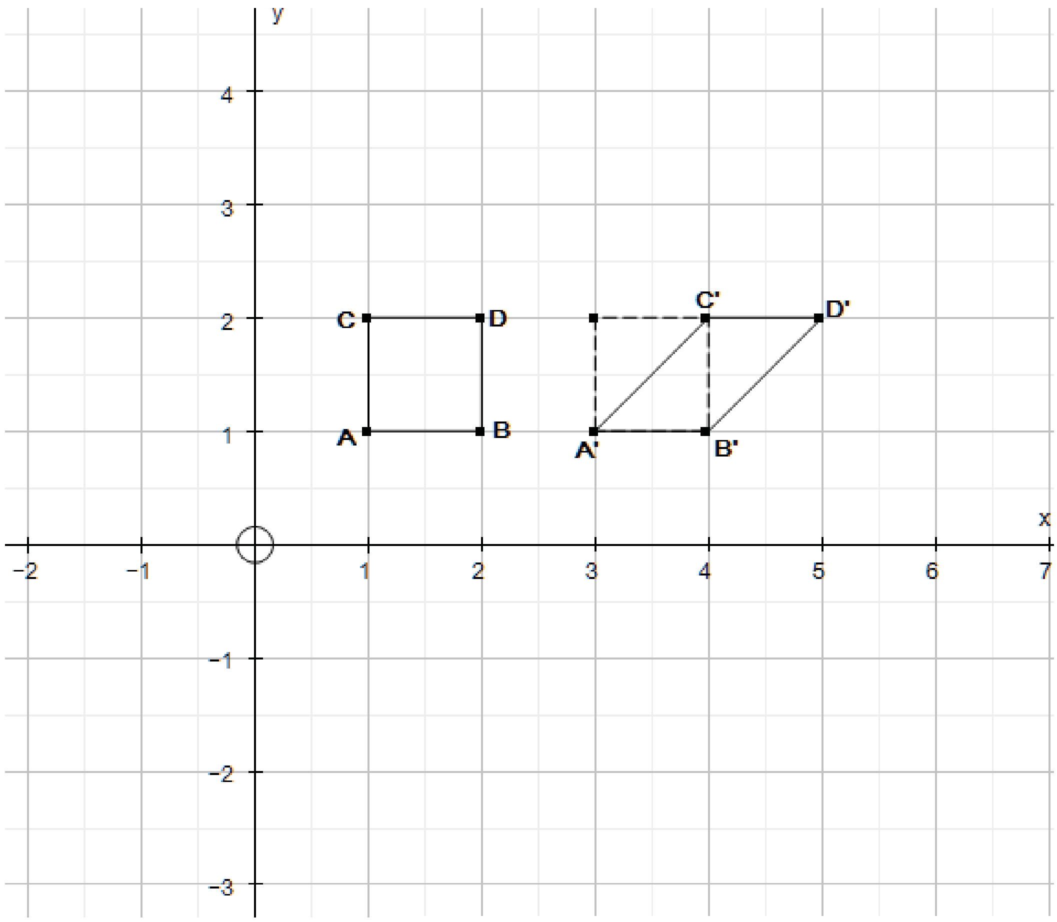


Figure 30

In figure 30, ABCD is the object and A'B'C'D' is the image of ABCD after a shear. The object has been shown with doted lines. The points on the line AB remain fixed and the points on the line CD are shifted parallel to the invariant line except those on the invariant line.

Now these two shapes ABCD and A'B'C'D' have the same area since they have the same bases and the same height.

Now consider another figure below. In figure 31 let us learn how to find the shear factor.

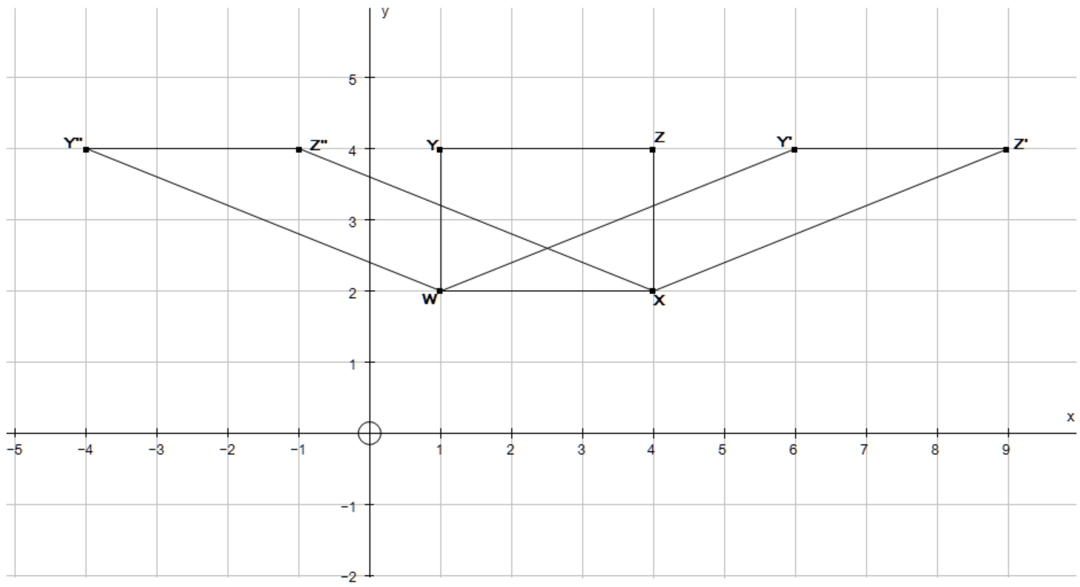


Figure 31

WXYZ is an object that has two images, W'X'Y'Z' and W"X"Y"Z". The line WX is the invariant line.

Now work out the following.

Find the length of YY', ZZ', and YW.

Compare your answer with the following:

YY'=+5

ZZ'=+5

YW=+5

Now Calculate: $\frac{YY'}{YW}$ and $\frac{ZZ'}{YW}$

____

____

____

Compare your answer with the following.

$\frac{YY'}{YW} = \frac{5}{2} = 2.5$ and $\frac{ZZ'}{YW} = \frac{5}{2} = 2.5$

This is an example of a positive shear factor.

$$Shear\ \ factor = \frac{distance\ moved\ by\ a\ point}{Perpendicular\ distance\ of\ \ a\ \ point\ \ from\ the\ invariant\ line}$$

Find the length of YY", ZZ", and YW.

____

____

____

____

____

__

Compare your answer with the following:

YY"=-5

ZZ"=-5

YW=+2

Now Calculate: $\frac{YY''}{YW}$ and $\frac{ZZ''}{YW}$

Compare your answer with the following.

$$\frac{YY''}{YW} = \frac{-5}{2} = -2.5$$

$$\frac{ZZ''}{YW} = \frac{-5}{2} = -2.5$$

This is an example of a negative shear factor. So in general:

$$Shear\ factor = \frac{distance\ moved\ by\ a\ point}{distance\ of\ a\ point\ from\ the\ invariant\ line}$$

Now consider the following examples on shear.

Example 1:

In figure 32 ABCD is the object transformed into A'B'C'D' by a shear. Find the invariant line and the shear factor.

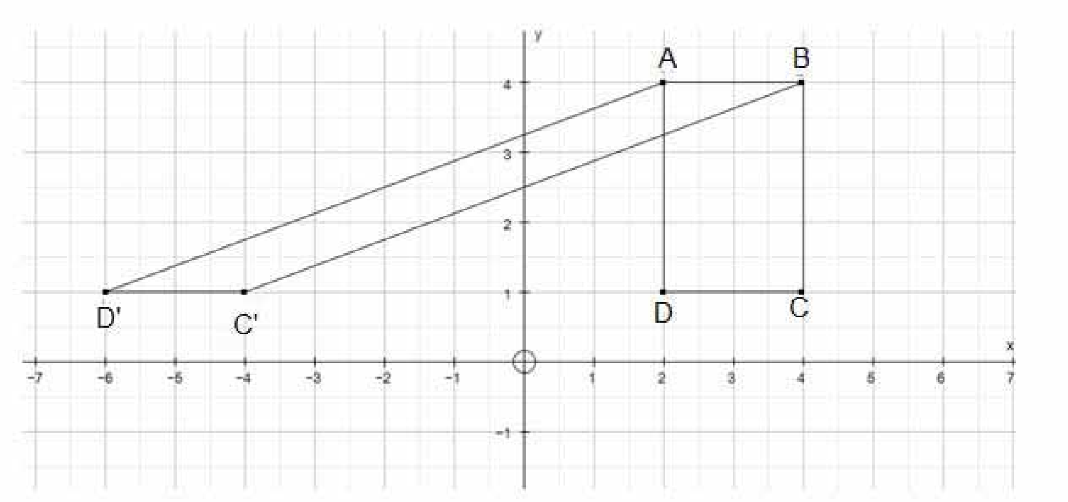


Figure 32

Solution

I believe you can see that the points A and B have not changed, so line AB is the invariant line.

$$Shear\ factor = \frac{distance\ moved\ by\ a\ point}{distance\ of\ a\ point\ from\ the\ invariant\ line}$$

$$= \frac{CC'}{AC} = \frac{DD'}{BD} = \frac{-8}{-3} = 2.7$$

CC' and DD' are negative because the motion has been towards the left.

AC and BD are negative because they are points below the invariant line.

So the shear factor is positive.

Example 2:

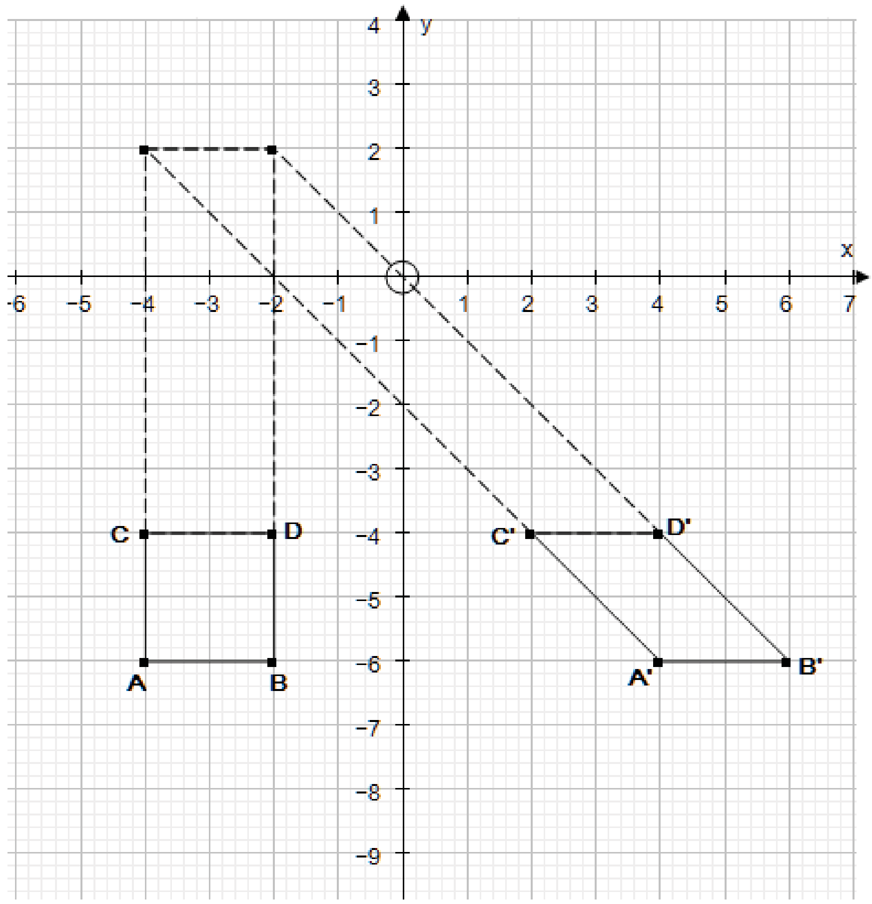


Figure 33

Figure 33 above shows that shape ABCD has been transformed into A'B'C'D' by a shear.

Find the invariant line and the shear factor.

Solution

The invariant line is found by extending the lines AC and A'C', where these two lines meet it is the invariant line. Or the lines BD and B'D' can be extended so that where they meet is the invariant line. So the invariant line is the line y=2.

$$Shear\ \ factor = \frac{\text{distance moved by a point}}{\text{distance of a point from the invariant line}}$$

let the invariant line be I.

$$Shear\ \ factor = \frac{AA'}{AI} = \frac{BB'}{BI} = \frac{CC'}{CI} = \frac{DD'}{DI} = \frac{+8}{-8} = -1$$

In general:

For a shear:

-Points move parallel to the invariant line, except those on the invariant line.

-An invariant line is always parallel to the line segments that do not change in length.

-To locate the invariant line, extent a pair of corresponding sides that are not parallel to the invariant line. They meet at a point on the invariant line.

$$\text{-}\ Shear\ \ factor = \frac{\text{distance moved by a point}}{\text{distance of a point from the invariant line}}$$

- Points on opposite sides of the invariant line move in opposite direction.

Now try activity 4 on the shear.

Activity

Activity 4

1. Identify and describe fully the transformation shown in each of the following diagrams.

a)

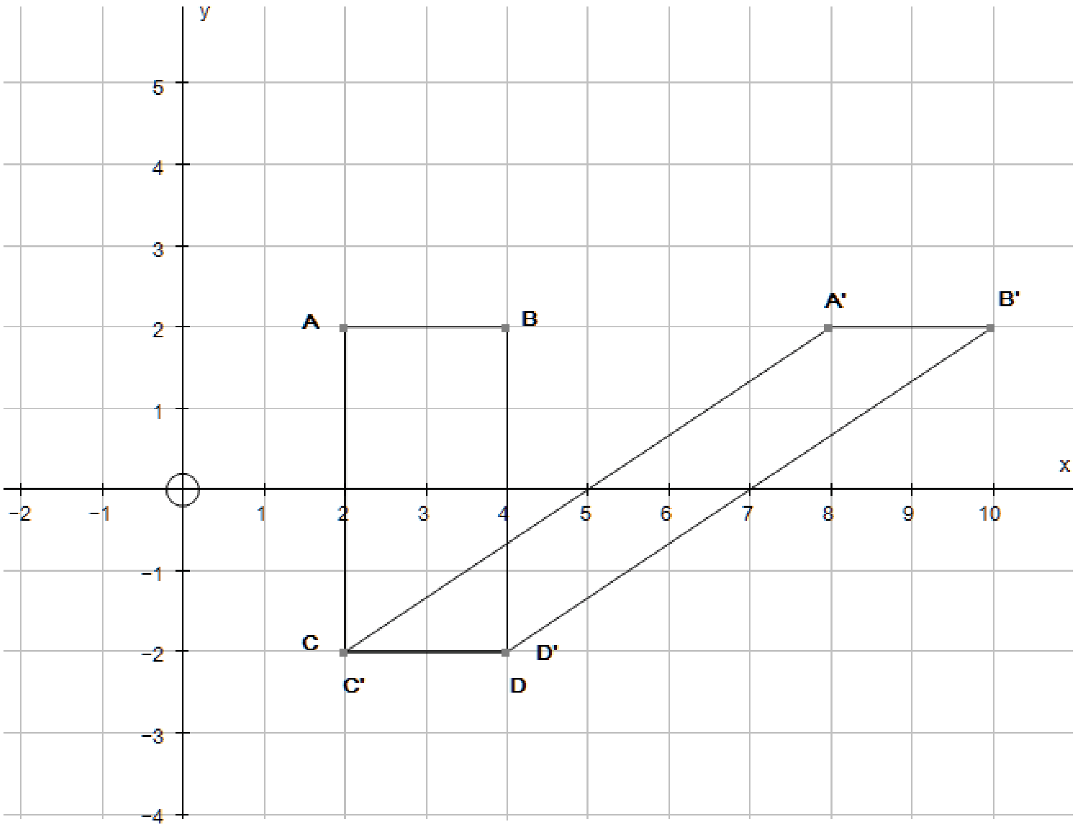


Figure 34

b)

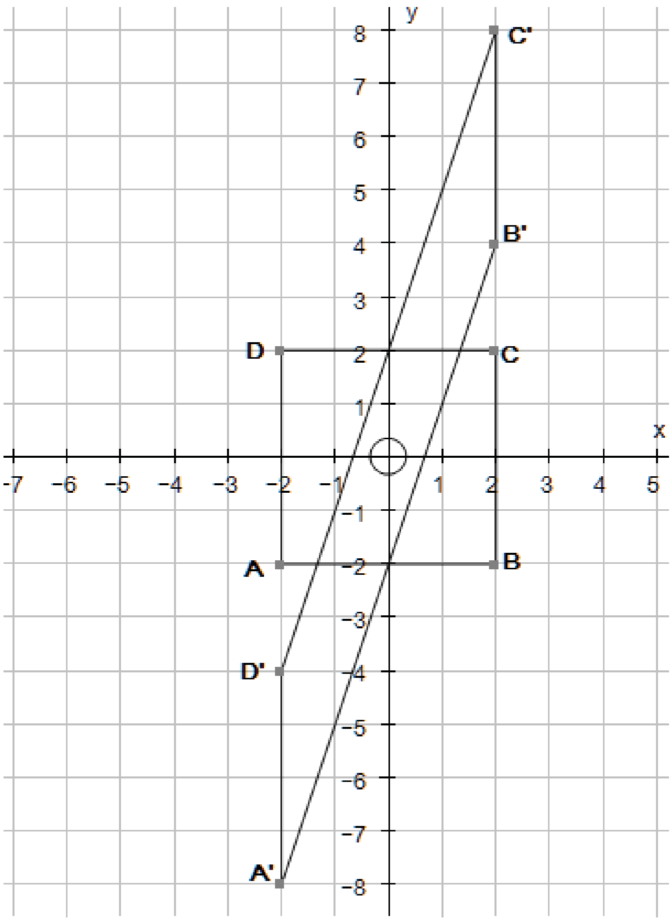


Figure 35

c)

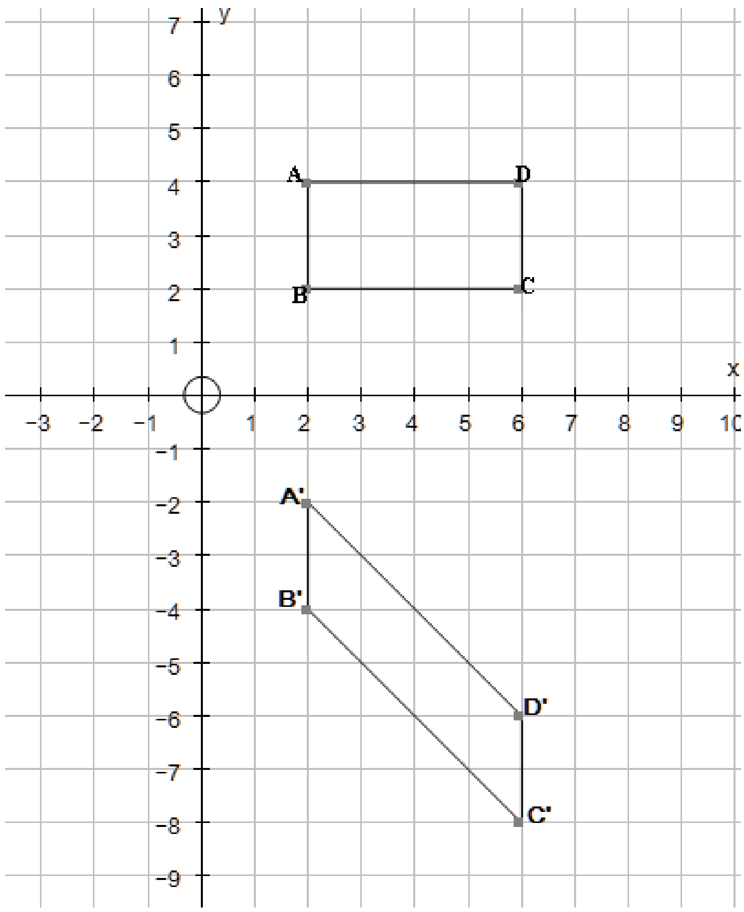
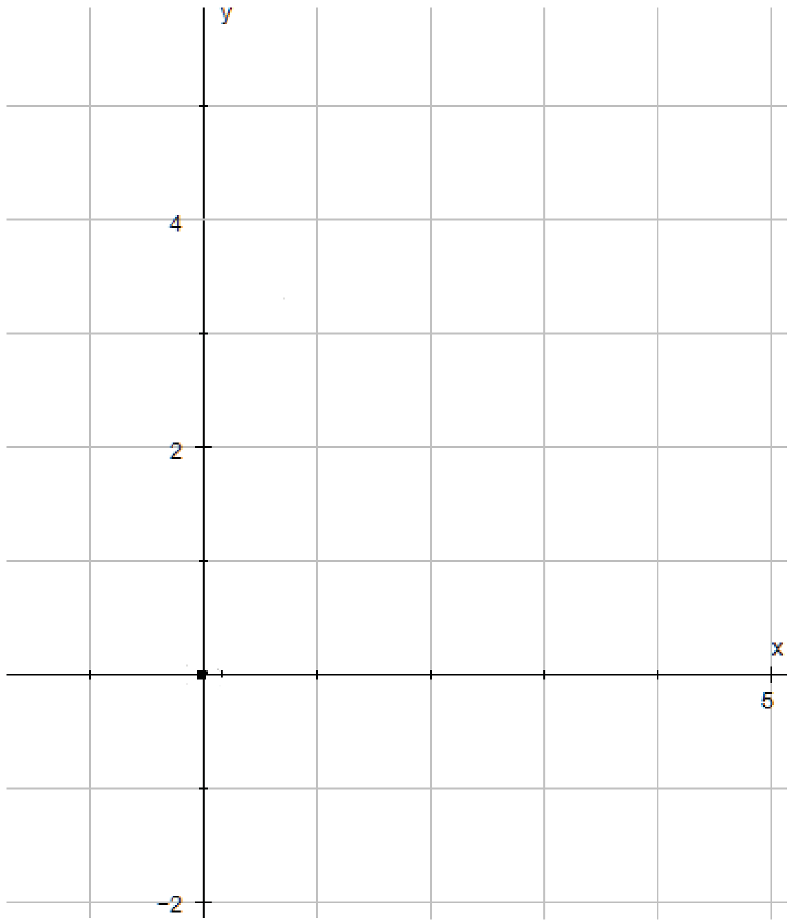
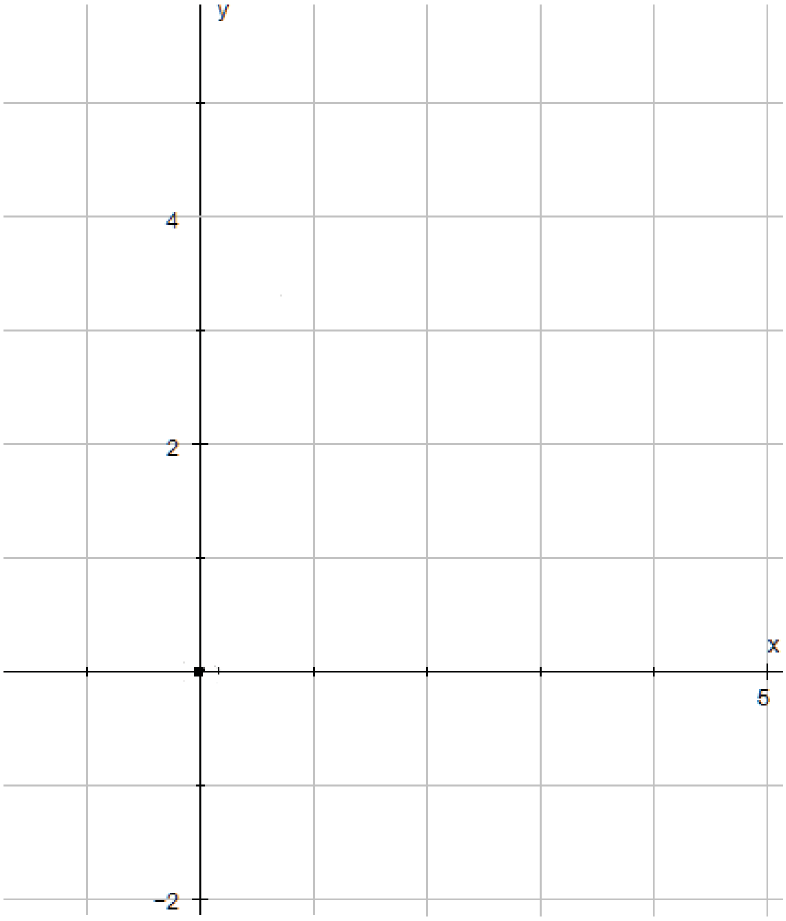


Figure 36

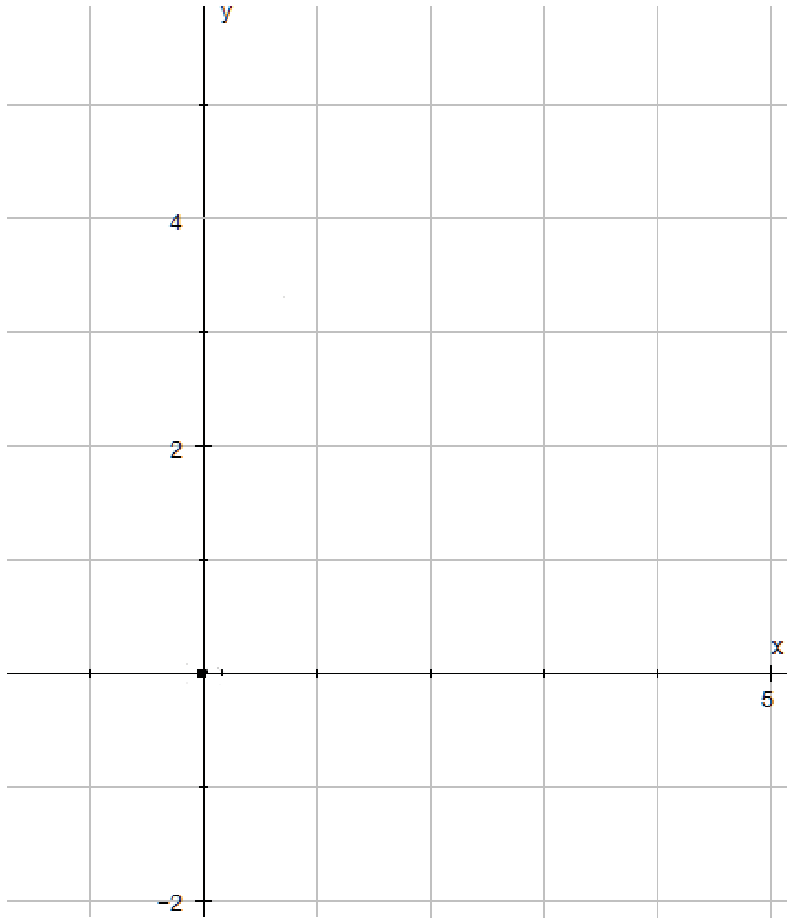
2. Triangle ABC has the following coordinates: A (2, 2), B (4, 2), C (2, 4). It has been transformed by a shear, shear factor 2 and the invariant line is the line AB. Draw both triangle ABC and its image A'B'C'.



3. ABCD is a trapezium with the following coordinates: A (2, 2), B (4, 2), C (4, 6), D (1, 6). It has been transformed by a shear, shear factor -3 and the invariant line is the line y=4. Draw the shape ABCD and its image A'B'C'D'.



4. A rectangle ABCD has coordinates: A (-6, 2), B (-4, 2), C (-4, 6), D (-6, 6). It has been transformed by a shear, shear factor 1.5 and the invariant line is the line y=-2. Draw both ABCD and its image A'B'C'D'.



I hope you were able to answer all the questions in this activity. Compare your answers with those given at the end of the subtopic. If you have got some wrong go back to the examples and see where you have missed the concept.

In this Unit You Have Learned that

- A transformation in which all points on the invariant line remain fixed while other points are shifted parallel to the invariant line by a distance proportional to their perpendicular distance from the invariant line is a shear.
- Shearing a plane figure does not change its area.

- A shear is described by:

i) An invariant line.

ii) A shear factor.

- $$Shear\ factor = \frac{distance\ moved\ by\ a\ point}{Perpendicular\ distance\ of\ a\ point\ from\ the\ invariant\ line}$$

Solutions to activity 4:

1.

a) It is a shear because it has an invariant line and the points on the line AB have moved parallel to invariant line.

The invariant line is the line CD.

$$Shear\ factor = \frac{distance\ moved\ by\ a\ point}{distance\ of\ a\ point\ from\ the\ invariant\ line}$$

$let\ the\ invariant\ line\ be\ I.$

$$Shear\ factor = \frac{AA'}{AI} = \frac{BB'}{BI} = \frac{+6}{+4} = 1.5$$

b) It is a shear because it has an invariant line and the points have moved parallel to the invariant line.

The invariant line is the y-axis, because the points (0, 2) and (0, -2) which are on the y-axis have not moved.

$$Shear\ factor = \frac{distance\ moved\ by\ a\ point}{distance\ of\ a\ point\ from\ the\ invariant\ line}$$

$let\ the\ invariant\ line\ be\ I.$

$$Shear\ factor = \frac{BB'}{BI} = \frac{CC'}{CI} = \frac{6}{2} = \frac{AA'}{AI} = \frac{DD'}{DI} = \frac{-6}{-2} = 3$$

c)

It is a shear because the points have moved parallel to the invariant line.

To find the invariant line we first have to extend a pair of corresponding sides that are not parallel to the invariant line. Where they meet is part of the invariant line.

This is shown in figure 37 below.

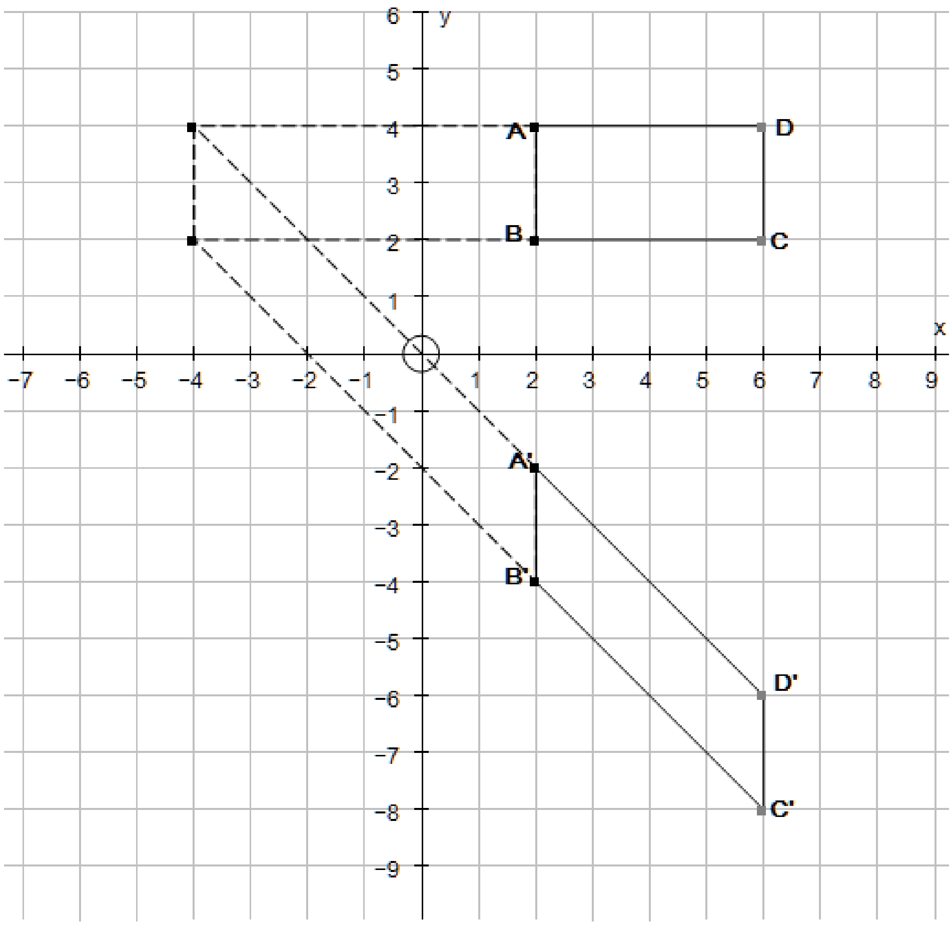


Figure 37

So the invariant line is the line x = -4.

Now that we know the invariant line we can find the shear factor.

$$Shear\ \ factor = \frac{distance\ moved\ by\ a\ \ point}{distance\ of\ \ a\ \ point\ \ from\ the\ invariant\ line}$$

$let\ the\ invariant\ line\ be\ I.$

$$Shear\ \ factor = \frac{AA'}{AI} = \frac{BB'}{BI} = \frac{-6}{6}\frac{CC'}{CI} = \frac{DD'}{DI} = \frac{-10}{10} = -1.$$

2.

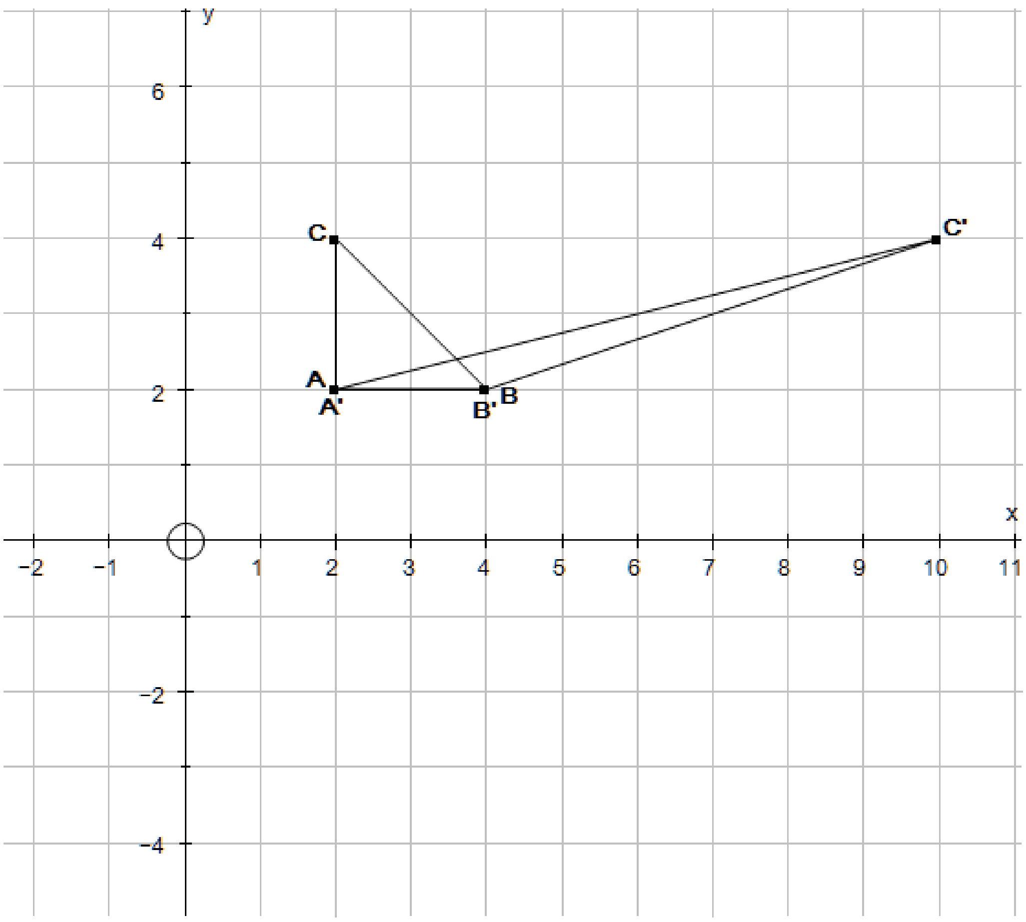


Figure 38

3.

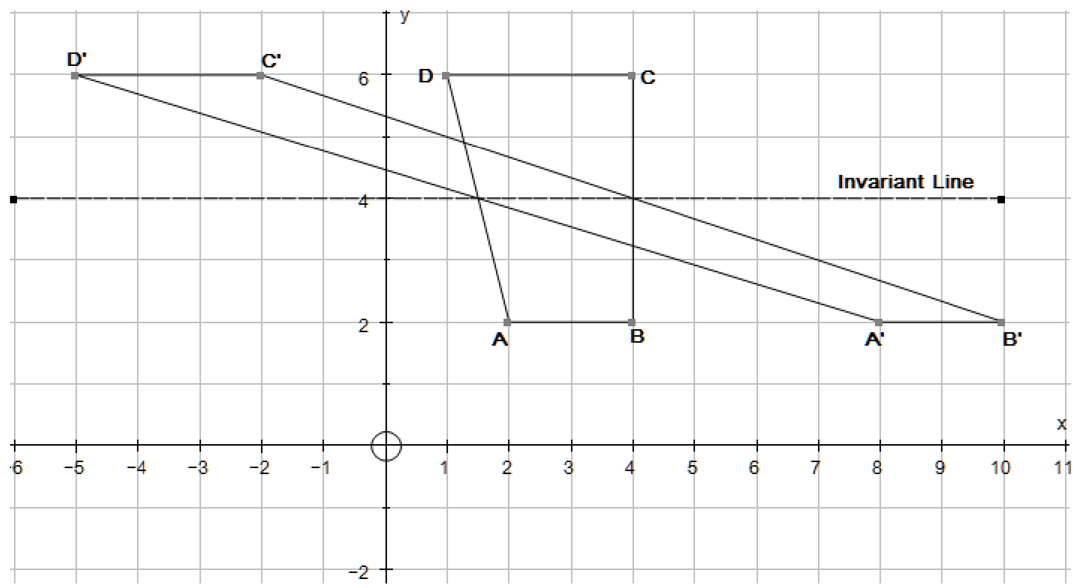


Figure 39

4.

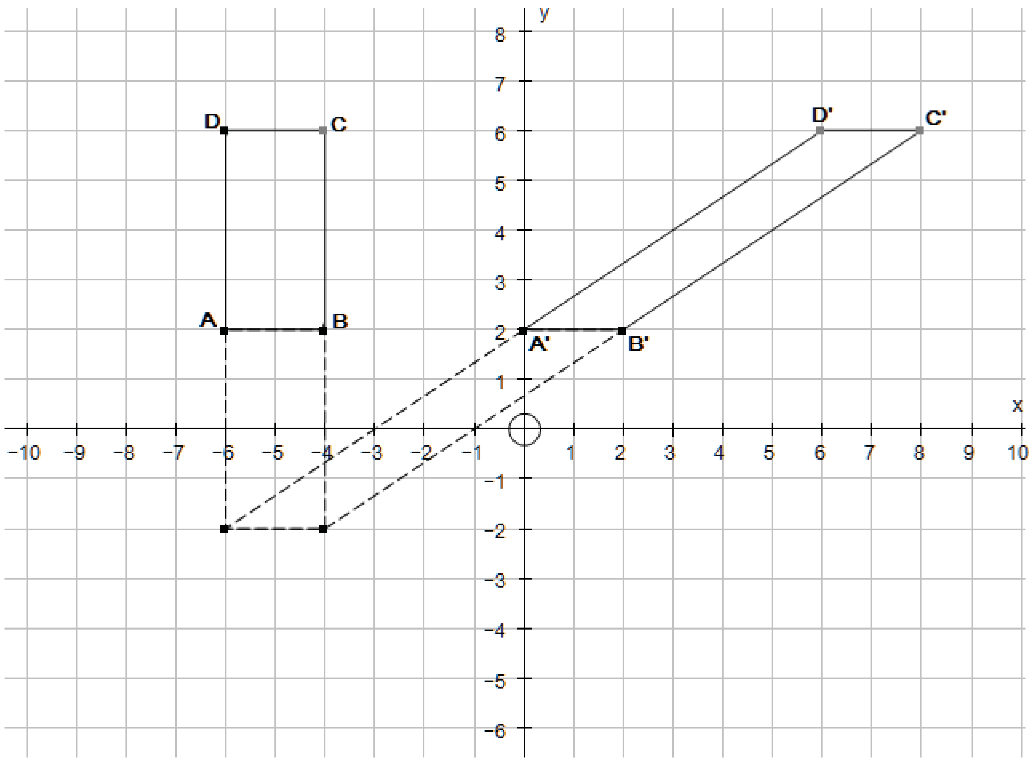


Figure 40

Lesson 4 Matrices and Transformations

At the end of this subunit you should be able to:

-Transform any given object with a given matrix.

-Find the matrix of a given transformation.

This subunit consists of about 18 pages.

Transforming By Matrices

Matrices are another way of defining transformations. In this subtopic we are treating matrices only in relation to transformation, so if you feel you have no idea of what matrices are, first consult the unit on matrices.

To transform something is to change it in some way. Consider the following point on a plane: (2, 4), if we multiply this point by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

we get: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$.

This means that the point (2, 4) has been transformed and has the image (-4, 2) after being transformed under the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Example 1:

Triangle ABC has the following coordinates: A (2, 2), B (4, 2), C (2, 4). Find the coordinates of the vertices of the image triangle A'B'C' under a transformation represented by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Then draw both triangles ABC and A'B'C'. Identify and describe fully the transformation.

Solution:

To get the image we multiply the points by the matrix.

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} -2 & -4 & -2 \\ 2 & 2 & 4 \end{pmatrix}$. Now the coordinates of the image are A' (-2, 2), B' (-4, 2), C' (-2, 4).

Below is Triangle ABC and its image A'B'C'.

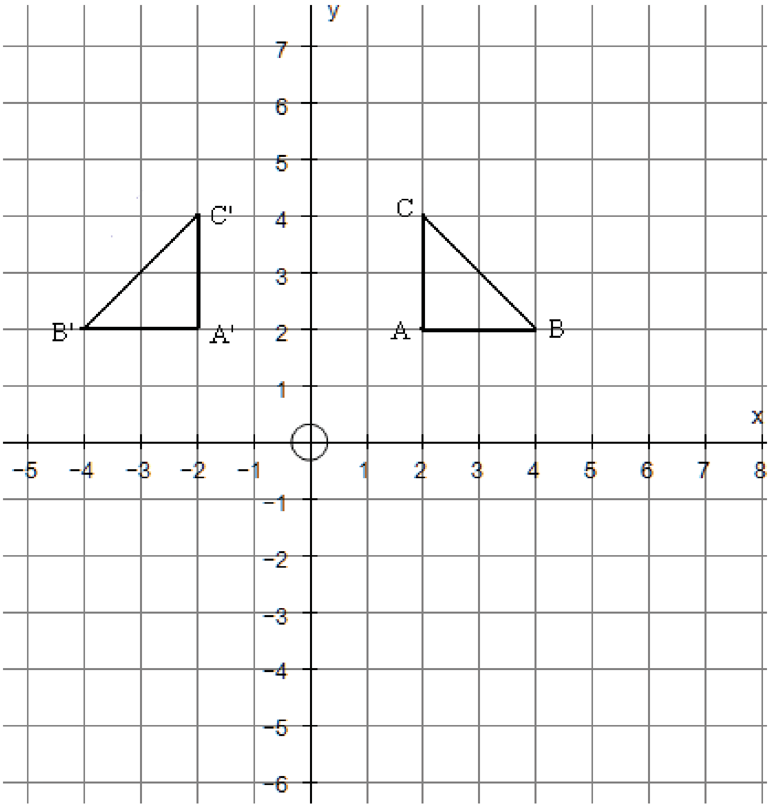


Figure 41

The transformation is a reflection through the y- axis as the mirror line.

Example 2:

Shape ABCD has the following coordinates: A (0, 2), B (2, 2), C (2, 0), D (0, 0). Find the coordinates of the vertices of the image A'B'C' D' under a transformation represented by the matrix $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$. Then draw both shapes ABCD and A'B'C'D' and identify and describe fully the transformation.

Solution:

To get the image we multiply the points by the matrix.

$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 8 & 2 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}$ Now the coordinates of the image are A' (6, 2), B' (8, 2), C' (2, 0), D (0, 0).

ABCD and its image A'B'C'D' are show in the following figure.

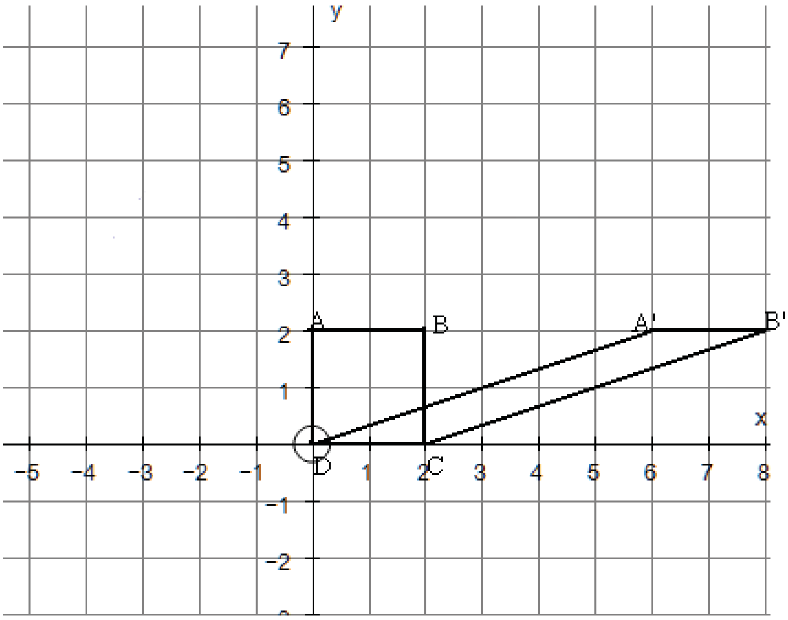


Figure 42

The transformation that maps ABCD onto A'B'C'D' is a shear. The invariant line is the x-axis.

$$Shear\ \ factor = \frac{dis\tan ce\ moved\ by\ a\ po\text{int}}{dis\tan ce\ of\ \ a\ \ po\text{int}\ \ from\ the\ in\text{var}iant\ line}$$

$$let\ the\ in\text{var}iant\ line\ be\ I.$$

$$Shear\ \ factor = \frac{AA'}{AI} = \frac{BB'}{BI} = \frac{6}{2} = 3.$$

This is how shapes and points are transformed with matrices. Now try activity 5 to have some practice on matrices and transformations.

Activity 5

Activity

1. ABC is a triangle with the following coordinates: A (1, 0), B (3, 0), C (1, 4). Transform triangle ABC using the following matrices. In each case draw the image and the object, identify and describe fully the transformation represented.

a) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Name the image A'B'C'.

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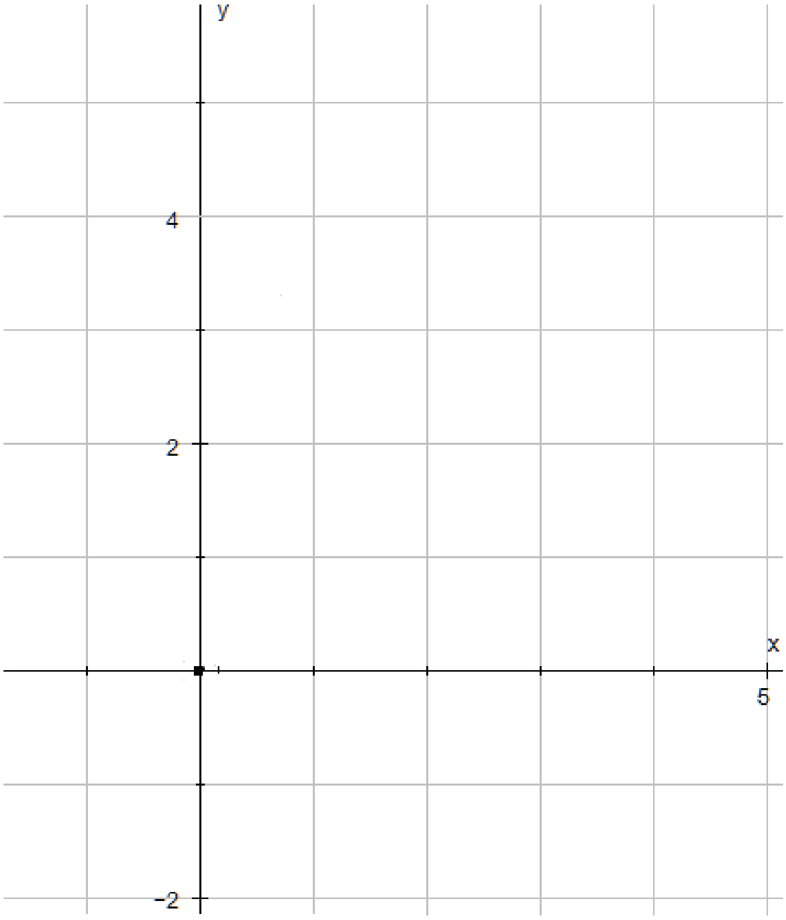

b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Name the image A"B"C".

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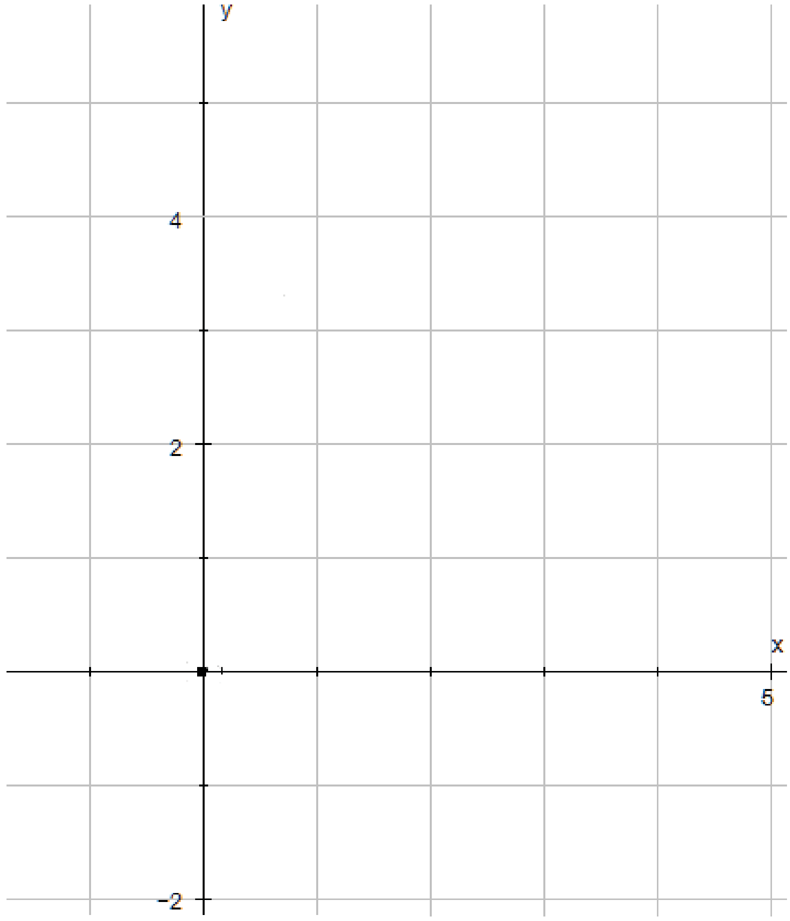
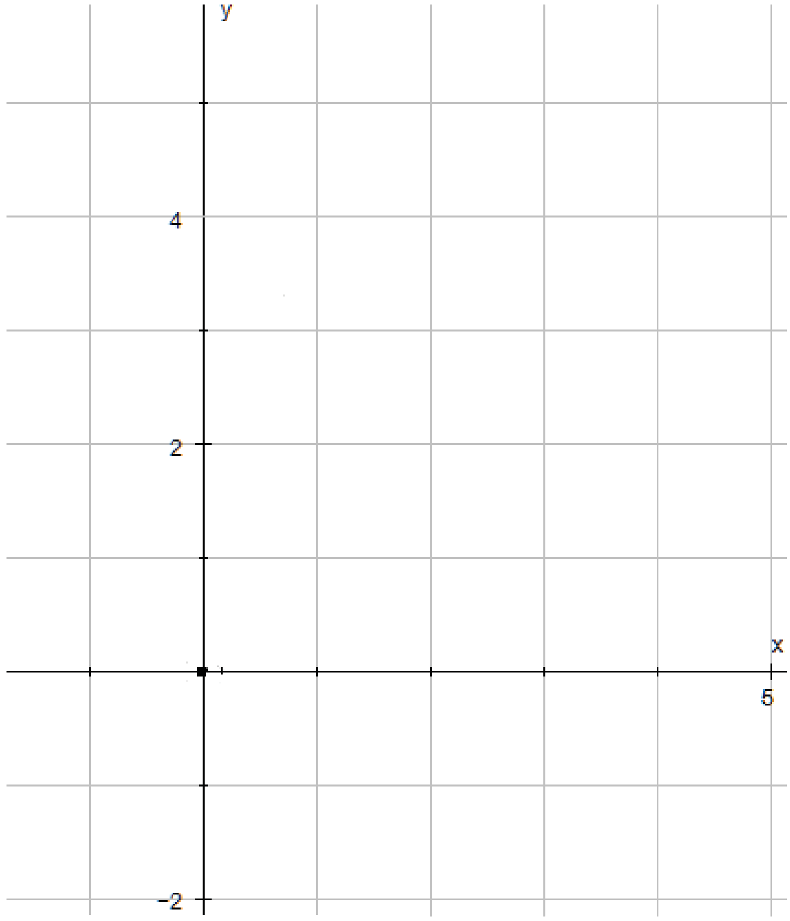
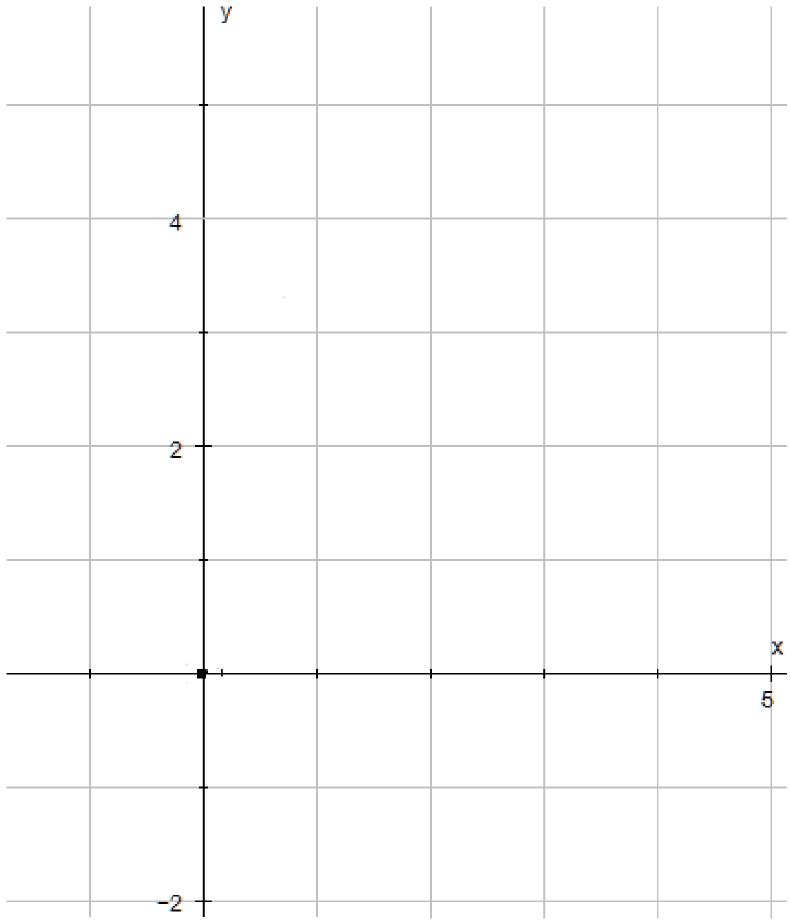


c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ Name the image A'''B'''C'''.

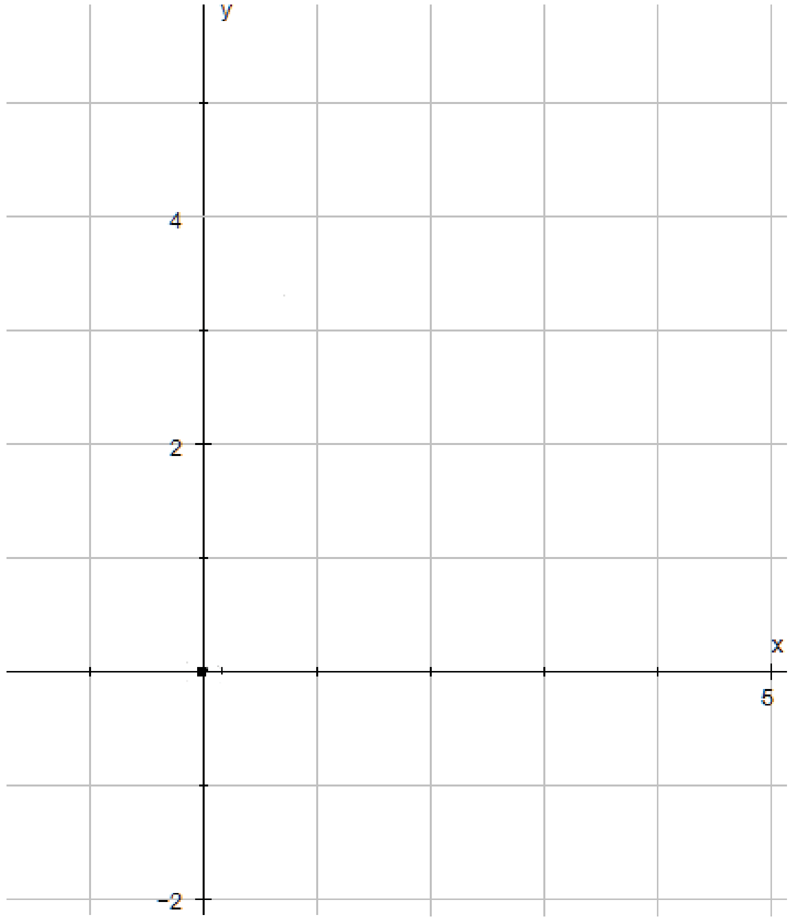


d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ Name the image A""B""C"".

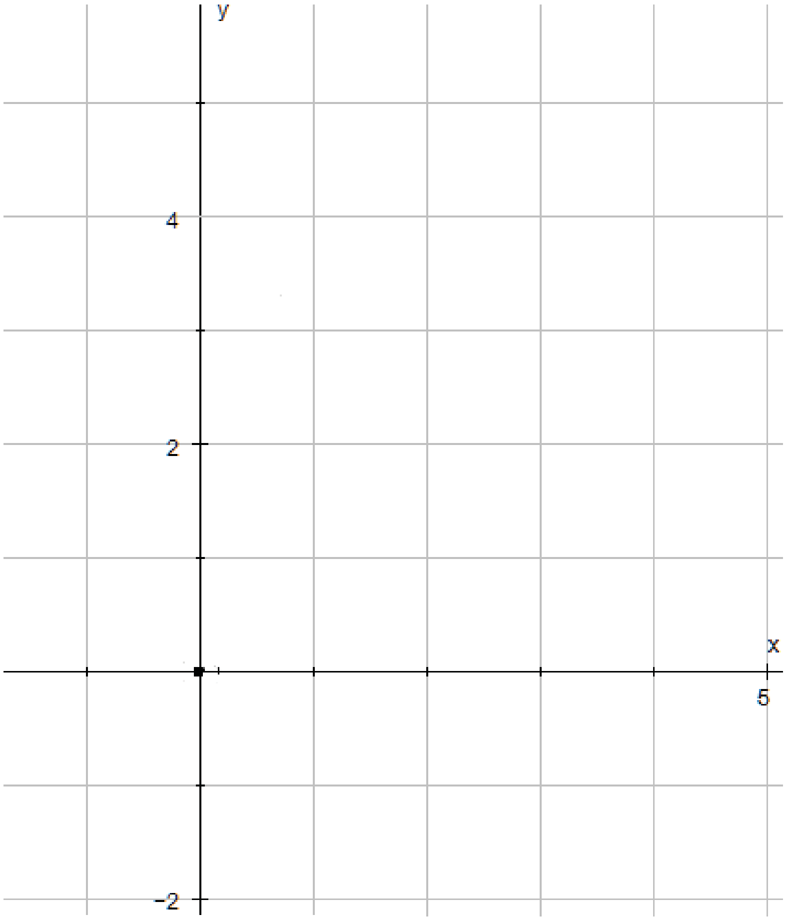


2. WXYZ is a rectangle with coordinates: W (1, 1), X (4, 1), Y (1, -2) and Z (4, -2).

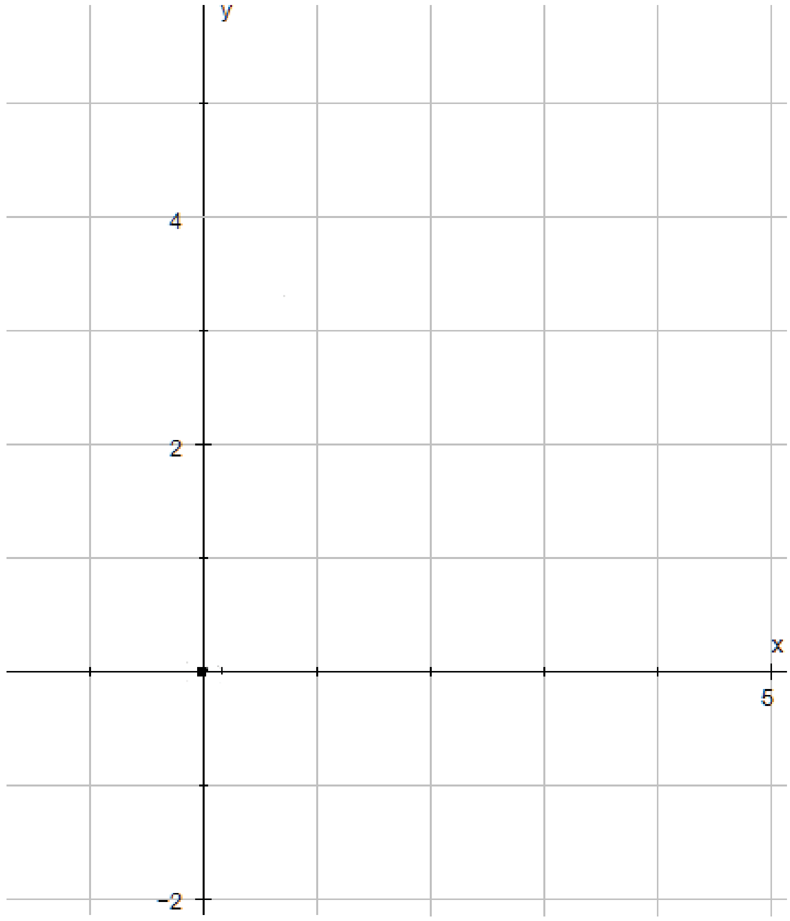
a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ Name the image W'X'Y'Z'.



b) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ Name the image W"X"Y"Z".



c) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ Name the image W'''X'''Y'''Z'''.



The answers to activity 5 have been provided at the end of the subunit; you can compare your answer and see how you have done.

Finding the Matrix of a Transformation

We have seen that transformations can be defined with matrices. This means that for a given transformation we can find the matrix of that transformation. In this section we are finding the matrices of given transformations.

The following examples are meant to help you learn how to find the matrices of transformations.

Example 1

Point P (1, 1) maps onto (-1, -1) and point Q (3, 1) maps onto (-1, -3), find the two by two matrix that represents the transformation.

Solution

We now know that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

And that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

When combining the two matrix equations we obtain:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -3 \end{pmatrix}$$

Now we can solve for a, b, c and d by multiplying both sides of the equation with the inverse of the matrix $\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$. To find how to get the inverse of a matrix consult the topic on matrices. The inverse is $\frac{1}{-2}\begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}$.

Now multiplying both sides of the equation by the inverse matrix we get:

$$\frac{1}{-2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{-2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$\Rightarrow a = 0, b = -1, c = -1 \text{ and } d = 0.$

Example 2

In figure 43 below, triangle A has been transformed into triangle D. Find the matrix representing the transformation shown in the following diagram.

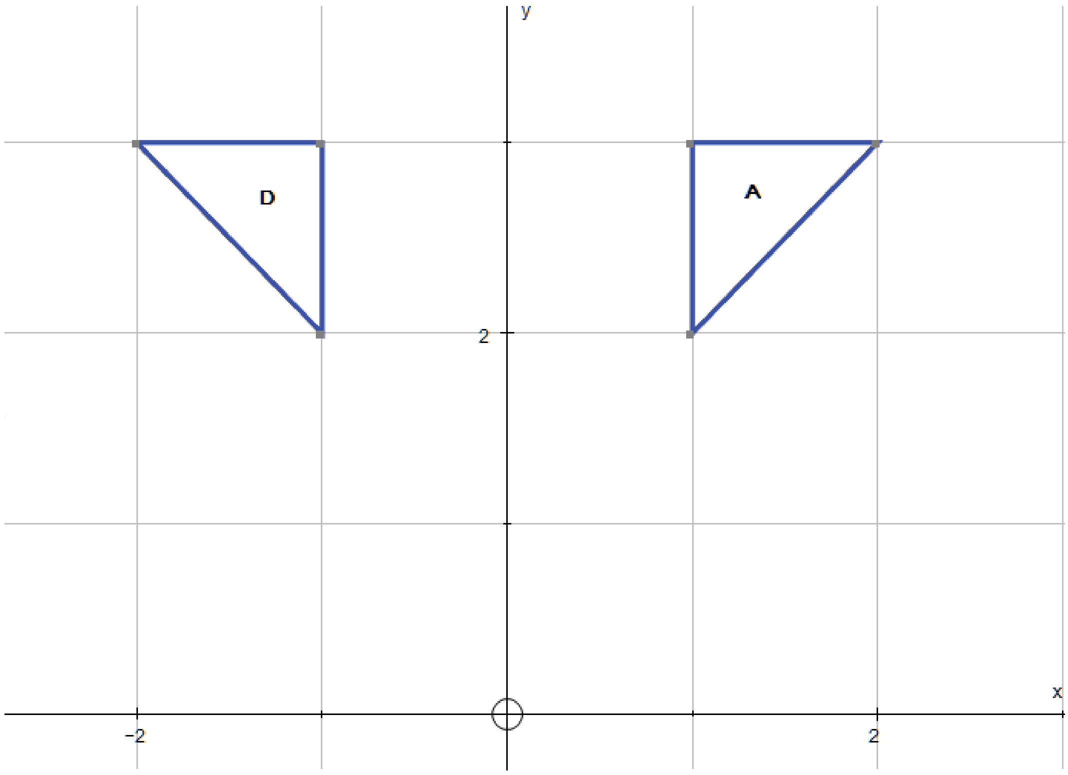


Figure 43

Solution:

As we see $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

And $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

When combining the two matrix equations we obtain:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$$

Now we can solve for a, b, c and d by multiplying both sides of the equation with the inverse of the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. The inverse is $-1\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$.

Now multiplying both sides of the equation by the inverse matrix we get:

$$-1\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} = -1\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\Rightarrow -1\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -1\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow a = 1, b = -1, c = 0$ *and* $d = 1$.

So that is how we find the matrix of a transformation.

Now work on activity 6 and see how much you got on finding the matrix representing a transformation.

Activity

Activity 6

1. Triangle ABC has Vertices A (6, 12), B (8, 4), and C (2, 2).Triangle ABC has been mapped onto triangle FGH with vertices F (-12, 6), G (-4, 8) and H (-2, 2) by a transformation.

Find the matrices representing the transformation.

2.

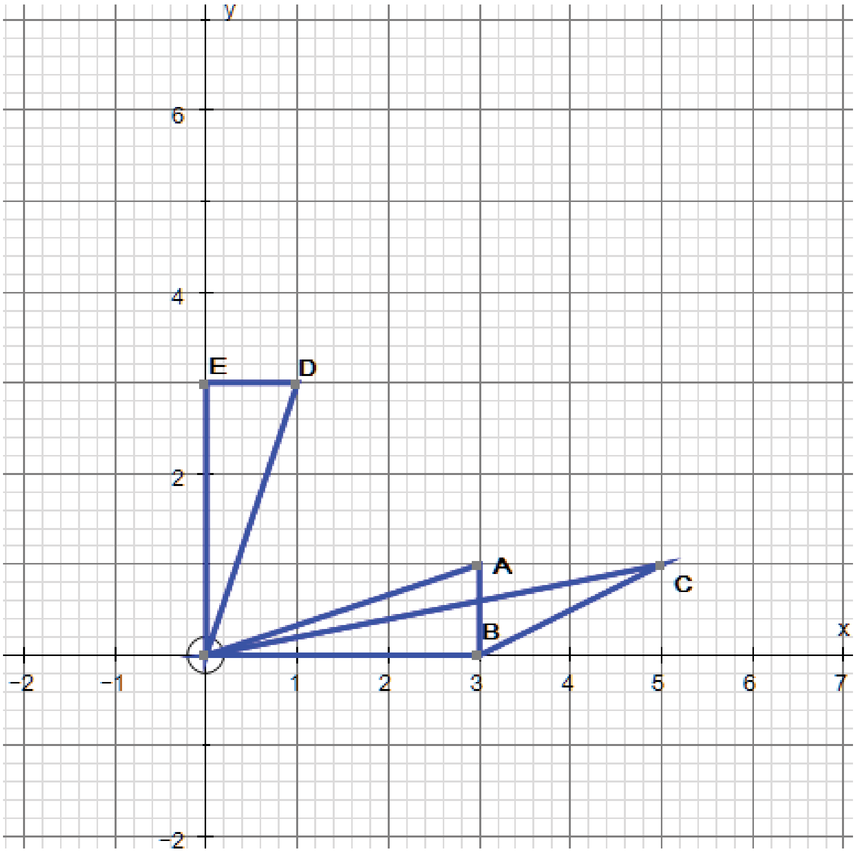


Figure 44

In the above diagram, O is the origin, A is the point (3, 1), B is (3, 0), C is (5, 1), D is (1, 3) and E is (0, 3).

The single transformation P maps triangle OAB onto triangle ODE.

The single transformation Q maps triangle OAB onto triangle OCB.

a)

i) Describe P Completely.

ii) Find the matrix which represents P.

b)

i) What kind of transformation is Q?

ii) The matrix which represents Q is $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

Find the value of n.

3. Find the matrix for the following transformation.

(-1, -1)→ (1, 0) and (2, 0) → (0, 1).

When you are done answering the questions on activity 6 compare your answers with those given at the end of the subunit. If there is anything that you have missed feel free to go back to examples and try to understand better.

In this Subunit You Have Learned that

-2×2 matrices can be used to define and represent matrices.

-To find the matrix of a transformation we take any two points of the object with corresponding two of the image and form a matrix equation such that the unknown matrix is multiplied with the matrix of the object to give the matrix of the image. To solve now for the unknown matrix the inverse of the matrix of the object multiplies both sides of the equation.

Solutions to Activities:

Activity 5

No.1

a) To get the image we multiply the points by the matrix.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & -4 \end{pmatrix}.$$

Now the coordinates of the image are A' (1, 0), B' (3, 0), C' (1, -4).

Below is Triangle ABC and its image A'B'C'.

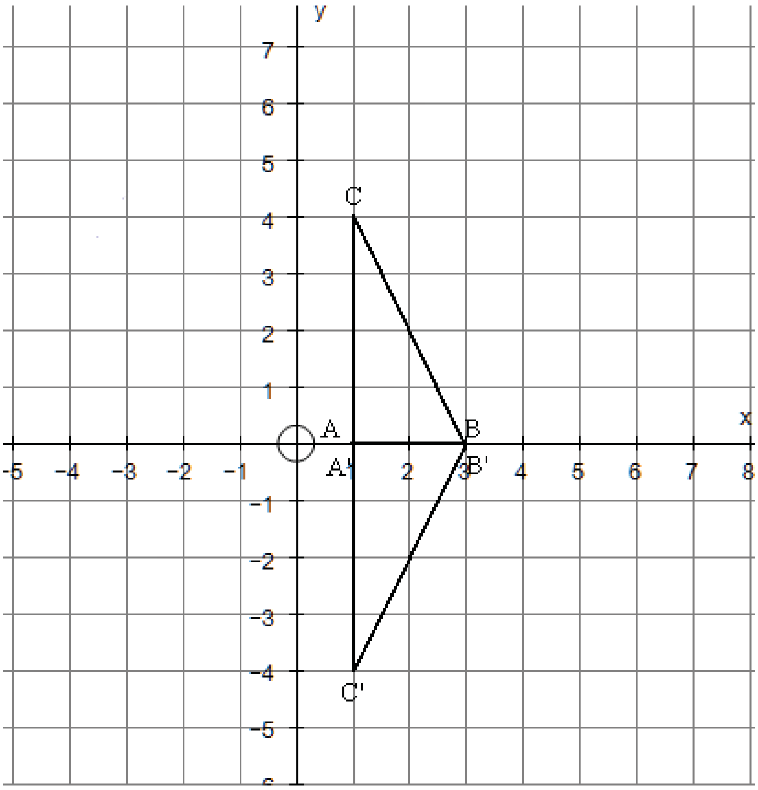


Figure 45

The transformation is a reflection in the x- axis

b) To get the image we multiply the points by the matrix.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 4 \\ 1 & 3 & 1 \end{pmatrix}.$$

Now the coordinates of the image are A" (0, 1), B" (0, 3), C" (4, 1).

Below is Triangle ABC and its image A"B"C".

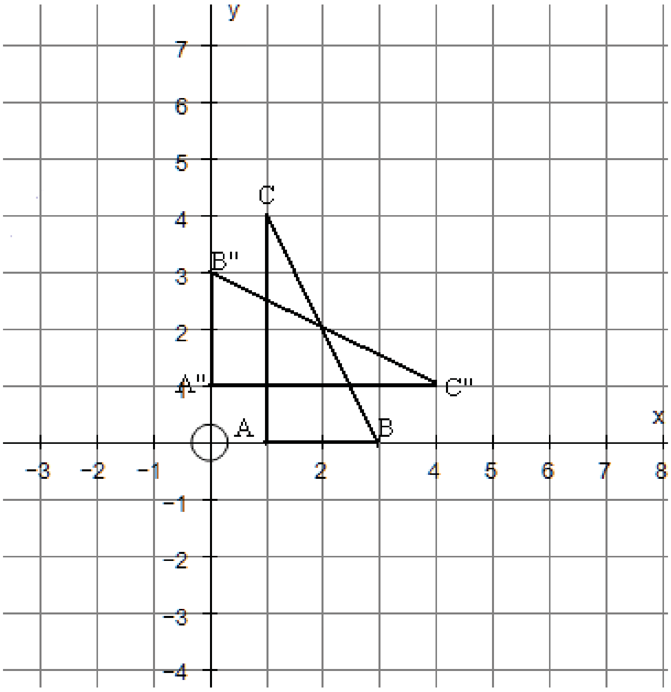


Figure 46

The transformation is a reflection through the line y= x.

c) To get the image we multiply the points by the matrix.

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 2 \\ 0 & 0 & 8 \end{pmatrix}.$$

Now the coordinates of the image are A''' (2, 0), B''' (6, 0), C''' (2, 8).

Below is Triangle ABC and its image A'''B'''C'''.

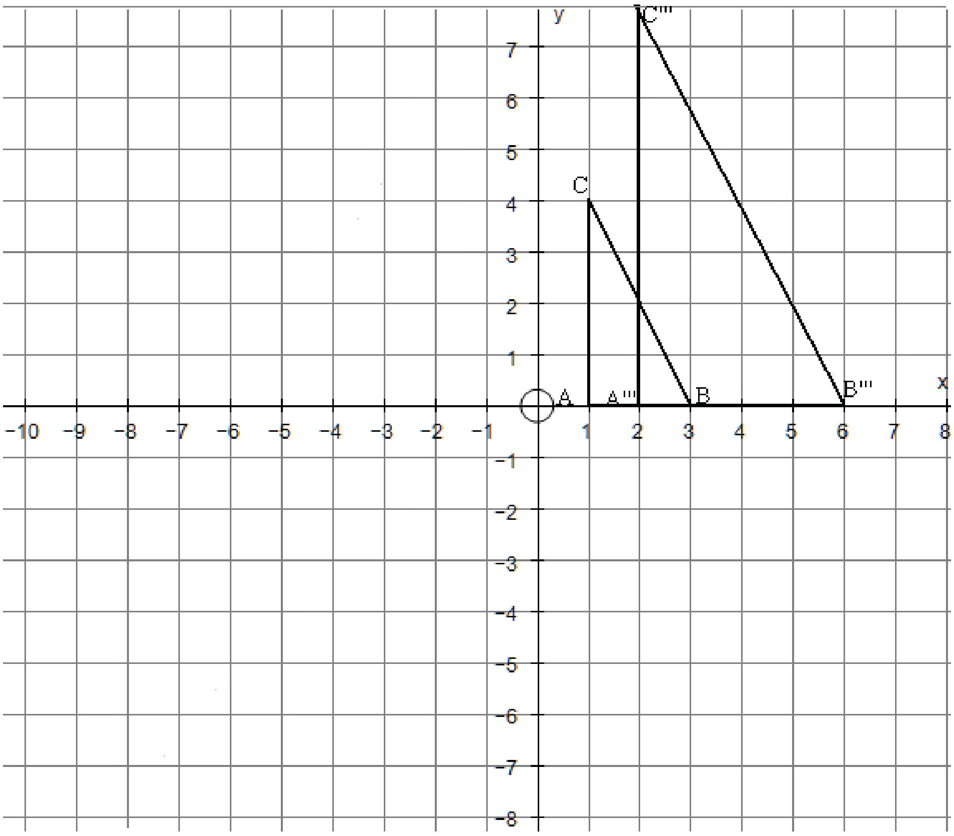


Figure 47

The transformation is an enlargement, centre (0, 0) and scale factor 2.

d) To get the image we multiply the points by the matrix. .

Now the coordinates of the image are A'''' (1, 0), B'''' (3, 0), C'''' (1, -4).

Below is Triangle ABC and its image A''''B''''C''''.

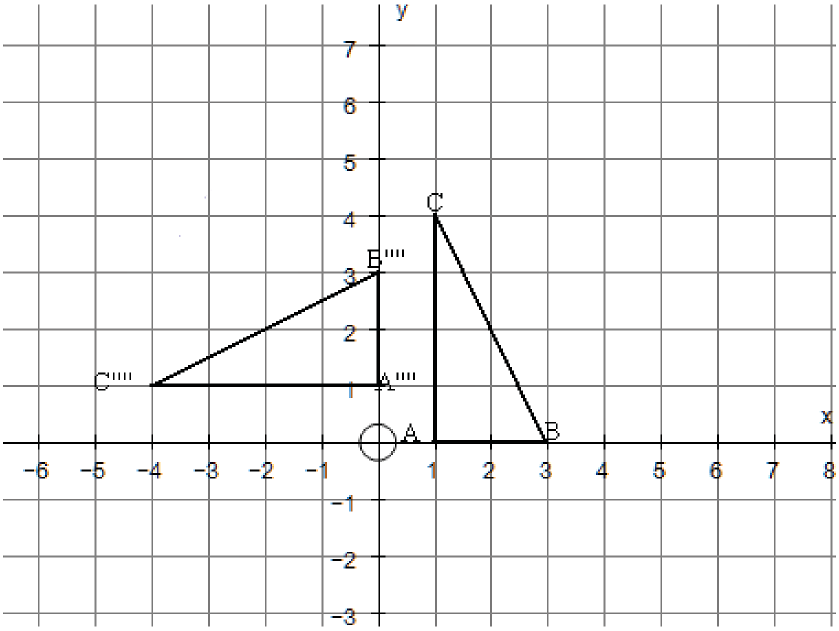


Figure 48

The transformation is a rotation about the point (0, 0) through 90^0 anticlockwise.

No.2

a) To get the image we multiply the points by the matrix.

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 1 & 4 \\ 1 & 1 & -2 & -2 \end{pmatrix} = \begin{pmatrix} -1 & -4 & -1 & -4 \\ -1 & -1 & 2 & 2 \end{pmatrix}.$$

Now the coordinates of the image are W' (1, 1), X' (4, 1), Y' (1, -2), Z' (4, -2).

Below is shape WXYZ and its image W'X'Y'Z'.

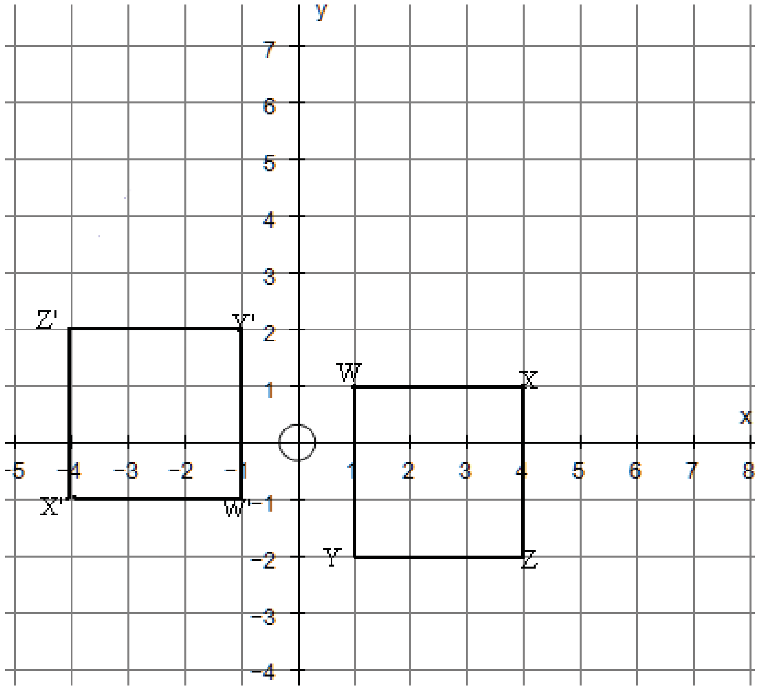


Figure 49

The transformation is a rotation through the point (0, 0) 180^0 anticlockwise.

b)) To get the image we multiply the points by the matrix.

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 1 & 4 \\ 1 & 1 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 8 & 2 & 8 \\ 1 & 1 & -2 & -2 \end{pmatrix}.$$

Now the coordinates of the image are W" (2, 1), X" (8, 1), Y" (2, -2), Z" (8, -2).

Below is the shape WXYZ and its image W"X"Y"Z".

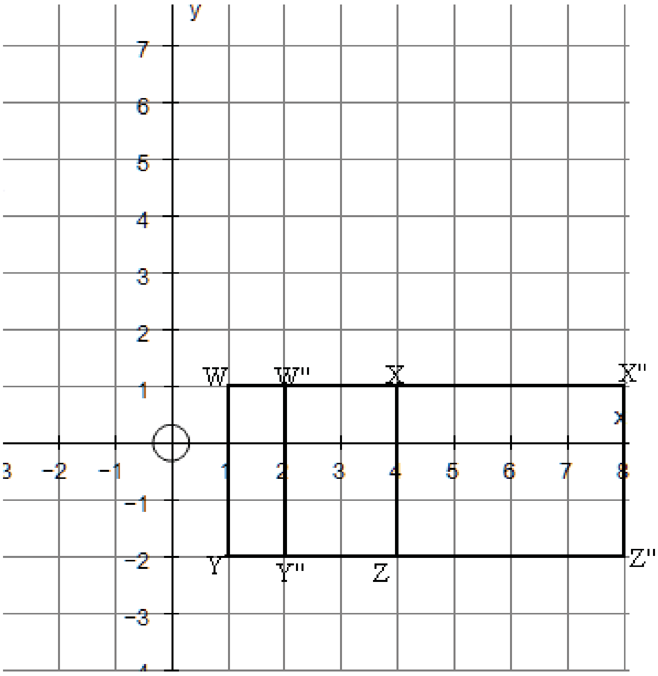


Figure 50

The transformation is a stretch, stretch factor=3 and the invariant line is the y-axis.

c) To get the image we multiply the points by the matrix.

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 1 & 4 \\ 1 & 1 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 7 & -5 & -2 \\ 1 & 1 & -2 & -2 \end{pmatrix}.$$

Now the coordinates of the image are W''' (4, 1), X''' (7, 1), Y''' (-5, -2), Z''' (-2, -2).

Below is the shape WXYZ and its image W'''X'''Y'''Z'''.

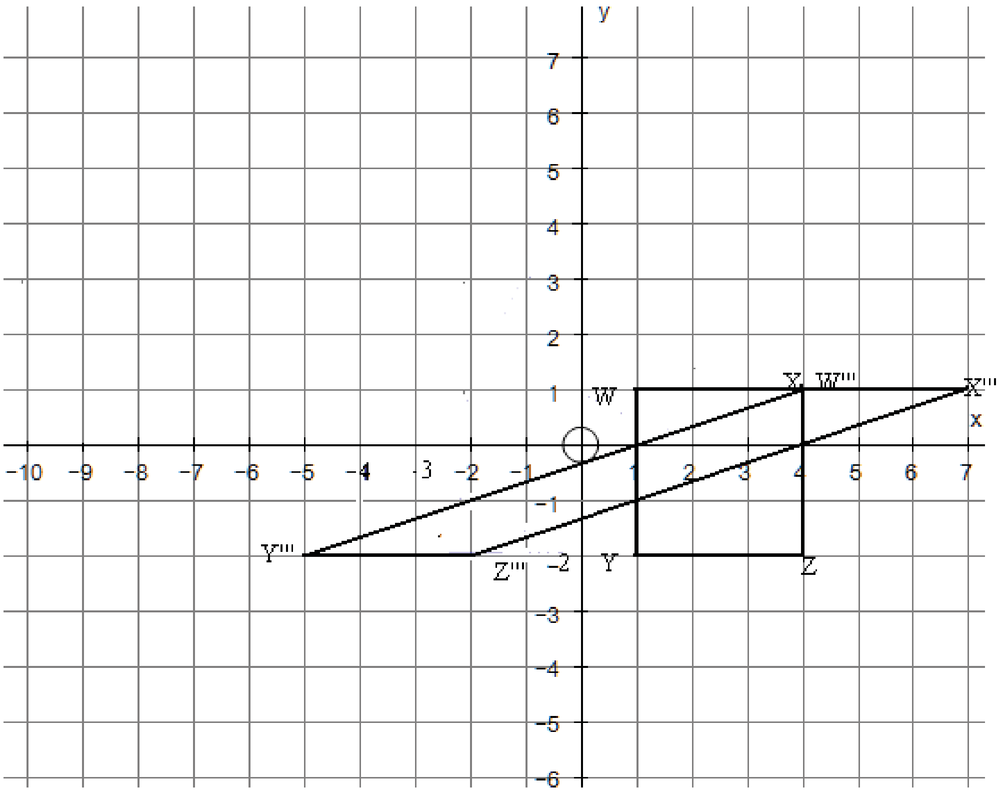


Figure 51

The transformation is a shear, shear factor = 3 and the invariant line is the x-axis.

Activity 6

1. To find the matrix we can use only two points of the object and corresponding two of the image.

Let the matrix representing the transformation be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 6 & 8 \\ 12 & 4 \end{pmatrix} = \begin{pmatrix} -12 & -4 \\ 6 & 8 \end{pmatrix}$$

Now we can solve for a, b, c and d by multiplying both sides of the equation with the inverse of the matrix $\begin{pmatrix} 6 & 8 \\ 12 & 4 \end{pmatrix}$. The inverse is $\frac{1}{-72}\begin{pmatrix} 4 & -8 \\ -12 & 6 \end{pmatrix}$.

Now multiplying both sides of the equation by the inverse matrix we get:

$$\frac{1}{-72}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 6 & 8 \\ 12 & 4 \end{pmatrix}\begin{pmatrix} 4 & -8 \\ -12 & 6 \end{pmatrix} = \frac{1}{-72}\begin{pmatrix} -12 & -4 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} 4 & -8 \\ -12 & 6 \end{pmatrix}$$

$$\Rightarrow \frac{1}{-72}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -72 & 0 \\ 0 & -72 \end{pmatrix} = \frac{1}{-72}\begin{pmatrix} 0 & 72 \\ -72 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow a = 0, b = -1, c = 1 \text{ and } d = 0.$$

2.

a)

i) P is a reflection, mirror line y=x.

ii) Let the matrix representing the transformation be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix}$$

Now we can solve for a, b, c and d by multiplying both sides of the equation with the inverse of the matrix $\begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}$. The inverse is $\frac{1}{3}\begin{pmatrix} 1 & -3 \\ 0 & 3 \end{pmatrix}$.

Now multiplying both sides of the equation by the inverse matrix we get:

$$\frac{1}{3}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 3 \end{pmatrix}$$

$$\Rightarrow \frac{1}{3}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = P$$

$\Rightarrow a = 0, b = 1, c = 1$ *and* $d = 0.$

b)

i) Q is a shear, invariant line is the x-axis. Shear factor:

$$\frac{AC}{AI} = \frac{2}{1} = 2$$

ii)

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3+n & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow 3 + n = 5$

$\Rightarrow n = 2$

3.

Let the matrix representing the transformation be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now we can solve for a, b, c and d by multiplying both sides of the equation with the inverse of the matrix $\begin{pmatrix} -1 & 2 \\ -1 & 0 \end{pmatrix}$. The inverse is $\frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$.

Now multiplying both sides of the equation by the inverse matrix we get:

$$\frac{1}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}$$

$$\Rightarrow a = 0,\ b = -1,\ c = \frac{1}{2}\ and\ d = \frac{-1}{2}.$$

Lesson 5 Combined Transformation

At the end of this subunit you should be able to:

-Transform an object by a number of similar or different transformations and matrices.

-Identify the single transformation that maps the object onto the final image.

This subunit consists of 19 pages.

With combined transformation, we are dealing with a transformation followed by another on the same object. The following examples are meant to help you learn how to solve transformation problems with combined transformation be it transformations of the same kind, different transformation or matrices with transformations.

Example 1

This is an example of transformations of the same kind.

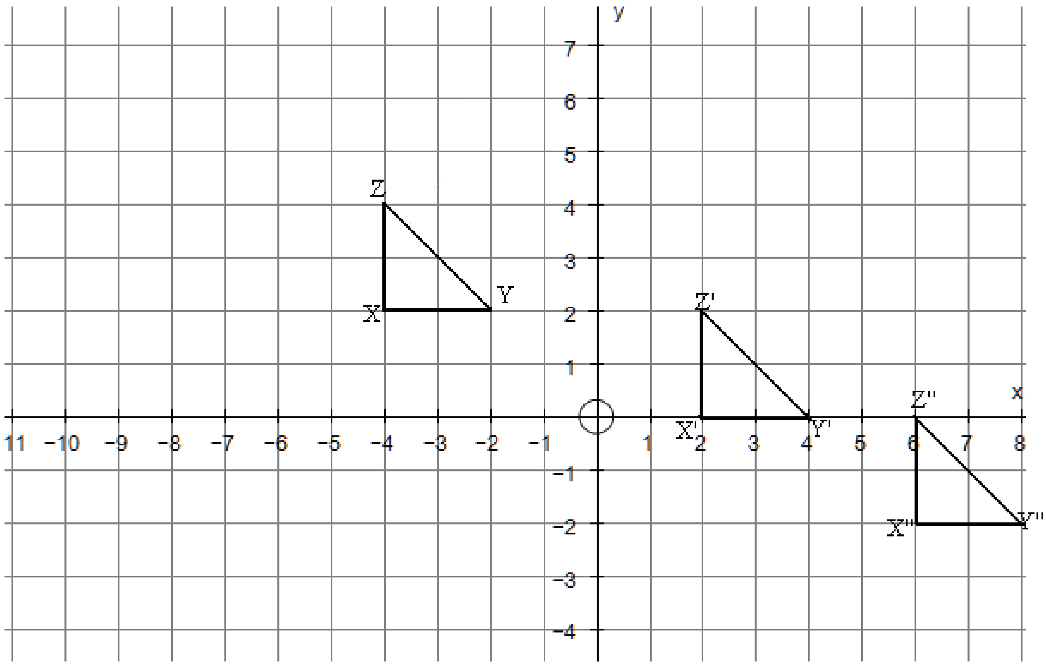


Figure 52

In the above diagram triangle XYZ ((-4, 2) (-2, 2) (-4, 4)) is translated first by the translation vector $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$,

To give the image X'Y'Z', then by the translation vector $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ to give the image X"Y"Z".

a) What is the final position of the of the triangle?

b) Describe fully the single transformation that maps triangle XYZ onto X"Y"Z".

Solution

a) The final position of the triangle is ((6, -2) (8, -2) (6, 0)).

b) The single transformation that maps triangle XYZ onto triangle X"Y"Z" is a translation, with translation vector $\begin{pmatrix} 10 \\ -4 \end{pmatrix}$ which could be obtained by adding the two translation vectors together. $\begin{pmatrix} 6 \\ -2 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \end{pmatrix}$.

So we can see that the combination of translations gives a translation.

This is an example of a combination of to transformations of the same kind. Let us do another example on transformations of the same kind.

Example 2

This also is another example of the transformations of the same kind.

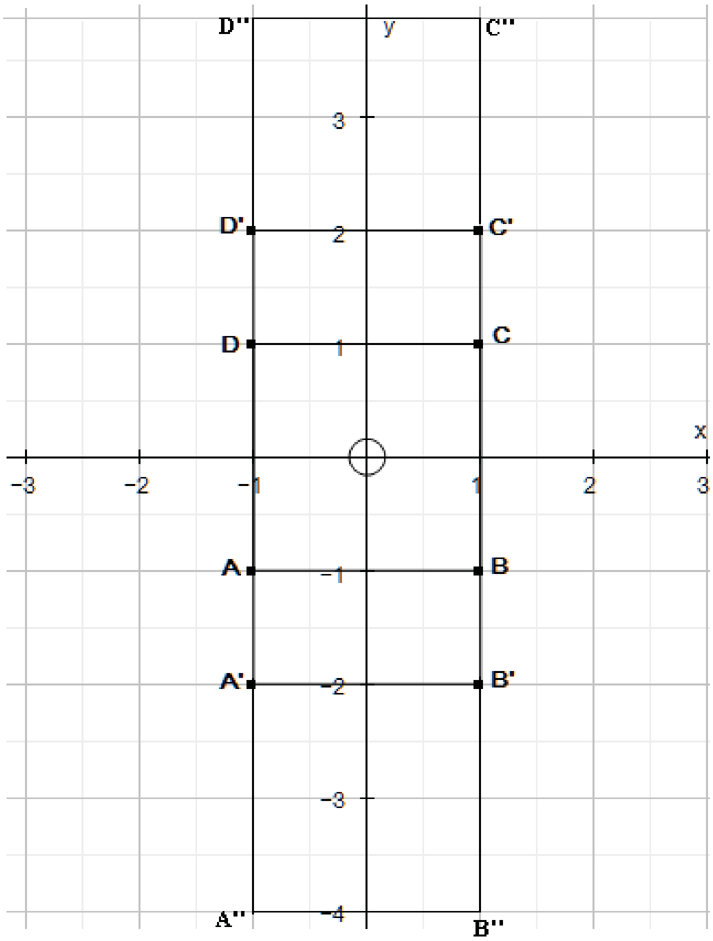


Figure 53

In the above diagram ABCD has been stretched to A'B'C'D', through the invariant line y=0 and stretch factor 2then A'B'C'D' has been stretched through the invariant line y=0 with the stretch factor 2 to get A"B"C"D".

Describe fully the single transformation that maps triangle ABCD onto triangle A"B"C"D".

Solution

The single transformation that maps triangle ABCD onto triangle A"B"C"D" is a stretch with stretch factor 4.I believe we now have an idea of what is meant by combined transformation. It could be a stretch followed by a stretch, a shear followed by a shear or anything.

Example 3

We said earlier in this subtopic that it also can happen that an object goes through a series of different transformation. This is an example of such.

Triangle A is transformed by an enlargement E with scale factor 2 and centre (-4, 0).

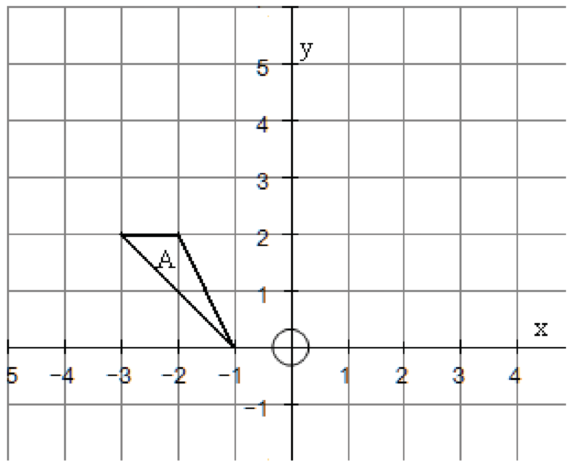


Figure 54

i) Draw E (A) and label it B.

ii) This is followed by a translation $T = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$. Draw TE (A) and label it C.

iii) Describe fully the single transformation which maps A onto C.

Solution:

i), ii)

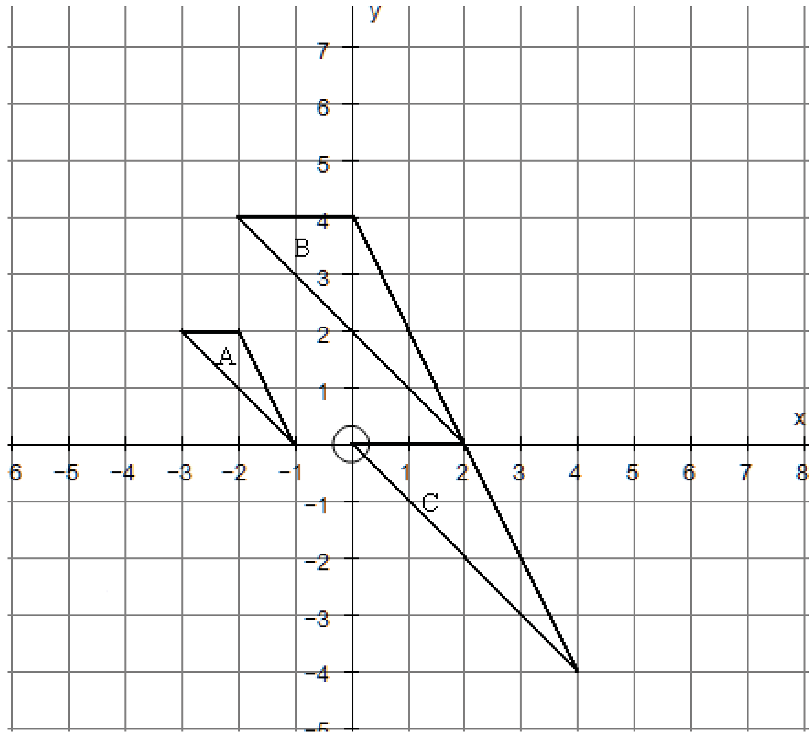


Figure 55

iii) The single transformation which maps triangle A onto triangle C is an enlargement, centre (-6, 4) and factor 2.

I believe you can see that triangle A could have still been directly mapped onto triangle C through enlargement.

Now consider example 4 and see an example of matrices and combined transformations.

Example 4

Triangle A in the diagram is transformed using a matrix T followed by a matrix H, so that T(A)=B and HT(A)=C.

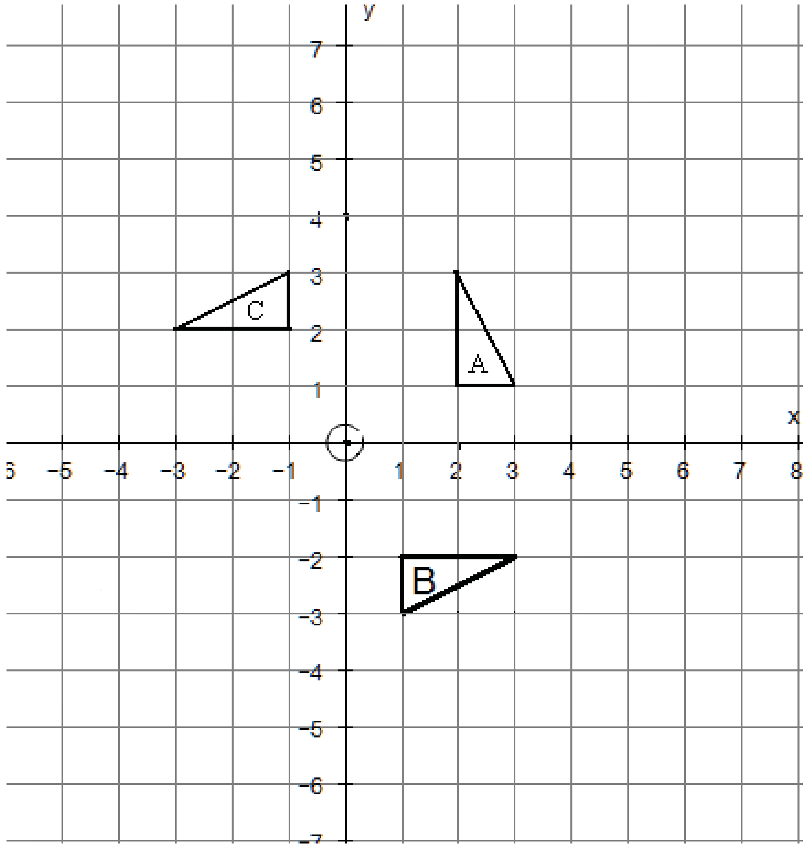


Figure 56

i) Identify the transformation T and write down the matrix of the transformation.

ii) Identify the transformation H and write down the matrix of the transformation.

iii) Find the matrix HT and apply it to the vertices of A.

Solution

i) The transformation T is a rotation, centre (0, 0), through 90˚ clockwise.

To find the matrix of the transformation we can use any two points of the object and corresponding two of the image.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix}$$

multiplying by the inverse of the matrix on both sides we get :

$$\frac{1}{4}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow \frac{1}{4}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{which is the matrix of the transformation.}$$

ii) Transformation H is a rotation about the point (0, 0), through 180°.

To find the matrix of the transformation we can use any two points of the object and corresponding two of the image.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} -1 & -3 \\ 2 & 2 \end{pmatrix}$$

multiplying by the inverse of the matrix on both sides we get :

$$\frac{1}{4}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -1 & -3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix}$$

$$\Rightarrow \frac{1}{4}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{which is the matrix of the transformation.}$$

iii)

$$\text{The matrix } HT = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Now applying the matrix to the vertices of triangle A we get :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -3 \\ 2 & 3 & 2 \end{pmatrix}$$

which we can see that it gives the coordinates of triangle C.

so triangle A is mapped onto triangle C by a rotation, centre (0,0) *through* 90^0 *anticlo*

Activity

I believe you are now ready to go through the activity and answer the question. If you meet any problems while trying to answer the questions feel free to go back to the example and try to understand better.

Activity 7

1. In the following diagram, triangle A has been mapped onto triangle B by a translation, translation vector $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$, then translated to triangle C by the translation vector $\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ and finally translated to triangle D by the translation vector $\begin{pmatrix} 8 \\ 0 \end{pmatrix}$.

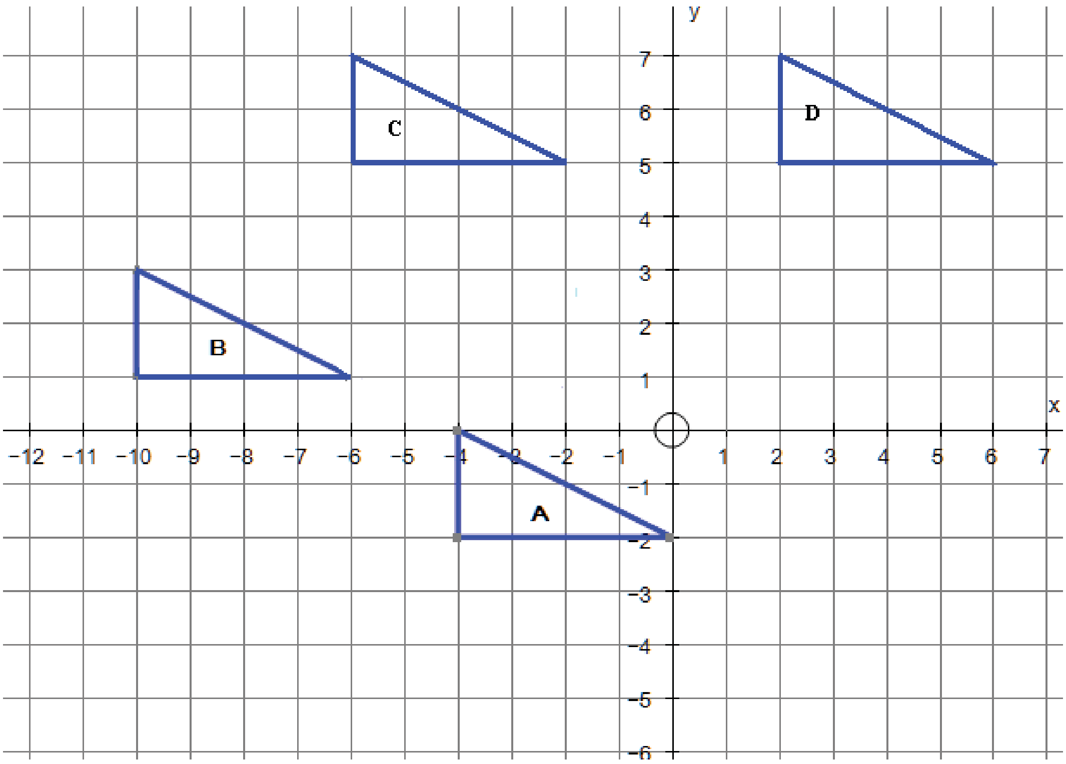


Figure 57

Describe fully the single transformation that maps triangle A onto Triangle D.

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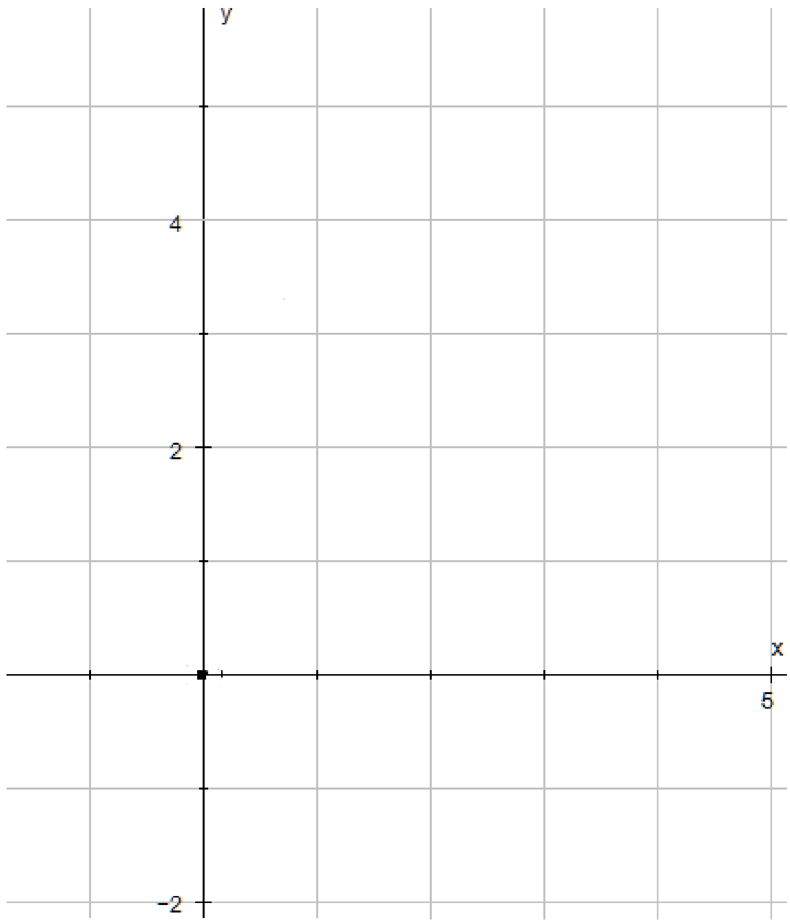
2. Triangle A ((-1, 2) (-1, 3) (-2, 3)) has been transformed by a reflection, with the mirror line as the y-axis to give triangle B, then reflected again through the line X=2 to give triangle C.

a) Draw Triangles A, B and C.



b) Identify and describe fully the single transformation that maps triangle A onto triangle C.

3. Plot the points P (-1, 3), Q (-1, 5), R (2, 5), and S (2, 3) on a pair of axes.

a) Draw the line y=x. Find the images of PQRS under a reflection in t he line y=x. Label the image P'Q'R'S'.

b) Draw P"Q"R"S" the image of P'Q'R'S' under a reflection in the y-axis.

c) Describe the rotation that will map PQRS directly onto P"Q"R"S".

4.

The diagram shows triangles A, B, C and D.

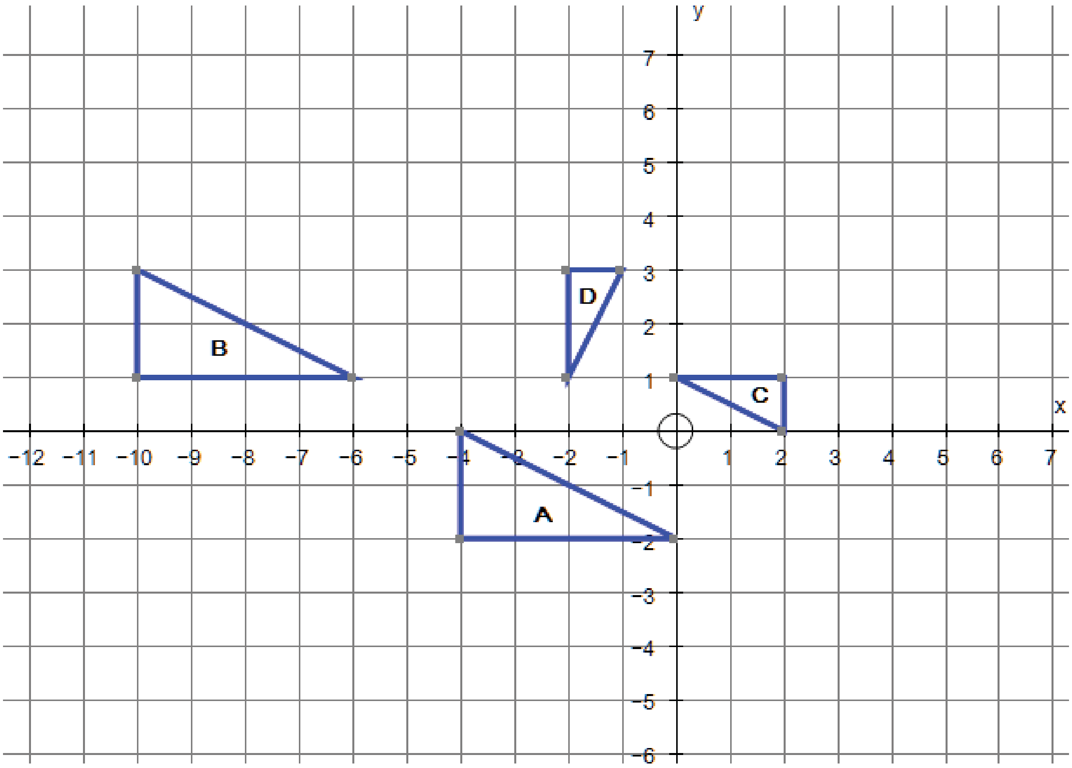


Figure 58

a) Describe fully the **single** transformation that maps triangle A onto triangle B.

b) Describe fully the **single** transformation that maps triangle B onto triangle C.

c) Describe fully the **single** transformation that maps triangle C onto triangle D.

d) Write down the matrix that represents the transformation which maps triangle C onto triangle A.

5.

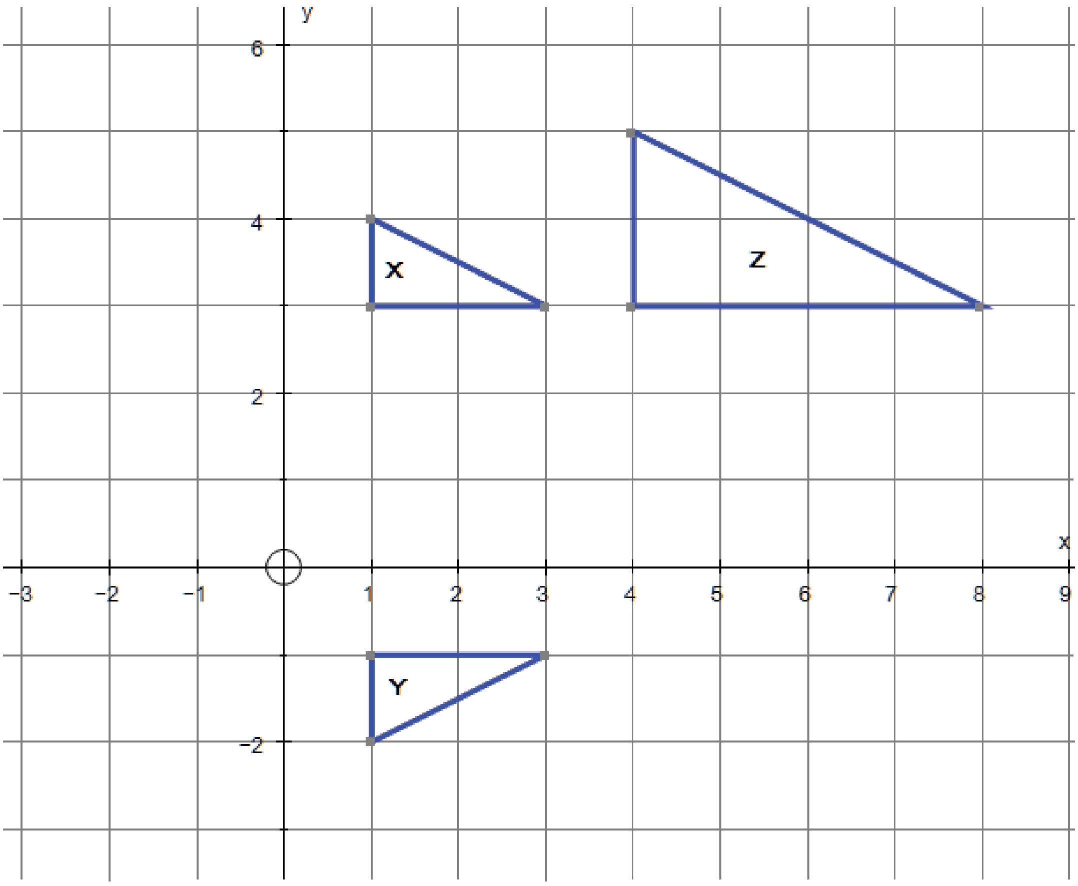


Figure 59

The diagram shows the triangles X, Y, and Z.

The single transformation D maps X onto Y.

Then enlargement E maps X onto Z.

i) Describe D completely.

ii) Find the coordinates of the centre of enlargement E.

iii) O is the origin. Find the coordinates of DE (O).

6.

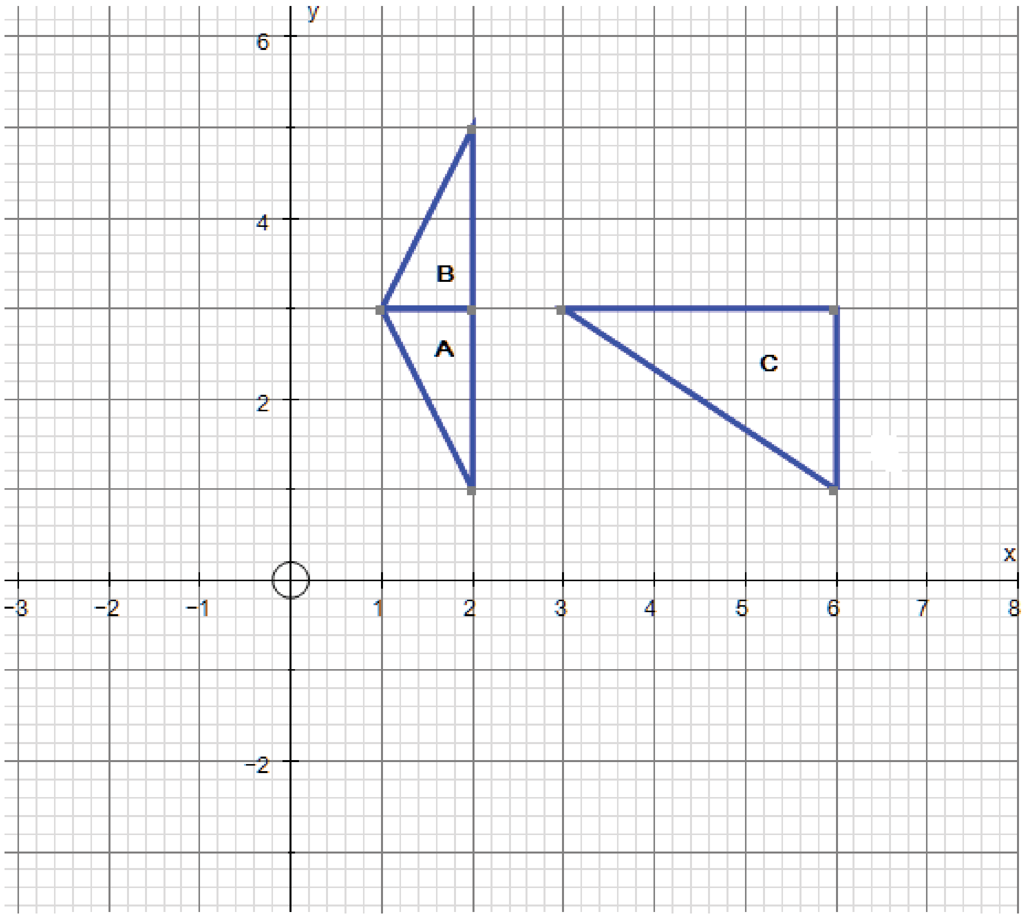


Figure 60

The diagram shows triangles A, B and C.

a) Describe fully the single transformation which maps A onto B.

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b) Describe fully the single transformation which maps A onto C.

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7.

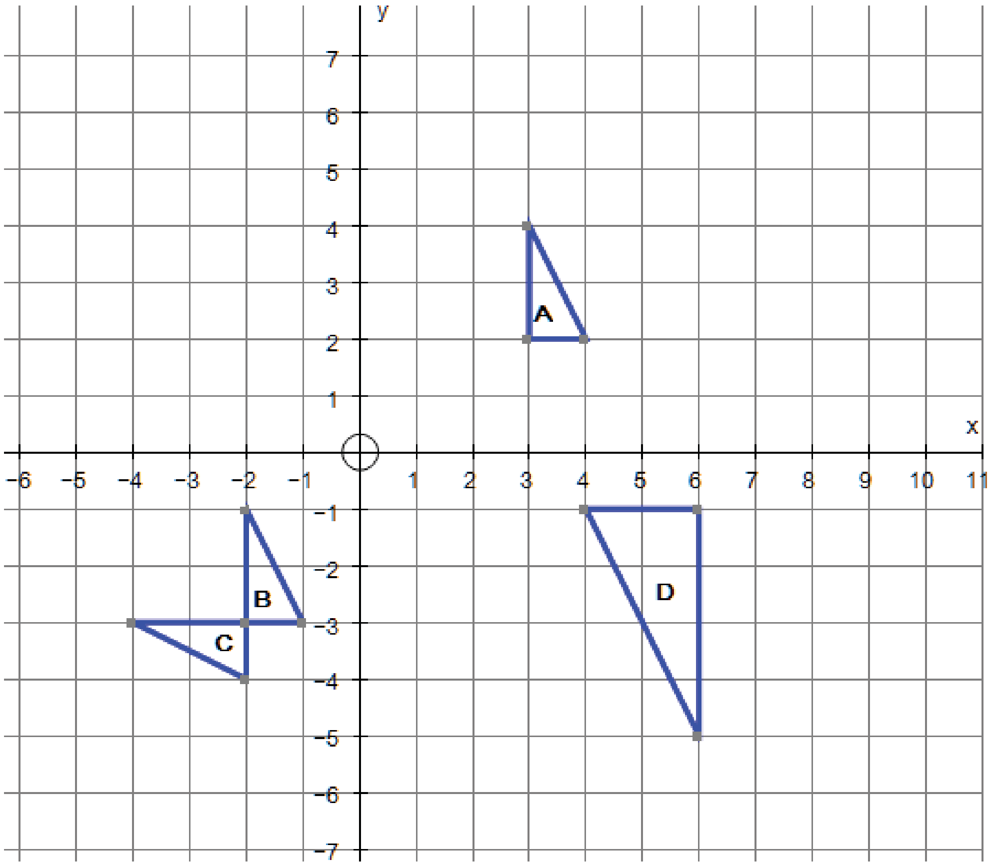

Figure 61

The diagram shows triangles A, B, C, D, and E.

i) Describe fully the single transformation that maps triangle A onto triangle B.

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ii) Describe fully the single transformation that maps triangle A onto triangle C.

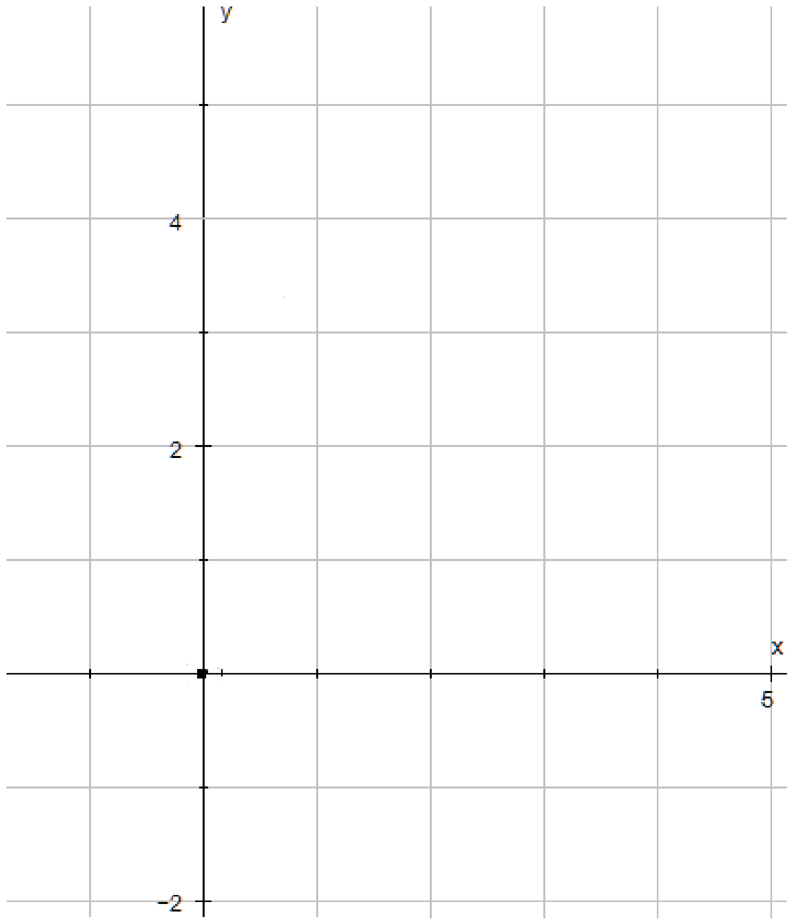
____

iii) Describe fully the single transformation that maps triangle B onto triangle C.

iv) Describe fully the single transformation that maps triangle A onto triangle D.

8. The vertices of triangle X have coordinates A (1, 1), B (3, 1) and C (1, 2).

a) Draw and label triangle X.



b) The matrix of the transformation P is $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, and is such that the images of the images of A, B and C under P are, respectively, A' (-2, -2), B' (-6, -2) and C' (-2, -4).

i) Draw triangle A'B'C' and label it P(X).

ii) Describe fully the single transformation P and write down the value of *a*.

c) The triangle A"B"C" is the image of triangle ABC under an enlargement E with centre (2, 1) and scale factor 4. Draw and label the triangle E(X), taking care to label the vertices A"B"C".

____

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d) Triangle A'B'C' can be mapped onto triangle A"B"C" by a single transformation F. Describe fully the single transformation F.

____

____

____

Compare your answers with those given at the end of the subunit .I believe you were able to get all the answers. If you got some wrong go back to the examples and see where you might have gone wrong.

In this Subunit You Have Learned That

-A single object can go through a series of transformations, of the same kind, different or even transformations with matrices.

- A transformations that maps the object to the final object can be found after an object has gone through the many transformations.

Solutions to Activity 7

1. The translation that maps triangle A onto triangle D is a translation, with translation vector $\begin{pmatrix} 6 \\ 7 \end{pmatrix}$. It can also be found

by $\begin{pmatrix} -6 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$.

2. a) The triangle are as follows.

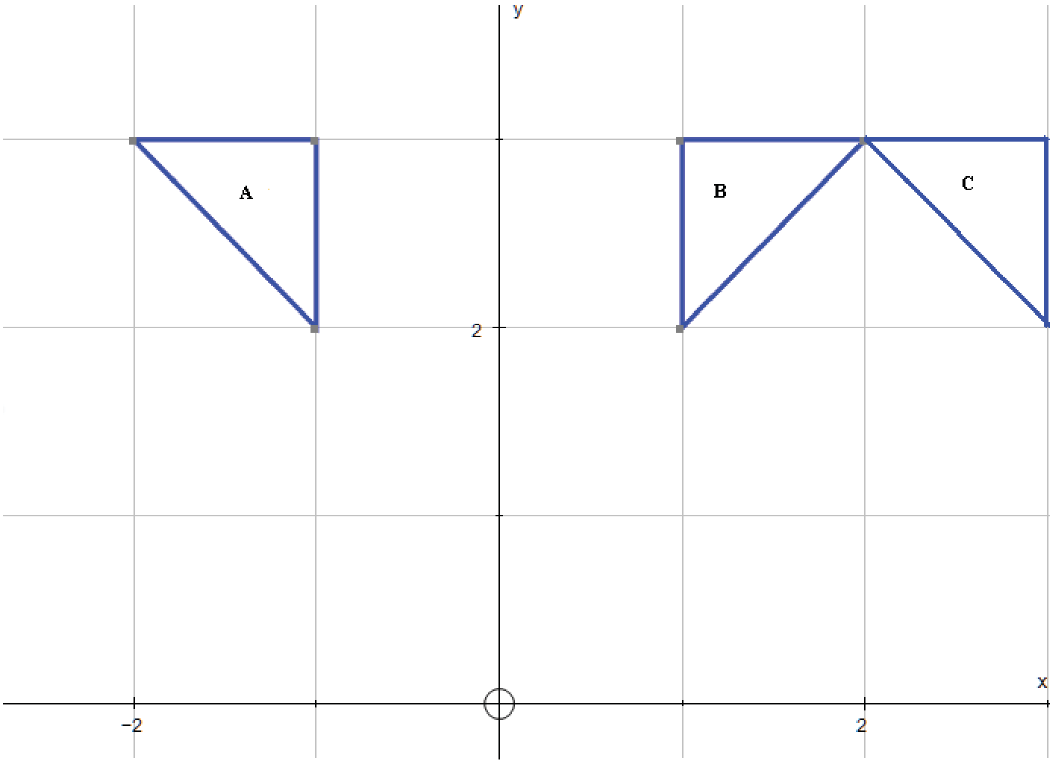


Figure 62

b) The single transformation that maps triangle A onto C is a translation, of translation vector $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

3.

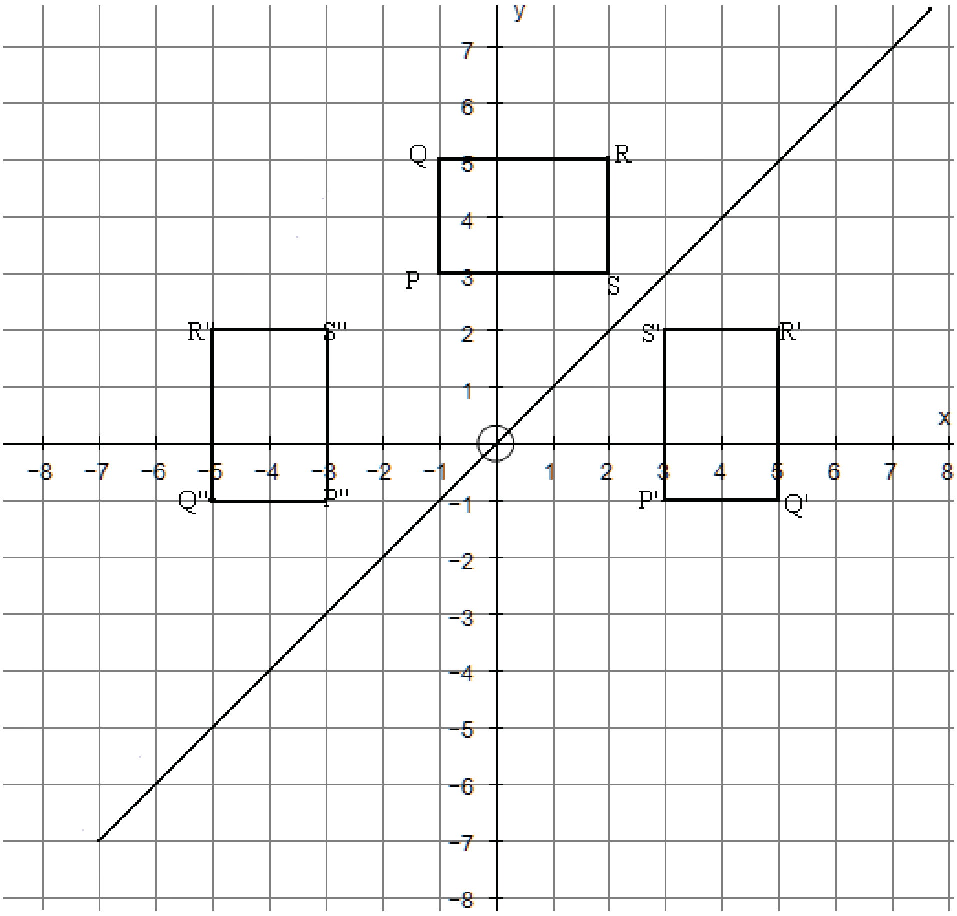


Figure 63

c) The rotation that maps PQRS onto P"Q"R"S" is a about the point (0, 0), through 90° anticlockwise.

4.

a) The single transformation that maps triangle A onto triangle B is a translation with translation vector $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$.

b) The single transformation that maps triangle B onto triangle C is an enlargement centre (-2, 1), factor $-\frac{1}{2}$. Remember that when the centre of enlargement is between the image and the object the scale factor is always negative.

c) The single transformation that maps triangle C onto triangle D is a rotation, centre (-1, 0), through 90° anticlockwise.

d) _To find the matrix of the transformation that maps triangle C onto triangle A we can use any two points of the object and corresponding two of the image.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ -2 & -2 \end{pmatrix}$$

multiplying by the inverse of the matrix on both sides we get:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 2 & -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 2 & -2 \end{pmatrix}$$

which is the matrix of the transformation.

5. i) D is a reflection through the line y=1.

ii) The centre of enlargement is (-2, 3)

iii) First E (O) is an enlargement of the point (0, 0) with factor 2 and centre (-2, 3). So E (O) = (4, -6).

DE (O) is a reflection the point (4, -6) through the line y=1 which gives (4, 8).

6.

a) The single transformation which maps triangle A onto triangle B is a reflection through the line y=3.

b) The single transformation which maps triangle A onto C is a stretch with the invariant line x=0 and factor 3.

7.

i) Triangle A is mapped onto triangle B by a translation, vector $\begin{pmatrix} -5 \\ -5 \end{pmatrix}$.

ii) Triangle A is mapped onto triangle C by a reflection, mirror line as the line y=-x.

iii) Triangle B is mapped triangle C by a reflection, mirror line y=-x-5.

iv) Triangle A is mapped onto triangle D by an enlargement, factor -2 and centre (4, 1).

8.

a)

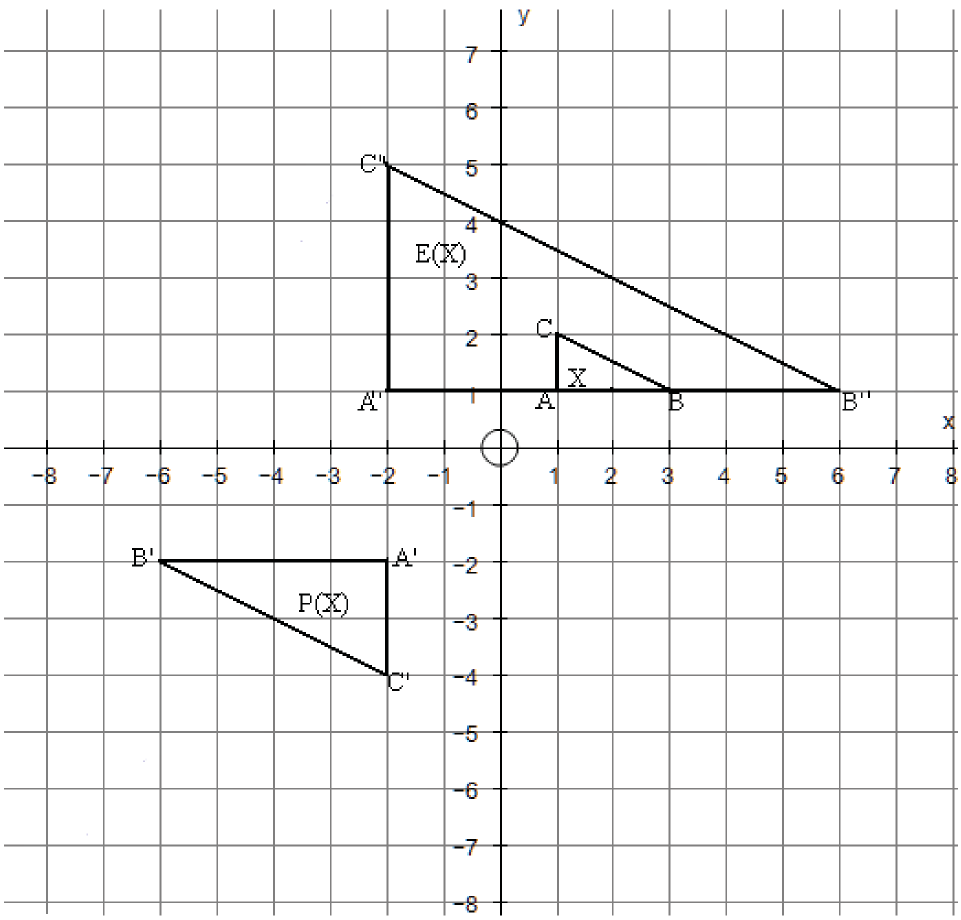


Figure 64

b) ii) P is an enlargement, centre (0,0) and factor -2.

iii) a=-2.

c) Triangle ABC can be mapped onto triangle ABC by an enlargement, centre (-2, -1), factor -2.

Unit Summary

Summary

In this unit on transformations you have learned that:

-An object can be transformed by translation, reflection, rotation, enlargement, shear, stretch and matrices.

- In Translation:

-The transformation is defined by the translation vector.

- After the translation the object is exactly the same as the object, the only difference is in the position.

-In Reflection:

-The transformation is defined by the mirror line.

-The image is exactly the same shape and size and the object but in the opposite sense.

-In Rotation:

-The transformation is defined by the centre of rotation, the angle and the direction.

-All the points of are turned through the same angle about the centre and appear to move along arcs of concentric circles.

-The image and the object are exactly the same shape and size.

In Enlargement:

-The transformation is defined by the centre of enlargement and the scale factor.

-The size of the object is changed, but not the shape.

In Stretch:

- The transformation is defined by an invariant line and a stretch factor.
- All points move at right angles to the invariant line, a distance proportional to their distance from the invariant line.

In Shear:

- The transformation is defined by an invariant line and a shear factor.
- All points move parallel to the invariant line such that the distance a point moves is proportional to its distance from that line or plane.

Assignment

Assignment

The assessment consists of 8 questions, answer all.

The marks for each question are shown. There are a total of 50 marks.

You are advised to spend no more than 50 minutes on this assessment.

Calculators may be used.

Show all the necessary working.

1. Describe fully the transformations shown in each of the following diagrams.

   a) The transformation that maps triangle A onto triangle B. (2 marks)

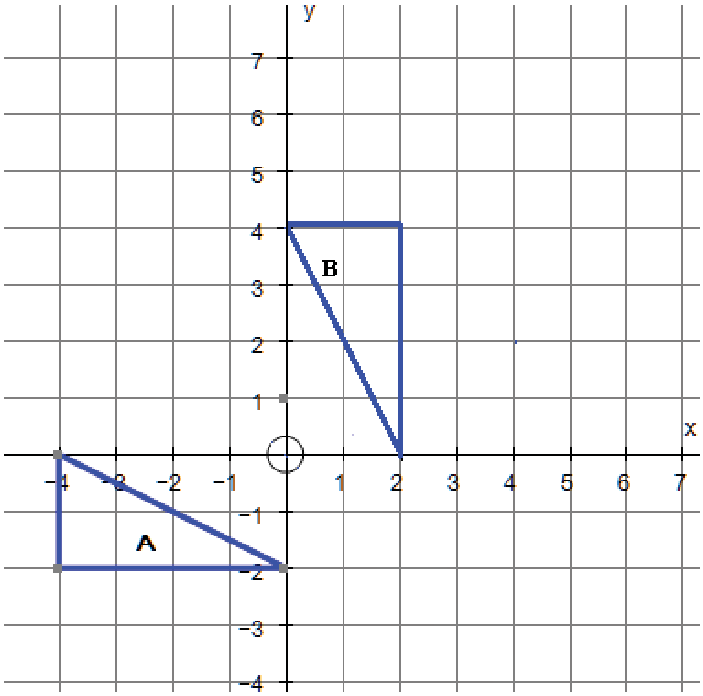


Figure 65

b) The transformation that maps triangle A onto triangle C. (2 marks)

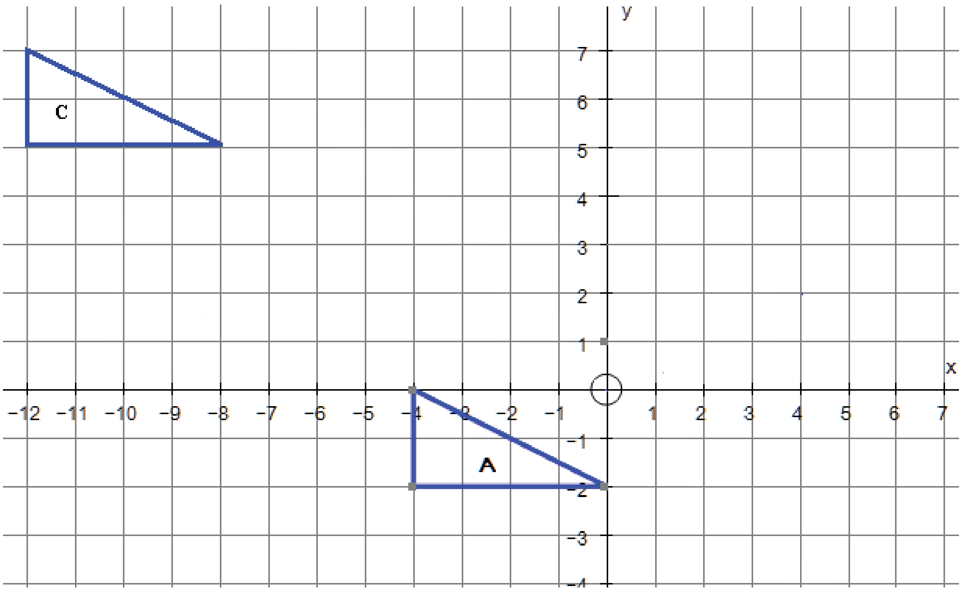


Figure 66

c) The transformation that maps triangle ABC onto A'B'C'. (2 marks)

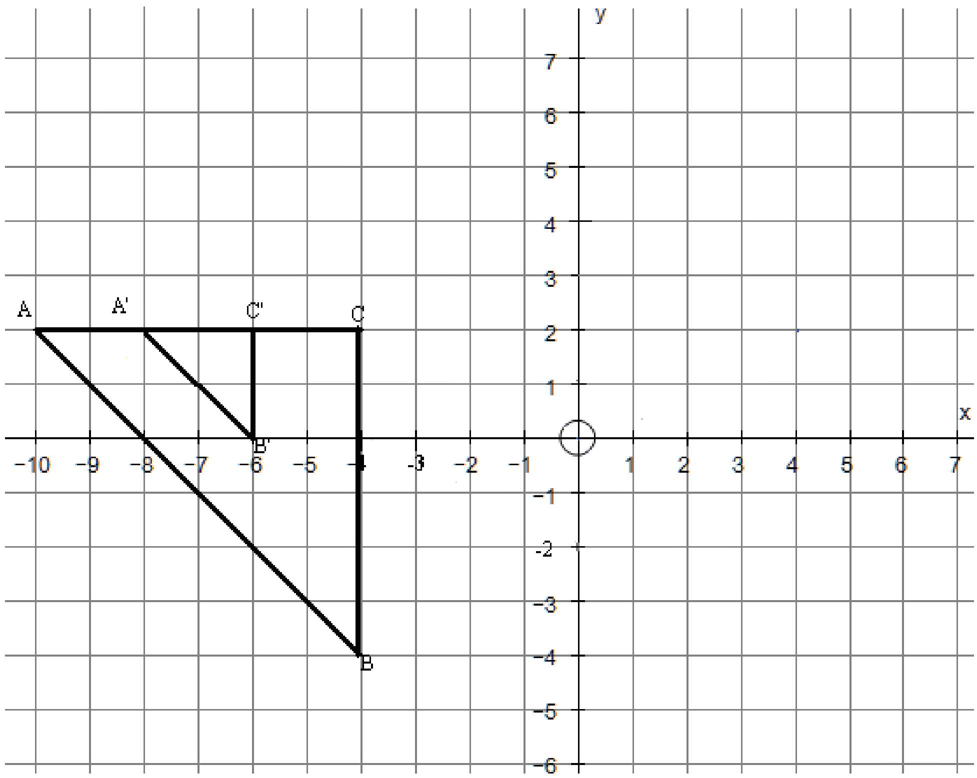


Figure 67

d) The transformation that maps triangle ABC onto triangle A'B'C'. (2 marks)

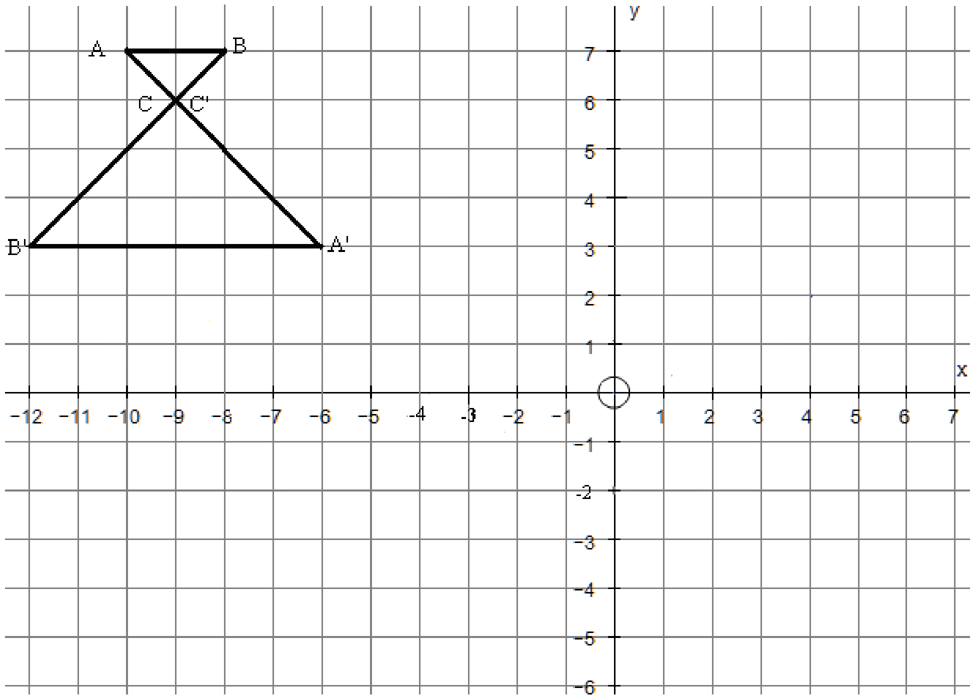


Figure 68

e) The transformation that maps triangle ABC onto triangle A'B'C'. (2 marks)

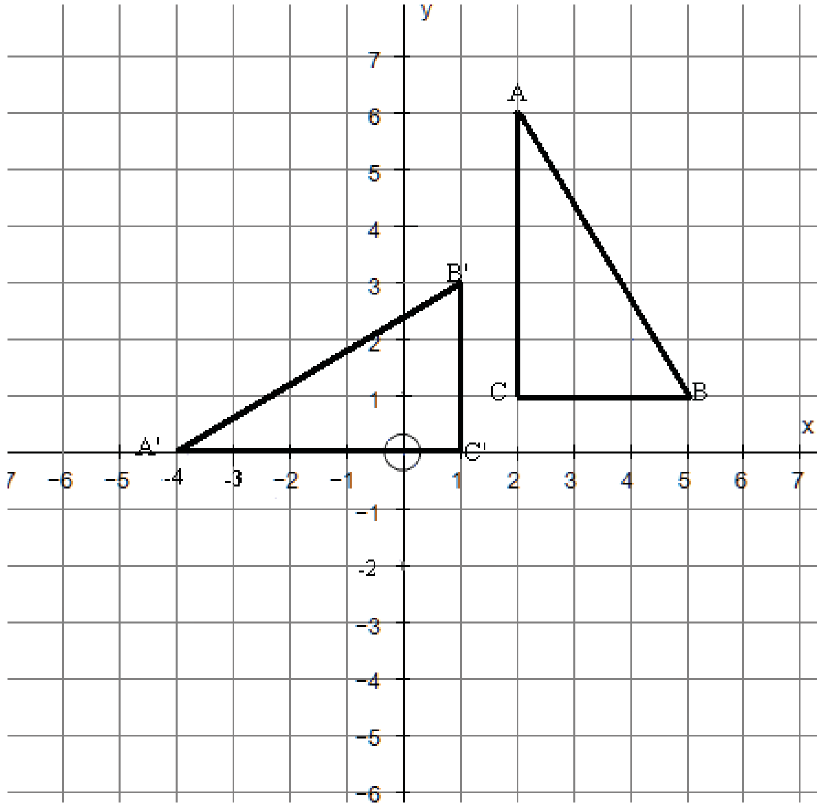


Figure 69

f) The transformation that maps triangle ABC onto triangle A'B'C'. (2 marks)

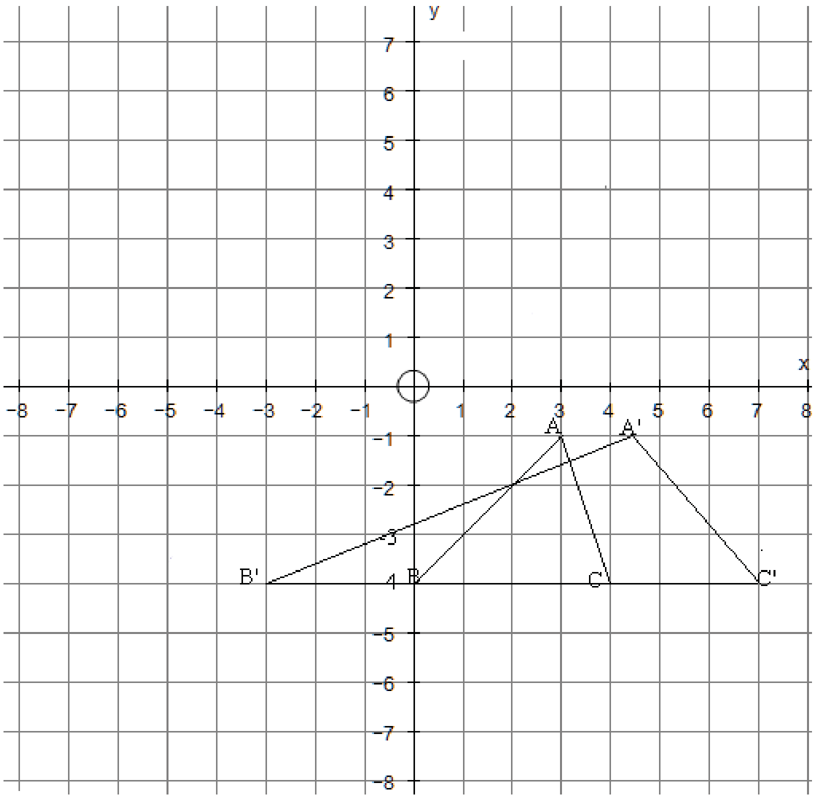


Figure 70

g) The transformation that maps shape ABCD onto shape A'B'C'D'. (2 marks)

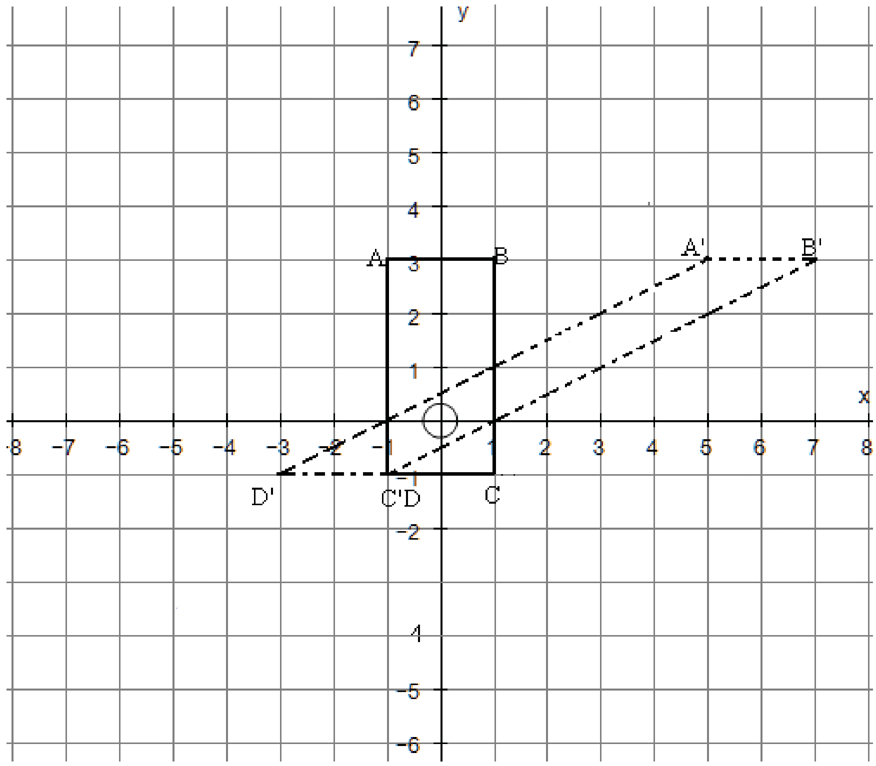
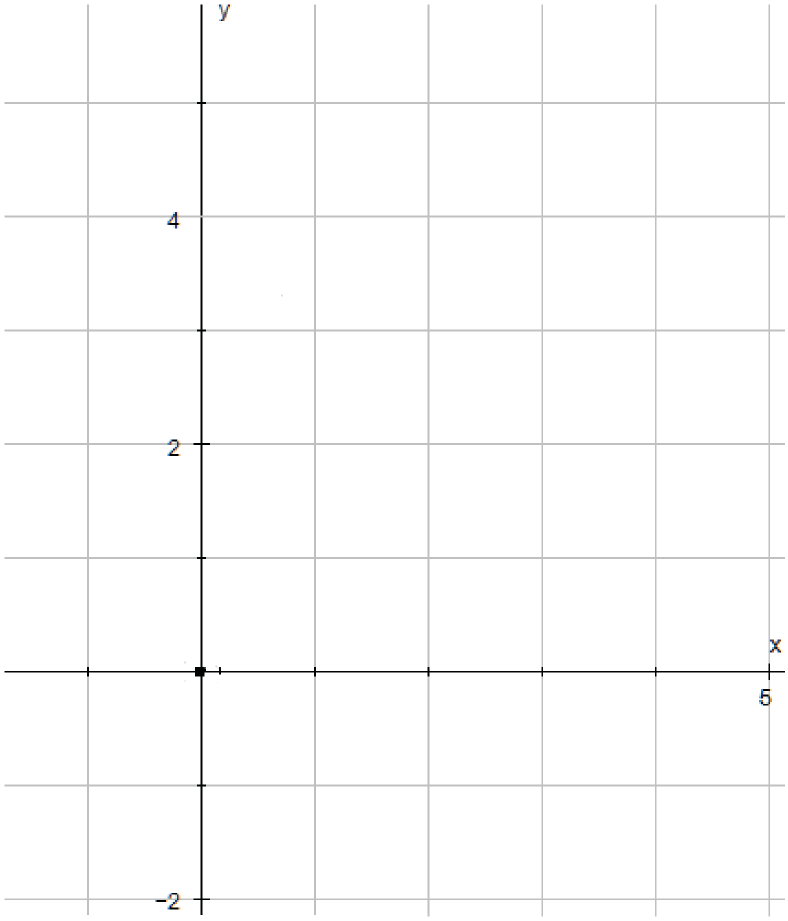


Figure 71

2. On a squared paper plot the points (5, 5), (7, 5), (7, 7) and (6, 7) and join the points in order to form a trapezium. Name the trapezium T. (1 mark)

a) Plot the point C (4, 4). (1 mark)



b) i) Draw T_1, the image of T under a translation , vector $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$. (2 marks)

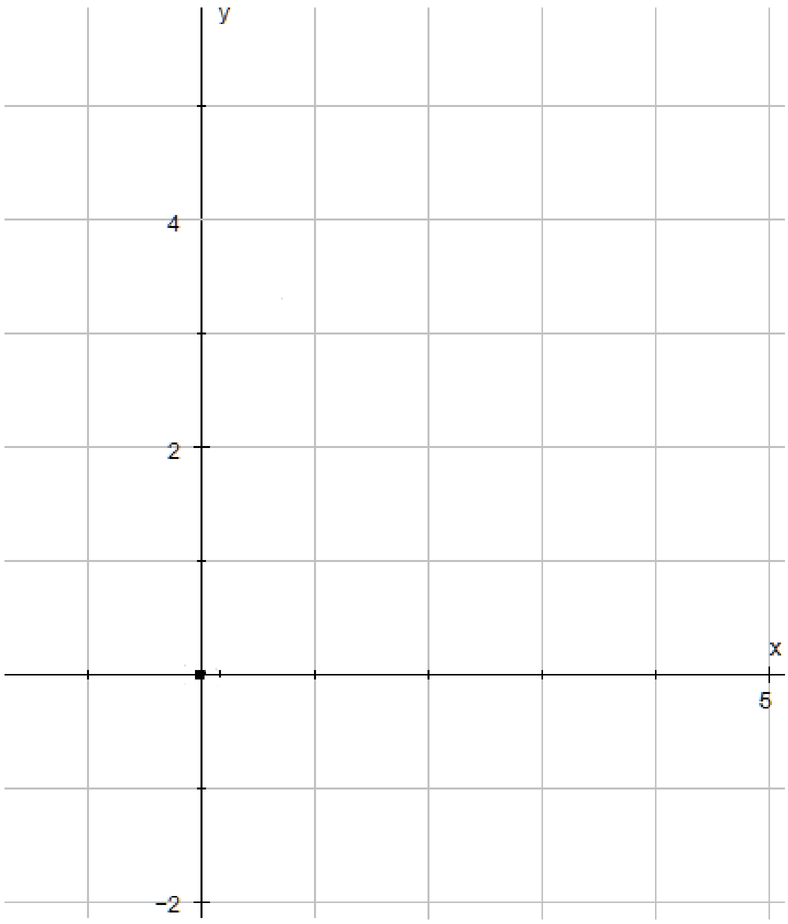
ii) Draw T_2, the image of T under an enlargement, centre C and scale factor -3.

(2 marks)

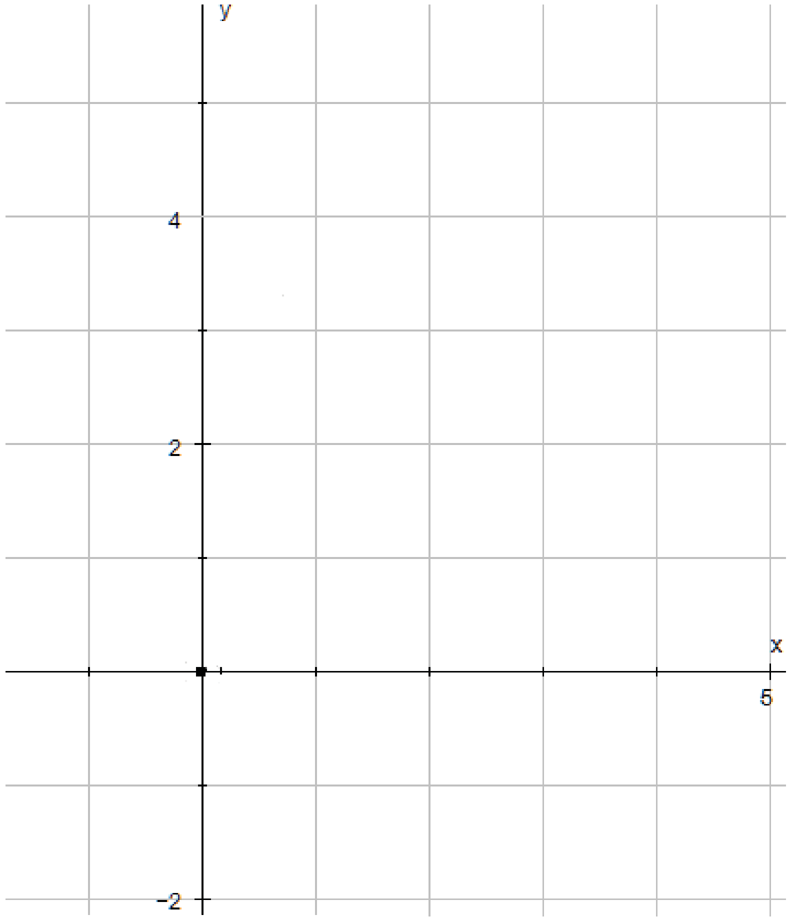
iii) Draw T_3, the image of T under a rotation of -90° about (0, 0). (2 marks)

iv) Draw T_4, the reflection of T in the y-axis. (2 marks)

3. The quadrilateral ABCD has vertices A (4, 0), B (6, -1), C (7, -2), and D (3, -3). The quadrilateral ABCD is mapped onto A'B'C'D' by a stretch, factor -2 and invariant line y=2. Draw and label the quadrilaterals ABCD and A'B'C'D'. (4 marks)



4. PQRS has vertices P (4, 1), Q (6, 1), R (6, -3) and S (4, -3). PQRS is mapped onto P'Q'R'S' by a shear, factor 1, and invariant line y=0. Draw and label PQRS and P'Q'R'S'.
(3 marks)

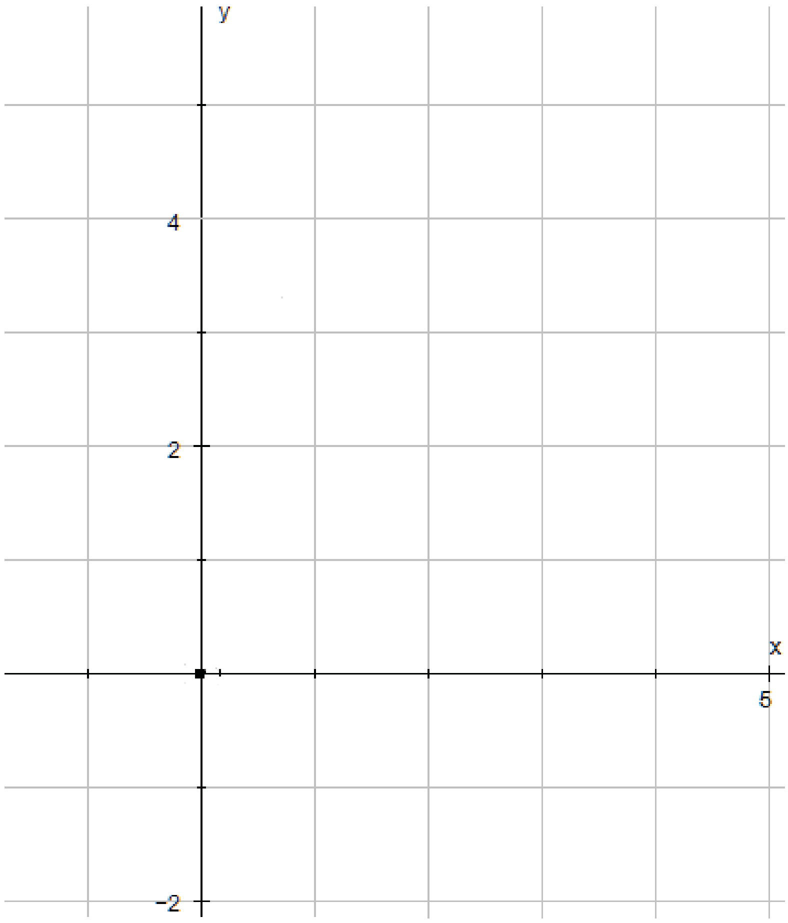

y
4
2
x
5
−2

5. ABCD is a rectangle with A (1, 1), B (4, 1), C (1, -2) and D (4, -2). Transform this rectangle using the following matrices. Plot A, B, C, and D and the images and identify the transformations: (12 marks)

$$a)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \textit{label the image } A'B'C'D'$$

$$b)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \textit{label the image } A''B''C''$$

$$c)\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \textit{label the image } A'''B'''C'''D'''$$



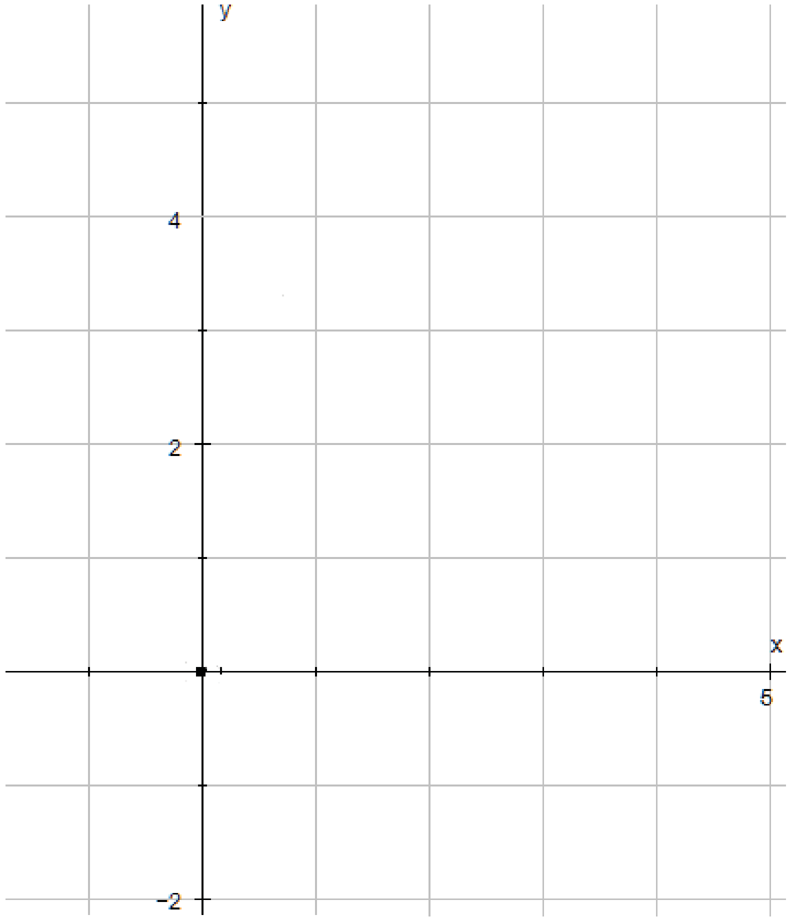
6. Find the matrices representing the following transformation.

(2, 3)→(5, 3) and (3, 5)→(8, 5) (3 marks)

____
____
____
____

7. Triangle XYZ has the following coordinates X (1, 1), Y (3, 2), Z (3, 4).

a) Draw triangle XYZ. (1 mark)



b) i) Reflect triangle XYZ in the x-axis. Name the image X'Y'Z'. (2 marks)

ii) Describe the transformation which maps triangle X'Y'Z' onto triangle XYZ.

(2 marks)

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c) i) Rotate triangle X'Y'Z' through -90˚ about (0,0). Label the image X"Y"Z".

(2 marks)

ii) Describe the single transformation which will map triangle XYZ onto triangle X"Y"Z". (2 marks)

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8.

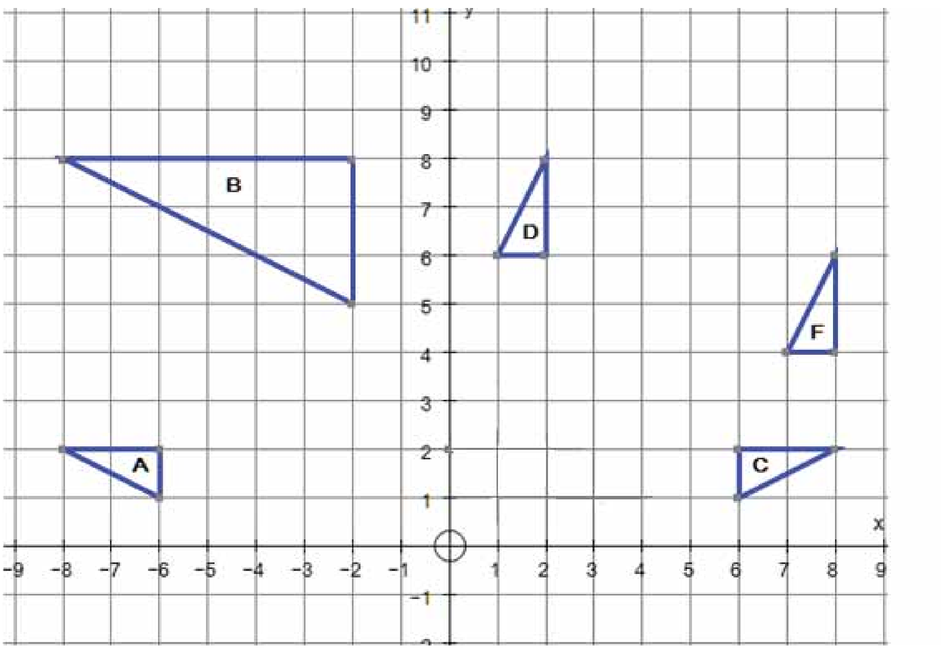

Figure 72

The diagram shows triangles A, B, C, D, E and F.

a) Triangle A is mapped onto triangle B by an enlargement.

Find

i) The scale factor , (2 marks)

____

____

ii) The coordinates of the centre of the enlargement. (1 mark)

____

____

b) Triangle A is mapped onto triangle C by a single transformation.

Find the matrix which represents this transformation. (2 marks)

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____

____

c) Triangle C is mapped onto triangle D by a single triangle transformation.

Describe fully this transformation. (2 marks)

____

____

d) Triangle A is mapped onto triangle F by a transformation V which is made up from separate transformations.

The first is a clockwise rotation of 90° about O; it is followed by a translation represented by column vector $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

Therefore the transformation V maps the point (x, y) onto (x', y') where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix}.$$

Show that V maps point (g, h) onto (h+6, -g-2). (2 mark)

____

____

Solutions to the assignment:

1.

a) The transformation that maps triangle A onto triangle B is a reflection, through the line y=

x.

b) The transformation that maps triangle A onto triangle C is a translation, vector $\begin{pmatrix} -8 \\ 7 \end{pmatrix}$.

c) The transformation that maps triangle ABC onto triangle A'B'C' is an enlargement, centre (-7, 2), and factor $\frac{2}{6} = \frac{1}{3}$.

d) The transformation that maps triangle ABC onto triangle A'B'C' is an enlargement, centre (-9, 6), and factor $\frac{6}{2} = 3$.

e) The transformation that maps triangle ABC onto triangle A'B'C' is a rotation, centre (2, 0), through 90° anticlockwise.

f) The transformation that maps triangle ABC onto triangle A'B'C' is a stretch, invariant line x = 2. Factor $\frac{IA'}{IA} = \frac{2.5}{1} = 2.5.$

g) The transformation that maps quadrilateral ABCD onto quadrilateral ABCD is a shear invariant line, y= 0. Factor $\frac{AA'}{IA}=\frac{6}{3}=2.$

2.

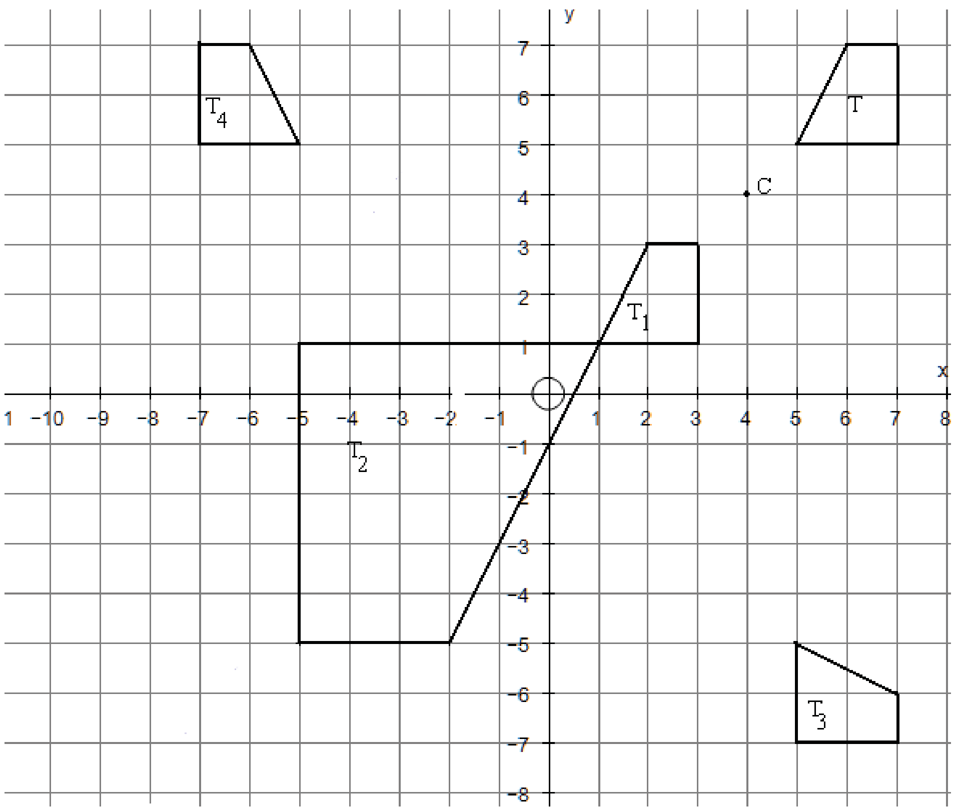


Figure 73

3.

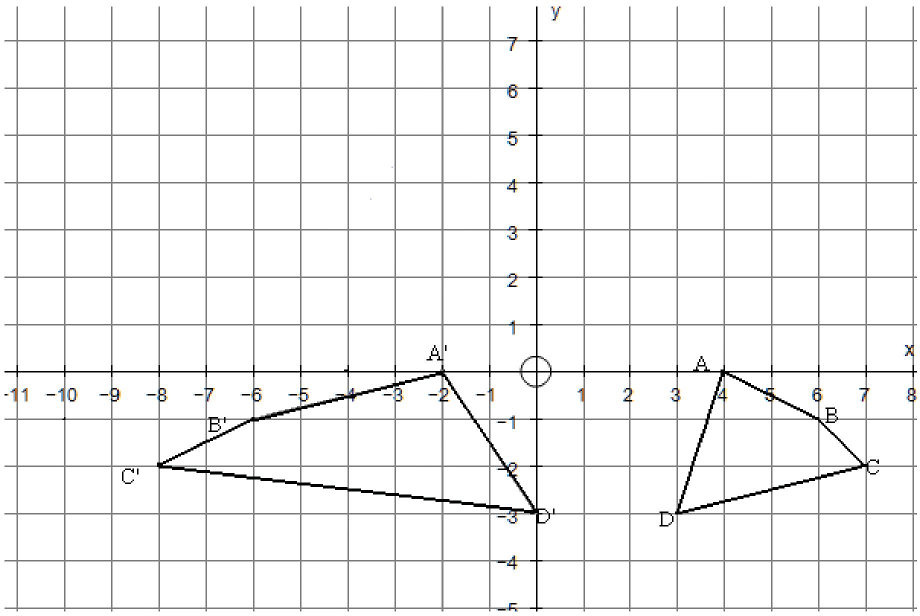


Figure 74

4.

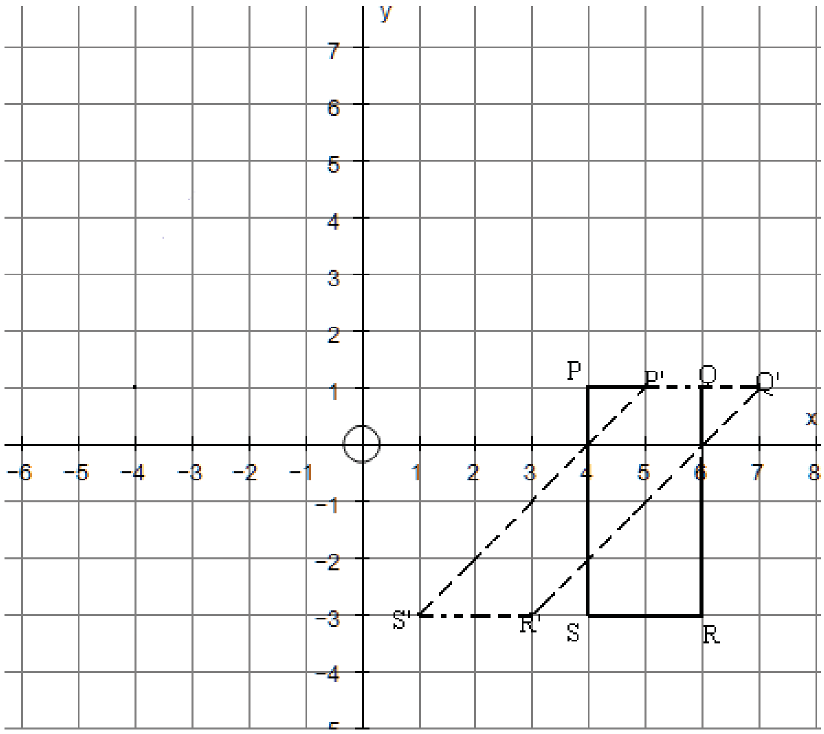


Figure 75

5.

Firs we will multiply the coordinates of the quadrilateral with the matrices in order to be able to draw the images.

$$a)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 4 & 1 & 4\\ 1 & 1 & -2 & -2\end{pmatrix}=\begin{pmatrix}-1 & -4 & -1 & -4\\ 1 & 1 & -2 & -2\end{pmatrix}=A'B'C'D'$$

$$b)\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 4 & 1 & 4\\ 1 & 1 & -2 & -2\end{pmatrix}=\begin{pmatrix}2 & 8 & 2 & 8\\ 1 & 1 & -2 & -2\end{pmatrix}=A''B''C''$$

$$c)\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & 4 & 1 & 4\\ 1 & 1 & -2 & -2\end{pmatrix}=\begin{pmatrix}1 & 4 & 1 & 4\\ -1 & -7 & -4 & -10\end{pmatrix}=A'''B'''C'''D'''$$

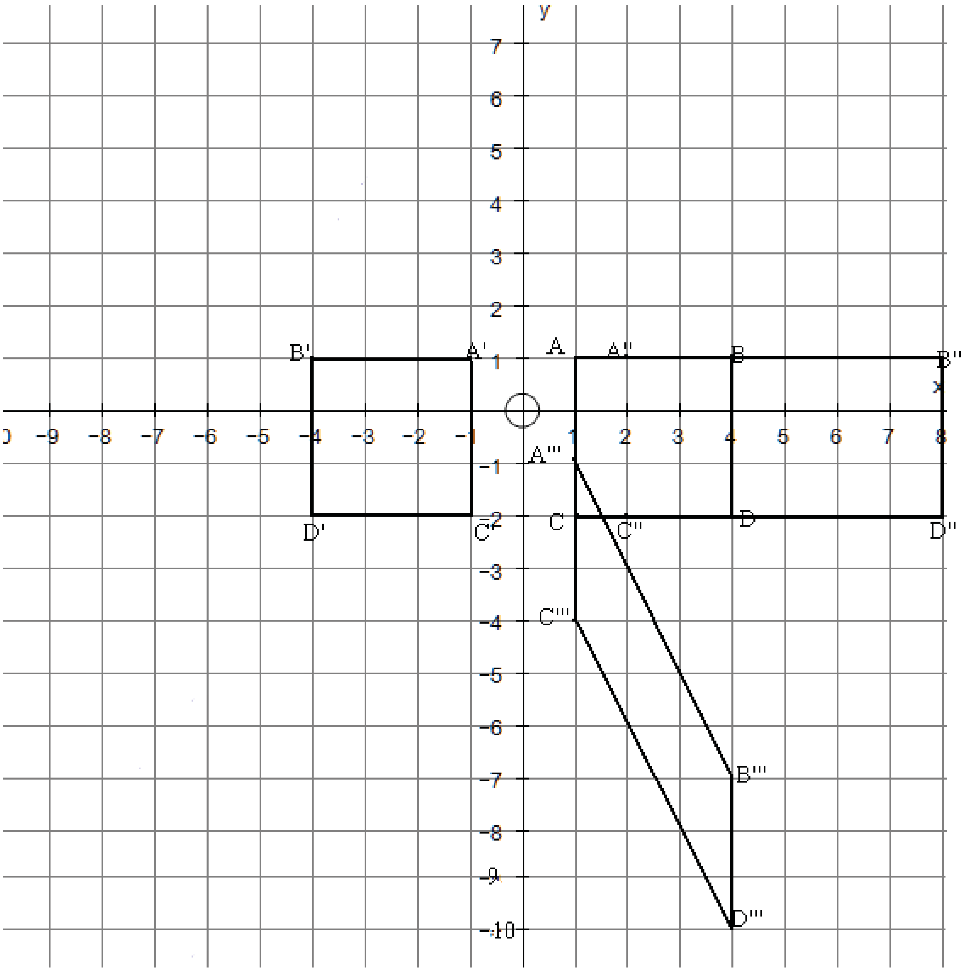


Figure 76

a) A'B'C'D' is a reflection of ABCD, the line x=0 as the mirror line.

b) A"B"C"D" is a stretch of ABCD, invariant line is the line x= 0, and the factor is 2.

c) A"'B"'C"'D"' is a shear, the invariant line is the line x= 0, and the shear factor= 2.

6.

To find the matrix of the transformation we use any two points of the object and corresponding two of the image.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 3 & 5 \end{pmatrix}$$

multiplying by the inverse of the matrix on both sides we get :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which is the matrix of the transformation.

7.

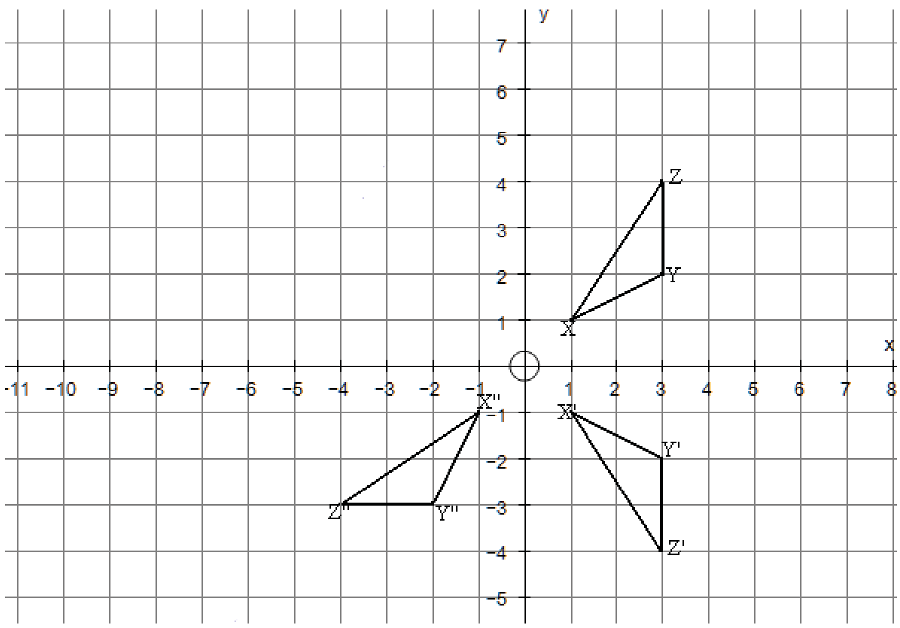


Figure 77

b) ii) Triangle X'Y'Z' is mapped onto triangle XYZ by a reflection through the x-axis.

c) ii) The single transformation that maps triangle XYZ onto X"Y"Z" is a reflection through the line y= -x.

8.

a) i) scale factor $= \frac{6}{2} = \frac{3}{1} = 3$.

ii) the centre of rotation is the point (-8, -1).

b)

To find the matrix of the transformation we use any two points of the object and corresponding two of the image.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -6 & -8 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 1 & 2 \end{pmatrix}$$

multiplying by the inverse of the matrix on both sides we get:

$$\frac{1}{-4}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -6 & -8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 8 \\ -1 & -6 \end{pmatrix} = \frac{1}{-4}\begin{pmatrix} 6 & 8 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 8 \\ -1 & -6 \end{pmatrix}$$

$$\Rightarrow \frac{1}{-4}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \frac{1}{-4}\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is the matrix of the transformation.

c) Triangle C is mapped onto triangle D by a reflection, through the line y=x.

d) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} g \\ h \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} h \\ -g \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} h+6 \\ -g-2 \end{pmatrix}$

So V maps (g,h) onto $(h+6,-g-2)$.

Assessment

The assessment consists of 5 questions, answer all.

The marks for each question are shown. There are a total of 30 marks.

You are advised to spend no more than 40 minutes on this assessment.

Calculators may be used.

Show all the necessary working.

Good luck!!

1.

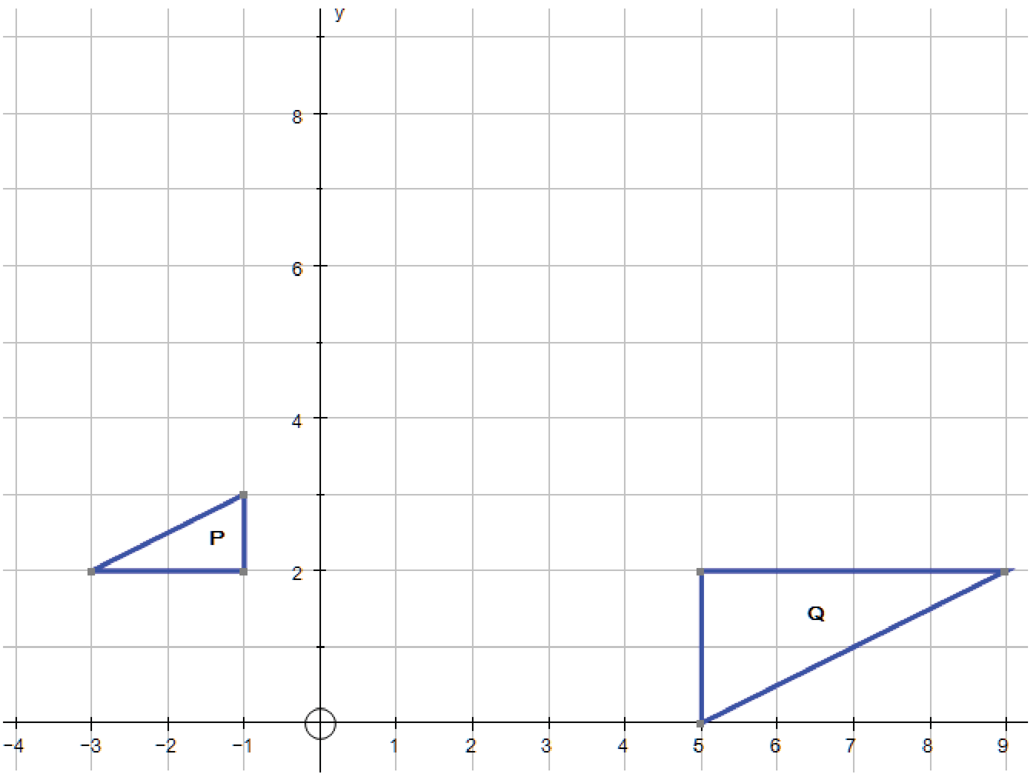


Figure 78

The diagram shows the triangles P and Q.

a) The enlargement E maps triangle P onto triangle Q.

For this enlargement:

i) Write down the scale factor,
(2 mark)

____

ii) Find the coordinates of the centre of enlargement.
(2 marks)

____

2. Figure --- shows triangle A, B, C and D.

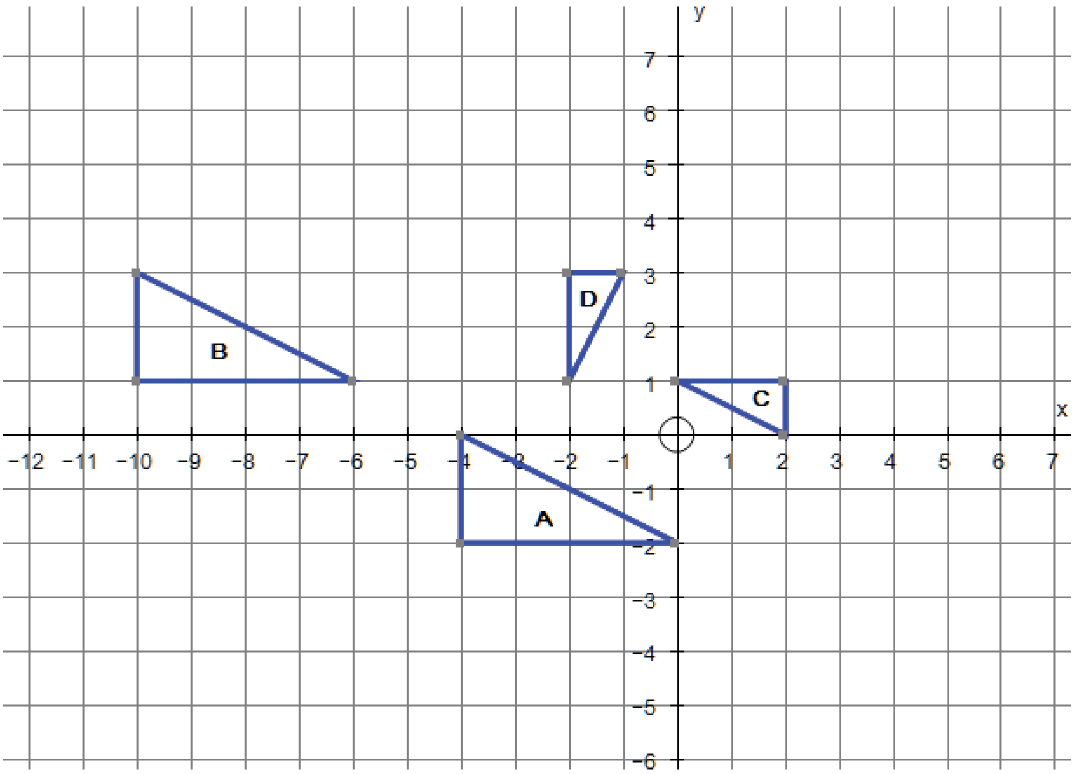


Figure 79

i) Describe fully the **single** transformation that maps triangle A onto triangle B. (2 marks)

____

ii) Describe fully the **single** transformation that maps triangle C onto triangle D. (2 marks)

iii) Write down the matrix that represents the transformation which maps triangle C onto triangle D.

(2 marks)

3.

a) The diagrams show triangles A, B, C and D.

i) The single transformation P maps triangle A onto triangle B.

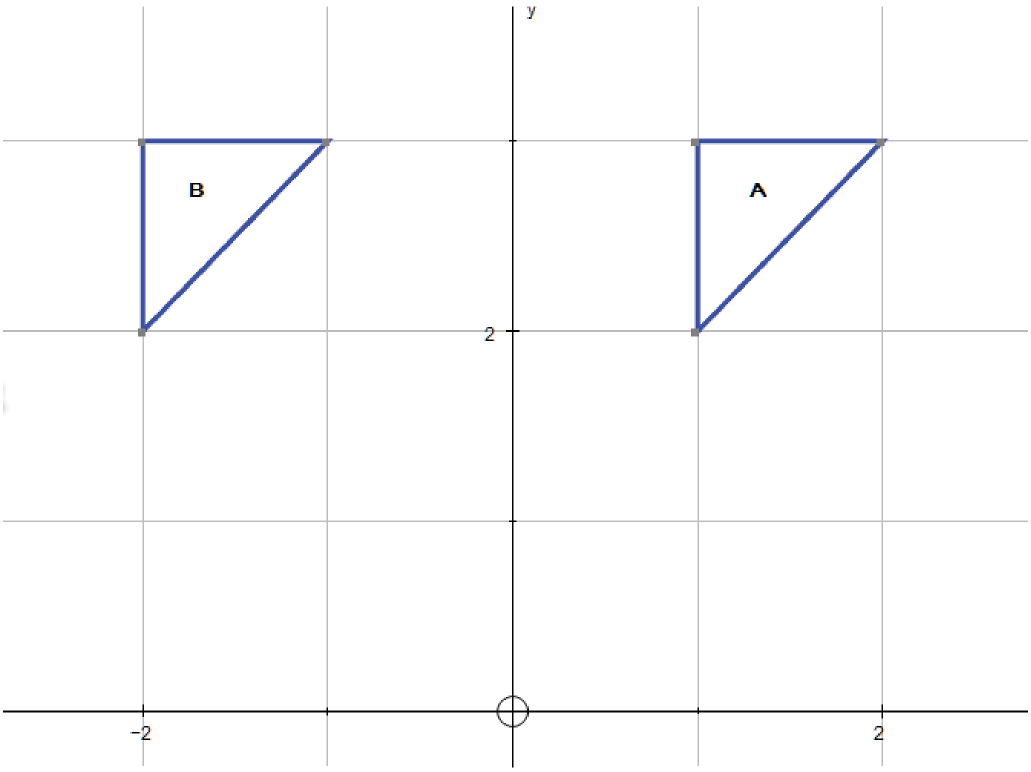


Figure 80

Describe fully the transformation P.
(2 marks)

ii) The single transformation Q maps triangle A onto triangle C.

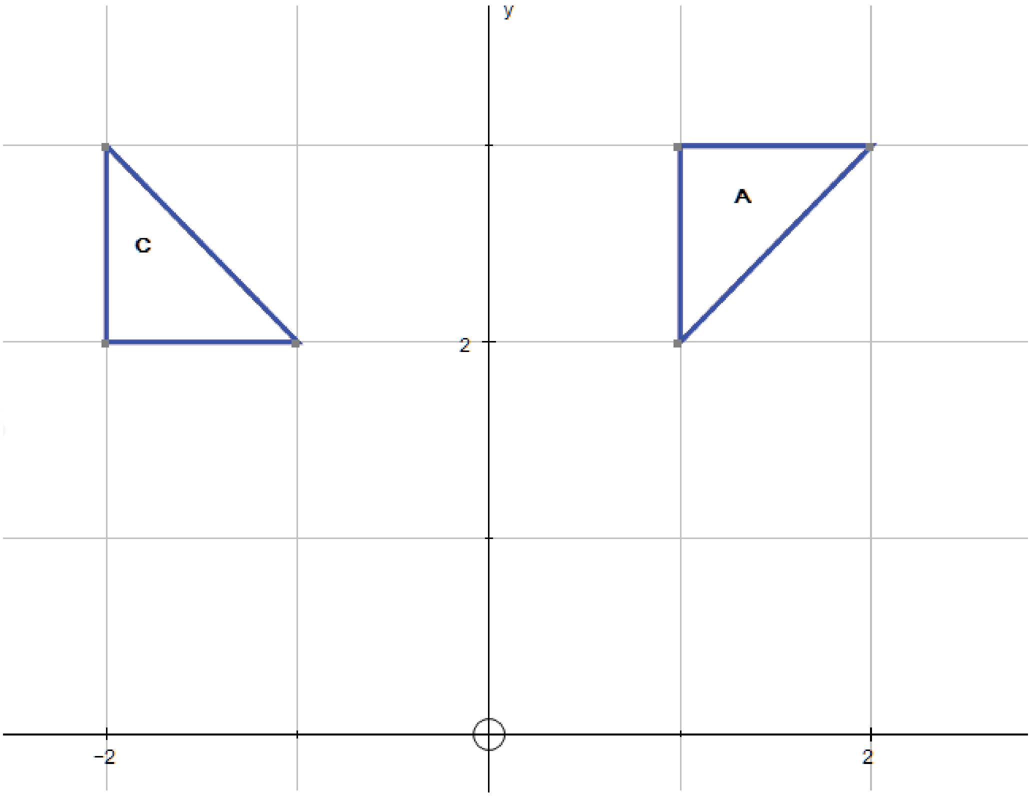


Figure 81

Describe fully the transformation Q.
(2 marks)

iii) The reflection R maps triangle A onto triangle D.

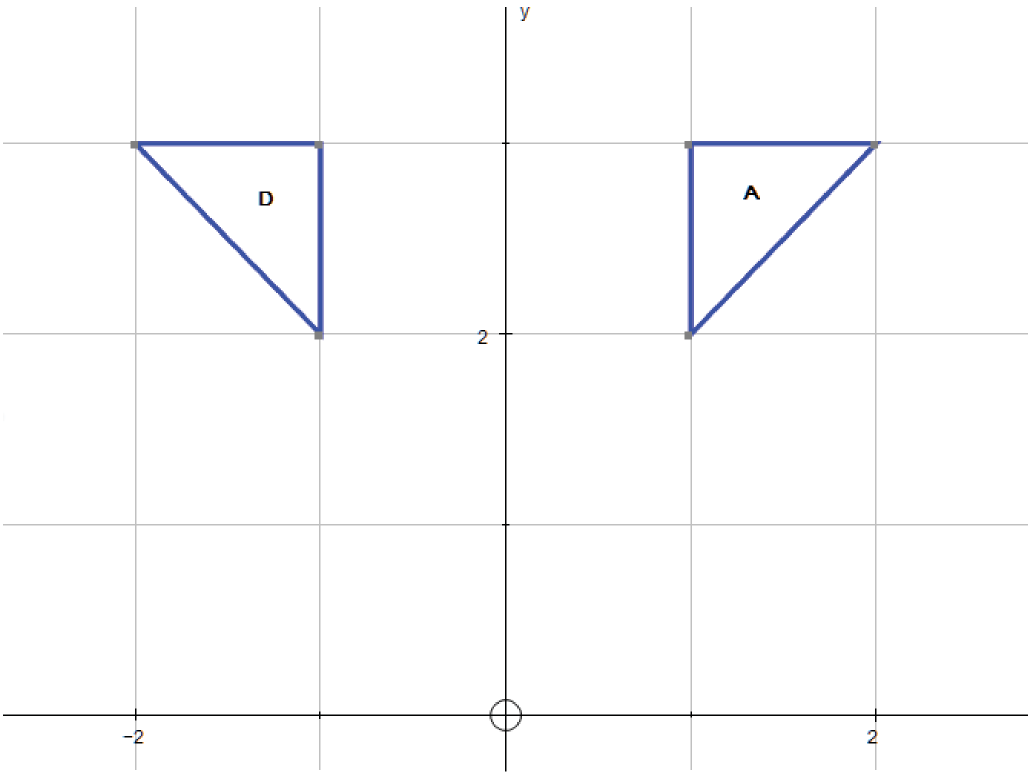


Figure 82

Find the matrix that represents the reflection R.
(2 marks)

b) The diagram shows the points E (1, 3), F (2, 3), and G (-1, 3). An enlargement, centre E, maps F onto G.

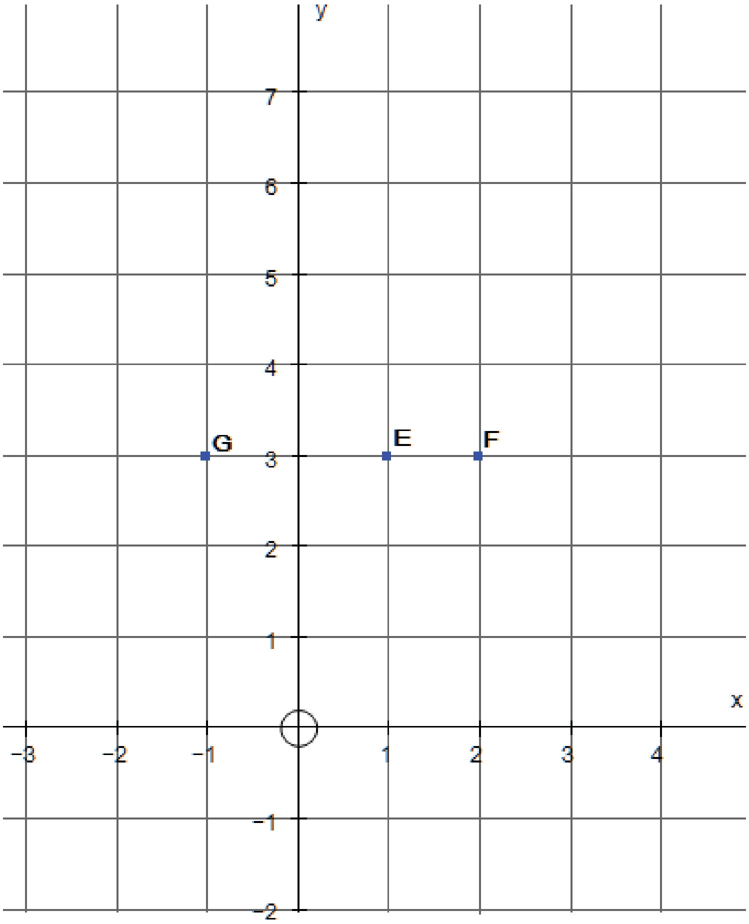


Figure 83

Write down:

i) The scale factor,
(1 mark)

ii) The coordinates of the image of (0, 4), under the same enlargement.
(1 mark)

c) $M = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$

i) Find the determinant of M.
(1 mark)

ii) Write down the inverse of M.
(1 mark)

iii) Find the matrix X, such that MX= $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

(2 marks)

4.

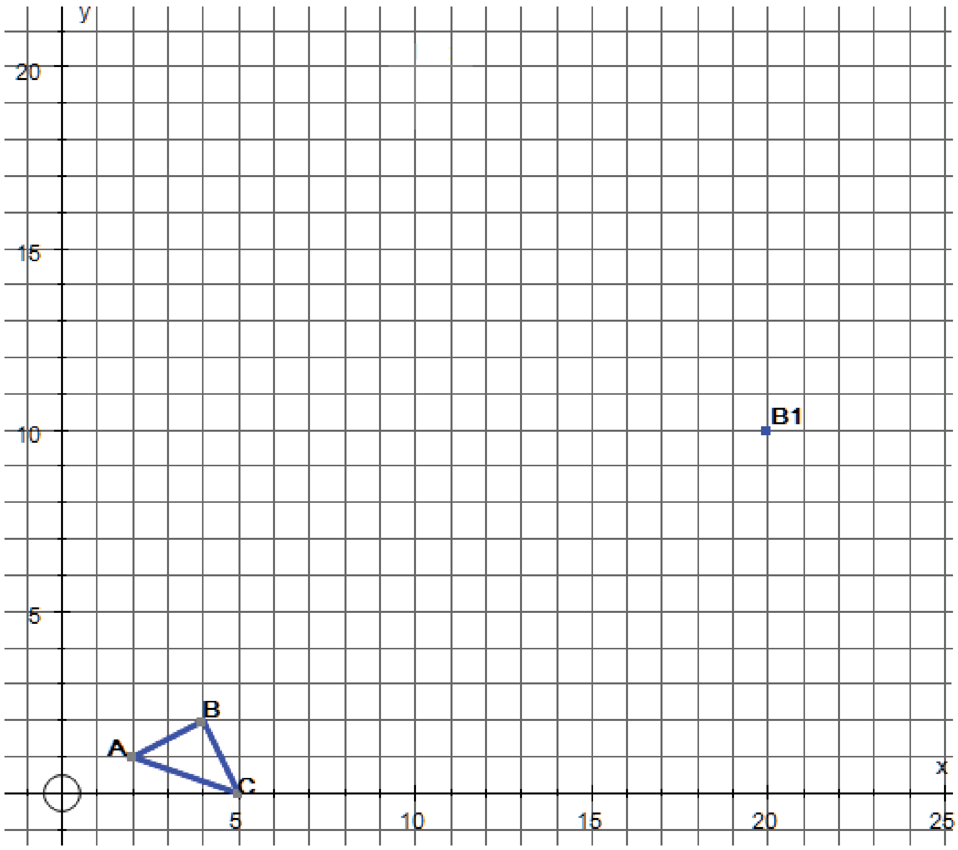


Figure 84

In the above diagram, a triangle has vertices A (2, 1), B (4, 2) and C (5, 0).

An enlargement, centre the origin maps triangle ABC onto triangle $A_1B_1C_1$ where B_1 is (20, 10).

Find

i) The scale factor of the enlargement.
(1 mark)

ii) The matrix which represents the enlargement.
(1 mark)

iii) The coordinates of C_1.
(1 mark)

5. Triangle A has vertices (6, -2) (8, -2) (6, -5).

a) Draw and label triangle A.
(1 mark)

b) The translation T is represented by the column vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

The translation T maps triangle A onto triangle B, so that T (A) =B.

Draw and label triangle B.
(1 mark)

c) The transformation R is a rotation through 90° clockwise, centre (3, 4).

The transformation R maps triangle A onto triangle C, so that R (A) = C.

Draw and label triangle C.
(2 marks)

d) Given that TR (A) = D, draw and label triangle D.
(2 marks)

e) Triangle E has vertices (2, -2), (4, -2), and (-4, -5).

The single transformation H maps triangle A onto triangle E.

Describe fully the transformation H.
(2 marks)

Unit Contents

Unit 20

Unit 20

Linear Graphs

Introduction

Graphs of linear functions!

Functions can be defined by equations such that the function $f(x)=2x+1$ can also be written as $y=2x+1$.

You have already learnt to draw the graphs of equations such as $y=2x+1$ by using ordered pairs (x, y) from a completed table like the one below:

x	-3	-2	-1	0	1	2	3
y	-5	-3	-1	1	3	5	7

The graph thereof is a straight line or a linear graph.

In this unit, you are going to learn how to draw linear graphs without using tables.

This unit consists of 67 pages. This is approximately 5 % of the whole course. As reference, you will need to devote 25 hours to work on this unit, 15 hours for formal study and 10 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Four Lessons:

Lesson 1 Calculating the Gradient (slope) of a Linear Graph
Lesson 2 Finding the X-intercept of Linear Graphs
Lesson 3 Finding the Y-intercept of Linear Graph
Lesson 4 Deriving the Equation of a Graph, from the Drawn Graph
Lesson 5 Drawing a Linear Graph, from the Given Equation

Upon completion of this unit you will be able to:

Outcomes

- *calculate* the gradient (slope) of linear graph.
- *find* the x-intercept of linear graph.
- *find* the y-intercept of linear graph.
- *derive* the equation of a graph, from the drawn graph.
- *draw* a linear graph, from the given equation.

Terminology

Constant	A number.
$ax + by = c$	The **standard form** of the equation of a linear graph, where a , b and c are constants.
$y = mx + c$	The **gradient-intercept form** of the equation of a linear graph, where m is the gradient and c is the y-intercept.
y-intercept	The y coordinate of a point where a graph crosses the y-axis on the x-y plane.
x-intercept	The x coordinate of a point where a graph crosses the x-axis on the x-y plane.
Linear graph	A graph which is a line.
Gradient	Vertical change divided by horizontal change in the x-y plane.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Calculating the Gradient (slope) of a Linear Graph

Introduction

By the end of this subunit, you should be able to:

- Find the gradient of the linear graph, using the coordinates of **two** points on the graph.

This subunit is about 7 pages in length.

Finding the gradient using the coordinates of two points on a linear graph.

A linear graph in an x-y plane is either a **horizontal**, **vertical** or **diagonal** line.

The gradient is calculated by subtracting the coordinates of one point from those of another point. It is defined as vertical change divided by horizontal change.

The vertical change is found by subtracting the y coordinate of one point from the y coordinate of the other point. Similarly, the horizontal change is found by subtracting x coordinate of one point from the x coordinate of the other, respectively.

Any two points on the graph can be used to calculate the gradient. If the points (x_1, y_1) and (x_2, y_2) lie on a linear graph, the gradient (m) is calculated using the formula: $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$.

Can the gradient be calculated from $m = \frac{(y_1 - y_2)}{(x_1 - x_2)}$? Give a reason for answer.

____

____

____

____

Compare your answer with:

Yes, it can be calculated. This is because multiplying the denominator and the numerator of $\frac{(y_1 - y_2)}{(x_1 - x_2)}$ by -1 and simplifying gives $\frac{(y_2 - y_1)}{(x_2 - x_1)}$:

$$\frac{-1(y_1 - y_2)}{-1(x_1 - x_2)} = \frac{-y_1 + y_2}{-x_1 + x_2} = \frac{(y_2 - y_1)}{(x_2 - x_1)}.$$

In other words, the gradient can be calculated by either subtracting the coordinates of point A from those of point B, or subtracting the coordinates of point B from those of point A.

Example 1

An insect crawls from the origin such that its position is at $(3,4)$ after two minutes. Starting at the same time as the insect, a worm moves from the same origin, and it is at $(5,6)$ five minutes later. A line is drawn from the new position of the insect to that of the worm. Find the gradient of the line.

Now, the line passes through $(3,4)$ and $(5,6)$. So, use $(3,4)$ and $(5,6)$ to calculate the gradient of the line.

Let $(x_1, y_1) = (3,4)$ and $(x_2, y_2) = (5,6)$.

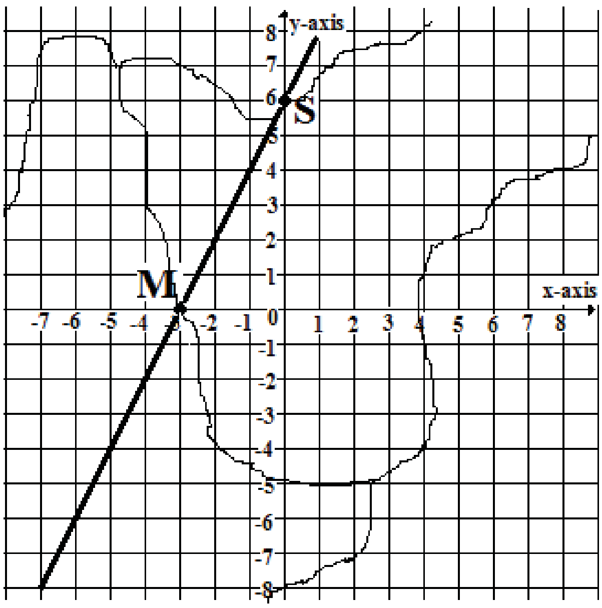
Now, substitute the coordinates in the formula, $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$:

$$m = \frac{(6-4)}{(5-3)} = \frac{2}{2} = 1.$$

Therefore, the gradient is 1.

Example 2

The distance between Maseru (M) and Seforong (S) is shown on a map by the line from M to S, as shown in the diagram.



Find the gradient of the line.

When the graph has been drawn, choose two points on the graph and substitute their coordinates in the formula for gradient, $m=\frac{(y_2-y_1)}{(x_2-x_1)}$.

The positions of Maseru and Seforong are at points $(-3,0)$ and $(0,6)$, respectively on the graph. Use these two points to calculate gradient.

Compare your answer with:

Let $(x_1, y_1)=(-3,0)$ and $(x_2, y_2)=(0,6)$.

Then, $m=\frac{(6-0)}{(0-{}^{-}3)}=\frac{6}{3}=2$.

Even if you chose different points on the graph from the points used above, you should still get the gradient as 2.

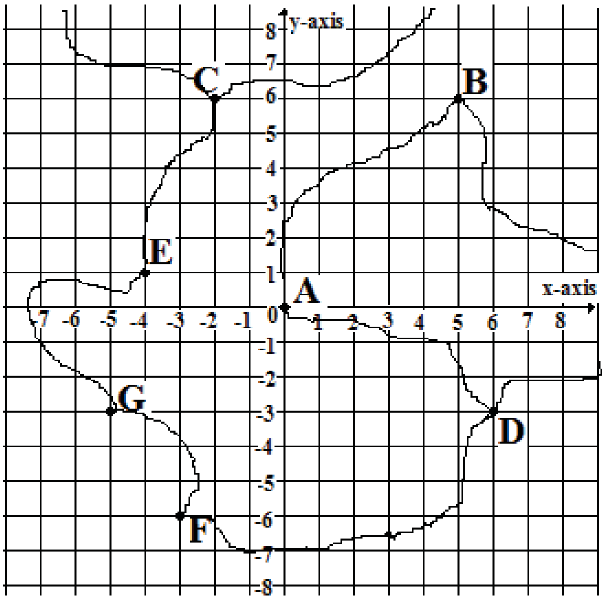
That means the gradient of a linear graph does not change. In other words, it is a **constant**.

Remember that, <u>any two points</u> can be used to calculate the gradient.

Activity 1

Activity 1

1. Calculate the gradients of the linear graphs that pass through the given points on the map.



a) A and B.

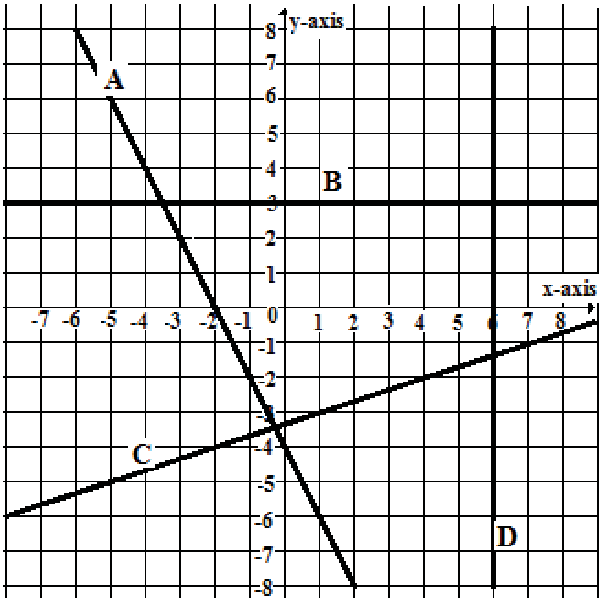
b) C and D.

c) E and G.

d) F and D.

e) A and D.

2. The linear graphs, A, B, C and D, are shown in the grid below.



a) Find the gradient of graph A.

b) Find the gradient of graph B.

c) Find the gradient of graph C.

d) Find the gradient of graph D.

Check your performance against the given solutions at the end of this subunit; and if you are satisfied with your performance continue, or otherwise review ***calculating the gradient (slope) of linear graph***.

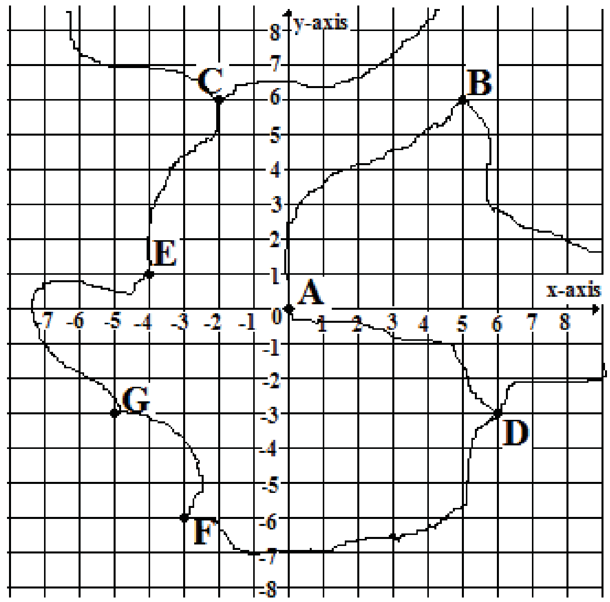
Note it!

Any points on a linear graph can be used to calculate the gradient.

The gradient of a linear graph is a constant.

Solutions to ACTIVITY 1:

1. Calculating the gradients of the linear graphs that pass through the given points on the map.



a) A and B.

A and B are $(0,0)$ and $(5,6)$, respectively.

Let $(x_1, y_1) = (0,0)$ and $(x_2, y_2) = (5,6)$.

Then, $m = \dfrac{(6-0)}{(5-0)} = \dfrac{6}{5}$.

b) C and D.

C and D are $(-2,6)$ and $(6,-3)$, respectively.

Let $(x_1, y_1) = (-2,6)$ and $(x_2, y_2) = (6,-3)$.

Then, $m = \dfrac{(-3-6)}{(6-^{-}2)} = \dfrac{-9}{8} = -\dfrac{9}{8}$.

c) E and G.

E and G are $(-4,1)$ and $(-5,-3)$, respectively.

Let $(x_1, y_1) = (-4,1)$ and $(x_2, y_2) = (-5,-3)$.

Then, $m = \dfrac{(-3-1)}{(-5-^{-}4)} = \dfrac{-4}{-1} = 4$.

d) F and D.

F and D are $(-3,-6)$ and $(6,-3)$, respectively.

Let $(x_1, y_1) = (-3, -6)$ and $(x_2, y_2) = (6, -3)$.

Then, $m = \frac{(-3 - {}^{-}6)}{(6 - {}^{-}3)} = \frac{3}{9} = \frac{1}{3}$.

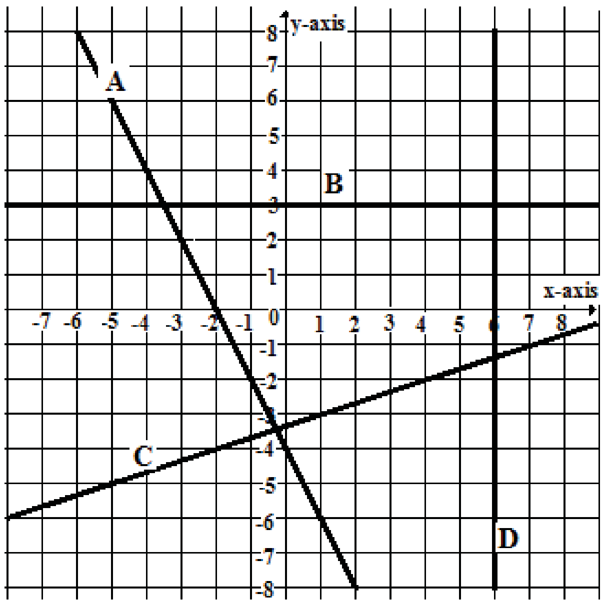
e) A and D.

A and D are $(0, 0)$ and $(6, -3)$, respectively.

Let $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (6, -3)$.

Then, $m = \frac{(-3 - 0)}{(6 - 0)} = \frac{-3}{6} = -\frac{1}{2}$.

2. The linear graphs, A, B, C and D.



a) Finding the gradient of graph A:

Graph A passes through $(-2, 0)$ and $(0, -4)$.

Let $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (0, -4)$.

Then, $m = \frac{(-4 - 0)}{(0 - {}^{-}2)} = \frac{-4}{2} = -2$.

b) Find the gradient of graph B.

Graph B passes through $(0, 3)$ and $(2, 3)$.

Let $(x_1, y_1) = (0,3)$ and $(x_2, y_2) = (2,3)$.

Then, $m = \dfrac{(3-3)}{(2-0)} = \dfrac{0}{2} = 0$.

The gradient of any horizontal line is zero.

c) Find the gradient of graph C.

Graph C passes through $(1,-3)$ and $(4,-2)$.

Let $(x_1, y_1) = (1,-3)$ and $(x_2, y_2) = (4,-2)$.

Then, $m = \dfrac{(-2-{}^{-}3)}{(4-1)} = \dfrac{1}{3}$.

d) Find the gradient of graph D.

Graph D passes through $(6,0)$ and $(6,4)$.

Let $(x_1, y_1) = (6,0)$ and $(x_2, y_2) = (6,4)$.

Then, $m = \dfrac{(4-0)}{(6-6)} = \dfrac{4}{0}$.

Since division by zero does not exist, the gradient is undefined.

All vertical lines have undefined gradient.

Lesson 2 Finding the X-intercept of Linear Graphs

Introduction

By the end of this subunit, you should be able to:

- write the x-intercept of a linear graph from the drawn graph.
- calculate the x-intercept of a linear graph from the equation of the graph.

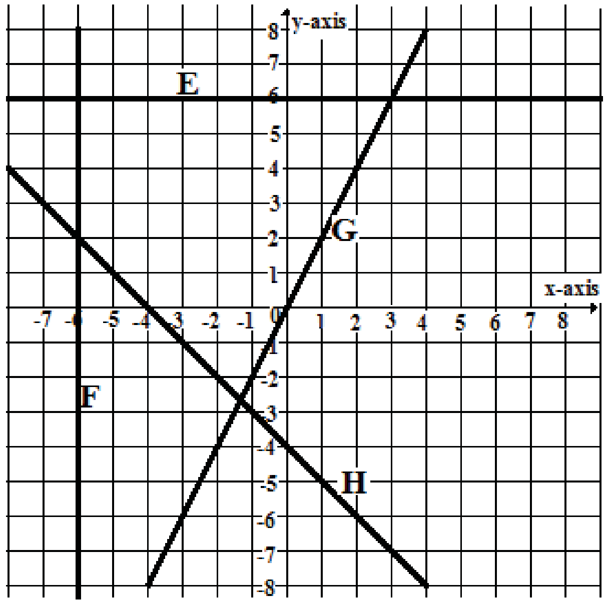
This subunit is about 10 pages in length.

Writing the x-intercept of a linear graph from the drawn graph.

The x-intercept is the x coordinate of a point where a graph crosses the x-axis on the x-y plane.
When a graph has been drawn, the x-intercept can be easily read if it is an integer.

Example 1

Look at the graphs on the grid below.



i. Graph E does not cross the x-axis. So it does not have the x-intercept. In fact all **horizontal lines** do not have the x-intercept.

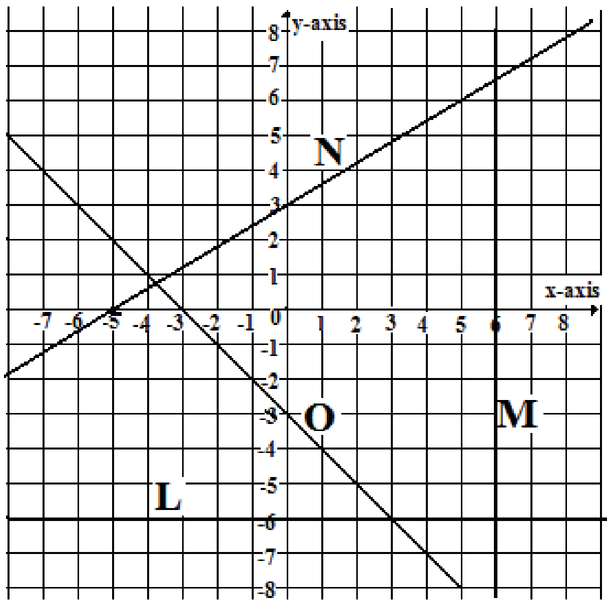
ii. Graph F crosses the x-axis at $(-6,0)$. So, -6 is the x-intercept.

iii. Graph G crosses the x-axis at $(0,0)$. So, 0 is the x-intercept.

iv. Graph H crosses the x-axis at $(-4,0)$. So, -4 is the x-intercept.

Example 2

The graphs of linear equations are shown on the grid below.



i. What is the x-intercept of graph L?

Compare your answer with:

Graph L has no x-intercept, because it is a horizontal line.

ii. What is the x-intercept of graph M?

Compare your answer with:

6 is x-intercept, because it meets x-axis at $(6,0)$.

iii. What is the x-intercept of graph N?

Compare your answer with:

-5 is x-intercept, because it meets x-axis at $(-5,0)$.

iv. What is the x-intercept of graph O?

Compare your answer with:

-5 is x-intercept, because it meets x-axis at $(-5,0)$.

Finding the x-intercept from the equation of a linear graph.

The x-intercept is the x coordinate of a point where a graph crosses the x-axis on the x-y plane. Since all the points on the x-axis have the y coordinate zero, the x-intercept is found by substituting zero for y in the equation and solving for x.

Example 1

The equation of a linear graph is $x+2y=4$ .What is its x-intercept?

Substitute zero for y in $x+2y=4$ and solve for x:

$x + 2(0) = 4$

$x + \ \ 0 \ \ = 4$

$x \qquad\ = 4$

Therefore, the x-intercept is 4.

Example 2

The linear graph has the equation $2y = 3x + 4$ .Find the x-intercept.

____

____

____

____

____

____

____

Compare your answer with:

The x-intercept is $-\frac{4}{3}$. It has been found by substituting zero for y in the equation and solving for x:

$$2(0) \ = 3x + 4$$

$$0 \quad = 3x + 4$$

$$\textit{Subtract } 4 \textit{ from both sides and simplify}:$$

$$0 - 4 = 3x + 4 - 4$$

$$-4 = 3x$$

$$\textit{Divide both sides by } 3:$$

$$\frac{-4}{3} = x$$

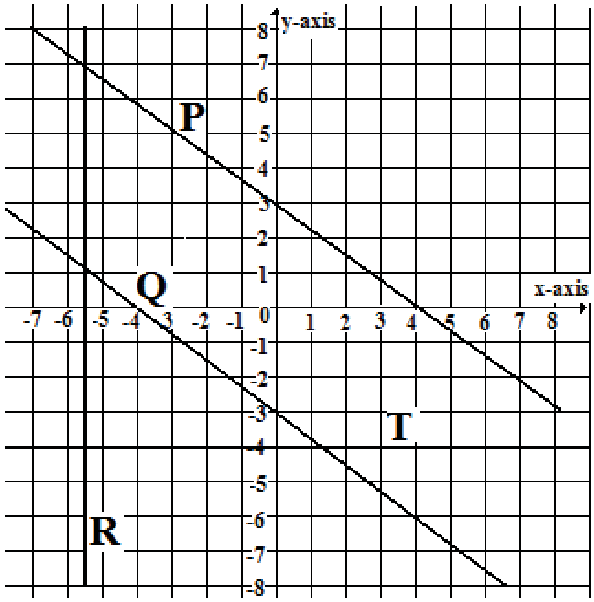
$$x = -\frac{4}{3}$$

So, substituting zero for y in a linear equation and solving for x gives the x-intercept.

Activity 2

Activity 2

1. Look at the linear graphs shown below.



a) Write down the x-intercept of graph P.

____

b) Write down the x-intercept of graph Q.

____

c) Write down the x-intercept of graph R.

____

d) Write down the x-intercept of graph T.

____

2. Find the x-intercepts of the linear graphs from their given equations.

a) $x + y = 3$

____

____

____

____

b) $y = x - 7$

c) $y = \dfrac{x+1}{5}$

d) $x = \dfrac{3}{4}$

e) $\dfrac{1}{2}y = \dfrac{2}{3}x + 2$

Check your performance against the given solutions at the end of this subunit; and if you are satisfied with your performance continue, or otherwise review ***calculating the x-intercept of linear graph***.

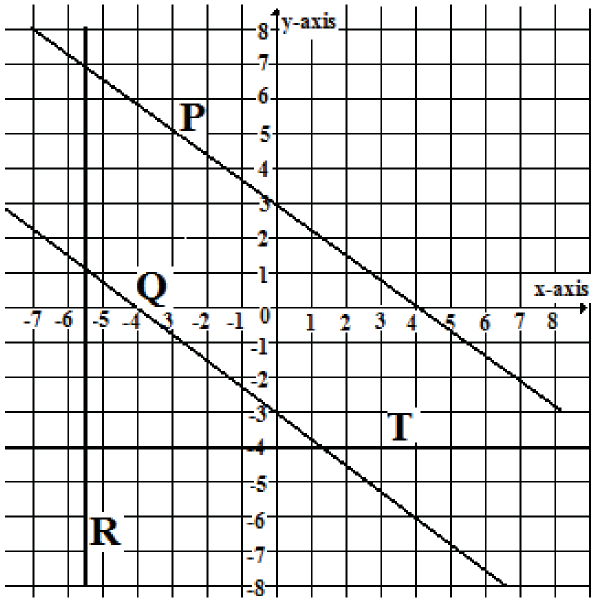
Note it!

The x-intercept of a graph is found by substituting zero for x in the equation and solving for x.

It can also be read, from a drawn graph, at the point where the graph crosses the x-axis.

Solutions to ACTIVITY 2:

1.



a) The x-intercept of graph P:

P crosses x-axis at $(4,0)$; 4 is x-intercept.

b) The x-intercept of graph Q:

Q crosses x-axis at $(-4,0)$; -4 is x-intercept.

c) The x-intercept of graph R:

R crosses x-axis at $(-5\frac{1}{2},0)$; $-5\frac{1}{2}$ is x-intercept.

d) The x-intercept of graph T:

T is parallel to the x-axis, so it does not have x-intercept.

2. The x-intercepts of the linear graphs from their given equations.

a) The x-intercept of $x+y=3$:

Substitute zero for y in $x+y=3$ and solve for x:

$x+0=3$

$x \quad =3$

Therefore, the x-intercept is 3.

b) The x-intercept of $y=x-7$:

Substitute zero for y in $y=x-7$ and solve for x:

$0 = x - 7$

$0 + 7 = x - 7 + 7$

$7 = x$

$x = 7$

Therefore, the x-intercept is 7.

c) The x-intercept of $y = \dfrac{x+1}{5}$:

Substitute zero for y in $y = \dfrac{x+1}{5}$ and solve for x:

$0 = \dfrac{x+1}{5}$

Multiply both sides by 5.

$0(5) = \dfrac{x+1}{5}(5)$

$0 = x + 1$

$0 - 1 = x + 1 - 1$

$-1 = x$

$x = -1$

Therefore, the x-intercept is -1.

d) The x-intercept of $x = \dfrac{3}{4}$:

It is $\dfrac{3}{4}$

e) The x-intercept of $\dfrac{1}{2}y = \dfrac{2}{3}x + 2$:

Substitute zero for y in $\dfrac{1}{2}y = \dfrac{2}{3}x + 2$ and solve for x:

$$\frac{1}{2}(0) = \frac{2}{3}x + 2$$

$$0 = \frac{2}{3}x + 2$$

Multiply both sides by 3.

$$0(3) = 3 \times \frac{2}{3}x + 2(3)$$

$$0 = 2x + 6$$

$$0 - 6 = 2x + 6 - 6$$

$$-6 = 2x$$

$$\frac{-6}{2} = \frac{2x}{2}$$

$$-3 = x$$

$$x = -3$$

Therefore, the x-intercept is -3.

Lesson 3 Finding the Y-intercept of Linear Graph

Introduction

By the end of this subunit, you should be able to:

- write the y-intercept of a linear graph from the drawn graph.
- calculate the y-intercept of a linear graph from the equation of the graph.

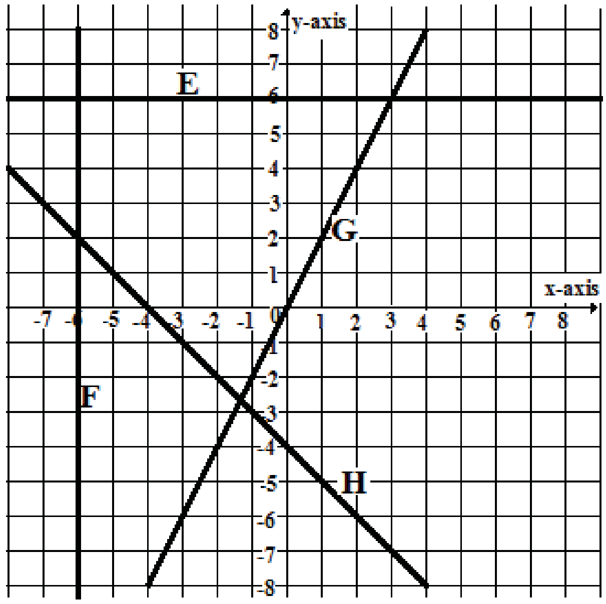
This subunit is about 8 pages in length.

Writing the y-intercept of a linear graph from the drawn graph.

The y-intercept is the y coordinate of a point where a graph crosses the y-axis on the x-y plane.
When a graph has been drawn, the y-intercept can be easily read if it is an integer.

Example 1

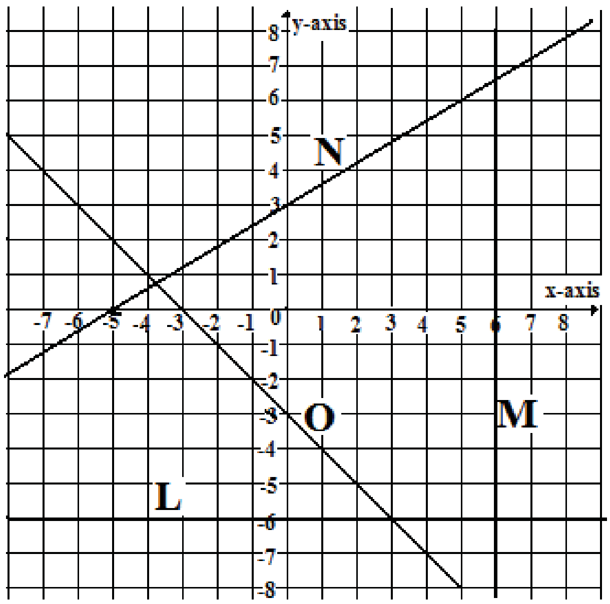
Look at the graphs on the grid below.



i. Graph E crosses the y-axis at $(0, 6)$. So, 6 is the y-intercept.
ii. Graph F does not cross the y-axis. So it does not have the y-intercept.
All **vertictal lines** do not have the y-intercept.
iii. Graph G crosses the y-axis at $(0, 0)$. So, 0 is the y-intercept.
iv. Graph H crosses the y-axis at $(0, -4)$. So, -4 is the y-intercept.

Example 2

The graphs of linear equations are shown on the grid below.



i. What is the y-intercept of graph L?

Compare your answer with:

-6 is y-intercept, because it meets y-axis at $(0,-6)$.

ii. What is the y-intercept of graph M?

Compare your answer with:

Graph M has no y-intercept, as it is a vertical line.

iii. What is the y-intercept of graph N?

Compare your answer with:

3 is y-intercept, because it meets y-axis at $(0,3)$.

iv. What is the y-intercept of graph O?

Compare your answer with:

-3 is y-intercept, because it meets y-axis at $(0,-3)$.

Finding a y-intercept from the equation of a linear graph.

The y-intercept is the y coordinate of a point where a graph crosses the y-axis on the x-y plane. Since all the points on the y-axis have the x coordinate zero, the y-intercept is found by substituting zero for x in the equation and solving for y.

Example 1

The equation of a linear graph is $x + 2y = 4$ .What is its y-intercept?

Substitute zero for x in $x + 2y = 4$ and solve for y:

$$0 + 2y = 4$$
$$2y = 4$$
$$\textit{Divide both sides by } 2.$$
$$\frac{2y}{2} = \frac{4}{2}$$
$$y = 2$$

Therefore, the y-intercept is 2.

Example 2

The linear graph has the equation $2y = 3x + 4$ .Find the y-intercept.

Compare your answer with:

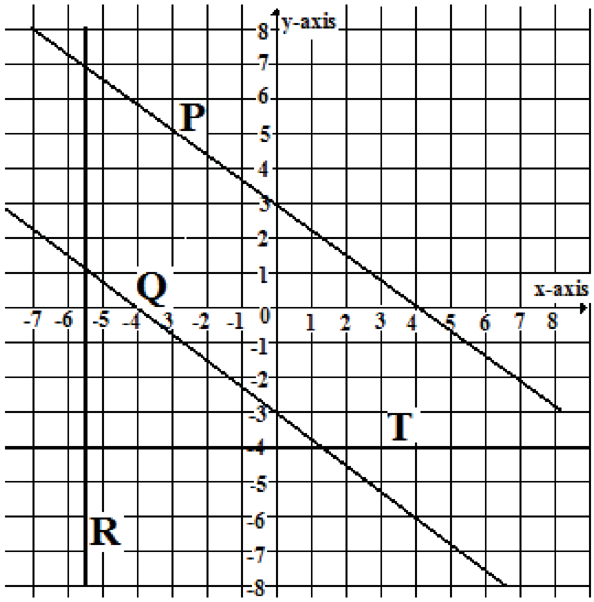
The y-intercept is 2 . It has been found by substituting zero for x in the equation and solving for y:

$$2y = 3(0) + 4$$
$$2y = 4$$
$$\textit{Divide both sides by } 2:$$
$$\frac{2y}{2} = \frac{4}{2}$$
$$y = 2$$

Activity 3

1. Look at the linear graphs shown below.

Activity 3



a) Write down the y-intercept of graph P.

---

b) Write down the y-intercept of graph Q.

---

c) Write down the y-intercept of graph R.

---

d) Write down the y-intercept of graph T.

---

2. Find the y-intercepts of the linear graphs from their given equations.

a) $x + y = 3$

---

b) $y = x - 7$

c) $y = \dfrac{x+1}{5}$

d) $x = \dfrac{3}{4}$

e) $\frac{1}{2}y = \frac{2}{3}x + 2$

Check your performance against the given solutions at the end of this subunit; and if you are satisfied with your performance continue, or otherwise review ***calculating the y-intercept of linear graph***.

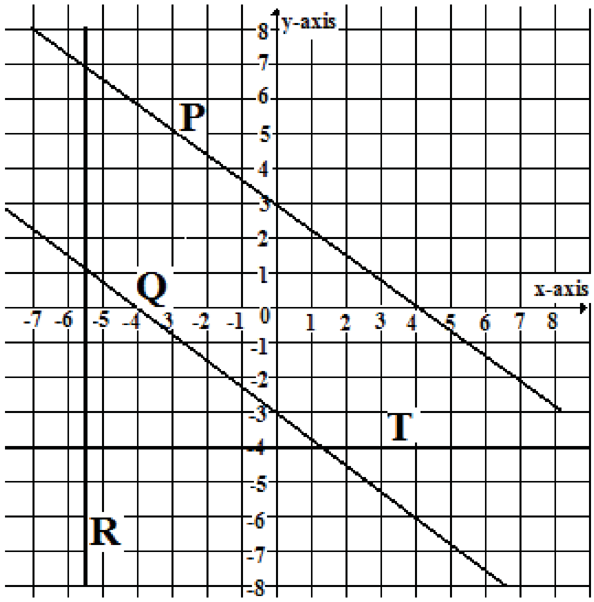
Note it!

The y-intercept of a graph is found by substituting zero for x in the equation and solving for y.

It can also be read, from a drawn graph, at the point where the graph crosses the y-axis.

Solutions to ACTIVITY 3:

1.



a) The y-intercept of graph P:

P crosses y-axis at $(0,3)$; 3 is y-intercept.

b) The y-intercept of graph Q:

Q crosses y-axis at $(0,-3)$; -3 is y-intercept.

c) The y-intercept of graph R:

R is parallel to y-axis, and therefore, it does not have y-intercept.

d) The y-intercept of graph T:

T crosses y-axis at $(0,-4)$; -4 is y-intercept.

2. The y-intercepts of the linear graphs from their given equations.

a) The y-intercept of $x+y=3$:

Substitute zero for x in $x+y=3$ and solve for y:

$$0+y=3$$
$$y=3$$

Therefore, the y-intercept is 3.

b) The y-intercept of $y=x-7$:

Substitute zero for x in $y = x - 7$ and solve for y:

$$y = 0 - 7$$
$$y = -7$$

Therefore, the y-intercept is -7.

c) The y-intercept of $y = \frac{x+1}{5}$:

Substitute zero for x in $y = \frac{x+1}{5}$ and solve for y:

$$y = \frac{0+1}{5}$$
$$y = \frac{1}{5}$$

Therefore, the y-intercept is $\frac{1}{5}$.

d) The y-intercept of $x = \frac{3}{4}$:

$x = \frac{3}{4}$ is parallel to y-axis, and therefore, there is no y-intercept.

e) The y-intercept of $\frac{1}{2}y = \frac{2}{3}x + 2$:

Substitute zero for x in $\frac{1}{2}y = \frac{2}{3}x + 2$ and solve for y:

$$\frac{1}{2}y = \frac{2}{3}(0) + 2$$
$$\frac{1}{2}y = 0 + 2$$
$$\frac{1}{2}y = 2$$
$$(2)\frac{1}{2}y = 2(2)$$
$$y = 4$$

Therefore, the y-intercept is 4.

Lesson 4 Deriving the Equation of a Graph, from the Drawn Graph

Introduction

By the end of this subunit, you should be able to
- find the equation of a drawn linear graph.

This subunit is about 10 pages in length.

Finding the equation of a drawn linear graph.

The standard form of the equation of a linear graph is $ax + by = c$,where a , b and c are constants; and a and b are **not both** equal to zero .

When a is zero, the equation, $ax + by = c$ is reduced to $by = c$ .That means $y = \frac{c}{b}$, and $\frac{c}{b}$ is y-intercept. The linear graph is a **horizontal** line in the x-y plane.

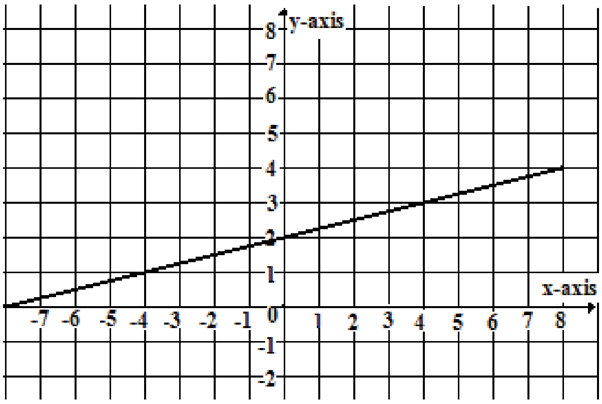
When b is zero, the equation, $ax + by = c$ is reduced to $ax = c$ .That means $x = \frac{c}{a}$, and $\frac{c}{a}$ is x-intercept. The linear graph is a **vertical** line in the x-y plane.

When a is not equal to zero and b is not equal to zero, the linear graph is a **diagonal** in the x-y plane.

Now, when y is made the subject in the standard form of a linear equation $ax + by = c$, the resulting equation is called the **gradient-intercept form** of the equation of a linear graph. It is $y = mx + c$,where m is the gradient and c is the y-intercept. The equation $y = mx + c$ is mostly used when deriving the equation of a graph, from the drawn graph.

Example 1

Find the equation of the linear graph shown in the grid.



First, choose two points on the graph, and then calculate the gradient with them.

Suppose $(0,2)$ and $(4,3)$ are chosen.

Let $(x_1, y_1) = (0,2)$ and $(x_2, y_2) = (4,3)$.

Now, substitute the coordinates in the gradient formula, $m = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}$:

$$m = \frac{(3-2)}{(4-0)} = \frac{1}{4}.$$

Therefore, the gradient is $\dfrac{1}{4}$.

So far, the equation is $y = \dfrac{1}{4}x + c$.

Second, find the y-intercept, c, **either** by looking at it on the graph (where possible) **or** by calculation.

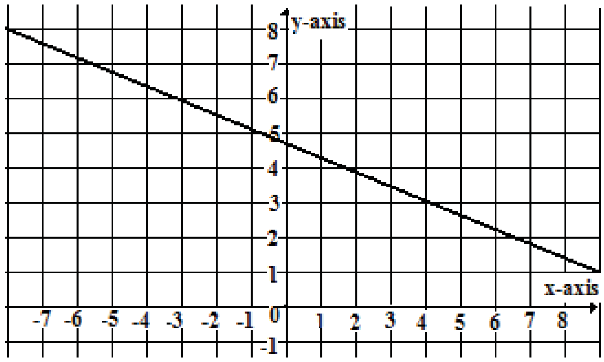
When looking at the graph above, it can be seen that the y-intercept is 2.

Therefore, in gradient-intercept form, $y = mx + c$, the equation is $y = \dfrac{1}{4}x + 2$.

Remember that m stands for the gradient and c stands for the y-intercept.

Example 2

Look at the linear graph shown in the grid.



Find the gradient of the line shown in the grid.

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Compare your answer with:

The gradient is $\frac{33}{7}$.

Suppose $(-3,6)$ and $(4,3)$ are chosen.

Let $(x_1, y_1) = (-3,6)$ and $(x_2, y_2) = (4,3)$.

Now, substitute the coordinates in the gradient formula, $m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$:

$$m = \frac{(3-6)}{(4-{}^{-}3)} = -\frac{3}{7}.$$

Therefore, the gradient is $-\frac{3}{7}$.

So far, the equation is $y = -\frac{3}{7}x + c$.

Looking at the graph, the y-intercept cannot be easily read, since it is not an integer. Therefore, the y-intercept is calculated by substituting the coordinates of any point on the graph in $y = -\frac{3}{7}x + c$ and solving for c .

Now, substitute the coordinates of $(4,3)$ in $y=-\frac{3}{7}x+c$ and solve for c.

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Compare your answer with:

$3=-\frac{3}{7}(4)+c$

Multiply both sides by 7 and simplify.

$3(7)=-\frac{3}{7}(4)(7)+(7)c$

$21 \quad =-12 \qquad +7c$

Add 12 to both sides and simplify.

$21+12=-12+12+7c$

$33 \qquad = \qquad 7c$

Divide by 7 on both sides and simplify.

$\frac{33}{7} \qquad = \qquad \frac{7c}{7}$

$\frac{33}{7} \qquad =c$

$c \qquad =\frac{33}{7}$

So, what is the equation in gradient-intercept form?

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Compare your answer with:

$y=-\frac{3}{7}x+\frac{33}{7}$

Therefore, two things are needed in order to write the linear equation in gradient-intercept form:

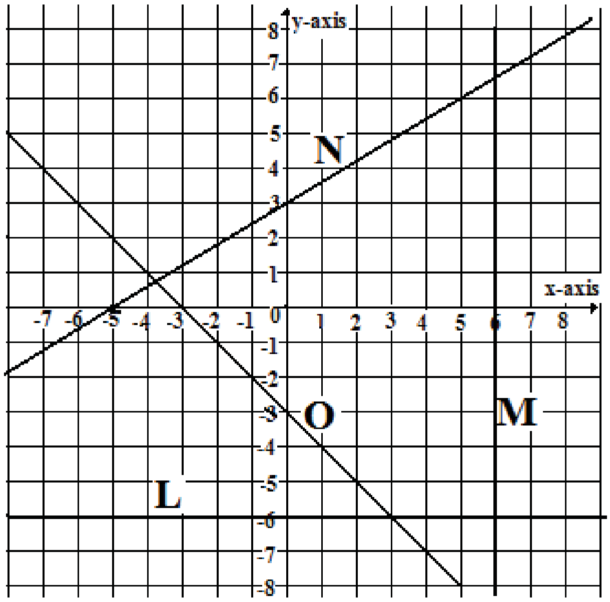
1. Choose two points, (x_1, y_1) and (x_2, y_2), on the graph, and then calculate the gradient with gradient formula, $m=\frac{(y_2-y_1)}{(x_2-x_1)}$.

2. Find the y-intercept, c, **either** by looking at it on the graph (where possible) **or** by substituting the coordinates of any point on the graph in $y = mx + c$ and solving for c.

Activity 4

Activity 4

1) Find the equations of the linear graphs in the grid.



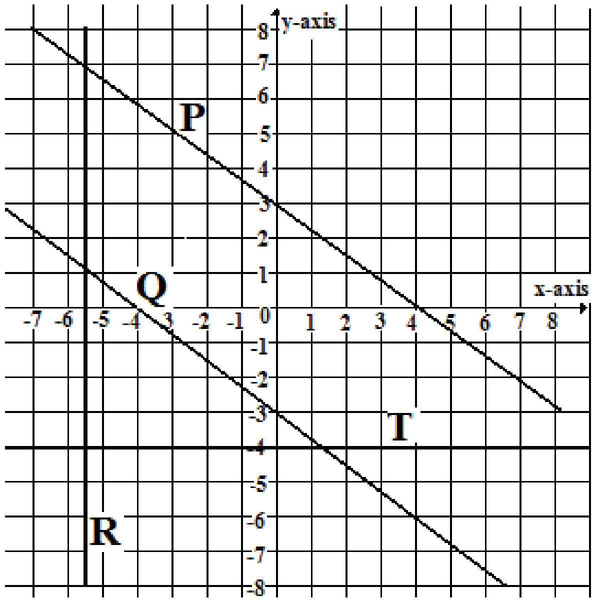
a. Equation of L:

b. Equation of M:

c. Equation of N:

d. Equation of O:

2) Work out the equations of the linear graphs shown in the grid.



a. Equation of P:

b. Equation of Q:

c. Equation of R:

d. Equation of T:

Check your performance against the given solutions at the end of this subunit; and if you are satisfied with your performance continue, or otherwise review ***deriving the equation of a graph, from the drawn graph***.

Note it!

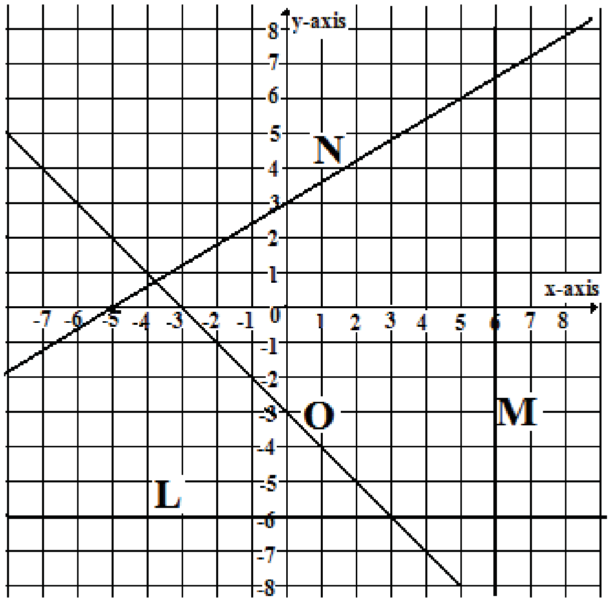
The standard form of the equation of a linear graph is $ax + by = c$, where a, b and c are constants; and a and b are **not both** equal to zero.

When y is made the subject in the standard form of a linear equation $ax + by = c$, the resulting equation is called the **gradient-intercept form** of the equation of a linear graph.

It is $y = mx + c$, where m is the gradient and c is the y-intercept. The equation $y = mx + c$ is mostly used when deriving the equation of a graph, from the drawn graph.

Solutions to ACTIVITY 4:

1)



a. Equation of L:

L is a horizontal line with -6 as y-intercept. So the equation is $y = -6$.

b. Equation of M:

M is a vertical line with 6 as x-intercept. So the equation is $x = 6$.

c. Equation of N:

When using $(-5, 0)$ and $(0, 3)$ from the graph, to calculate the gradient:

Let $(x_1, y_1) = (-5, 0)$ and $(x_2, y_2) = (0, 3)$.

Then, $m = \dfrac{(3-0)}{(0-{}^-5)} = \dfrac{3}{5}$.

So far, the equation is $y = \dfrac{3}{5}x + c$.

Looking at the graph, the y-intercept can be read as 3.

So the equation is $y = \dfrac{3}{5}x + 3$.

d. Equation of O:

When using $(-3, 0)$ and $(0, -3)$ from the graph, to calculate the gradient:

Let $(x_1, y_1) = (-3, 0)$ and $(x_2, y_2) = (0, -3)$.

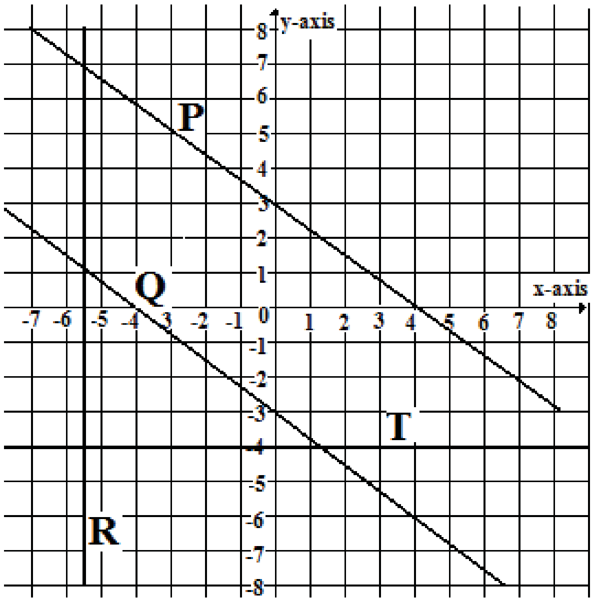
Then, $m = \frac{(-3-0)}{(0-^{-}3)} = \frac{-3}{3} = -1$.

So far, the equation is $y = -x + c$.

Looking at the graph, the y-intercept can be read as -3.

So the equation is $y = -x - 3$.

2) The equations of the linear graphs.



a. Equation of P:

When using $(4,0)$ and $(0,3)$ from the graph, to calculate the gradient:

Let $(x_1, y_1) = (4,0)$ and $(x_2, y_2) = (0,3)$.

Then, $m = \frac{(3-0)}{(0-4)} = \frac{3}{-4} = -\frac{3}{4}$.

So far, the equation is $y = -\frac{3}{4}x + c$.

Looking at the graph, the y-intercept can be read as 3.

So the equation is $y = -\frac{3}{4}x + 3$.

b. Equation of Q:

When using $(-4,0)$ and $(0,-3)$ from the graph, to calculate the gradient:

Let $(x_1, y_1) = (-4,0)$ and $(x_2, y_2) = (0,-3)$.

Then, $m = \dfrac{(-3-0)}{(0-{}^{-}4)} = \dfrac{-3}{4} = -\dfrac{3}{4}$.

So far, the equation is $y = -\dfrac{3}{4}x + c$.

Looking at the graph, the y-intercept can be read as -3.

So the equation is $y = -\dfrac{3}{4}x - 3$.

c. Equation of R:

R is a vertical line with $-5\frac{1}{2}$ as x-intercept. So the equation is $x = -5\frac{1}{2}$.

d. Equation of T:

T is a vertical line with -4 as y-intercept. So the equation is $y = -4$.

Lesson 5 Drawing a Linear Graph, from the Given Equation

Introduction

By the end of this subunit, you should be able to

- draw a linear graph , which is horizontal in the x-y plane.
- draw a linear graph , which is vertical in the x-y plane.
- draw a linear graph , which is diagonal in the x-y plane.

This subunit is about 15 pages in length.

A linear graph, which is horizontal in the x-y plane.

From the standard form of the equation of a linear graph, $ax + by = c$,where a , b and c are constants, for a **horizontal** linear graph, a is equal to zero.

When a is zero, the equation, $ax + by = c$ is reduced to $by = c$ .That means $y = \dfrac{c}{b}$, and $\dfrac{c}{b}$ is y-intercept.

Simply put, y is equal to a number for a **horizontal** linear graph.

Example 1

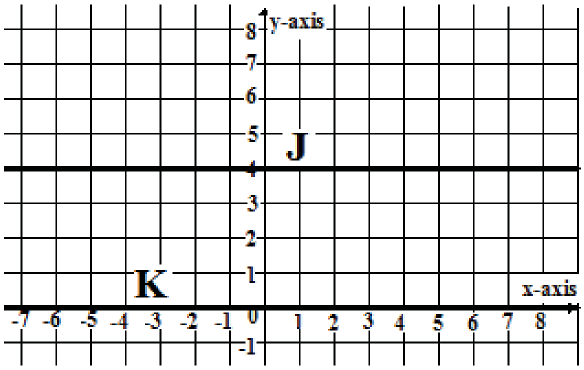
Draw the following linear graphs:

J is a linear graph whose equation is $y = 4$.

K is a linear graph whose equation is $y = 0$.

Now, J and K must be **horizontal** linear graphs, crossing the y-axis at 4 *and* 0 respectively because y is equal to a number in both cases.

So, the graphs are as shown in the diagram below.



A linear graph, which is vertical in the x-y plane.

From the standard form of the equation of a linear graph, $ax + by = c$,where a , b and c are constants, for a **vertical** linear graph, a is equal to zero.

When b is zero, the equation, $ax + by = c$ is reduced to $ax = c$ .That means $x = \frac{c}{a}$, and $\frac{c}{a}$ is x-intercept.

Simply put, x is equal to a number for a **vertical** linear graph.

Example 1

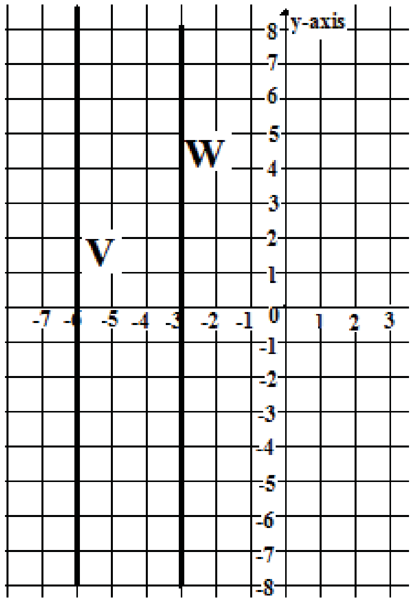
Draw the following graphs in a grid:

V is a linear graph whose equation is $x = -6$.

W is a linear graph whose equation is $x = -3$.

Now, V and W must be **vertical** linear graphs, crossing the x-axis at -6 *and* -3 respectively because x is equal to a number in both cases.

So, the graphs are as shown in the diagram below.



A linear graph, which is diagonal in the x-y plane.

The standard form of the equation of a linear graph is $ax + by = c$,where a , b and c are constants; and a and b are **not both** equal to zero .

When a is not equal to zero and b is not equal to zero, the linear graph is a **diagonal** in the x-y plane.

When the gradient is negative, the graph goes diagonally down; and when the gradient is positive, the graph goes diagonally up.

One **point** and the **gradient** can be used to draw a linear graph, which is diagonal in the x-y plane.

Example 1

Draw the graph of $x + y = 5$.

The equation is reduced to gradient-intercept form, $y = mx + c$:

$$x + y = 5$$

Subtract x from both sides.

$$x - x + y = 5 - x$$

$$y = 5 - x$$

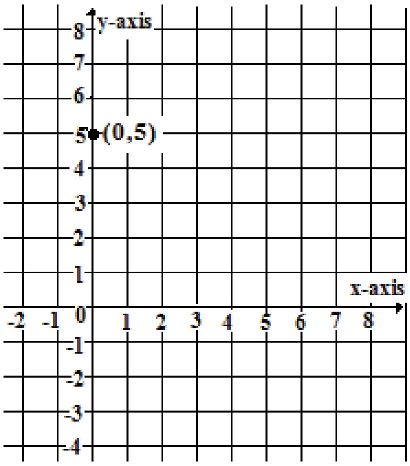
$$y = -x + 5$$

All points on $x + y = 5$ are ordered pairs (x, y), of the form $(x, -x+5)$ since y *is equal to* $-x+5$.

Points on the graph are obtained by letting x be any real number.

One such a point is $(0,5)$, obtained by letting x be 0 in $(x, -x+5)$.

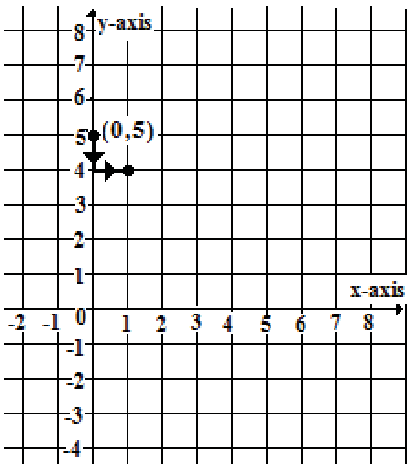
Now, $(0,5)$ can be plotted in the grid.



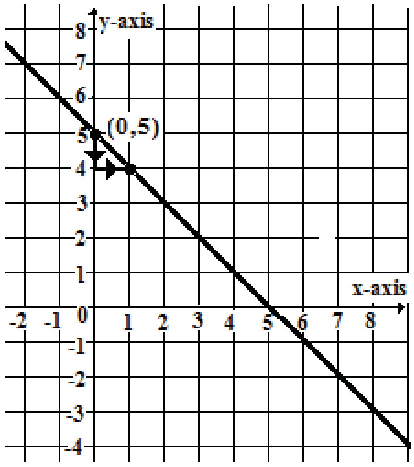
From the gradient-intercept form, $y = -x + 5$, the gradient is -1.

The gradient, -1 which is $-\frac{1}{1}$, means go 1 unit down from $(0,5)$, then 1 unit right.

The end point is on the graph of $x + y = 5$.



Now, the end point and $(0,5)$ can be joined with straight line, which represents the graph of $x+y=5$.The line is extended as far as possible.



Example 2

Draw the graph of $3y = x+6$.

Reduce the equation to gradient-intercept form, $y = mx + c$:

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Compare your answer with:

$3y = x+6$

Divide both sides by 3 and simplify.

$$\frac{3y}{3} = \frac{x+6}{3}$$

$$y = \frac{x}{3} + \frac{6}{3}$$

$$y = \frac{1}{3}x + 2$$

All points on $3y = x + 6$ are ordered pairs (x, y), of the form $\left(x, \frac{1}{3}x + 2\right)$ since y *is equal to* $\frac{1}{3}x + 2$.

Points on the graph are obtained by letting x be any real number.

Which point is on the graph, when x is 3?

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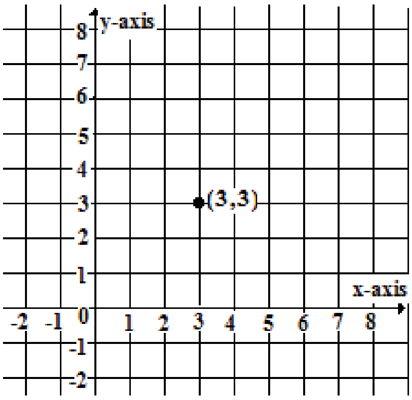
Compare your answer with:

The point is $(3, 3)$:

Letting x be 3 in $\left(x, \frac{1}{3}x + 2\right)$ gives $\left(3, \frac{1}{3}(3) + 2\right) = (3, 1 + 2) = (3, 3)$.

Now, $(3, 3)$ can be plotted in the grid.



What is the gradient of the graph?

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Compare your answer with:

The gradient-intercept form of $3y = x + 6$ is $y = \frac{1}{3}x + 2$.

So, the gradient is $\frac{1}{3}$.

How can the next point be plotted using the gradient?

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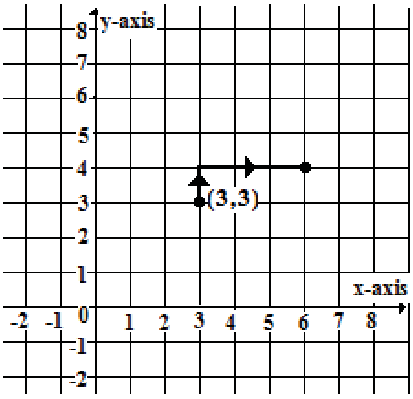
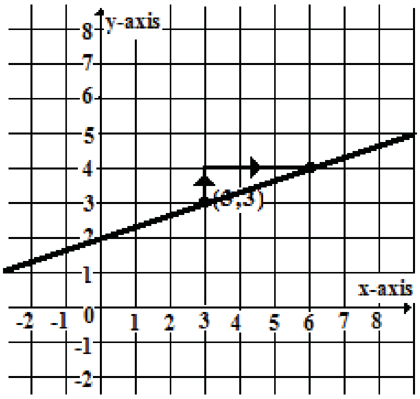
Compare your answer with:

The gradient, $\frac{1}{3}$, means go 1 unit up from $(3,3)$, then 3 units right.

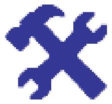
The end point is the next point on the graph of $3y = x + 6$.



Now, the end point and $(3,3)$ can be joined with straight line, which represents the graph of $3y = x + 6$. The line is extended as far as possible.



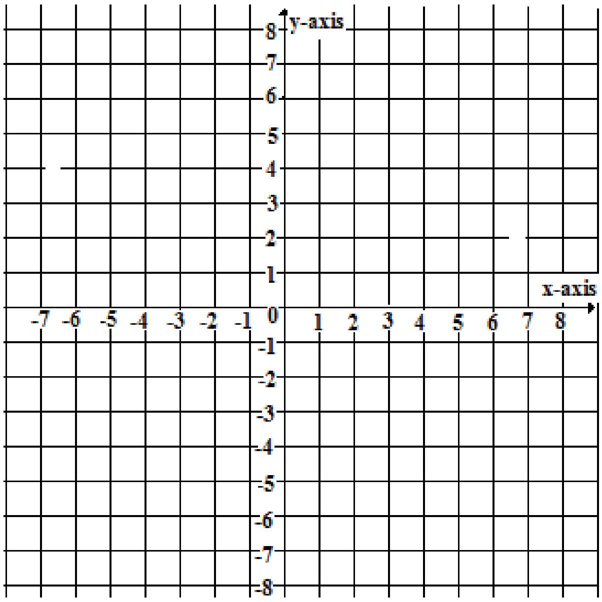
Activity 5



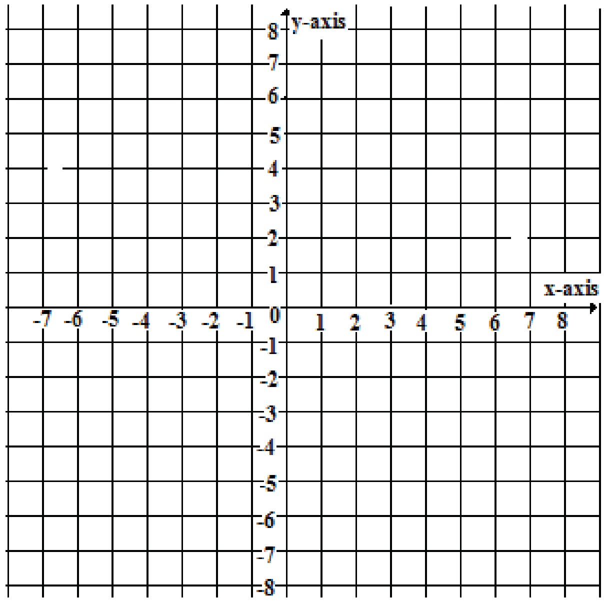
Activity 5

Draw the graphs of the given linear equations.

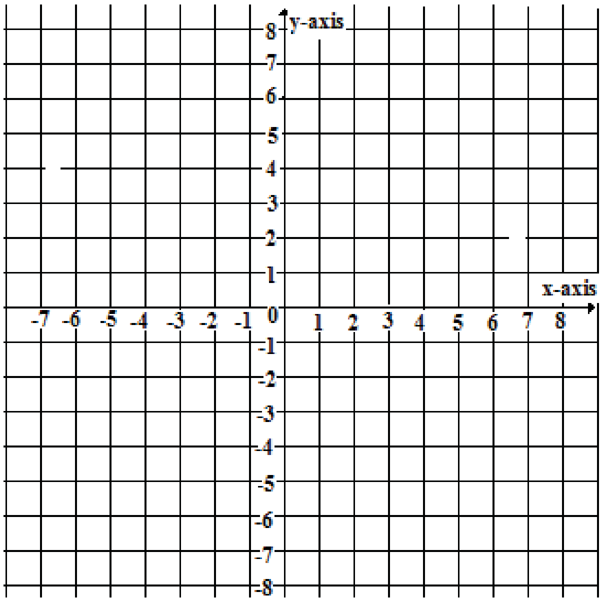
a) $3y = 6$.



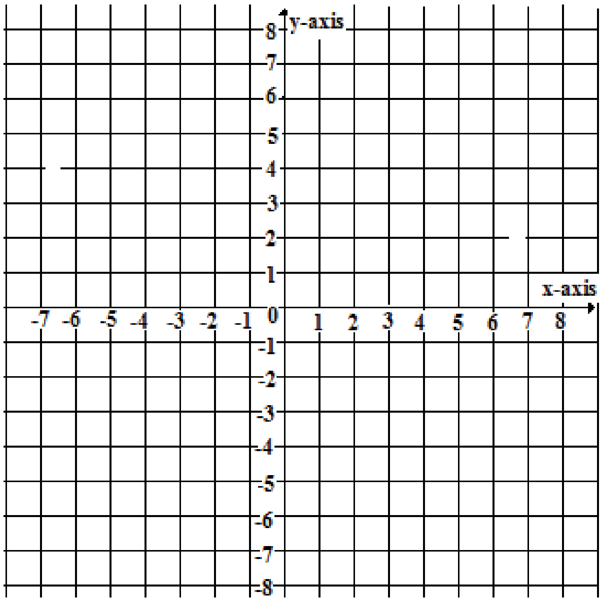
b) $-3 = x + 1$.



c) $y = \frac{3}{2}x - \frac{1}{2}$.


y-axis
x-axis
8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
-7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 8

d) $2y = x - 5$.


y-axis
x-axis
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-4
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1 2 3 4 5 6 7 8

e) $3y + 2x = x + 4$.

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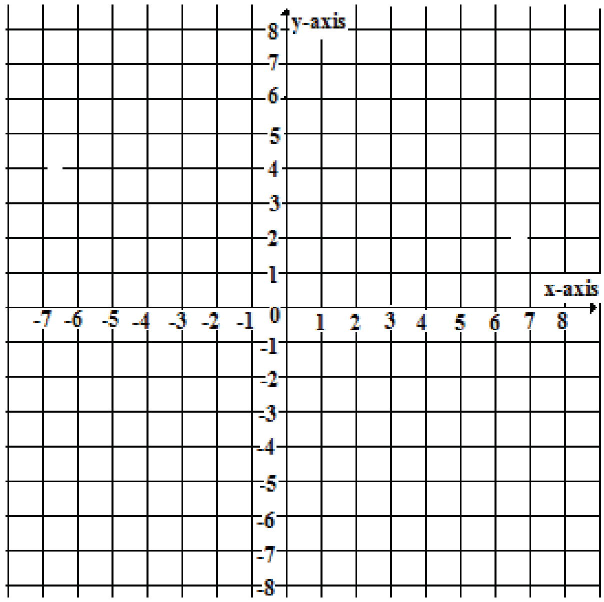


Check your performance against the given solutions at the end of this subunit; and if you are satisfied with your performance continue, or otherwise review ***drawing a linear graph, from the given equation***.

Note it!

A **horizontal** linear graph has an equation: y is equal to a number.

A **vertical** linear graph, has an equation: x is equal to a number.

One **point** and the **gradient** can be used to draw a linear graph, which is diagonal in the x-y plane.

Solutions to ACTIVITY 1:

Drawing the graphs of the given linear equations.

a) $3y = 6$.

Reduce the equation to gradient-intercept form, $y = mx + c$:

$3y = 6$

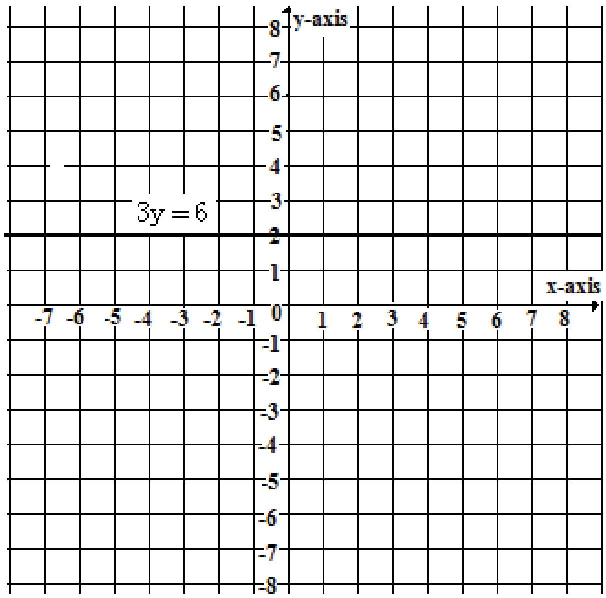
Divide both sides by 3 *and simplify.*

$$\frac{3y}{3} = \frac{6}{3}$$

$$y = \frac{6}{3}$$

$$y = 2$$

This is a horizontal line with 2 as y-intercept.



b) $-3 = x + 1$.

Reduce the equation to gradient-intercept form, $y = mx + c$:

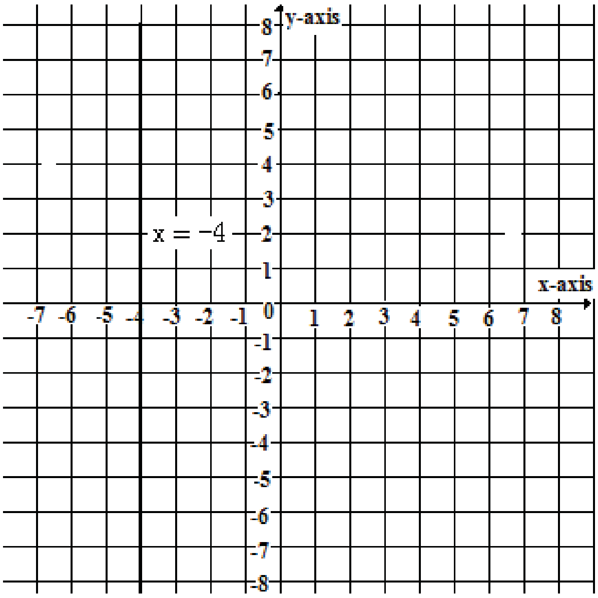
$-3 = x + 1$

$-3 - 1 = x + 1 - 1$

$-4 = x$

$x = -4$

This is a vertical line with -4 as x-intercept.



c) $y = \frac{3}{2}x - \frac{1}{2}$.

All points on $y = \frac{3}{2}x - \frac{1}{2}$ are ordered pairs (x, y), of the form $\left(x, \frac{3}{2}x - \frac{1}{2}\right)$.

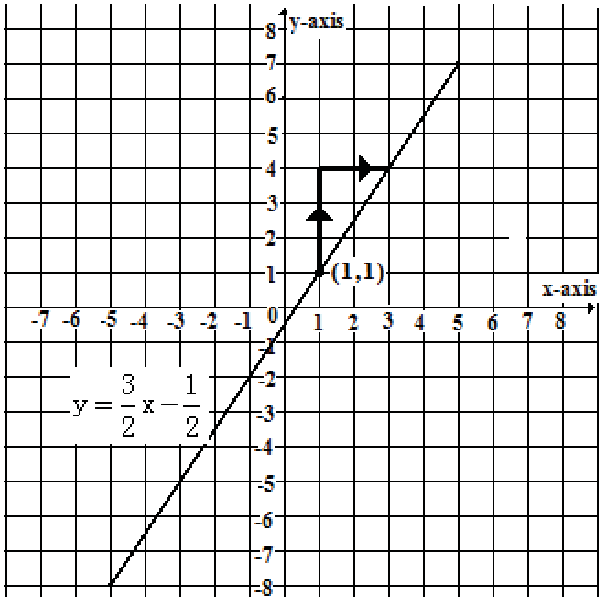
Points on the graph are obtained by letting x be any real number.

One such a point is $(1,1)$, obtained by letting x be 1 in $\left(x, \frac{3}{2}x - \frac{1}{2}\right)$.

The gradient of the line is $\frac{3}{2}$. It means, go 3 units up from $(1,1)$, then 2 units right.

The end point is the next point on the graph of $y = \frac{3}{2}x - \frac{1}{2}$.

Join $(1,1)$ and the end point to draw the graph:



d) $2y = x - 5$.

Reduce the equation to gradient-intercept form, $y = mx + c$:

$$\frac{2y}{2} = \frac{x-5}{2}$$

$$y = \frac{1}{2}x - \frac{5}{2}$$

All points on $2y = x - 5$ are ordered pairs (x, y), of the form $\left(x, \frac{1}{2}x - \frac{5}{2}\right)$.

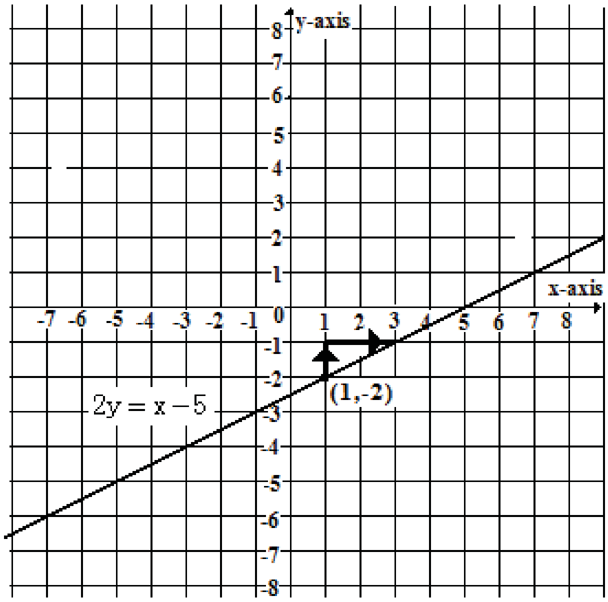
Points on the graph are obtained by letting x be any real number.

One such a point is $(1, -2)$, obtained by letting x be 1 in $\left(x, \frac{1}{2}x - \frac{5}{2}\right)$.

The gradient of the line is $\frac{1}{2}$. It means, go 1 unit up from $(1, -2)$, then 2 units right.

The end point is the next point on the graph of $2y = x - 5$.

Join $(1, -2)$ and the end point to draw the graph:



e) $3y + 2x = x + 4$.

Reduce the equation to gradient-intercept form, $y = mx + c$:

$3y + 2x = x + 4$

Subtract $2x$ from both sides and simplify.

$3y + 2x - 2x = x - 2x + 4$

$3y = -x + 4$

Divide both sides by 3.

$$\frac{3y}{3} = \frac{-x+4}{3}$$

$$y = -\frac{1}{3}x + \frac{4}{3}$$

All points on $3y + 2x = x + 4$ are ordered pairs (x, y), of the form $\left(x, -\frac{1}{3}x + \frac{4}{3}\right)$.

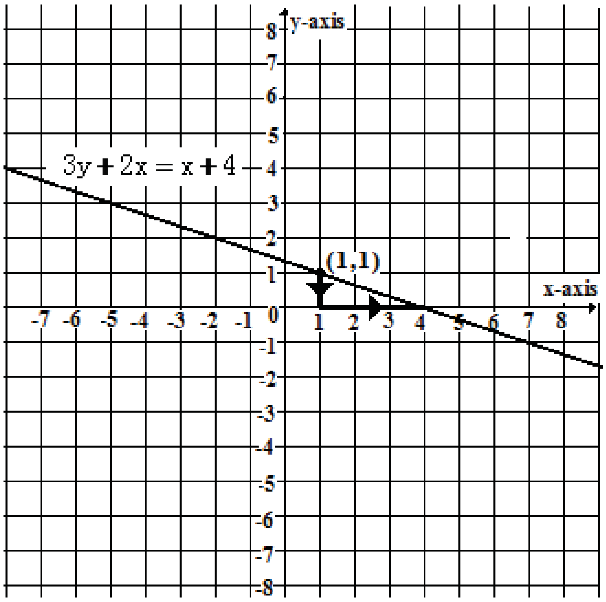
Points on the graph are obtained by letting x be any real number.

One such a point is $(1,1)$, obtained by letting x be 1 in $\left(x, -\frac{1}{3}x + \frac{4}{3}\right)$.

The gradient of the line is $-\frac{1}{3}$. It means, go 1 unit down from $(1,1)$, then 3 units right.

The end point is the next point on the graph of $3y+2x=x+4$.

Join $(1,1)$ and the end point to draw the graph:



Unit Summary

Summary

Any points on a linear graph can be used to calculate the gradient.

The gradient of a linear graph is a constant.

The x-intercept of a graph is found by substituting zero for x in the equation and solving for x.

It can also be read, from a drawn graph, at the point where the graph crosses the x-axis.

The y-intercept of a graph is found by substituting zero for x in the equation and solving for y.

It can also be read, from a drawn graph, at the point where the graph crosses the y-axis.

The standard form of the equation of a linear graph is $ax+by=c$,where a , b and c are constants; and a and b are **not both** equal to zero .

When y is made the subject in the standard form of a linear equation $ax+by=c$, the resulting equation is called the **gradient-intercept**

form of the equation of a linear graph.

It is $y = mx + c$,where m is the gradient and c is the y-intercept. The equation $y = mx + c$ is mostly used when deriving the equation of a graph, from the drawn graph.

A **horizontal** linear graph has an equation: y is equal to a number.

A **vertical** linear graph, has an equation: x is equal to a number.

One **point** and the **gradient** can be used to draw a linear graph, which is diagonal in the x-y plane.

You have completed the material for this unit on linear graphs. You should now spend some time reviewing the content. Once you feel that you can successfully write an exam that covers each of the learning outcomes, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

When you work on this assignment, please observe the time allocated and show your working or a reason for your answer.

TOTAL MARKS: 20

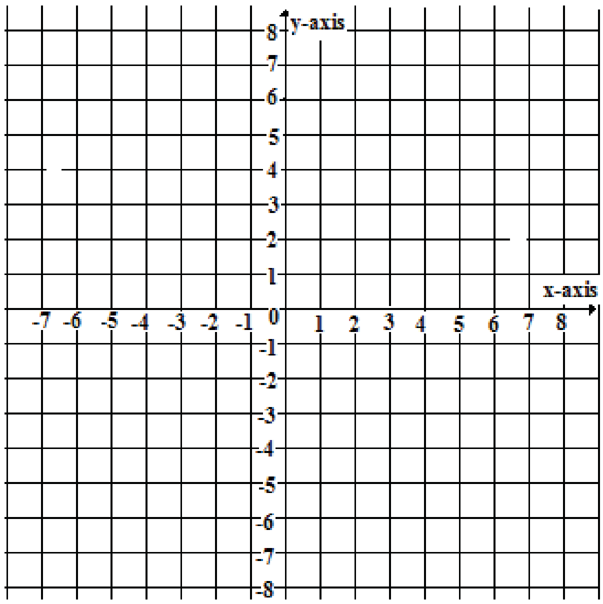
TIME: 25 minutes

Assignment

1. Draw the following graphs in a grid:

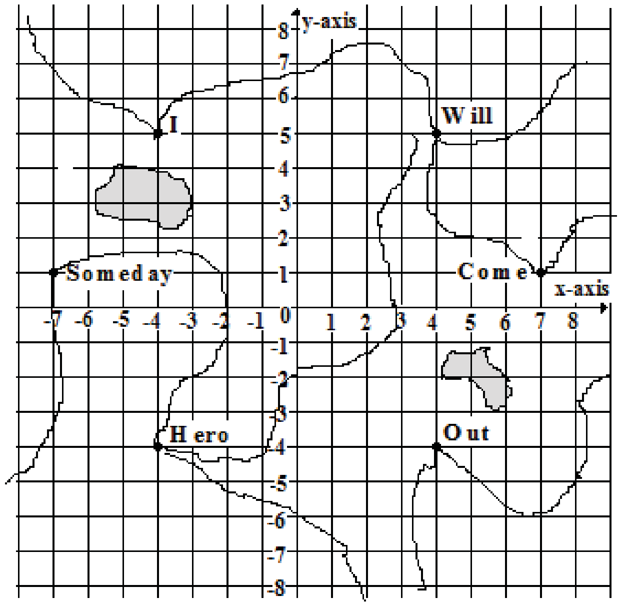
S is a linear graph whose equation is $x = 0$.

T is a linear graph whose equation is $x = 3$.



(2 marks)

2. Use the map below to answer the questions.



a) Write down the y-intercept of the line between I and Will.

____

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(1 mark)

b) Write down the x-intercept of the line between Will and Out.

____

____

(1 mark)

c) Find the gradient of the line joining Someday and I.

____

____

____

____

(2 marks)

d) What is the gradient of the line joining Will and Come?

(2 marks)

e) Find the equation of line between Out and Hero.

(2 marks)

f) Find the equation of line between Will and Hero.

(5 marks)

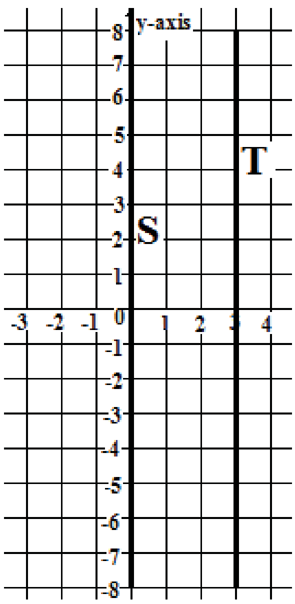
g) What is the equation of line joining Out and I?

(5 marks)

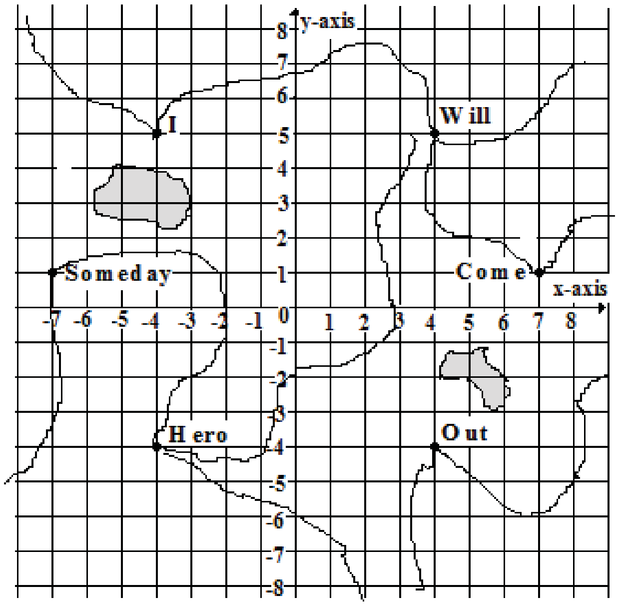
Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Solutions to the ASSIGNMENT:

1.



2.



a) The y-intercept of the line between I and Will.

The y-intercept is 5.

b) The x-intercept of the line between Will and Out.

The x-intercept is 4.

c) The gradient of the line joining Someday and I.

Someday is at $(-7,1)$ and I is at $(-4,5)$.

Let $(x_1, y_1) = (-7,1)$ and $(x_2, y_2) = (-4,5)$.

Then, $m = \dfrac{(5-1)}{(^-4-^-7)} = \dfrac{4}{3}$.

d) The gradient of the line joining Will and Come.

Will is at $(4,5)$ and Come is at $(7,1)$.

Let $(x_1, y_1) = (4,5)$ and $(x_2, y_2) = (7,1)$.

Then, $m = \dfrac{(1-5)}{(7-4)} = -\dfrac{4}{3}$.

e) The equation of line between Out and Hero.

The equation of line is $y = -4$.

f) The equation of line between Will and Hero.

Will is at $(4, 5)$ and Hero is at $(-4, -4)$.

Let $(x_1, y_1) = (4, 5)$ and $(x_2, y_2) = (-4, -4)$.

Then, $m = \frac{(-4-5)}{(-4-4)} = -\frac{-9}{-8} = \frac{9}{8}$.

Therefore, the gradient is $\frac{9}{8}$.

So far, the equation is $y = \frac{9}{8}x + c$.

Now, substitute the coordinates of $(4, 5)$ in $y = \frac{9}{8}x + c$ and solve for c.

$$5 = \frac{9}{8}(4) + c$$

Multiply both sides by 8 and simplify.

$$5(8) = \frac{9}{8}(4)(8) + (8)c$$

$$40 = 36 + 8c$$

$$40 - 36 = 36 - 36 + 8c$$

$$4 = 8c$$

Divide by 8 on both sides and simplify.

$$\frac{4}{8} = \frac{8c}{8}$$

$$\frac{1}{2} = c$$

$$c = \frac{1}{2}$$

Now, the equation is:

$$y = \frac{9}{8}x + \frac{1}{2}$$

g) The equation of line joining Out and I.

Out is at $(-4, 5)$ and I is at $(4, -4)$.

Let $(x_1, y_1) = (-4, 5)$ and $(x_2, y_2) = (4, -4)$.

Then, $m = \frac{(-4-5)}{(4-^-4)} = -\frac{-9}{8} = -\frac{9}{8}$.

Therefore, the gradient is $-\frac{9}{8}$.

So far, the equation is $y = -\frac{9}{8}x + c$.

Now, substitute the coordinates of $(-4, 5)$ in $y = -\frac{9}{8}x + c$ and solve for c .

$$5 = -\frac{9}{8}(-4) + c$$

Multiply both sides by 8 and simplify.

$$5(8) = -\frac{9}{8}(-4)(8) + (8)c$$
$$40 = 36 + 8c$$
$$40 - 36 = 36 - 36 + 8c$$
$$4 = 8c$$

Divide by 8 on both sides and simplify.

$$\frac{4}{8} = \frac{8c}{8}$$
$$\frac{1}{2} = c$$
$$c = \frac{1}{2}$$

Now, the equation is:

$$y = -\frac{9}{8}x + \frac{1}{2}$$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Attempt all the questions. Show your working or a reason for each answer.

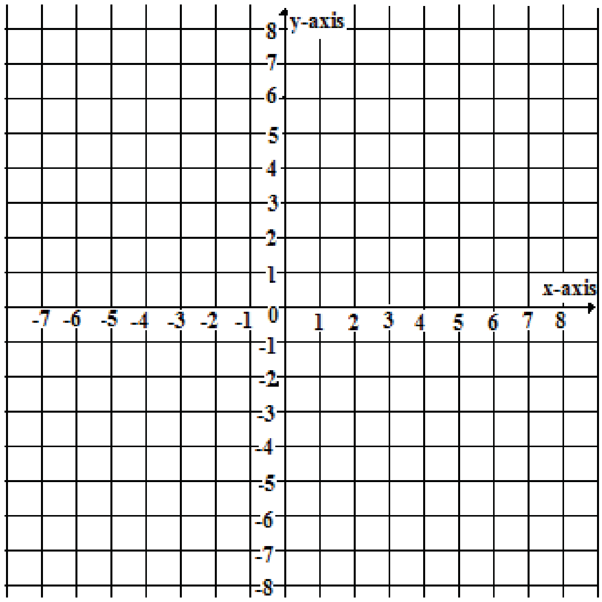
TOTAL MARKS: 20

TIME: 25 minutes

1. Draw the following graphs in a grid:

S is a linear graph whose equation is $x = 0$.

T is a linear graph whose equation is $x = 3$.



(2 marks)

2. The line passes through $(2, 5)$ and $(8, -4)$.

   i. Find the gradient of the line.

____

____

____

____

(2 marks)

   ii. Find the y-intercept of the line.

(2 marks)

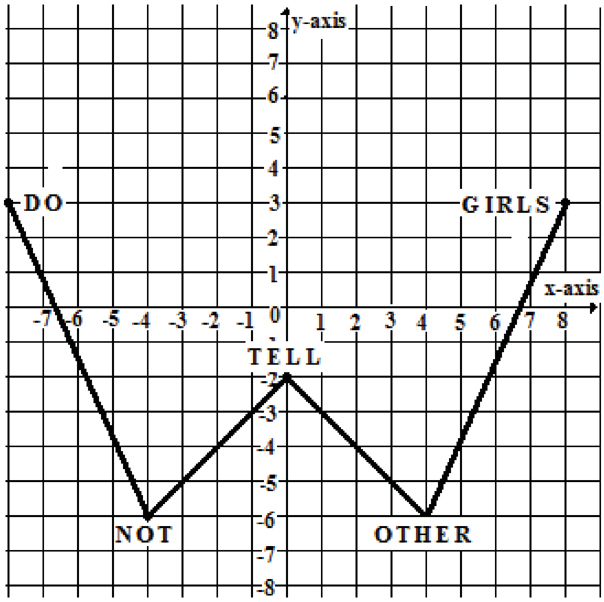
iii. Write down the equation of the line.

(2 marks)

iv. Write down the equation of the line in standard form.

(2 marks)

3. Use the diagram to answer the questions that follow.



a) What is the equation of the line through Do and Girls?

____

____

(2 marks)

b) Write down y-intercept of the line from Not through Tell.

____

____

(2 marks)

c) Calculate the gradient of linear graph from Other to Girls.

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(2 marks)

d) Find the equation of the linear graph that goes through Tell and Other.

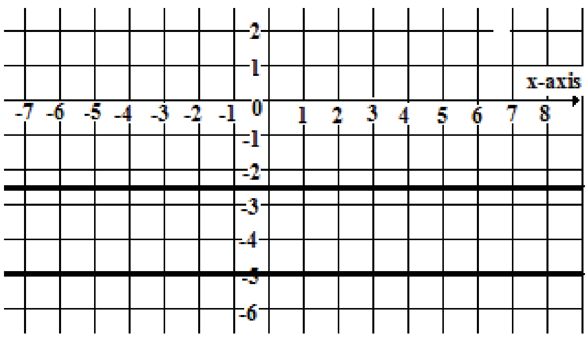
____

____

(4 marks)

Solutions to assessment questions.

1.



2. The line that passes through $(2, 5)$ and $(8, -4)$.

   i. The gradient of the line.

Let $(x_1, y_1) = (2,5)$ and $(x_2, y_2) = (8,-4)$.

Then, $m = \frac{(-4-5)}{(8-2)} = \frac{-9}{6} = -\frac{3}{2}$.

   ii. The y-intercept of the line.

The equation of the line that passes through $(2, 5)$ and $(8, -4)$ is $y = -\frac{3}{2}x + c$.

Substitute the coordinates of $(2,5)$ in $y=-\frac{3}{2}x+c$ and solve for c, which is the y-intercept.

$5=-\frac{3}{2}(2)+c$

Multiply both sides by 8 and simplify.

$5(2)=-\frac{3}{2}(2)(2)+(2)c$

$10 \quad =-6 \qquad +2c$

$10+6=-6+6+2c$

$14 \qquad = \qquad 2c$

Divide by 2 on both sides and simplify.

$\frac{14}{2} \qquad = \qquad \frac{2c}{2}$

$7 \qquad =c$

$c \qquad =7$

The y-intercept is 7.

iii. The equation of the line.

The gradient is $-\frac{3}{2}$ and y-intercept is 7. Therefore, the equation is $y=-\frac{3}{2}x+7$.

iv. The equation of the line in standard form.

$y=-\frac{3}{2}x+7$

$y(2)=-\frac{3}{2}x(2)+7(2)$

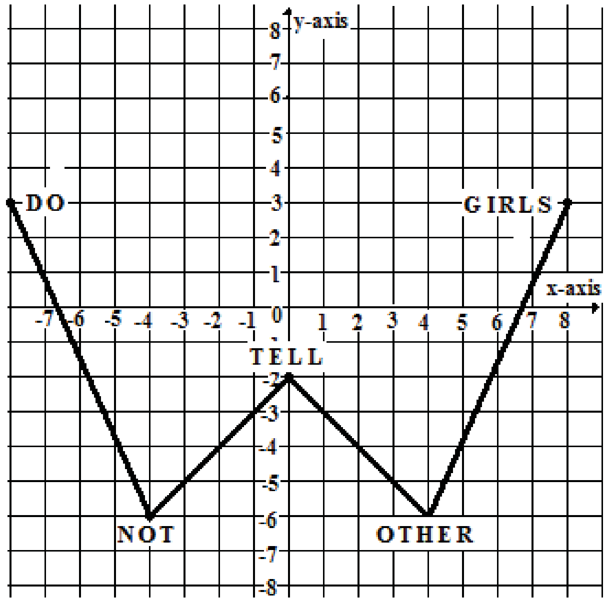
$2y=-3x+14$.

$2y+3x=-3x+3x+14$

$2y+3x=14$

The equation is $2y+3x=14$.

3.



a) The equation of the line through Do and Girls.

The equation of the line through Do and Girls is $y = 3$.

b) The y-intercept of the line from Not through Tell.

The y-intercept of the line from Not through Tell is -2.

c) The gradient of linear graph from Other to Girls.

Girls is at $(8, 3)$ and Other is at $(4, -6)$.

Let $(x_1, y_1) = (8, 3)$ and $(x_2, y_2) = (4, -6)$.

Then, $m = \frac{(-6-3)}{(4-8)} = \frac{-9}{-4} = \frac{9}{4}$.

d) The equation of the linear graph that goes through Tell and Other.

Tell is at $(0, -2)$ and Other is at $(4, -6)$.

Let $(x_1, y_1) = (0, -2)$ and $(x_2, y_2) = (4, -6)$.

Then, $m = \frac{(-6-^{-}2)}{(4-0)} = -\frac{-4}{4} = -1$.

Therefore, the gradient is -1.

From the diagram, y-intercept is -2.

Therefore, the equation is $y = -x - 2$.

Unit Contents

Unit 21

Unit 21

Quadratics and Other Non-linear Graphs

Introduction

Mathematical Models Of Practical Situations !

Mathematical relations result in mappings, whereby an input into the relation produces a corresponding output. Functions can be derived from these mappings, and the equations thereof written. These equations are graphed; or sometimes sketch graphs, which show important points such as **critical points** and the **intercepts**, are shown.

This unit consists of 115 pages. It covers approximately 5% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 30 hours to work on this unit, 20 hours for formal study and 10 hours for self-study and completing assessments/assignments.

When reading the following learning outcomes, think about them as a guide to what you should focus on while studying this unit.

This Unit is Comprised of Five Lessons:

Lesson 1 Constructing Tables of Variables
Lesson 2 Drawing Graphs Based on Data in a Table
Lesson 3 Finding Corresponding Values of x and y in Graphs
Lesson 4 Solving Equations and Inequalities Using Graphs
Lesson 5 Solving Problems Using Sketch Graphs

Outcomes

Upon completion of this unit you will be able to:

- *construct* tables of variables.
- *draw* graphs based on data in a table.
- *find* corresponding values of x and y in graphs.
- *solve* equations and inequalities using graphs.
- *solve* problems using sketch graphs.

Terminology

Absolute maximum: The largest value of a function.

Absolute minimum: The smallest value of a function.

Asymptote: A straight line that a graph approaches but never connects with.

Critical point: A point at which absolute maximum, absolute minimum, relative maximum, relative minimum, or point of inflection exists.

Inflection point: A point on a curve at which curvature changes from convex to concave or vice versa.

Parabola: The graph of a quadratic function.

Rational function: A function which can be written as a quotient of two polynomials.

Relative maximum: The largest value of a function in a given interval.

Relative minimum: The smallest value of a function in a given interval.

Vertex: The lowest or highest point of a parabola.

X-intercept: X value at which a graph crosses x-axis.

Y-intercept: Y value at which a graph crosses y-axis.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Constructing Tables of Variables

Introduction

By the end of this subunit, you should be able to:

- find the corresponding variable value, to another given variable, when the two variables are connected by an equation.

This subunit is about 3 pages in length.

Completing the tables of the variables of equations.

When given an incomplete table of variables, you should be able to complete it, with the help of the equation of those variables.

Example 1

To complete a given table, as shown in figure 1, you find the corresponding values of one variable given the values of the other.

In the equation: $y = x^2 - 1$, the variables **x** and **y** have a relationship. If the values of x are given, you substitute each value of x into the equation to calculate the corresponding y value. These values complete the table, as shown in figure 2. In this course, we will only address equations where y equals a function of x.

x	-4	-3	-2	-1	0	1	2	3	4
y									

Figure 1 – Given table for $y = x^2 - 1$

When x = - 4, y = $(-4)^2$ - 1 = 16 – 1 = 15

When x = - 3, y = $(-3)^2$ - 1 = 9 – 1 = 8

When x = - 2, y = $(-2)^2$ - 1 = 4 – 1 = 3

When x = - 1, y = $(-1)^2$ - 1 = 1 – 1 = 0

When x = 0, y = $(0)^2$ - 1 = 0 – 1 = –1

When x = 1, y = $(1)^2$ - 1 = 1 – 1 = 0

When x = 2, y = $(2)^2$ - 1 = 4 – 1 = 3

When x = 3, y = $(3)^2$ - 1 = 9 – 1 = 8

When x = 4, y = $(4)^2$ - 1 = 16 – 1 = 15

x	-4	-3	-2	-1	0	1	2	3	4

y	15	8	3	0	-1	0	3	8	15

Figure 2 – Completed table for $y = x^2 - 1$

Now, the graph can be drawn using these ordered pairs from this table. This graph, which is a curve, should be drawn freehand and smoothly. It takes practice to be good at it.

Activity 1

Activity 1

You can use a calculator to evaluate the expressions for y!

1.The variables **x** and **y** are connected by the equation $y = (2x+3)(x-4)$. Complete the given table.

x	-3	-2	-1	0	1	2	3	4	5	6
y										

2. The variables **x** and **y** are connected by the equation $y = x^3 - 3$. Complete the given table.

x	-4	-3	-2	-1	0	1	2	3
y								

3. The variables **x** and **y** are connected by the equation $y = \frac{1}{x^2}$. Complete the given table, by writing the values of y to 2 decimal places.

x	-4	-3	-2	-1	1	2	3	4
y								

Check your performance against the given solutions at the end of this subunit. If you are satisfied with your performance continue to the next subunit. Otherwise, review the *Constructing tables of variables* subunit and try the activity again.

Note it!

Remember:

The given table is completed by finding the corresponding values of y to the given values of x.(*These values of y are found by substituting the values of x in the equation.*).

Solutions to ACTIVITY 1:

1.The completed table of $y = (2x+3)(x-4)$

x	-3	-2	-1	0	1	2	3	4	5	6
y	21	6	-3	-12	-15	-14	-9	0	13	30

2. The completed table of $y = x^3 - 3$

x	-4	-3	-2	-1	0	1	2	3
y	-61	-24	-5	-4	-3	-2	5	24

3. The completed table of $y = \frac{1}{x^2}$

x	-4	-3	-2	-1	1	2	3	4
y	0.06	0.11	0.25	1.00	1.00	0.25	0.11	0.06

Lesson 2 Drawing Graphs Based on Data in a Table

By the end of this subunit, you should be able to:

- plot the points with the ordered pairs from your completed table.
- join the plotted points with a curve.

This subunit is about 14 pages in length.

Constructing non-linear graphs from the tables.

When you have completed a table for the variables **x** and **y**, which are connected by an equation, you can draw the graph after plotting the points based on data in a table.

Example 1

Look at, once again, the completed table of $y = x^2 - 1$.

x	-4	-3	-2	-1	0	1	2	3	4
y	15	8	3	0	-1	0	3	8	15

You generally, should plot the points on the graph paper, using a convenient scale; not unless the scale is specified in the question.

The x-axis should, at least, range from -4 to 4 in order to accommodate all the values of x in the table, while the y-axis should, at least, be from -1 to 15.

The plotted points are shown in figure 3:

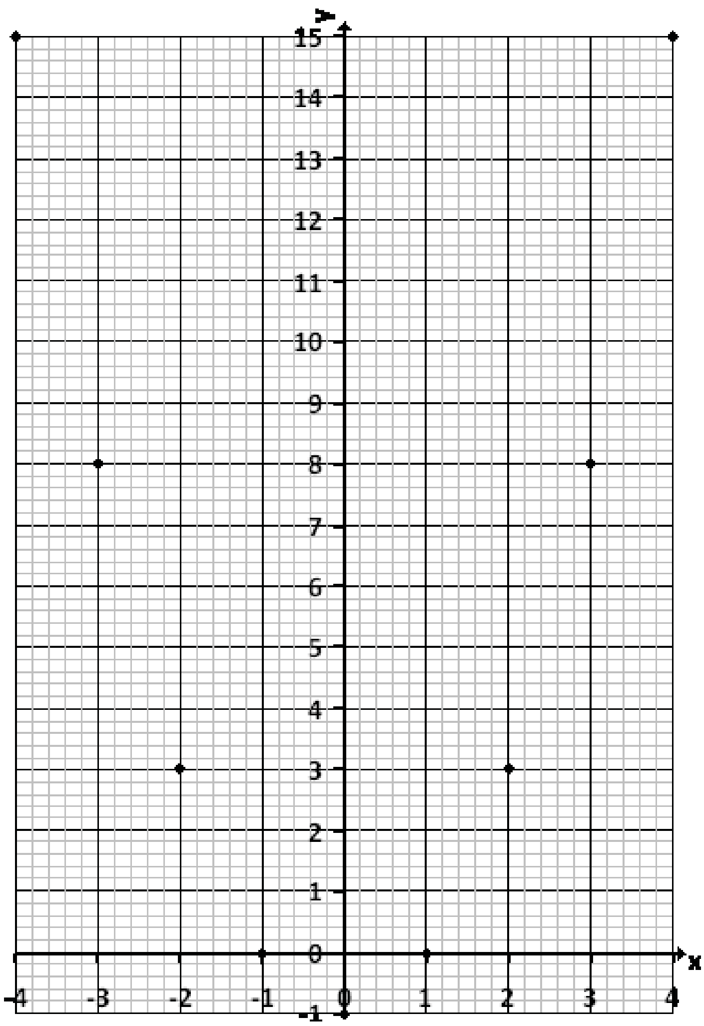


Figure 3 – Plotted points of $y = x^2 - 1$

Now, the plotted points are joined by a smooth curve as shown in figure 4:

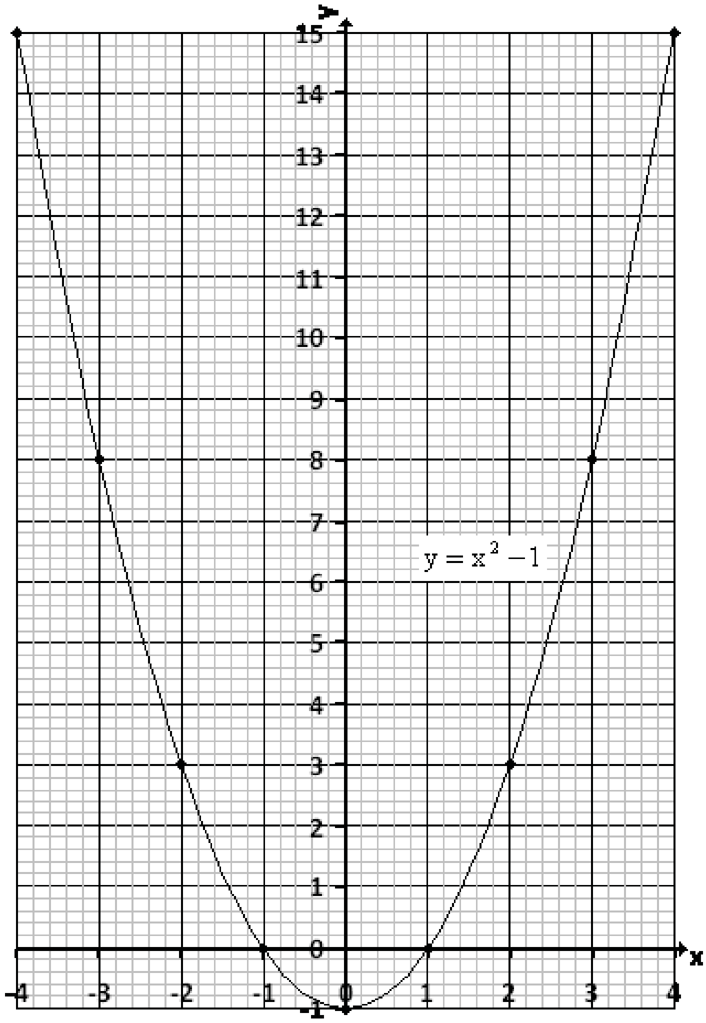


Figure 4 – Graph of $y = x^2 - 1$

Example 2

Look at the completed table of $\mathbf{y = 1 - \frac{1}{2}x^2}$.

x	-4	-3	-2	-1	0	1	2	3	4

y	-7	-3.5	-1	0.5	1	0.5	-1	-3.5	-7

The x-axis should, at least, range from -4 to 4 in order to accommodate all the values of x in the table, while the y-axis should, at least, be from -7 to 1.

The plotted points are shown in figure 5:

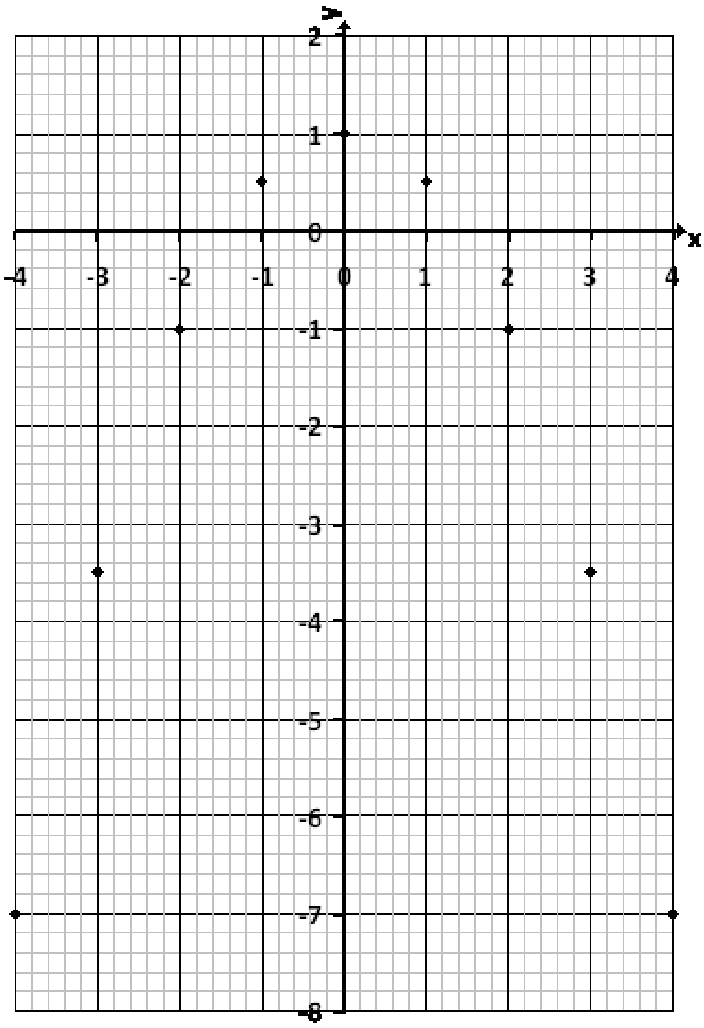


Figure 5 – Plotted points of $\mathbf{y} = 1 - \frac{1}{2}\mathbf{x}^2$

Now, the plotted points are joined by a smooth curve as shown in figure 6:

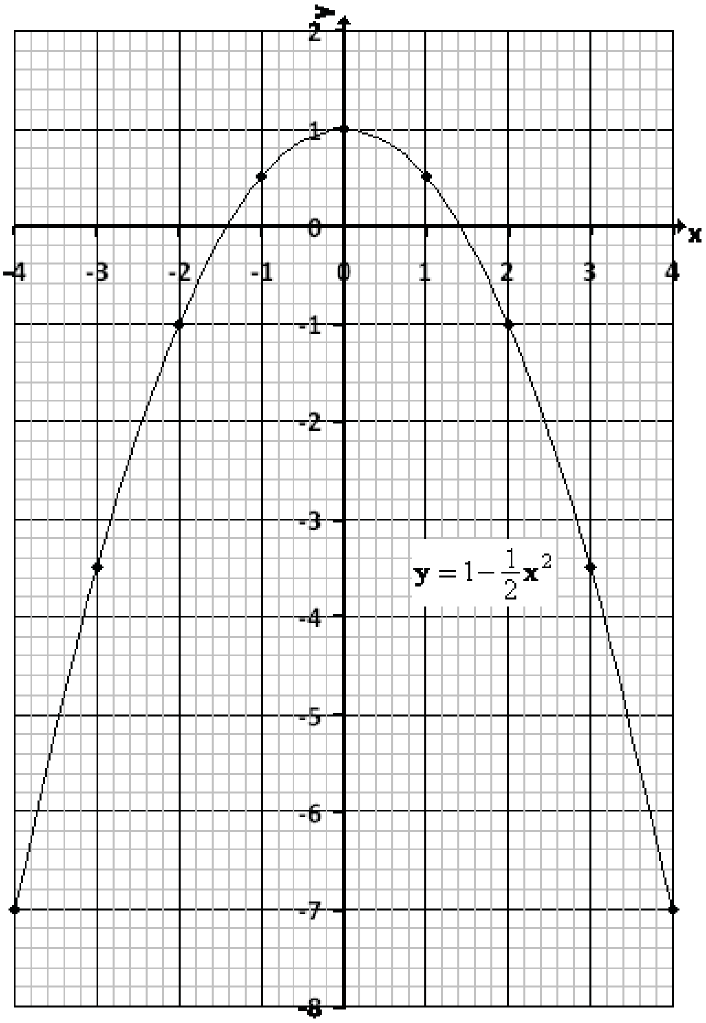


Figure 6 – Graph of $y = 1 - \frac{1}{2}x^2$

Example 3

In this example, the graph of $y = \frac{1}{2}x^3$ is drawn from two sets of points.

Look at the first completed table of $y = \frac{1}{2}x^3$.

x	-3	-2	0	2
y	-13.5	-4	0	4

The x-axis should, at least, range from -3 to 2 in order to accommodate all the values of x in the table, while the y-axis should, at least, be from -13.5 to 4.

Now, the plotted points which are joined by a smooth curve are shown in figure 7:

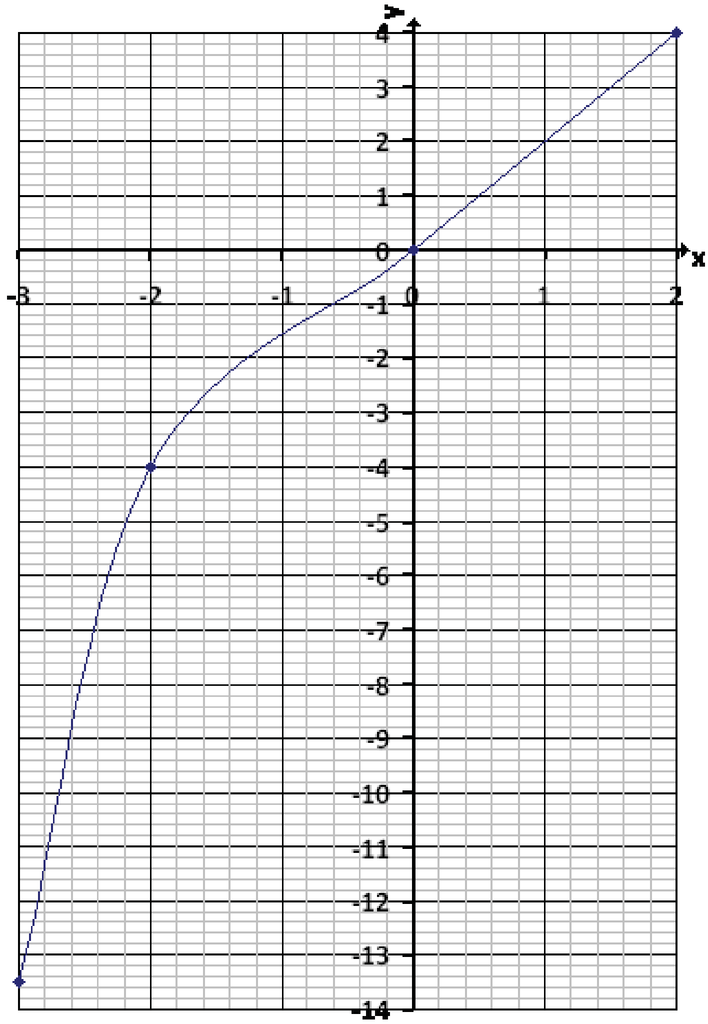


Figure 7 – Graph of $\mathbf{y} = \frac{1}{2}\mathbf{x}^3$

Look at the second completed table of $\mathbf{y} = \frac{1}{2}\mathbf{x}^3$.

x	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	-13.5	-7.8	-4	-1.7	-0.5	-0.1	0	0.1	0.5	1.7	4

Now, the plotted points which are joined by a smooth curve are shown in figure 8:

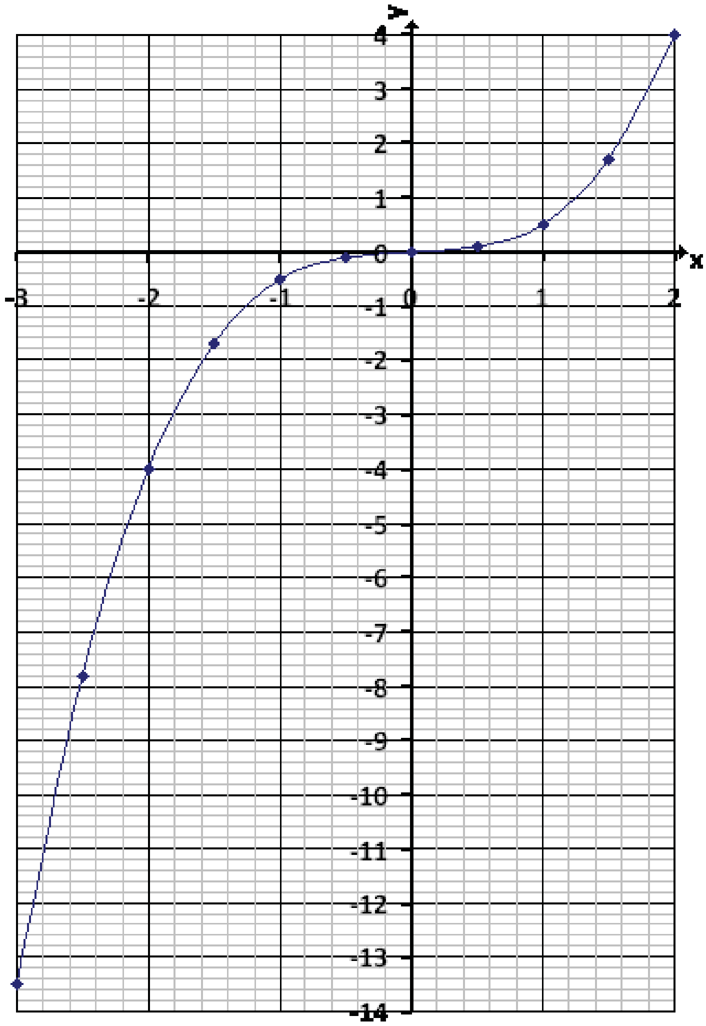


Figure 8 – Graph of $\mathbf{y} = \frac{1}{2}\mathbf{x}^3$

What can you say about the two graphs, from figure 7 and figure 8?

___

___

Compare your answer with:

The two graphs are not exactly the same.

What is the reason for this observation, since they are both graphs of $\mathbf{y} = \frac{1}{2}\mathbf{x}^3$.

___

___

Compare your answer with:

Well, they are not plotted from the same number of points.

The graph in figure 8 is more accurate as it is drawn from more points in the same range of the values of x.

In general, the more points used to draw a non-linear graph and the closer the points are to each other, the more accurate the graph.

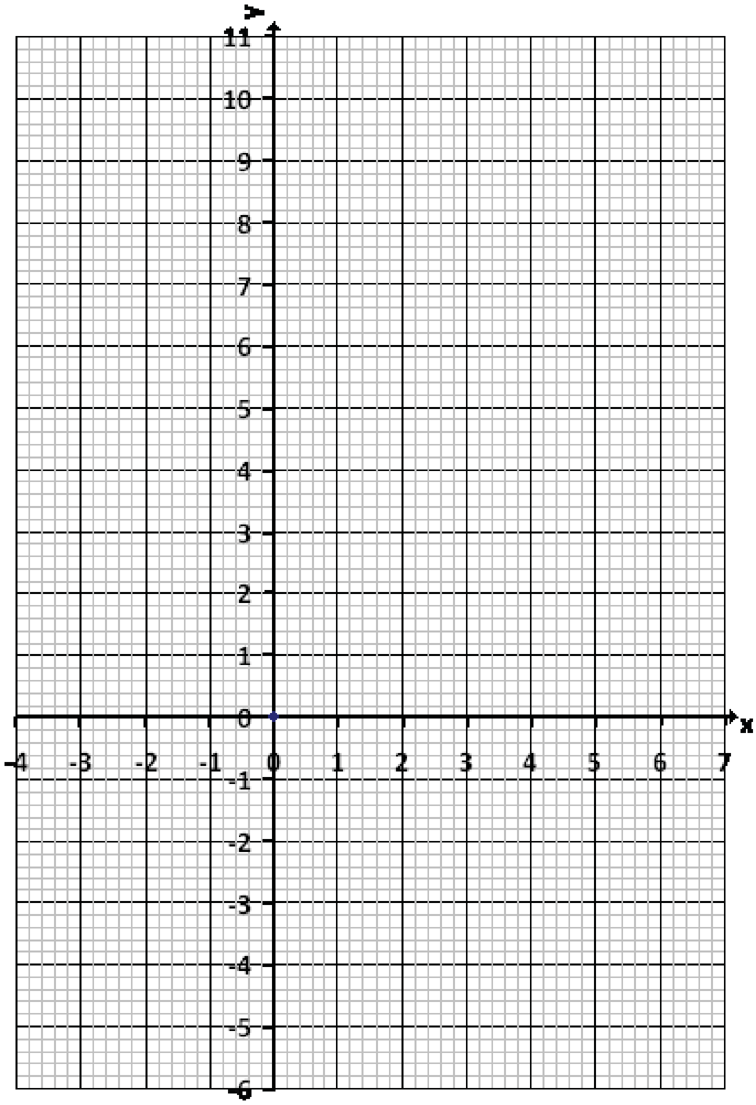
Activity 2

Activity 2

1.Plot the points given in the tables and join them with smooth curves:

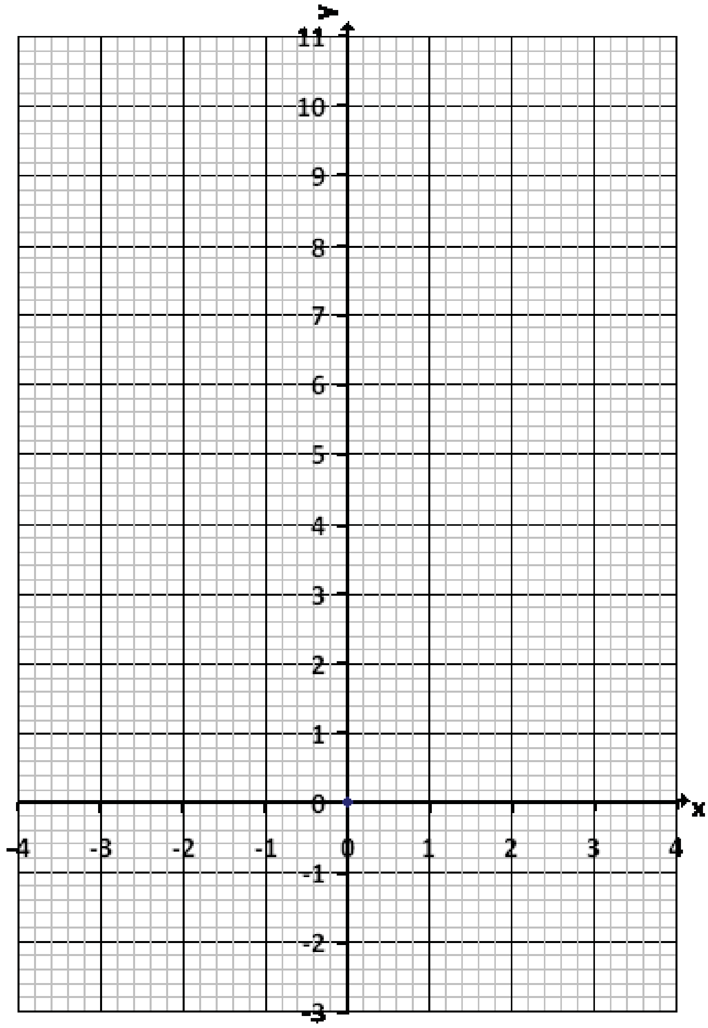
a) The completed table of $\mathbf{y} = \frac{(2\mathbf{x}+3)(\mathbf{x}-4)}{3}$

x	-3	-2	-1	0	1	2	3	4	5	6
y	7	2	-1.7	-4	-5	-4.7	-3	0	4.3	10



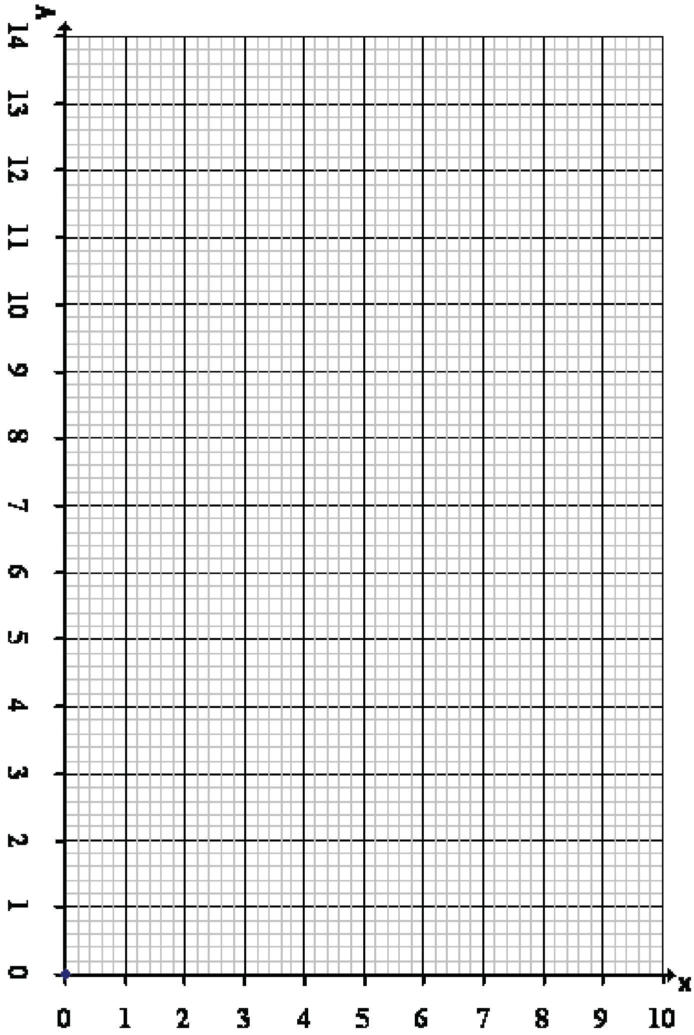
b) A partial table of values of x and the corresponding values of y which are connected by $\mathbf{y} = \frac{1}{3}\mathbf{x}^3 - \mathbf{x} + 4$.

x	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
y	-2	1.3	3.3	4.4	4.7	4.5	4	3.5	3.3	3.6	4.7	6.7	10



c) A table of x and the corresponding values of y, correct to one decimal place, which are connected by $\mathbf{y} = \frac{\mathbf{x}^2}{8} + \frac{18}{\mathbf{x}} - \frac{51}{10}$.

x	1	1.5	2	2.5	3	4	5	6	7	8
y	13.0	7.2	4.4	2.9	2.0	1.4	1.6	2.4	3.6	5.2



Check your performance against the given solutions at the end of this subunit. If your graphs are reasonably accurate continue with the next subunit. If your graphs have mistakes, review this subunit entitled, *Drawing graphs based on data in a table.*

Note it!

Remember:

When drawing a non-linear graph, the accuracy does not entirely depend on the smoothness of the curve. The nearer the points which are used to each other, the better the graph.

Solutions to ACTIVITY 2:

1.Plot the points given in the tables and join them with smooth curves:

a) The completed table of $\mathbf{y} = \frac{(2\mathbf{x}+3)(\mathbf{x}-4)}{3}$

x	-3	-2	-1	0	1	2	3	4	5	6
y	7	2	-1.7	-4	-5	-4.7	-3	0	4.3	10

Compare your graph with figure 9:

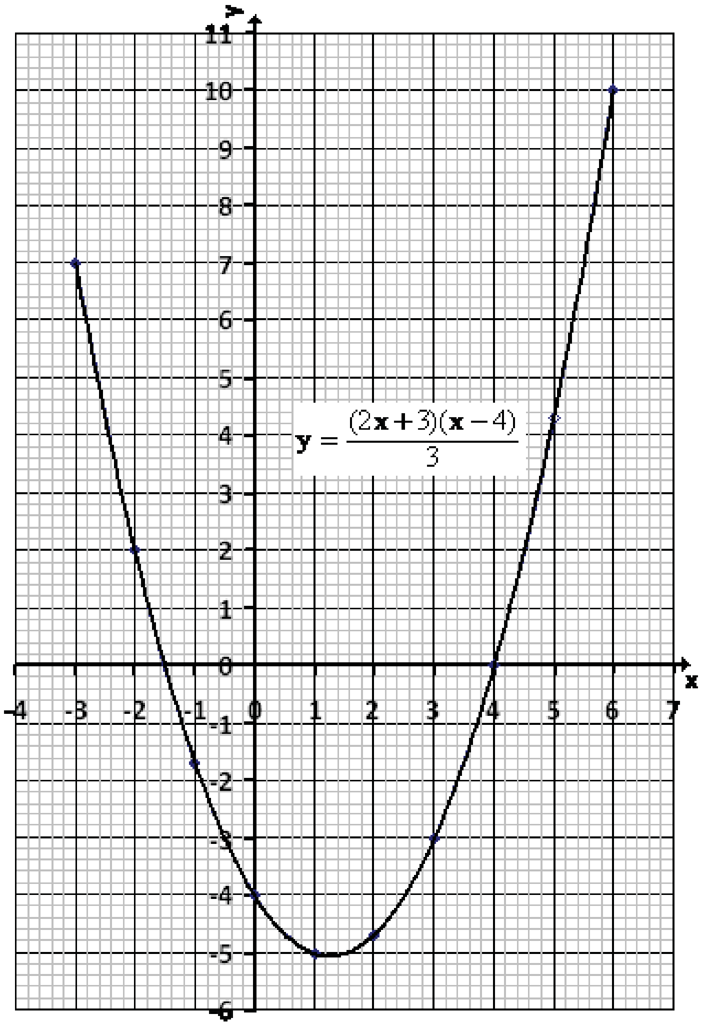


Figure 9 – Graph of $\mathbf{y} = \frac{(2\mathbf{x}+3)(\mathbf{x}-4)}{3}$

b) Some values of x and the corresponding values of y which are connected by $\mathbf{y} = \frac{1}{3}\mathbf{x}^3 - \mathbf{x} + 4$ are shown below.

x	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
y	-2	1.3	3.3	4.4	4.7	4.5	4	3.5	3.3	3.6	4.7	6.7	10

Compare your graph with figure 10:

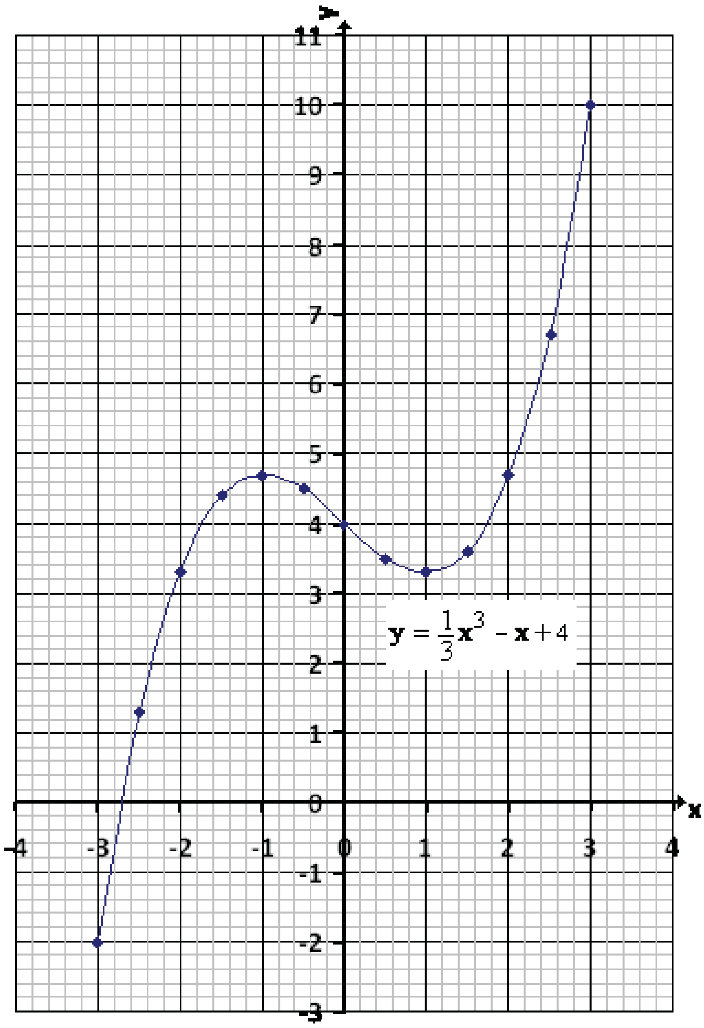


Figure 10 – Graph of $\mathbf{y} = \frac{1}{3}\mathbf{x}^3 - \mathbf{x} + 4$

c) Some values of x and the corresponding values of y, correct to one decimal place, which are connected by $y = \frac{x^2}{8} + \frac{18}{x} - \frac{51}{10}$ are given below.

x	1	1.5	2	2.5	3	4	5	6	7	8
y	13.0	7.2	4.4	2.9	2.0	1.4	1.6	2.4	3.6	5.2

Compare your graph with figure 11:

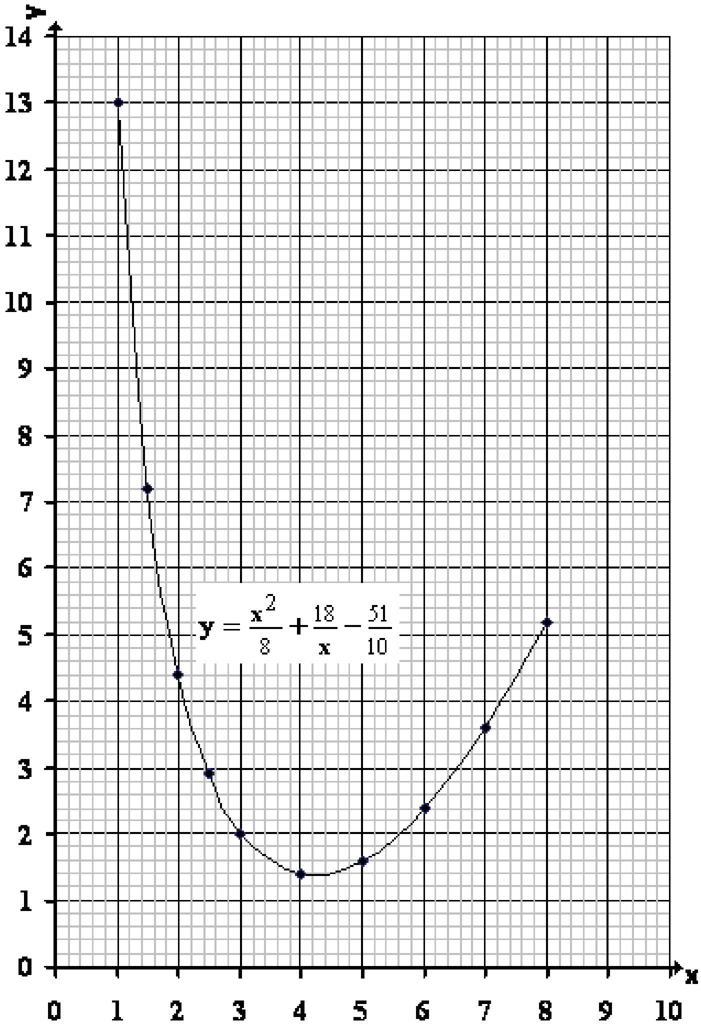


Figure 11 – Graph of $y = \frac{x^2}{8} + \frac{18}{x} - \frac{51}{10}$

Lesson 3 Finding Corresponding Values of x and y in Graphs

By the end of this subunit, you should be able to:
- find the value or values of x on the graph, when given any value of y, which is within the part of the drawn graph.
- also find the value of y, which corresponds to a given value of x.
- find x-intercepts and y-intercept.

This subunit is about 12 pages in length.

Corresponding values of x and y estimated from a graph.

The drawn graphs can be used to find other corresponding values of x and y which satisfy the equation of the graph. They will be estimates based on how well the graph is drawn.

Example 1

Look at the graph of $y = x^2 - 1$ in figure 12 below.

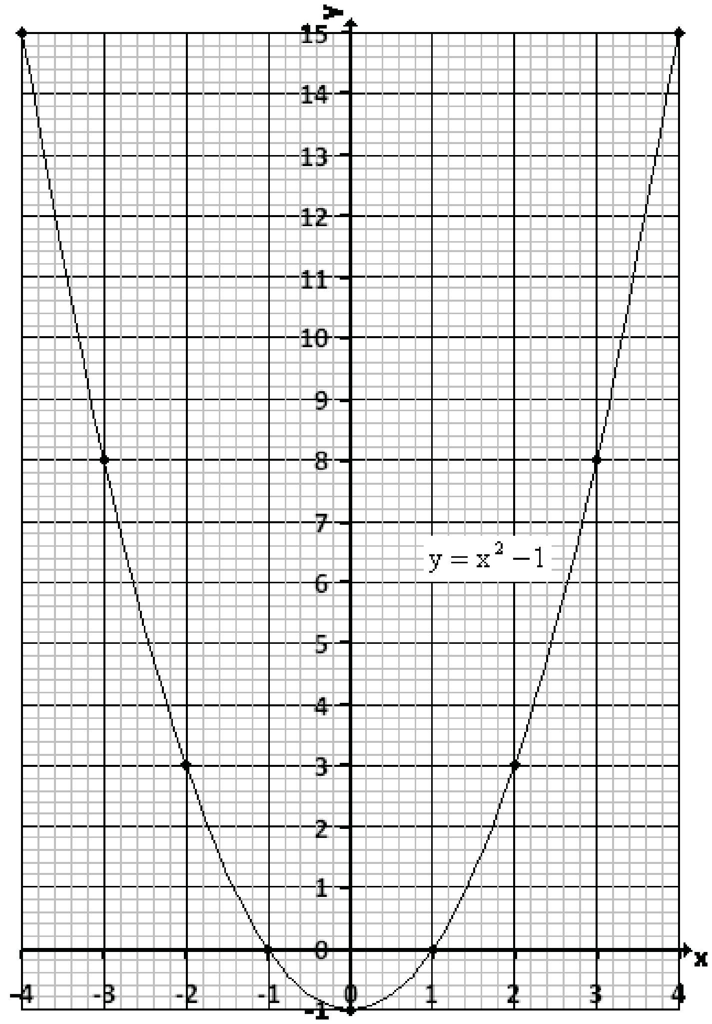


Figure 12 – Graph of $y = x^2 - 1$

a) Suppose you are to find y when x is 2.5:

Draw the line **x** = 2.5 in the graph and note the point of intersection of **x** = 2.5 with $y = x^2 - 1$. This is illustrated in figure 13.

The y coordinate of the point of intersection is the required y.

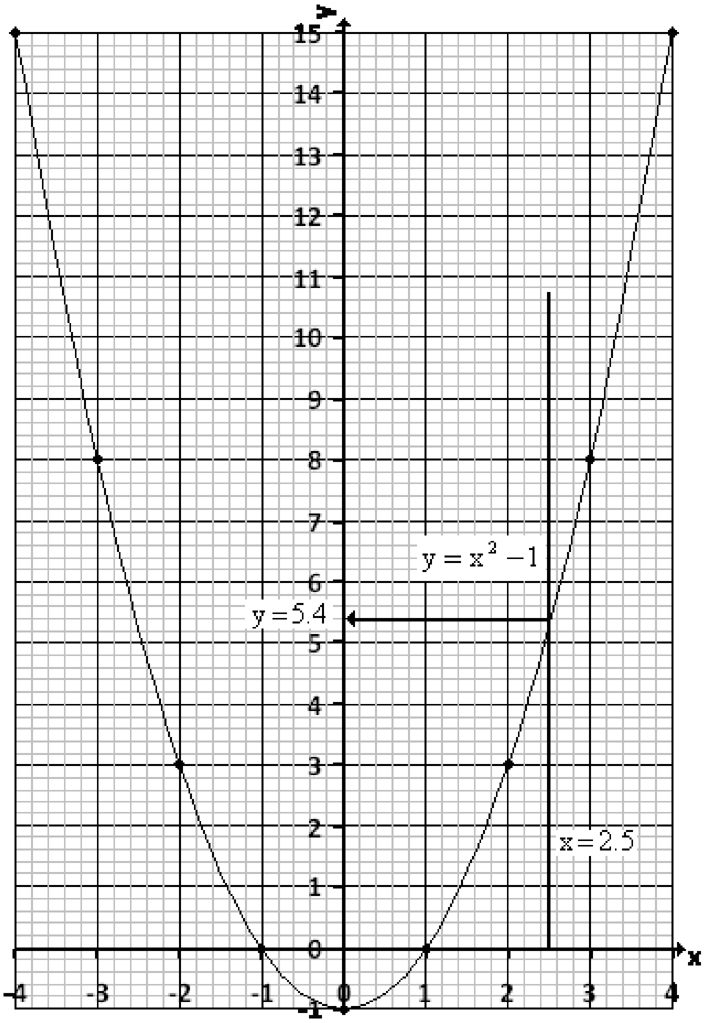


Figure 13 – Graph of $y = x^2 - 1$ with x = 2.5 and y estimated.

As seen in figure 13, y is approximately 5.4 when x is 2.5.

b) Suppose you are to find y when x is -3.5:

Draw the line $\mathbf{x} = -3.5$ in the graph and note the point of intersection of $\mathbf{x} = -3.5$ with $y = x^2 - 1$. This is highlighted in figure 14.

The y coordinate of the point of intersection is the required y.

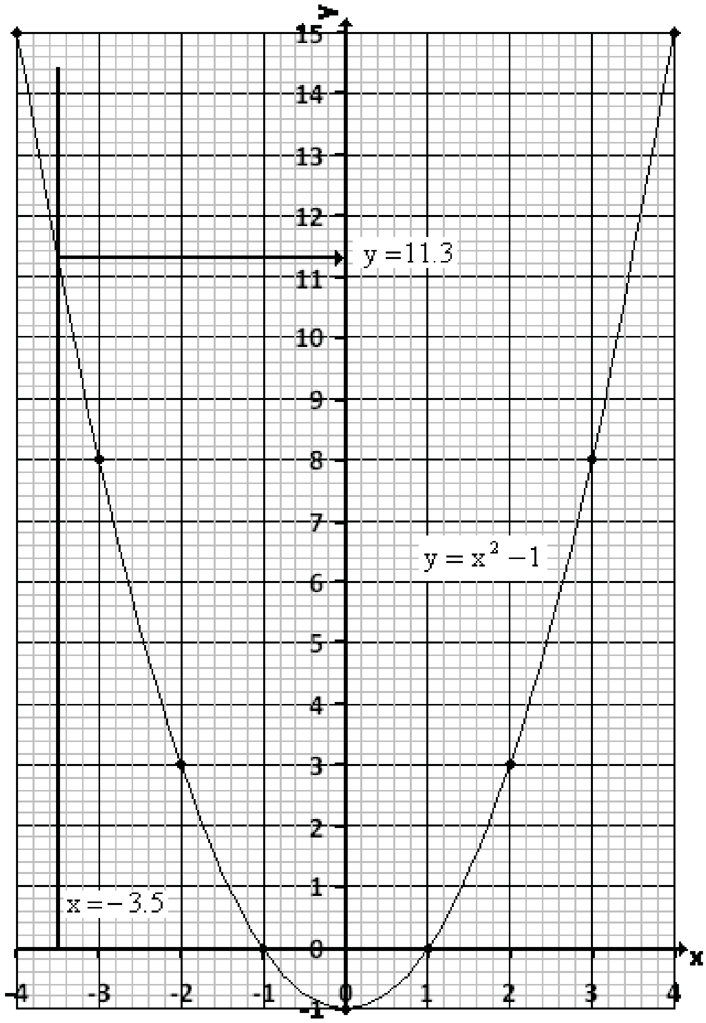


Figure 14 – Graph of $y = x^2 - 1$ with x = -3.5 and y estimated.

As seen in figure 14, y is approximately 11.3 when x is -3.5.

c) Suppose you are to find values of x when y is 6:

Draw the line $\mathbf{y} = 6$ in the graph and note the points of intersection of $\mathbf{y} = 6$ with $y = x^2 - 1$. This is illustrated in figure 15.

The x coordinates of the points of intersection are the required values of x.

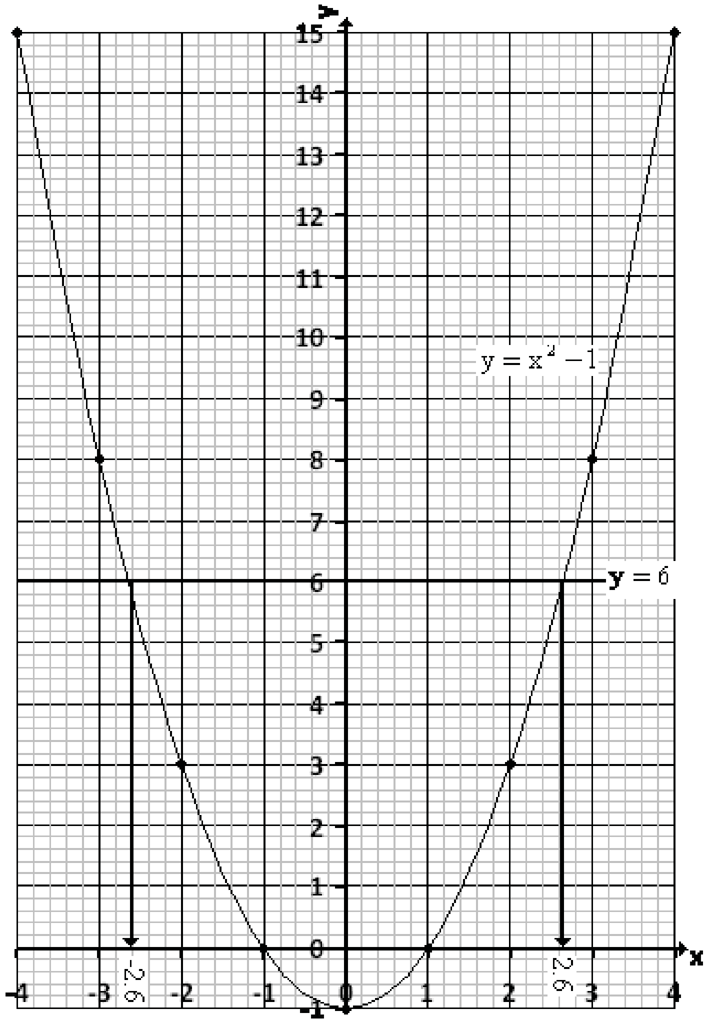


Figure 15 – Graph of $y = x^2 - 1$ with y = 6 and x estimated

As seen in figure 15, the values of x are approximately 2.6 and -2.6 when y is 6.

d) Suppose you are to find x-intercepts of $y = x^2 - 1$:

Draw the line $\mathbf{y} = 0$ in the graph and note the points of intersection of $\mathbf{y} = 0$ with $y = x^2 - 1$, or read the x coordinates of the points where $y = x^2 - 1$ crosses the x-axis. This is shown in figure 16. Remember that $\mathbf{y} = 0$ is the x-axis.

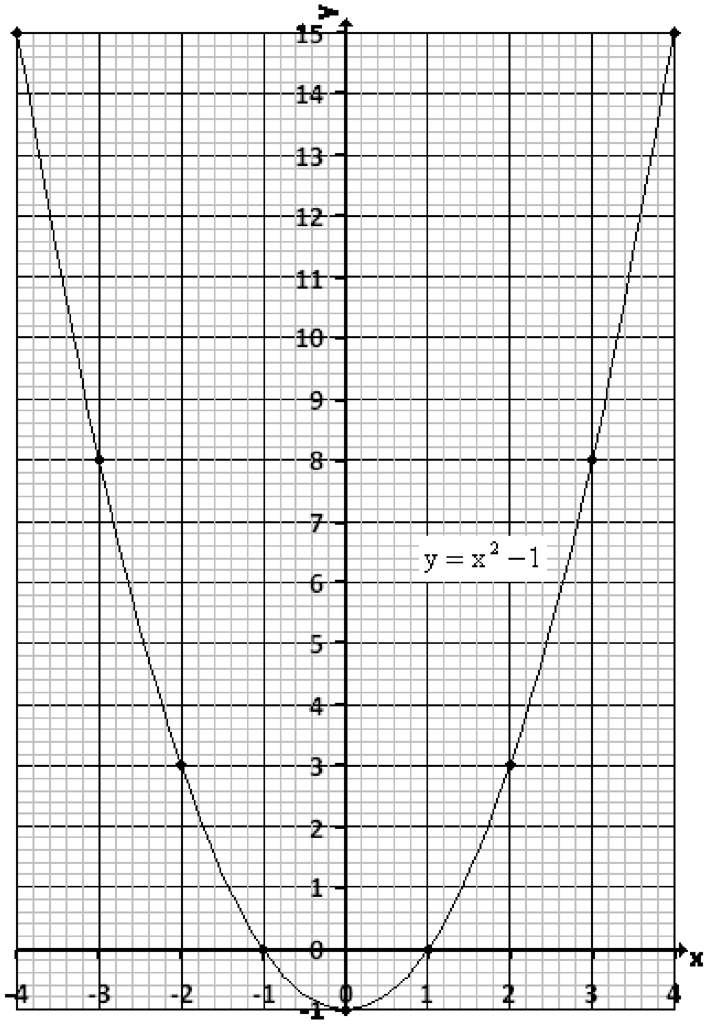


Figure 16 – Graph of $y = x^2 - 1$ with y = 0 and x-intercepts estimated.

As seen in figure 16, the x-intercepts are 1 and -1.

Example 2

Earlier, in activity 2, you drew the graph of $\mathbf{y} = \frac{1}{3}\mathbf{x}^3 - \mathbf{x} + 4$, which is shown here in figure 17:

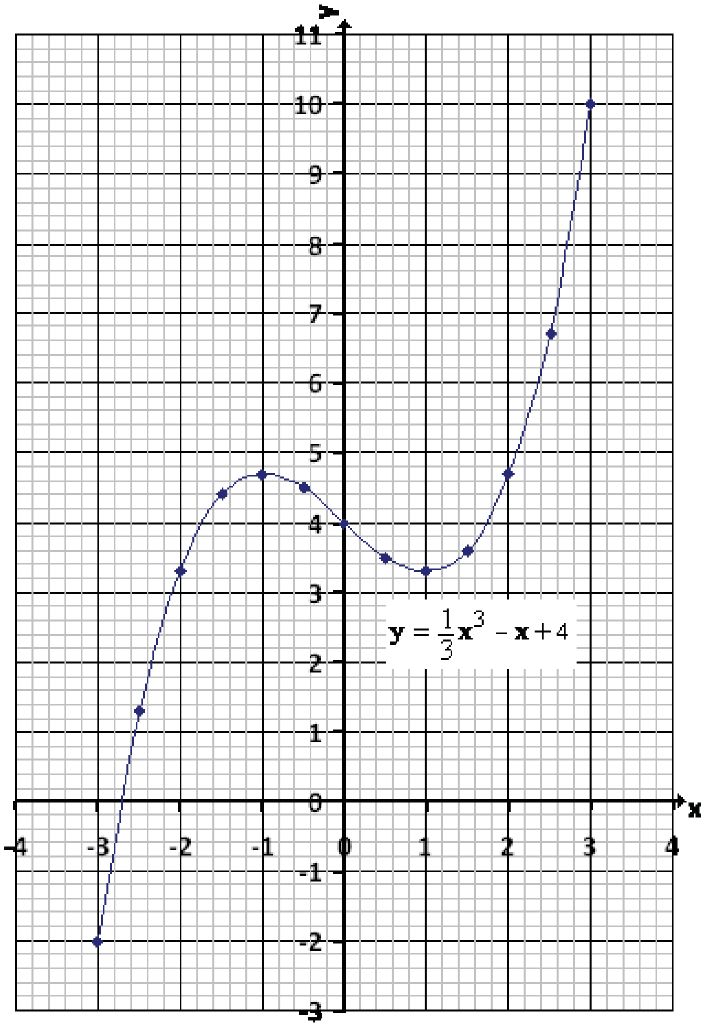


Figure 17 – Graph of $y = \frac{1}{3}x^3 - x + 4$.

a) What is x when $y = 0$?

Compare your answer with:

x = -2.7, which is read on the x-axis(y = 0) where the graph intersects y = 0.

b) Estimate, from the graph, the values of x for which $y = 4$.

Compare your answer with:

x = -1.8, x = 0, and x = 1.8.These are obtained at the points of intersections of $y = \frac{1}{3}x^3 - x + 4$ and the line $y = 4$.

c) What is y when $x = -2.2$?

Compare your answer with:

y = 2.6. It is the y coordinate of the point of intersection of $\mathbf{y} = \frac{1}{3}\mathbf{x}^3 - \mathbf{x} + 4$ and the line $\mathbf{x} = -2.2$.

d) Write down the value of y when $\mathbf{x} = 2.8$.

Compare your answer with:

y = 8.6. This is y coordinate, where line $\mathbf{x} = 2.8$ meets the graph of figure 17.

Activity 3

Activity 3

1. The following is the graph of $\mathbf{y} = \frac{4}{\mathbf{x}}$, for the values of x starting from -4 to -0.5.

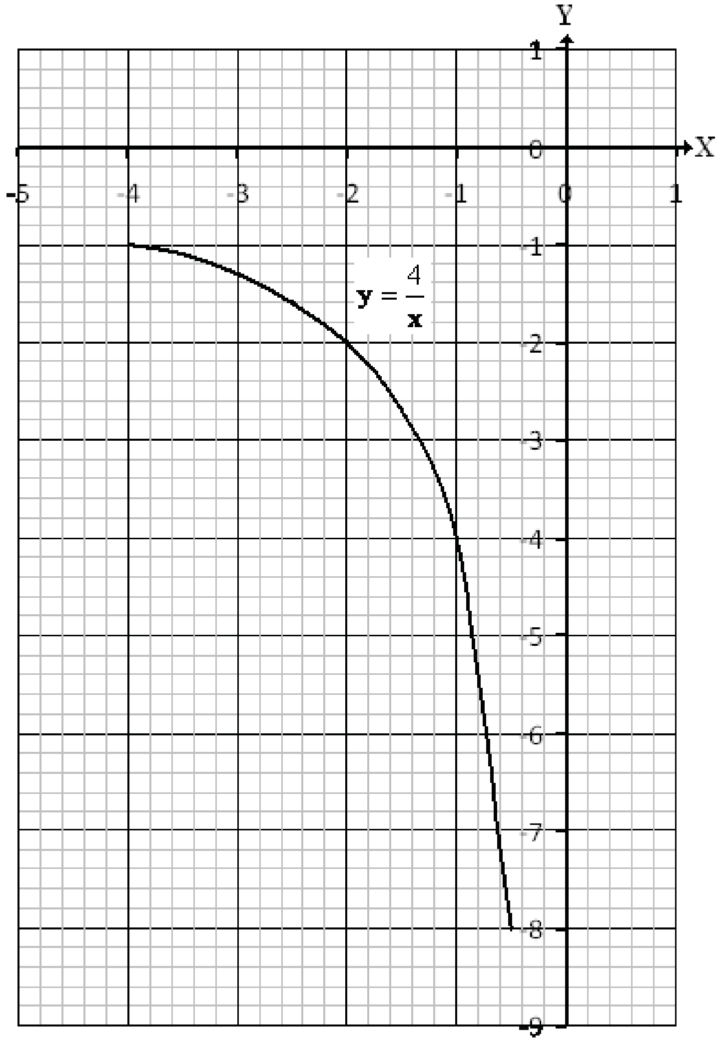


Figure 18 – Graph of $\mathbf{y} = \frac{4}{\mathbf{x}}$.

a) What value of x corresponds to y = -4?

---

b) What is y when x = -0.6?

---

c) Given that (-3.2 , y) lies on the graph of $\mathbf{y} = \frac{4}{\mathbf{x}}$,find the value of y.

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2. Use the graph in figure 19 to answer the questions that follow.

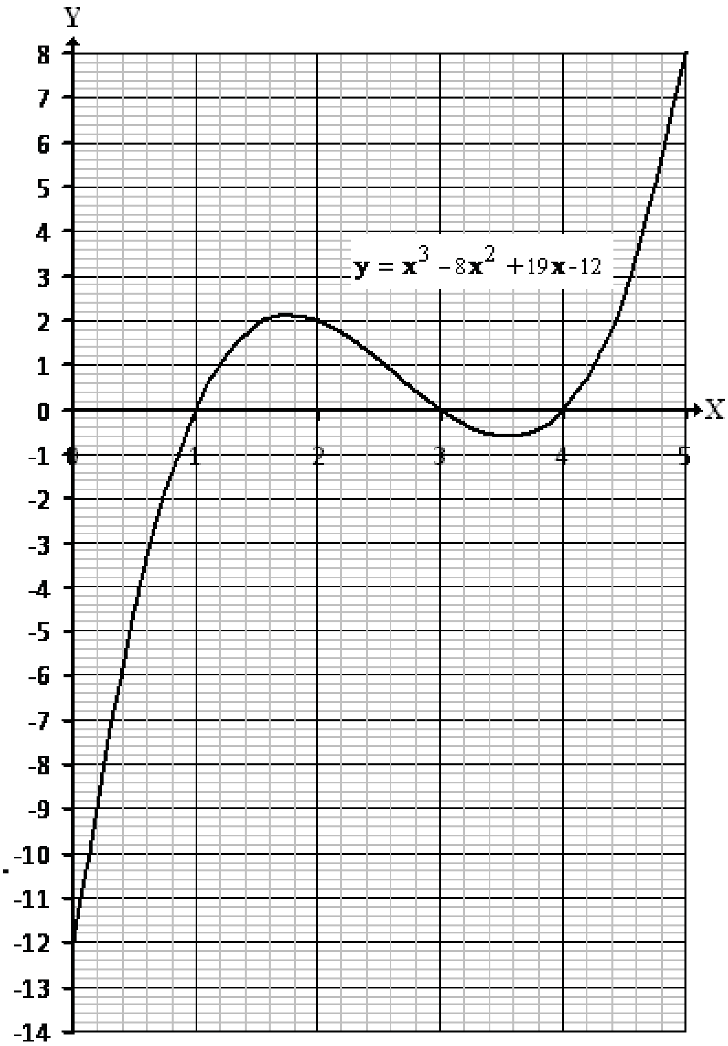


Figure 19 – Graph of $\mathbf{y} = \mathbf{x}^3 - 8\mathbf{x}^2 + 19\mathbf{x} - 12$.

a) What is the y-intercept of the graph?

____

b) Write down the x-intercepts of $\mathbf{y} = \mathbf{x}^3 - 8\mathbf{x}^2 + 19\mathbf{x} - 12$.

c) Estimate the values of x, from the graph, for which y = 1.

d) Find the approximate value of x, from the graph, for which y = -3.

Check your performance against the given solutions at the end of this subunit. Always, make it a point that you read the scales of the axes properly. If you are satisfied with your performance continue to the next subunit.

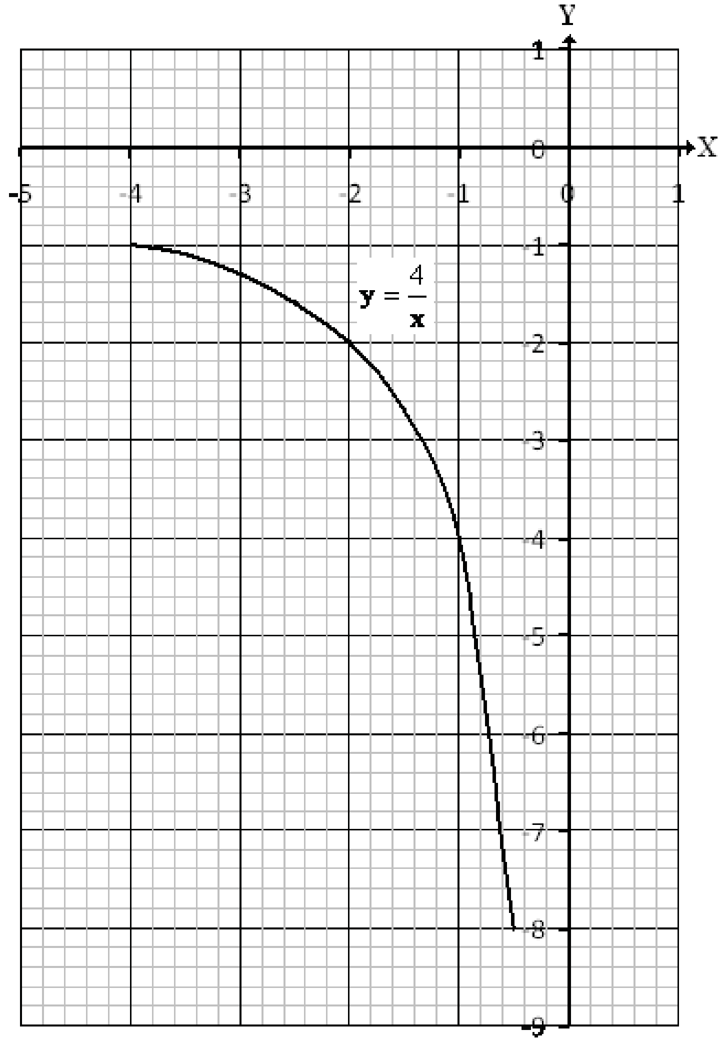
Note it!

Remember:

The drawn graphs can be used to find other corresponding values of x and y, which are not part of the completed table, which satisfy the equation of the graph. They will be estimates based on how well the graph is drawn.

Solutions to ACTIVITY 3:

1. The graph of $\mathbf{y} = \frac{4}{\mathbf{x}}$.

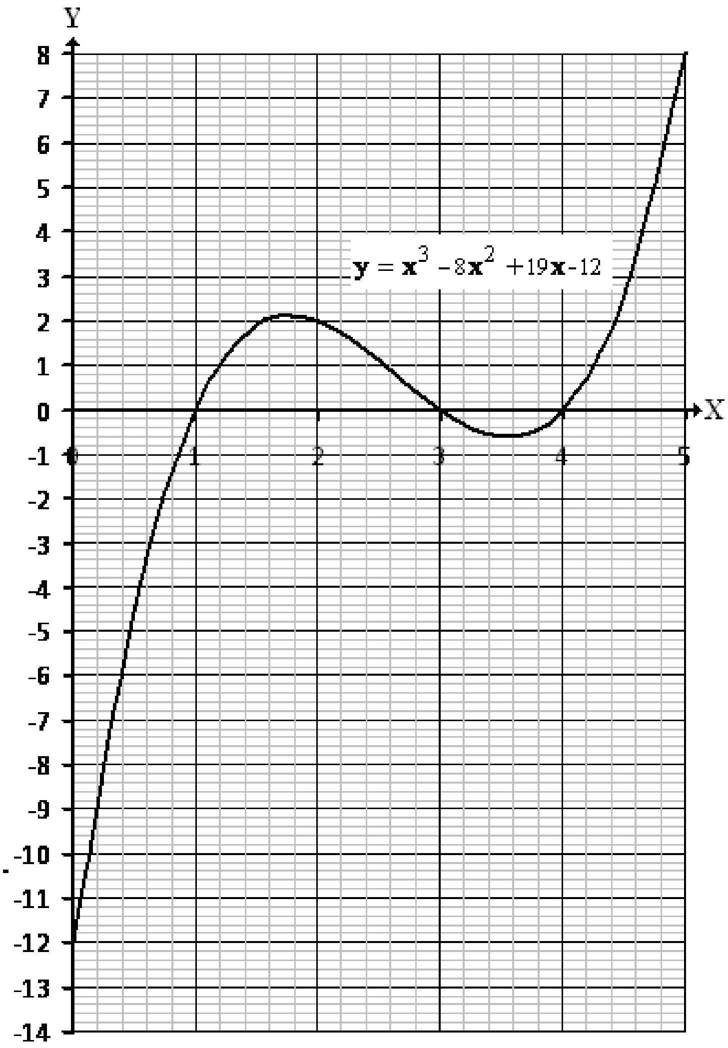


a) The value of x which corresponds to y = -4 is -1, which is the x coordinate of the point where the line y = -4 intersects the graph of $\mathbf{y} = \frac{4}{\mathbf{x}}$.

b) The value of y when x = -0.6 is found at intersection point of line x = -0.6 and $\mathbf{y} = \frac{4}{\mathbf{x}}$, which is (-0.6 , -7). Therefore, y = -7.

c) If (-3.2 , y) lies on the graph of $\mathbf{y} = \frac{4}{\mathbf{x}}$, when line x = -3.2 is drawn to intersect the graph of $\mathbf{y} = \frac{4}{\mathbf{x}}$, the y coordinate at the intersection point is the required y. This is -1.2.

2. The graph of $\mathbf{y} = \mathbf{x}^3 - 8\mathbf{x}^2 + 19\mathbf{x} - 12$.



a) The y-intercept of the graph is the y coordinate of point of intersection of y-axis and the graph, which is -12.

b) The x-intercepts of $\mathbf{y} = \mathbf{x}^3 - 8\mathbf{x}^2 + 19\mathbf{x} - 12$ are the x coordinates of the points on the x-axis . These points are (1, 0), (2 , 0) and (4 , 0). So, 1, 3 and 4 are x-intercepts.

c) Draw the line y = 1. It intersects the graph at (1.2 , 1), (2.6 , 1) and (4.2 ,1), approximately. 1.2, 2.6 and 4.2 are the values of x.

d) The line of y = -3 intersects the graph at (0.6 , -3). Therefore, x = 0.6.

Lesson 4 Solving Equations and Inequalities Using Graphs

By the end of this subunit, you should be able to:

- solve equations, using the graph, which are related to the equation of the graph.
- solve inequalities, using the graph, which are related to the equation of the graph.

This subunit is about 26 pages in length.

Solutions of Equations and Inequalities from the Graph.

After drawing the graph, you can be asked to solve some equations and inequalities using your graph. The important thing in this is to know that those equations and inequalities will always be related to the equation of your graph.

Example 1

Use the graph of $\mathbf{y} = 1 - \frac{1}{2}\mathbf{x}^2$ to solve the following equations:

a) $1 - \frac{1}{2}\mathbf{x}^2 = 0$

When comparing this equation with the equation of $\mathbf{y} = 1 - \frac{1}{2}\mathbf{x}^2$, y in the equation of the graph has been replaced by 0. The graph of $\mathbf{y} = 0$ needs to be drawn on top of the $\mathbf{y} = 1 - \frac{1}{2}\mathbf{x}^2$ graph. The solutions are found by reading the x coordinates of points of intersection of $\mathbf{y} = 1 - \frac{1}{2}\mathbf{x}^2$ and $\mathbf{y} = 0$.

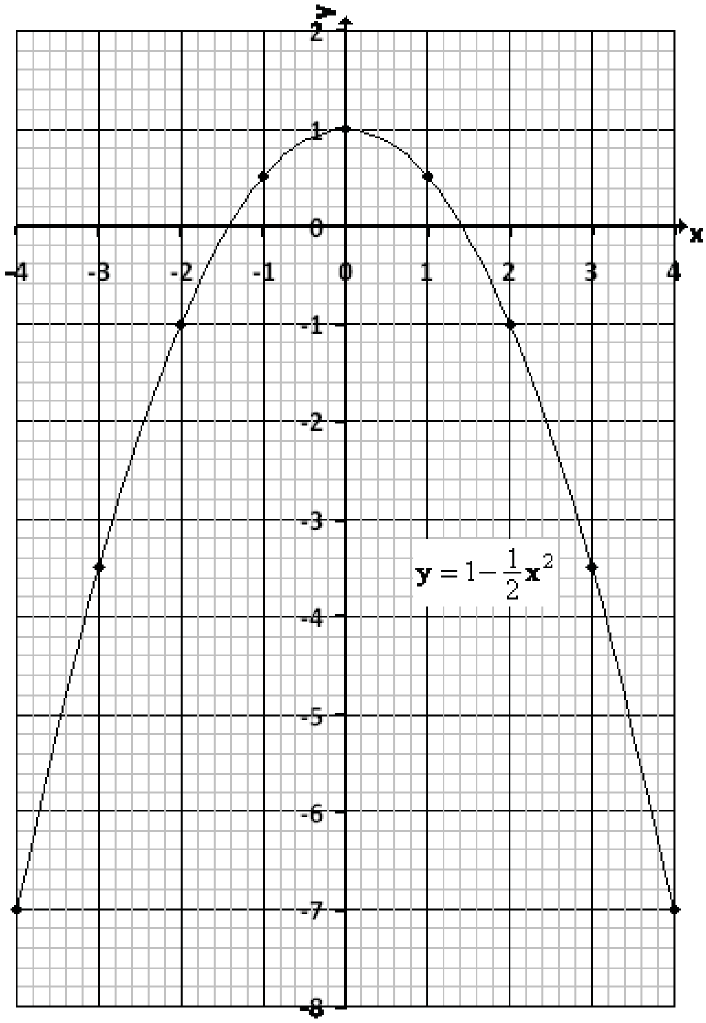


Figure 20 – Graph of $y = 1 - \frac{1}{2}x^2$ and $y = 0$ ($y = 0$ is the x-axis.).

From figure 20, the solutions are approximately 1.4 and -1.4.

b) $1 - \frac{1}{2}x^2 = -5$

When comparing this equation with the equation of $y = 1 - \frac{1}{2}x^2$, y in the equation of the graph has been replaced by -5.The graph of $y = -5$ needs to be drawn on top of the $y = 1 - \frac{1}{2}x^2$ graph. The solutions are found by reading the x coordinates of points of intersection of $y = 1 - \frac{1}{2}x^2$ and $y = -5$.

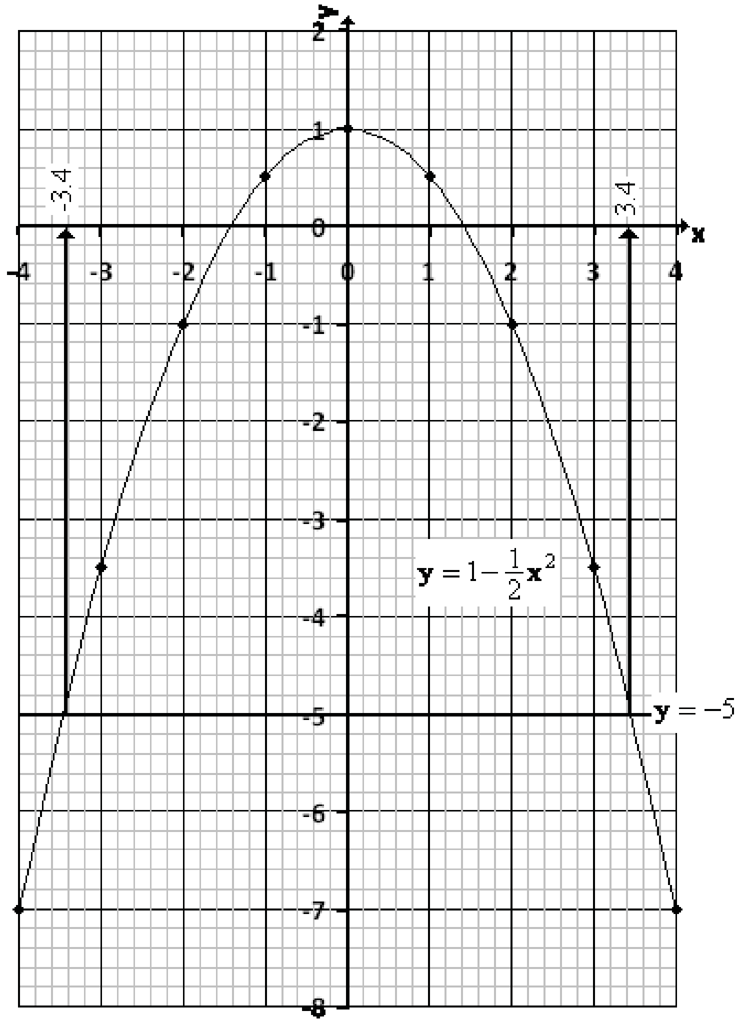


Figure 21 – Graph of $y = 1 - \frac{1}{2}x^2$ and $y = -5$.

As seen in figure 21, the solutions are approximately 3.4 and -3.4.

c) $6 - \frac{1}{2}x^2 = 0$

Remember that the question asks us to base our solutions on the $1 - \frac{1}{2}x^2$ graph. So, the equation $6 - \frac{1}{2}x^2 = 0$ must be rewritten so that the expression $1 - \frac{1}{2}x^2$ is obtained on one side:

$$6 - \frac{1}{2}x^2 = 0$$

Subtract 5 from each side:

$$6-\frac{1}{2}\mathbf{x}^2-5=0-5$$

$$1-\frac{1}{2}\mathbf{x}^2=-5$$

Now, comparing this equation with the equation of $\mathbf{y}=1-\frac{1}{2}\mathbf{x}^2$, y in the equation of the graph has been replaced by -5.The graph of $\mathbf{y}=-5$ needs to be drawn on top of the $\mathbf{y}=1-\frac{1}{2}\mathbf{x}^2$ graph. The solutions are found by reading the x coordinates of points of intersection of $\mathbf{y}=1-\frac{1}{2}\mathbf{x}^2$ and $\mathbf{y}=-5$. This is the same as the previous question and the solutions are approximately 3.4 and -3.4 as seen in figure 18.

d) $2\mathbf{x}=2-\mathbf{x}^2$

This equation needs to be rewritten so that the expression $1-\frac{1}{2}\mathbf{x}^2$ is obtained on one side:

$$2\mathbf{x}=2-\mathbf{x}^2$$

Divide both sides by 2:

$$\frac{2\mathbf{x}}{2}=\frac{2-\mathbf{x}^2}{2}$$

$$\mathbf{x}=1-\frac{1}{2}\mathbf{x}^2$$

Now, comparing this equation with the equation of $\mathbf{y}=1-\frac{1}{2}\mathbf{x}^2$, y in the equation of the graph has been replaced by x. The graph of $\mathbf{y}=\mathbf{x}$ needs to be drawn on top of the $\mathbf{y}=1-\frac{1}{2}\mathbf{x}^2$ graph. The solutions are found by reading the x coordinates of points of intersection of $\mathbf{y}=1-\frac{1}{2}\mathbf{x}^2$ and $\mathbf{y}=\mathbf{x}$.

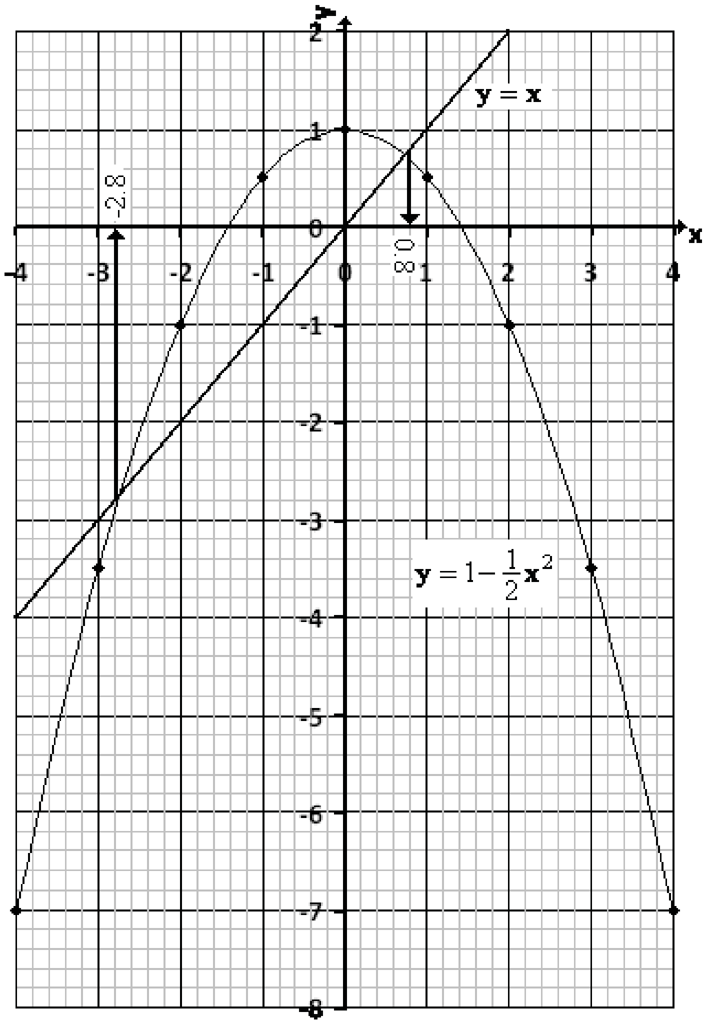


Figure 22 – Graph of $\mathbf{y} = 1 - \frac{1}{2}\mathbf{x}^2$ and $\mathbf{y} = \mathbf{x}$.

From figure 22, the solutions are approximately 0.8 and -2.8.

e) $\mathbf{x}^2 + \mathbf{x} = 8$

Rewrite this equation so that the expression $1 - \frac{1}{2}\mathbf{x}^2$ is obtained on one side:

$\mathbf{x}^2 + \mathbf{x} = 8$

Subtract $\mathbf{x}^2$ from each side:

$\mathbf{x}^2 + \mathbf{x} - \mathbf{x}^2 = 8 - \mathbf{x}^2$

$\mathbf{x} = 8 - \mathbf{x}^2$

Divide both sides by 2:

$$\frac{x}{2} = \frac{8 - x^2}{2}$$

$$\frac{x}{2} = 4 - \frac{1}{2}x^2$$

Subtract 3 from each side:

$$\frac{x}{2} - 3 = 4 - \frac{1}{2}x^2 - 3$$

$$\frac{1}{2}x - 3 = 1 - \frac{1}{2}x^2$$

Now, comparing this equation with the equation of $y = 1 - \frac{1}{2}x^2$, y in the equation of the graph has been replaced by $\frac{1}{2}x - 3$. The graph of $y = \frac{1}{2}x - 3$ needs to be drawn on top of the $y = 1 - \frac{1}{2}x^2$ graph. The solutions are found by reading the x coordinates of points of intersection of $y = 1 - \frac{1}{2}x^2$ and $y = \frac{1}{2}x - 3$.

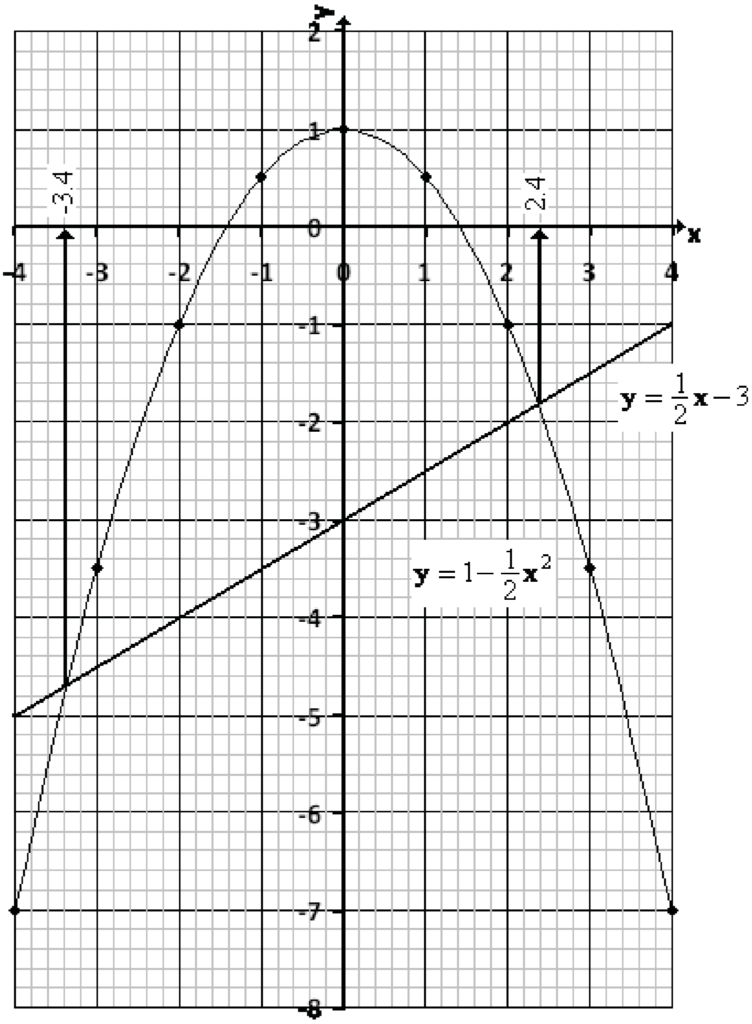


Figure 23 – Graph of $\mathbf{y} = 1 - \frac{1}{2}\mathbf{x}^2$ and $\mathbf{y} = \frac{1}{2}\mathbf{x} - 3$.

As seen in figure 23, the solutions are approximately 2.4 and -3.4.

Example 2

Use the graph of $y = x^2 - 1$ to solve the following inequalities:

a) $\mathbf{x}^2 - 1 \leq 3$

Compare the expressions in the inequality, with the expressions in $y = x^2 - 1$.

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Compare your answer with:

When comparing this inequality with the equation of $y = x^2 - 1$, y in the equation of the graph has been replaced by 3.The graph of $\mathbf{y} = 3$ needs to be drawn on top of the $y = x^2 - 1$ graph. The solution is found from the part or parts of the graph of $y = x^2 - 1$ where $y \leq 3$.

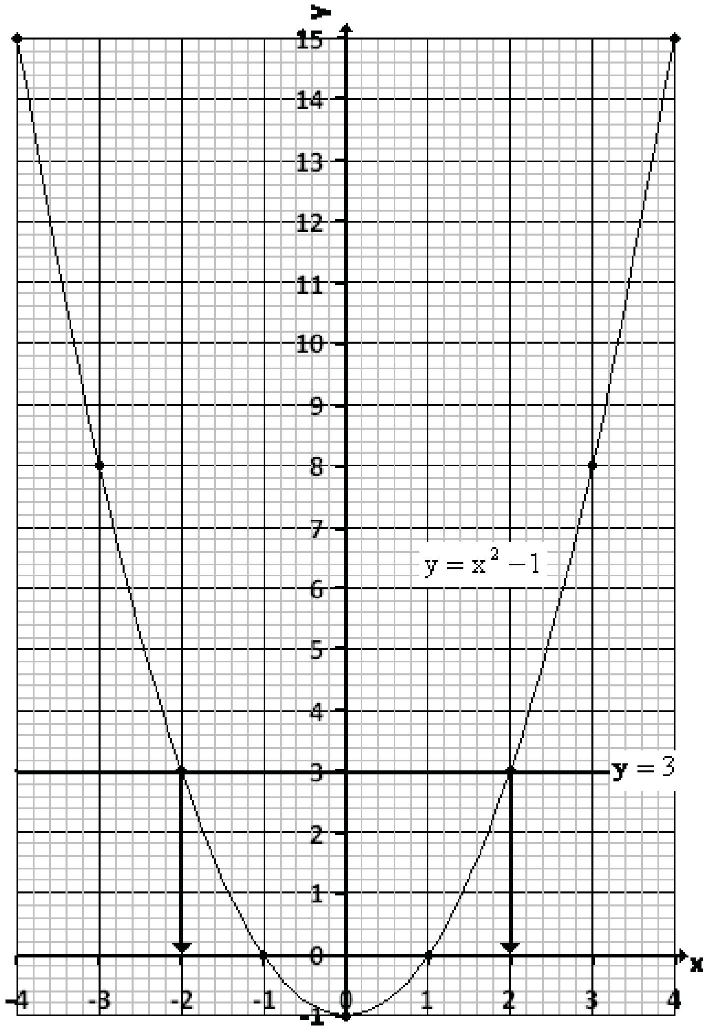

Figure 24 – Graph of $y = x^2 - 1$ and $\mathbf{y} = 3$

From figure 24, the solution of the inequality is given by the approximate values of x which lie between -2 and 2 inclusive.

Therefore, the solution is approximately $-2 \leq \mathbf{x} \leq 2$.

b) $\mathbf{x}^2 - 1 \geq 1 - \frac{1}{2}\mathbf{x}$

Compare the expressions in the inequality, with the expressions in $y = x^2 - 1$.

Compare your answer with:

When comparing this inequality with the equation of $y = x^2 - 1$, y in the equation of the graph has been replaced by $1 - \frac{1}{2}\mathbf{x}$ .The graph of $\mathbf{y} = 1 - \frac{1}{2}\mathbf{x}$ needs to be drawn on top of the $y = x^2 - 1$ graph. The solution is found from part or parts of the graph of $y = x^2 - 1$ where $y \geq 1 - \frac{1}{2}x$.

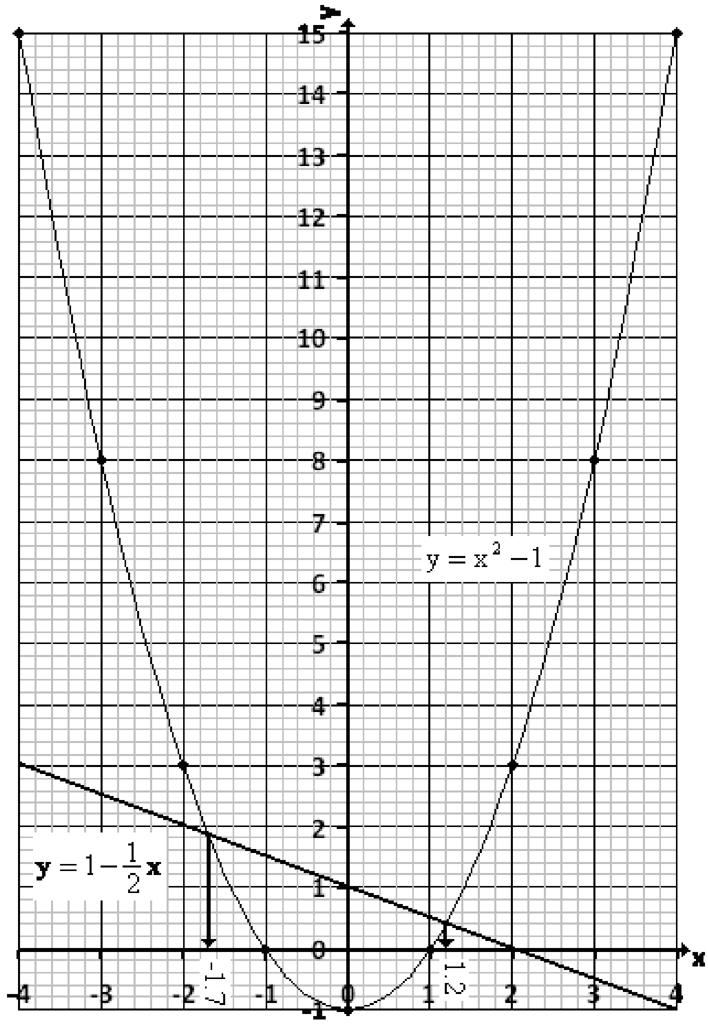

Figure 25 – Graph of $y = x^2 - 1$ and $\mathbf{y} = 1 - \frac{1}{2}\mathbf{x}$

As seen in figure 25, the solution of the inequality is given by the approximate values of x that are less than -1.7 or values of x that are greater than1.2 inclusive.

Therefore, the solution is approximately $x \le -1.7$ and $\mathbf{x} \ge 1.2$.

c) $\mathbf{x}^2 < 6 - \frac{1}{2}\mathbf{x}$

Rewrite this inequality so that the expression $\mathbf{x}^2 - 1$ is obtained on one side:

Compare your answer with:

$$\mathbf{x}^2 < 6 - \frac{1}{2}\mathbf{x}$$

Subtract 1 from each side:

$$\mathbf{x}^2 - 1 < 6 - \frac{1}{2}\mathbf{x} - 1$$

$$\mathbf{x}^2 - 1 < 5 - \frac{1}{2}\mathbf{x}$$

Then, compare the expressions in $\mathbf{x}^2 - 1 < 5 - \frac{1}{2}\mathbf{x}$

, with the expressions in $y = x^2 - 1$.

Compare your answer with:

Now, comparing this inequality with the equation of $y = x^2 - 1$, y in the equation of the graph has been replaced by $5 - \frac{1}{2}\mathbf{x}$. The graph of $\mathbf{y} = 5 - \frac{1}{2}\mathbf{x}$ has to be drawn broken to show that the points on the line are excluded. The solution is found from part or parts of the graph of $y = x^2 - 1$ where $y < 5 - \frac{1}{2}x$.

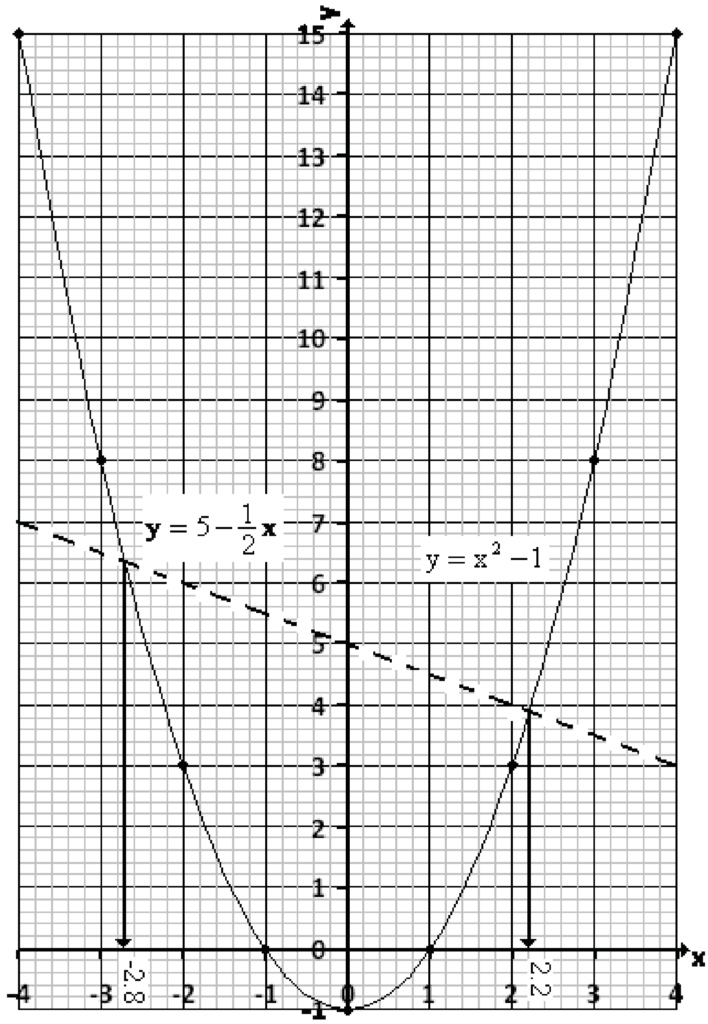

Figure 26 – Graph of $y = x^2 - 1$ and $\mathbf{y = 5 - \frac{1}{2}x}$

From figure 26, the solution of the inequality is given by the approximate values of x which lie between -2.8 and 2.2 exclusive.

Therefore, the solution is approximately -2.8 < x < 2.2.

d) $\mathbf{x}^2 - 1 < 4 - \mathbf{x}$

Compare the expressions in the inequality, with the expressions in $y = x^2 - 1$.

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Compare your answer with:

When comparing this inequality with the equation of $y = x^2 - 1$, y in the equation of the graph has been replaced by $4 - \mathbf{x}$. The graph of $\mathbf{y} = 4 - \mathbf{x}$ needs to be drawn broken to show that the points on the line are excluded. The solution is found from part or parts of the graph of $y = x^2 - 1$ where $y < 4 - x$.

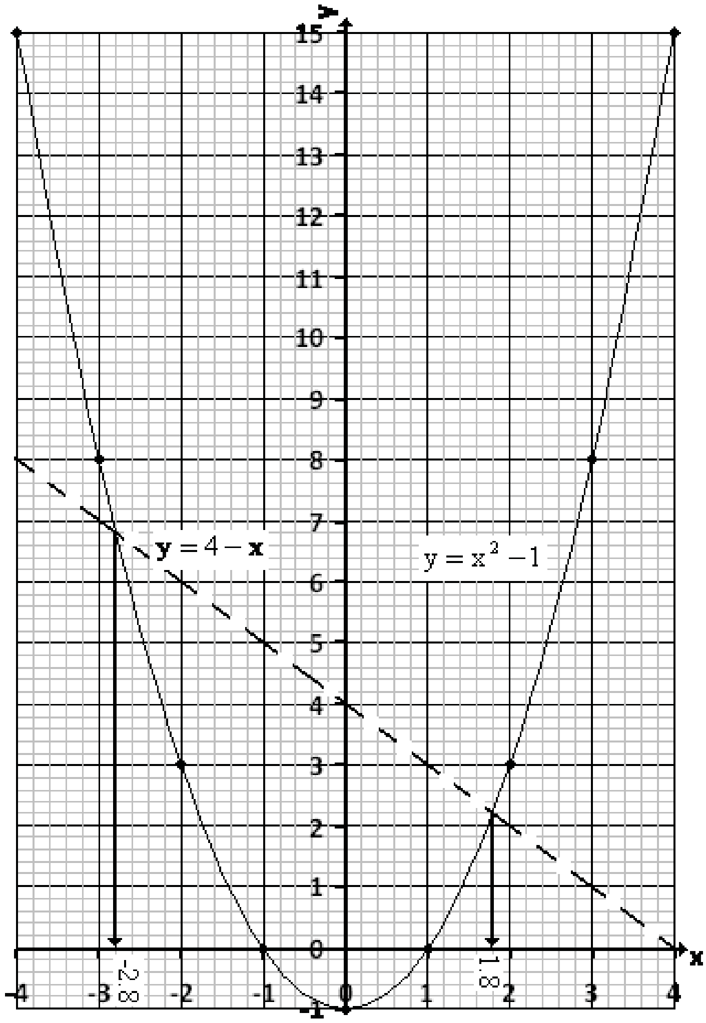

Figure 27 – Graph of $y = x^2 - 1$ and $\mathbf{y} = 4 - \mathbf{x}$

As shown in figure 27, the solution of the inequality is given by the approximate values of x which lie between -2.8 and 1.8 exclusive.

Therefore, the solution is approximately $-2.8 < x < 1.8$.

e) $2\mathbf{x}^2 - 1 > 3 + \mathbf{x}^2$

Rewrite this inequality so that the expression $\mathbf{x}^2 - 1$ is obtained on one side:

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Compare your answer with:

$2\mathbf{x}^2 - 1 > 3 + \mathbf{x}^2$

Subtract $\mathbf{x}^2$ from each side:

$2\mathbf{x}^2 - 1 - \mathbf{x}^2 > 3 + \mathbf{x}^2 - \mathbf{x}^2$

$\mathbf{x}^2 - 1 > 3$

Then, compare the expressions in the inequality, with the expressions in $y = x^2 - 1$.

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Compare your answer with:

Now, comparing this inequality with the equation of $y = x^2 - 1$, y in the equation of the graph has been replaced by 3. The graph of $\mathbf{y} = 3$ needs to be drawn broken to show that the points on the line are excluded. The solution is found from part or parts of the graph of $y = x^2 - 1$ where $y > 3$.

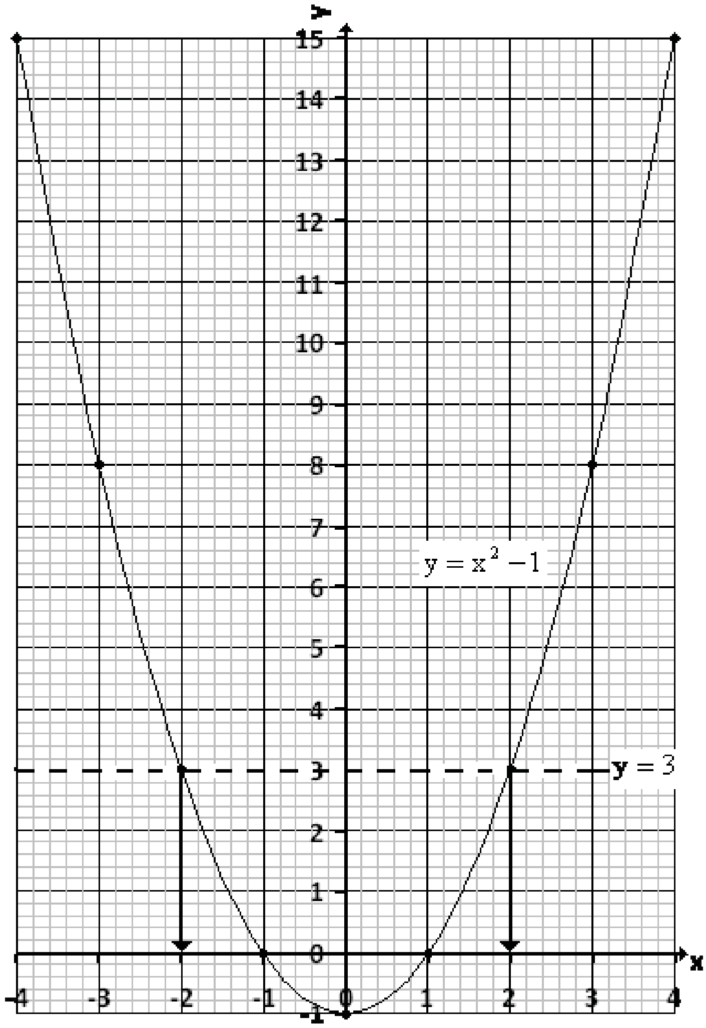


Figure 28 – Graph of $y = x^2 - 1$ and $\mathbf{y} = 3$

From figure 28, the solution of the inequality is given by the approximate values of x that are less than -2 or values of x that are greater than 2 exclusive.

Therefore, the solution is approximately $x < -2$ and $x > 2$.

Activity 4

Activity 4

1. Use the following graph of $y = -x^2 + 6x - 8$, to solve the given equations and inequalities.

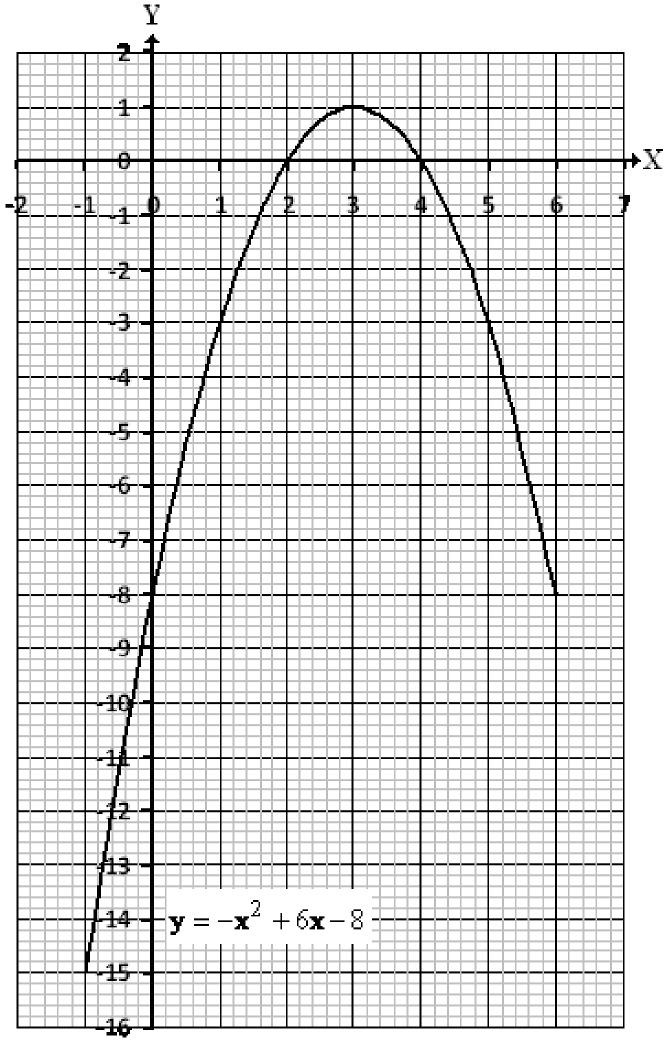
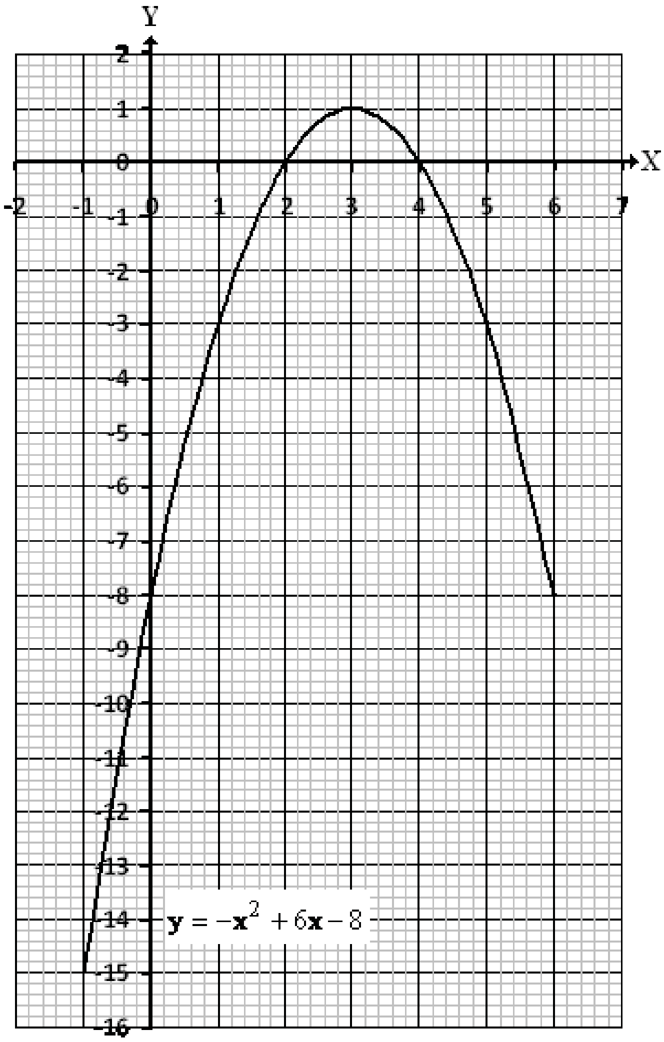
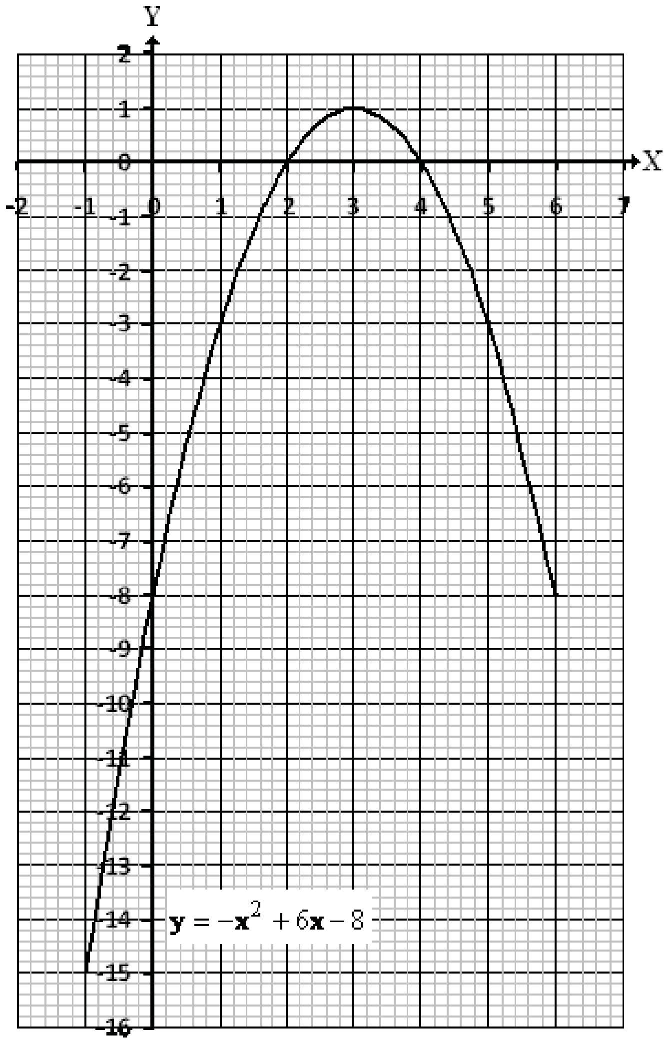


Figure 29 – Graph of $y = -x^2 + 6x - 8$.

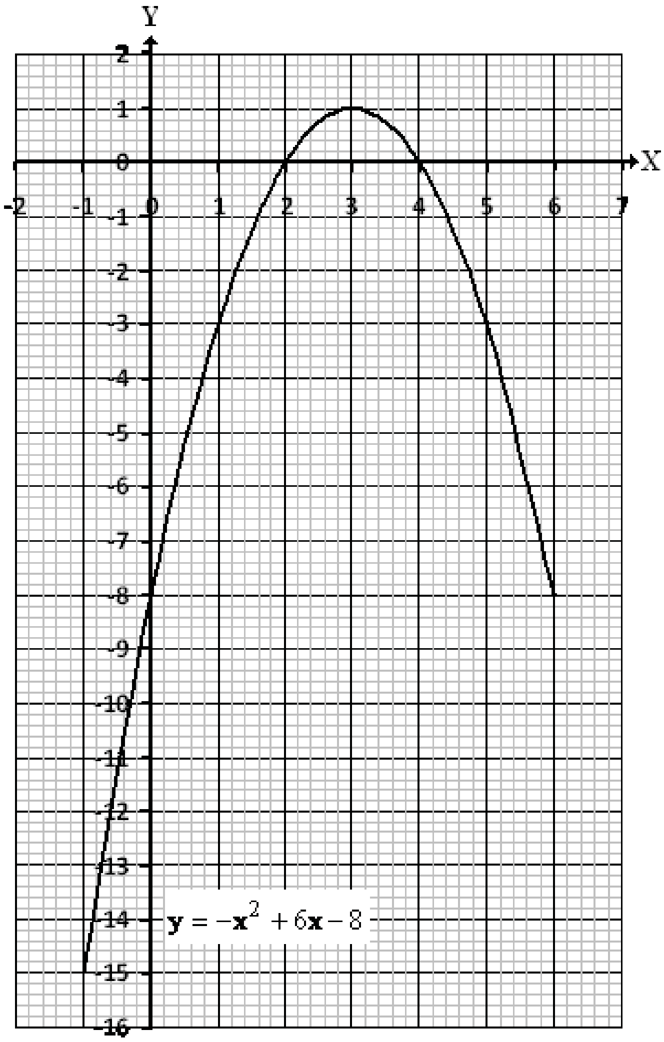
a) $0 = -x^2 + 6x - 8$



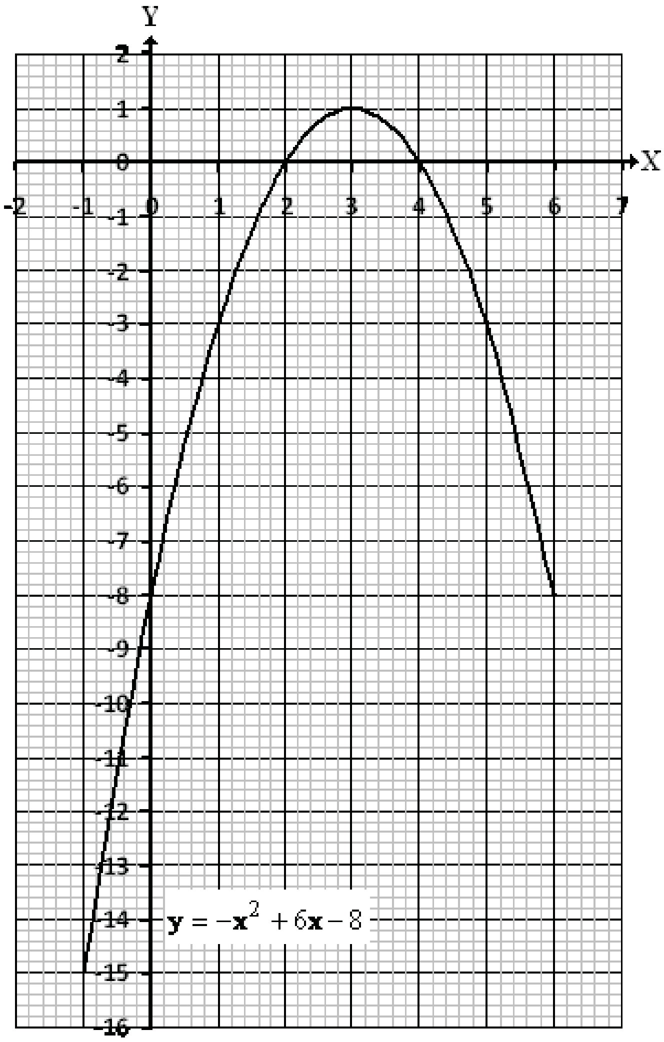
b) $-4 = -x^2 + 6x - 8$



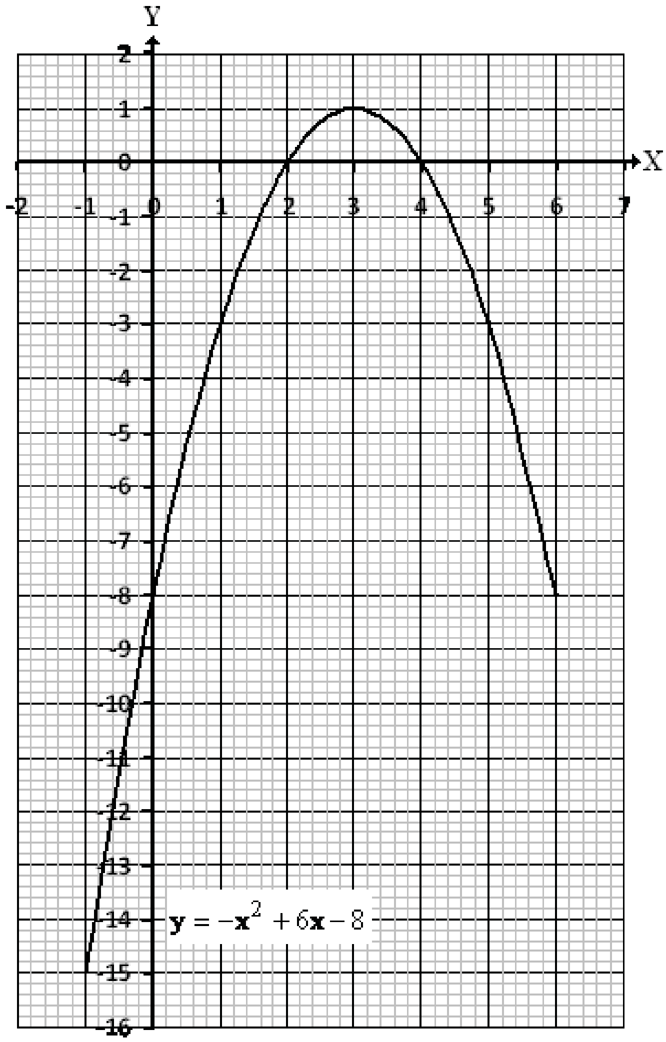
c) $6 = -x^2 + 6x$



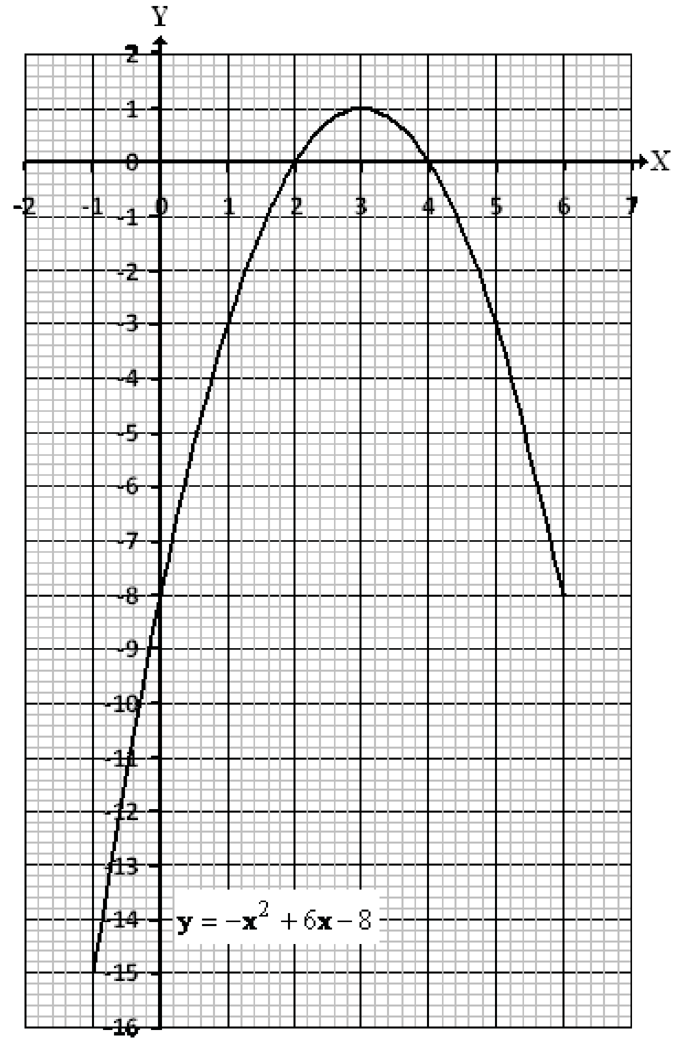
d) $2x - 8 = -x^2 + 6x - 8$



e) $0 \geq -x^2 + 6x - 8$



f) $2\mathbf{x} - 8 \leq -\mathbf{x}^2 + 6\mathbf{x} - 8$



Check your performance against the given solutions at the end of this subunit. If you are satisfied with your performance continue to the next subunit.

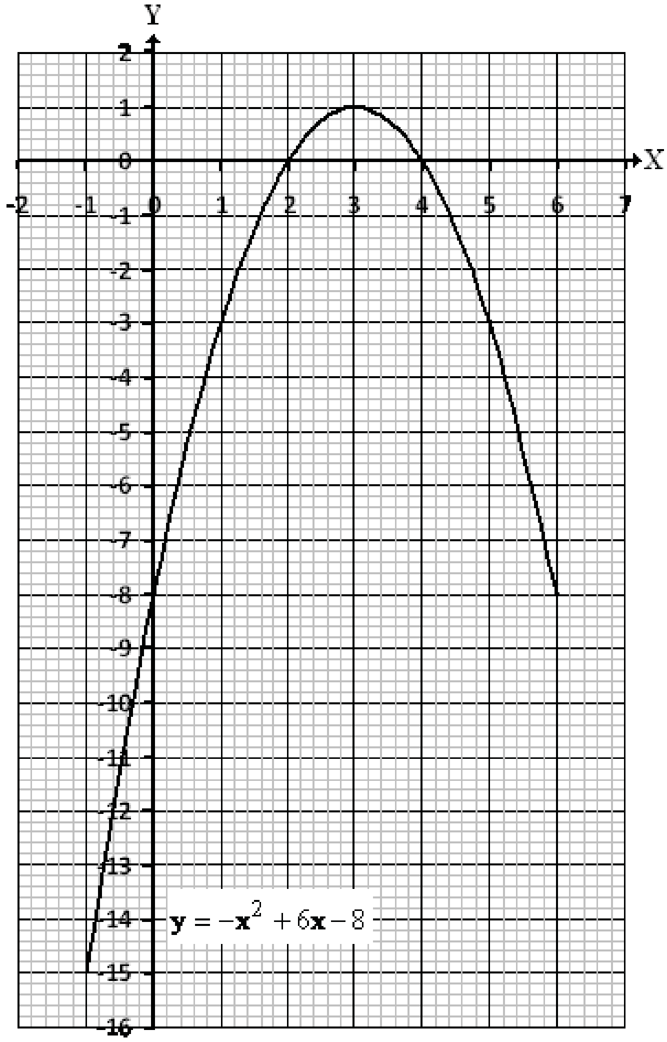
Note it!

Remember:

Equations and inequalities can be solved, using graphs. The important thing in this is to know that those equations and inequalities are always related to the equation of your graph.

Solutions to ACTIVITY 4:

1. The graph of $y = -x^2 + 6x - 8$.

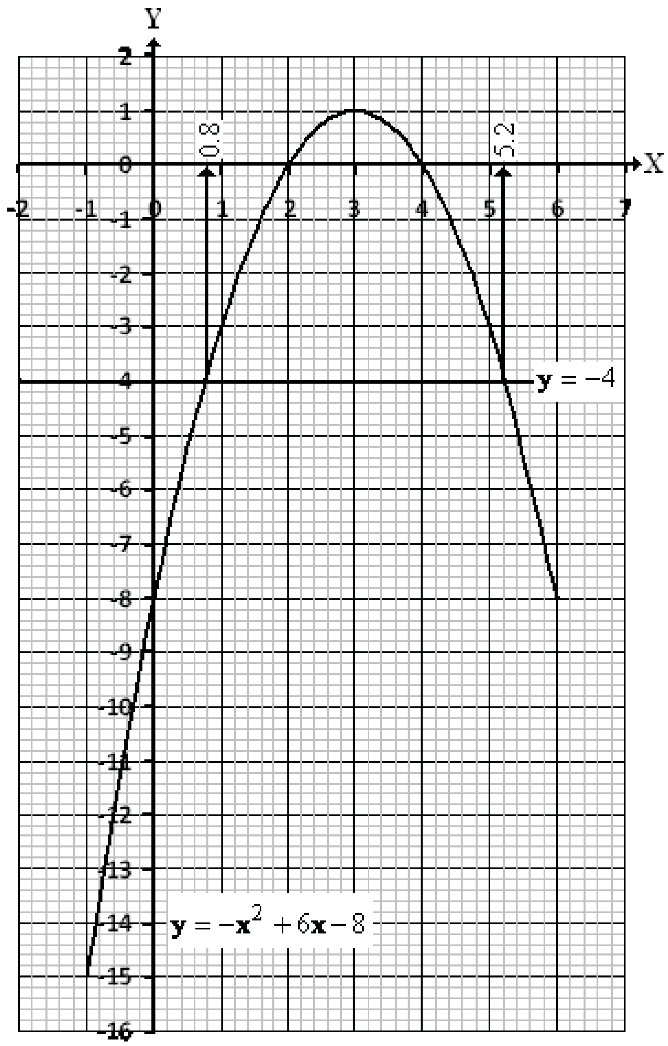


a) $0 = -x^2 + 6x - 8$

When comparing $0 = -x^2 + 6x - 8$ with $y = -x^2 + 6x - 8$, y in the equation of the graph has been replaced by 0. The graph of $y = 0$ needs to be drawn on top of the $y = -x^2 + 6x - 8$ graph. The solutions are found by reading the x coordinates of points of intersection of $y = -x^2 + 6x - 8$ and $y = 0$. These are 2 and 4. Remember that the graph of $y = 0$ is the x-axis.

b) $-4 = -x^2 + 6x - 8$

When comparing $-4 = -x^2 + 6x - 8$ with $y = -x^2 + 6x - 8$, y in the equation of the graph has been replaced by -4. The graph of $y = -4$ needs to be drawn on top of the $y = -x^2 + 6x - 8$ graph. The solutions are found by reading the x coordinates of points of intersection of $y = -x^2 + 6x - 8$ and $y = -4$.

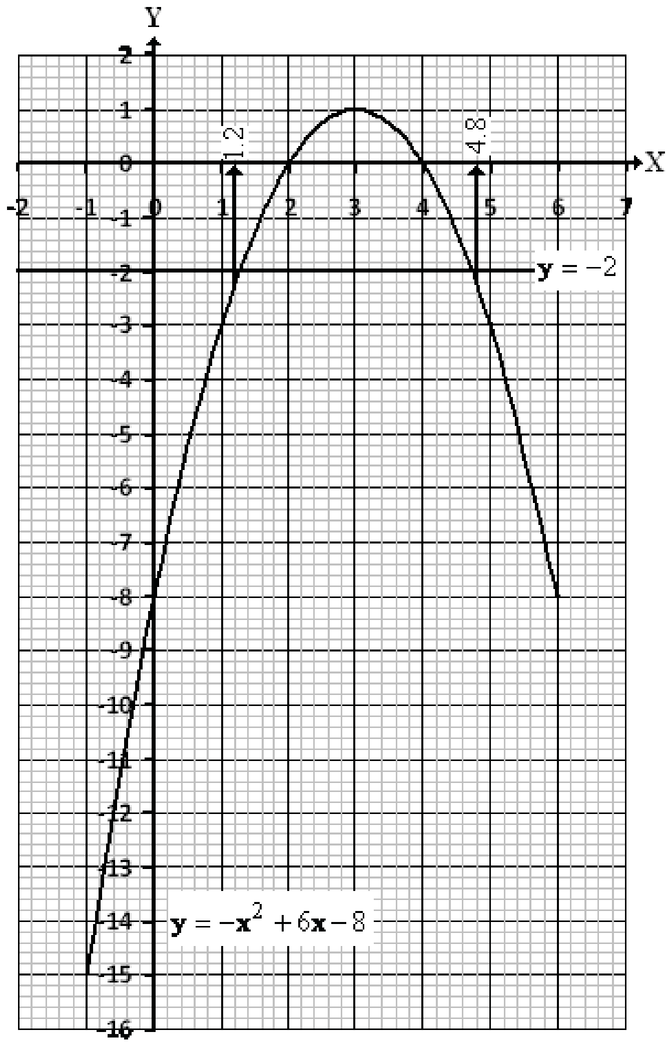


The solutions are 0.8 and 5.2.

c) $6 = -\mathbf{x}^2 + 6\mathbf{x}$

$6 = -\mathbf{x}^2 + 6\mathbf{x}$ can be written as $-2 = -\mathbf{x}^2 + 6\mathbf{x} - 8$, when subtracting 8 from both sides.

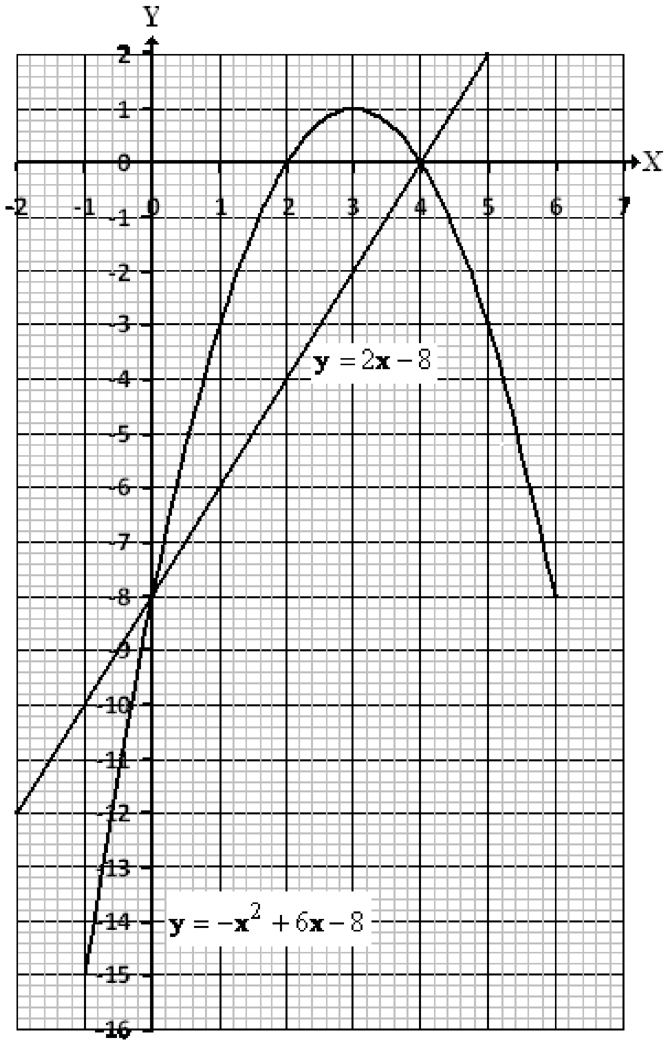
When comparing $-2 = -\mathbf{x}^2 + 6\mathbf{x} - 8$ with $\mathbf{y} = -\mathbf{x}^2 + 6\mathbf{x} - 8$, y in the equation of the graph has been replaced by -2. The graph of $\mathbf{y} = -2$ needs to be drawn on top of the $\mathbf{y} = -\mathbf{x}^2 + 6\mathbf{x} - 8$ graph. The solutions are found by reading the x coordinates of points of intersection of $\mathbf{y} = -\mathbf{x}^2 + 6\mathbf{x} - 8$ and $\mathbf{y} = -2$.



The solutions are 1.2 and 4.8.

d) $2x - 8 = -x^2 + 6x - 8$

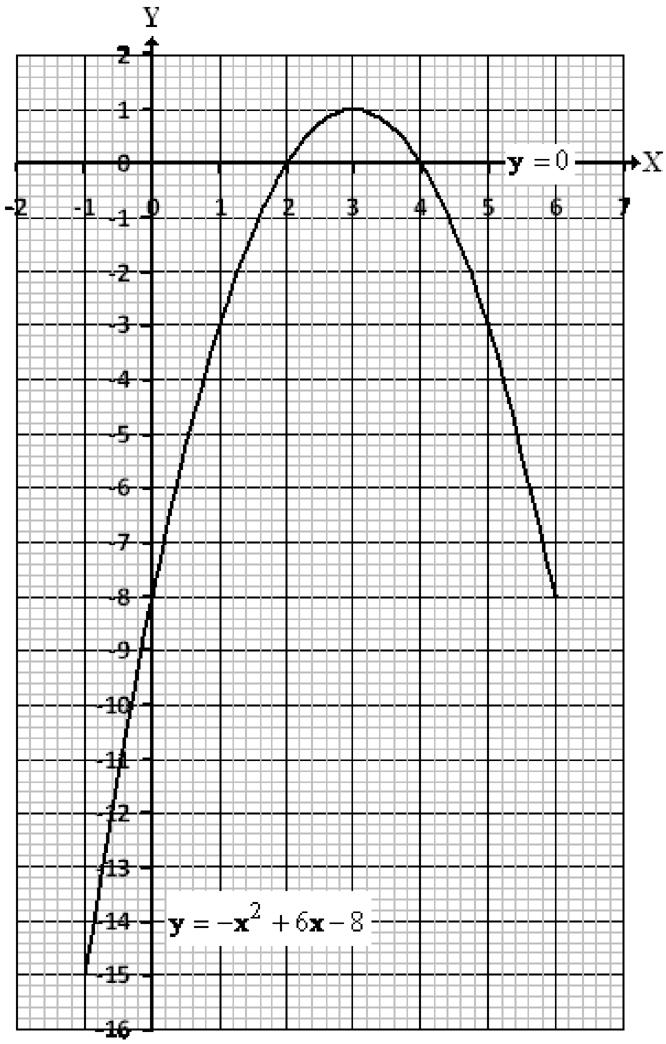
When comparing $2x - 8 = -x^2 + 6x - 8$ with $y = -x^2 + 6x - 8$, y in the equation of the graph has been replaced by $2x - 8$. The graph of $y = 2x - 8$ needs to be drawn on top of the $y = -x^2 + 6x - 8$ graph. The solutions are found by reading the x coordinates of points of intersection of $y = -x^2 + 6x - 8$ and $y = 2x - 8$.



The solutions are 0 and 4.

e) $0 \geq -x^2 + 6x - 8$

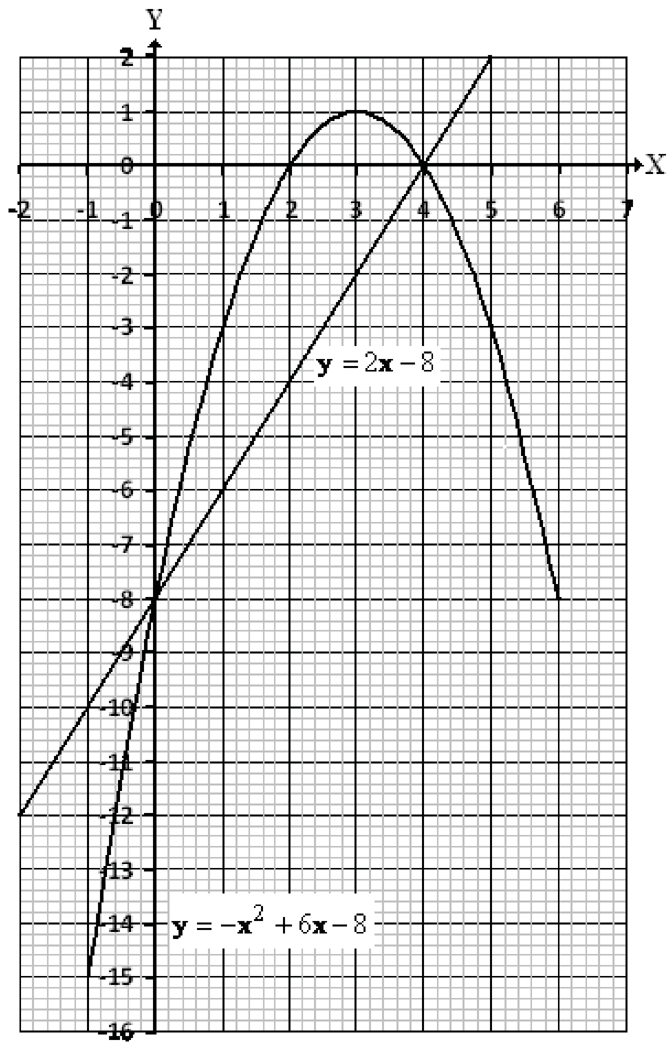
Draw the line $y = 0$ as a continuous line. The solution is found from the parts of the graph of $y = -x^2 + 6x - 8$ where $y \leq 0$.



The solution comes from x ≤ 2 and x ≥ 4.

f) $2\mathbf{x} - 8 \le -\mathbf{x}^2 + 6\mathbf{x} - 8$

Draw the line $\mathbf{y} = 2\mathbf{x} - 8$ as a continuous line. The solution is found from the part of the graph of $\mathbf{y} = -\mathbf{x}^2 + 6\mathbf{x} - 8$ where $\mathbf{y} \ge 2\mathbf{x} - 8$.



The solution is $0 \leq x \leq 4$.

Lesson 5 Solving Problems Using Sketch Graphs

By the end of this subunit, you should be able to:

- draw a sketch graph of a parabola.
- find the coordinates of the point, at absolute maximum, of a sketch graph of a parabola.
- find the coordinates of the point, at absolute minimum, of a sketch graph of a parabola.
- find the coordinates of the points of x-intercepts, of a sketch graph of a parabola.

- find the coordinates of the point of y-intercept, of a sketch graph of a parabola.
- find the equation of the line of symmetry of a parabola.

This subunit is about 30 pages in length.

Solutions from a sketch graph.

After completing this subunit, you should be able to solve problems using sketch graphs of parabolas.

When sketch graphs are drawn, important points like **critical points** and the **intercepts**, are shown. Critical points of parabolas are points at which absolute maximum and absolute minimum exists. Intercepts are y and x- intercepts.

Drawing sketch graphs:

Example 1

Suppose the sketch graph of $\mathbf{y} = \mathbf{x}^2 - \mathbf{x} - 6$ is to be drawn:

It is a parabola since $\mathbf{y} = \mathbf{x}^2 - \mathbf{x} - 6$ is quadratic.

The y-intercept is equal to constant term in $\mathbf{y} = \mathbf{x}^2 - \mathbf{x} - 6$:

Therefore, the y-intercept is -6.

The graph crosses y-axis at (0,-6).

The x-intercept is found by substituting zero for y in the equation and solving for x:

$0 = \mathbf{x}^2 - \mathbf{x} - 6$

Factorising quadratic expression on the right:

$0 = \mathbf{x}^2 - \mathbf{x} - 6$

$0 = (\mathbf{x} - 3)(\mathbf{x} + 2)$

By using the zero factor property:

$(\mathbf{x} - 3) = 0 \ldots \mathbf{or} \ldots (\mathbf{x} + 2) = 0$

$\mathbf{x} - 3 + 3 = 0 + 3 \ldots \mathbf{or} \ldots \mathbf{x} + 2 - 2 = 0 - 2$

$\mathbf{x} = 3 \ldots \mathbf{or} \ldots \mathbf{x} = -2$

Therefore, x-intercepts are $\mathbf{x} = 3$ and $\mathbf{x} = -2$.

The graph crosses x-axis at (3,0) and (-2,0).

Since the coefficient of $\mathbf{x}^2$ is positive (it is 1), the parabola has a point of absolute minimum. The equation of the line of symmetry is found from:

$$x = -\frac{b}{2a}$$

where a and b are the coefficients of $\mathbf{x}^2$ and $\mathbf{x}$ respectively.

For $\mathbf{y} = \mathbf{x}^2 - \mathbf{x} - 6$, $\mathbf{b} = -1$ and $\mathbf{a} = 1$.

By substituting b = -1 and a = 1 into $x = -\frac{b}{2a}$, we get:

$$\mathbf{x} = -\frac{(-1)}{2(1)}$$

$$\mathbf{x} = \frac{1}{2}$$

The line of symmetry passes through the point of absolute minimum, and $\mathbf{x} = \frac{1}{2}$ is used to find the corresponding y by substituting it in $\mathbf{y} = \mathbf{x}^2 - \mathbf{x} - 6$:

$$\mathbf{y} = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 6$$

$$\mathbf{y} = \frac{1}{4} - \frac{1}{2} - 6$$

$$\mathbf{y} = -6\frac{1}{4}$$

Therefore, the point of absolute minimum is $\left(\frac{1}{2}, -6\frac{1}{4}\right)$

The sketch graph is shown in figure 30:

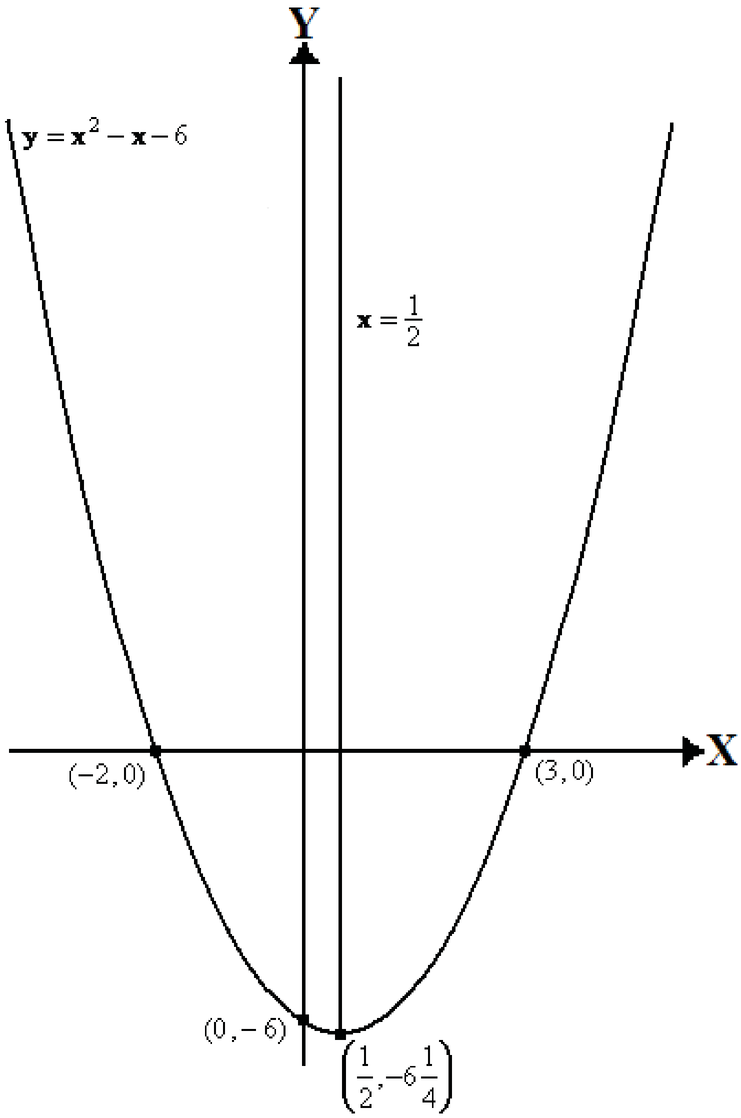


Figure 30 – Sketch graph of $y = x^2 - x - 6$

Example 2

Sketching the graph of $y = (x + 1)(3 - x)$:

Based on whether the equation is quadratic or not, predict whether the graph is a parabola or not.

____

Compare your answer with:

It is a parabola, as is proven below.

$y = (x+1)(3-x)$

$y = x(3-x)+1(3-x)$

$y = 3x - x^2 + 3 - x$

$y = -x^2 + 2x + 3$

What is the y-intercept?

____

Compare your answer with:

The y-intercept is equal to the constant term in $y = -x^2 + 2x + 3$:

Therefore, the y-intercept is 3.

At which point does the graph cross y-axis?

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Compare your answer with:
The graph crosses y-axis at (0,3).

Find the x-intercepts:

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Compare your answer with:
The x-intercept is found by substituting zero for y in the equation and solving for x:

$0 = (x+1)(3-x)$

By using the zero factor property:

$(\mathbf{x}+1)=0\ldots\mathbf{or}\ldots(3-\mathbf{x})=0$

$\mathbf{x}+1-1=0-1\ldots\mathbf{or}\ldots3-\mathbf{x}+\mathbf{x}=0+\mathbf{x}$

$\mathbf{x}=-1\ldots\mathbf{or}\ldots\mathbf{x}=3$

Therefore, the x-intercepts are $\mathbf{x}=3$ and $\mathbf{x}=-1$.

At which points does the graph cross x-axis?

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Compare your answer with:

The graph crosses x-axis at (3,0) and (-1, 0).

Based on whether the parabola opens upwards or downwards, what type of critical point does the graph have?

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Compare your answer with:

Since the coefficient of $\mathbf{x}^2$ is negative, which is -1, the parabola has a point of absolute maximum.

Find the equation of the line of symmetry:

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Compare your answer with:

The equation of the line of symmetry is found from:

$$x=-\frac{b}{2a}$$

where a and b are coefficients of $\mathbf{x}^2$ and $\mathbf{x}$ respectively.

For $\mathbf{y}=-\mathbf{x}^2+2\mathbf{x}+3$, $\mathbf{b}=2$ and $\mathbf{a}=-1$.

$$\mathbf{x}=-\frac{(2)}{2(-1)}$$

$$\mathbf{x}=1$$

Find the y coordinate of the point of absolute maximum:

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Compare your answer with:

The line of symmetry passes through the point of absolute maximum, and $\mathbf{x} = 1$ is used to find the corresponding y by substituting it in $\mathbf{y} = (\mathbf{x}+1)(3-\mathbf{x})$:

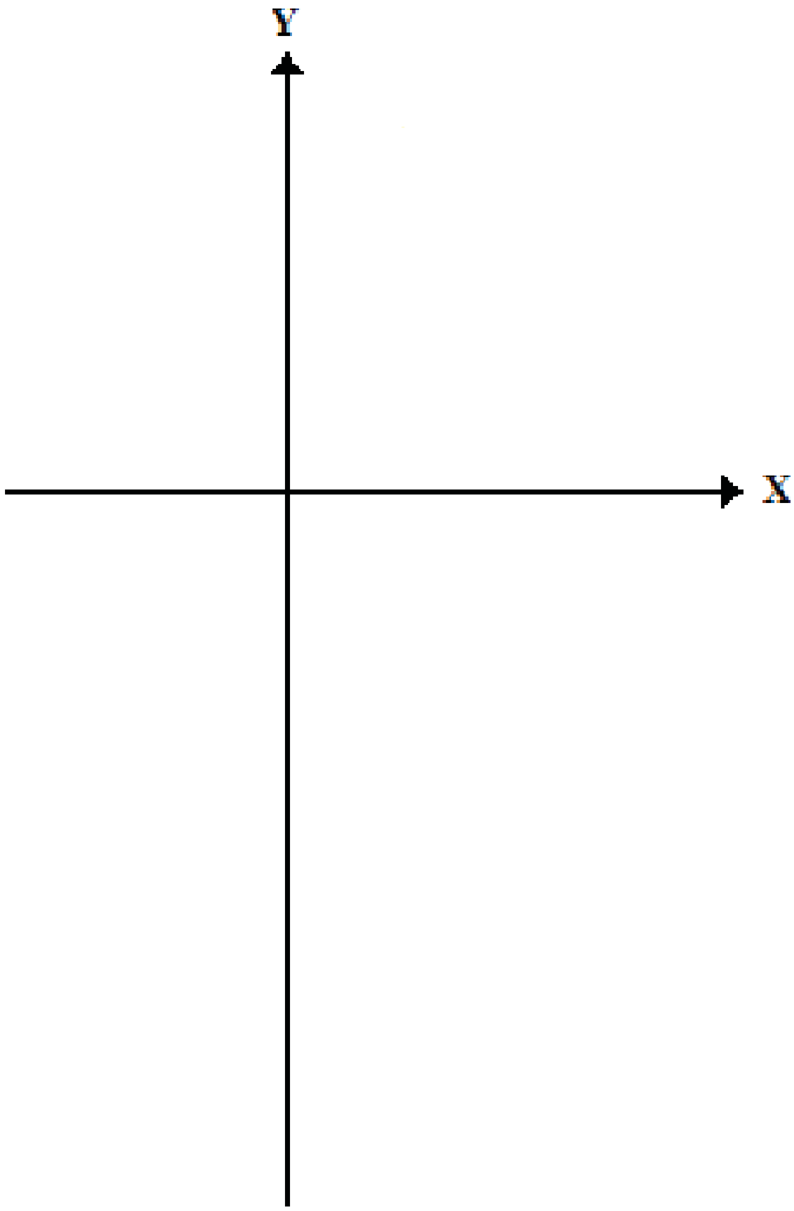
$\mathbf{y} = (1+1)(3-1)$

$\mathbf{y} = 2 \times 2$

$\mathbf{y} = 4$

Therefore, the point of absolute minimum is $(1,4)$

Now, sketch the graph:



Compare your sketch graph to that of figure 31. Be sure that you understand the reason for any differences before you continue.

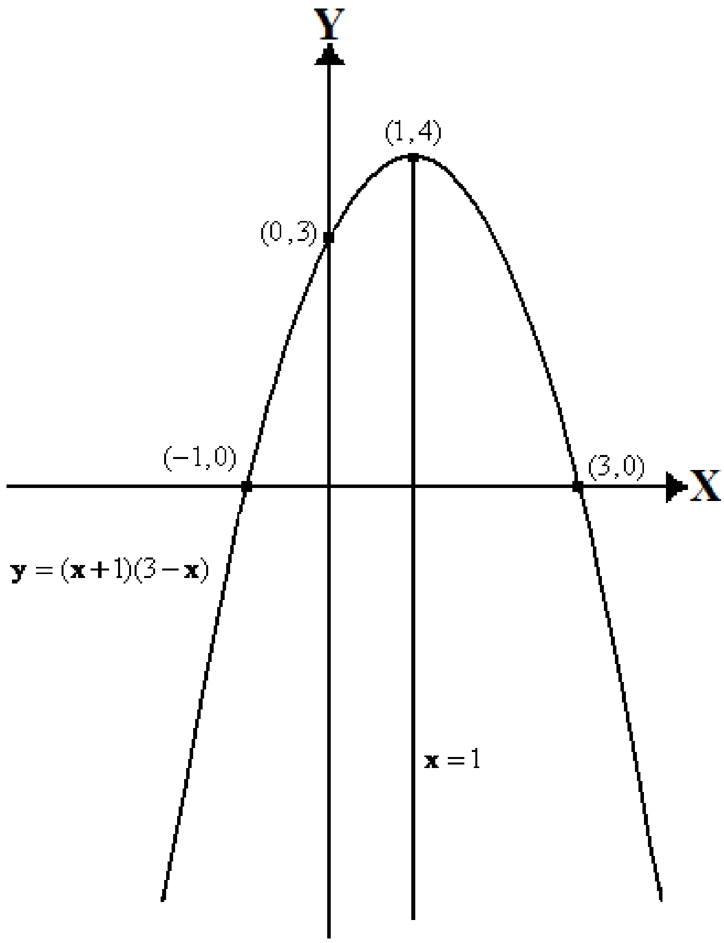


Figure 31 – Sketch graph of $\mathbf{y} = (\mathbf{x}+1)(3-\mathbf{x})$

Using sketch graphs:

You can determine the coordinates of the points labelled with letters, from a drawn sketch graph. Some values are determined in the same way as above. However, you can only determine other values by referring to the sketch graph itself.

Example 1

Figure 32 shows the sketch graph: $\mathbf{y} = \mathbf{x}^2 - 6\mathbf{x} + 8$. Study it as this example is based on the graph.

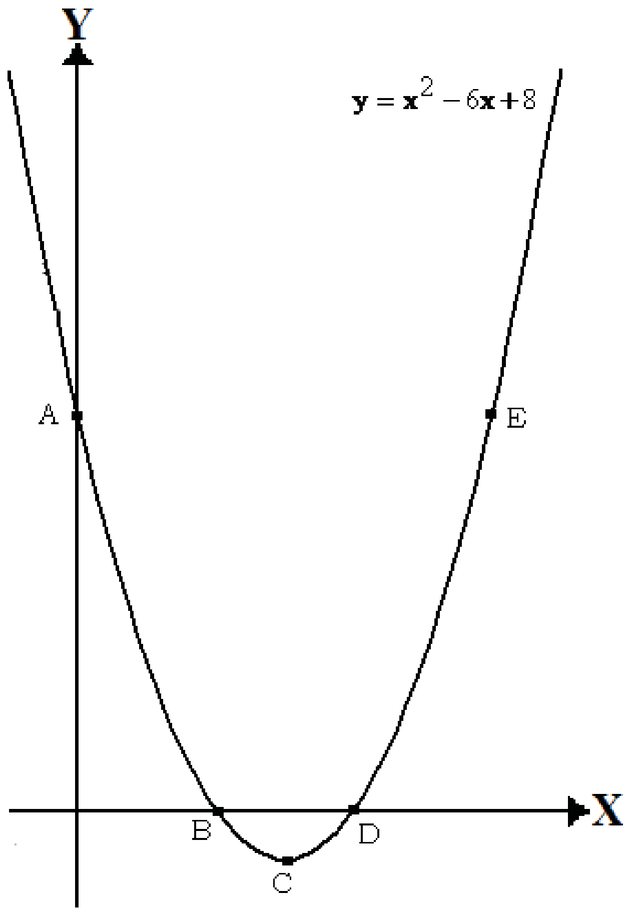


Figure 32 – Sketch graph of $y = x^2 - 6x + 8$

a) As you did previously, find the coordinates of A.

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Compare your answer with:

A is a point on the y-axis; so it has x coordinate equal to 0.The corresponding y is 8, which is the y-intercept:

A is (0,8).

b) As you did previously, find the coordinates of B and D.

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Compare your answer with:

B and D are points on the x-axis; so their y coordinates are equal to 0. Their x coordinates are x-intercepts.

The x-intercepts are found by substituting zero for y in the equation and solving for x:

$$0 = \mathbf{x}^2 - 6\mathbf{x} + 8$$

Factorise the quadratic expression on the right:

$$0 = \mathbf{x}^2 - 6\mathbf{x} + 8$$

$$0 = (\mathbf{x} - 2)(\mathbf{x} - 4)$$

By using the zero factor property:

$$(\mathbf{x} - 2) = 0 \ldots \mathbf{or} \ldots (\mathbf{x} - 4) = 0$$

$$\mathbf{x} - 2 + 2 = 0 + 2 \ldots \mathbf{or} \ldots \mathbf{x} - 4 + 4 = 0 + 4$$

$$\mathbf{x} = 2 \ldots \mathbf{or} \ldots \mathbf{x} = 4$$

Therefore, x-intercepts are $\mathbf{x} = 2$ and $\mathbf{x} = 4$.

$\mathbf{x} = 2$ is the x coordinate of B, as it is nearer the origin than D:

So, B is (2,0) and D is (4,0).

c) As you did previously, find the equation of the axis of symmetry.

Compare your answer with the following:

The equation of the axis of symmetry is found from:

$$x = -\frac{b}{2a}$$

where a and b are coefficients of x^2 and x respectively.

$x = -\frac{b}{2a}$

For $y = x^2 - 6x + 8$, $b = -6$ and $a = 1$.

By substituting b = -6 and a = 1 into $x = -\frac{b}{2a}$. we get:

$x = -\frac{(-6)}{2(1)}$

$x = 3$

d) Based on the coordinates for the axis of symmetry, find the coordinates of C.

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Compare your answer with:

C has x coordinate equal to the equation of the axis of symmetry; that is $x = 3$.

Find the corresponding y by substituting it $x = 3$ in $y = x^2 - 6x + 8$:

$y = (3)^2 - 6(3) + 8$

$y = 9 - 18 + 8$

$y = -1$

C is (3,-1).

e) Given the reflection of A in the axis of symmetry coincides with E, find the coordinates of E.

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Compare your answer with the following:

A and E have the same y coordinate; that is $\mathbf{y} = 8$.

Find the corresponding y coordinate of E by substituting it $\mathbf{y} = 8$ in $\mathbf{y} = \mathbf{x}^2 - 6\mathbf{x} + 8$:

$8 = \mathbf{x}^2 - 6\mathbf{x} + 8$

Subtract 8 from each side:

$8 - 8 = \mathbf{x}^2 - 6\mathbf{x} + 8 - 8$

$0 = \mathbf{x}^2 - 6\mathbf{x}$

Factorise the quadratic expression on the right:

$0 = \mathbf{x}^2 - 6\mathbf{x}$

$0 = \mathbf{x}(\mathbf{x} - 6)$

By using the zero factor property:

$\mathbf{x} = 0 \ldots \mathbf{or} \ldots \mathbf{x} - 6 = 0$

$\mathbf{x} = 0 \ldots \mathbf{or} \ldots \mathbf{x} = 6$

The y coordinate of E is 6 since 0 is the y coordinate of A, which is on the y-axis.

E is (6,8).

Example 2

Study figure 33 that shows a sketch graph of $\mathbf{y} = 2\mathbf{x} - \frac{1}{2}\mathbf{x}^2$.

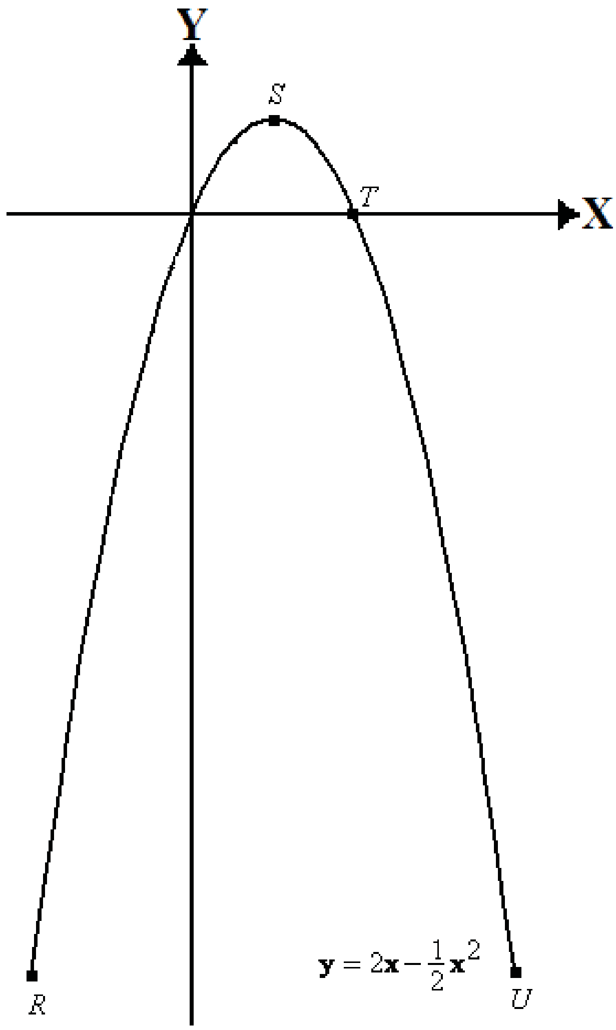

Figure 33 – Sketch graph of $y = 2x - \frac{1}{2}x^2$

a) Given that the point R has coordinates (-4, m), find the value of m.

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Compare your answer with the following:

R is a point on the graph of $y = 2x - \frac{1}{2}x^2$; so m is found by substituting -4 for x in $y = 2x - \frac{1}{2}x^2$:

$$m = 2(-4) - \frac{1}{2}(-4)^2$$

$$m = -8 - 8$$

$$m = -16$$

T is (-4,-16).

b) Find the coordinates of T.

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Compare your answer with the following:

T is a point on the x-axis; so it has y coordinate equal to 0. The x coordinate is one of the x-intercepts:

The x-intercept is found by substituting zero for y in the equation and solving for x:

$$0 = 2x - \frac{1}{2}x^2$$

Factorise the quadratic expression on the right:

$$0 = x(2 - \frac{1}{2}x)$$

By using the zero factor property:

$$x = 0 \ldots \text{or} \ldots 2 - \frac{1}{2}x = 0$$

$$x = 0 \ldots \text{or} \ldots 2 = \frac{1}{2}x$$

$$x = 0 \ldots \text{or} \ldots x = 4$$

$x = 4$ is x coordinate of T, as T is not at the origin:

T is (4,0).

c) Find the equation of the axis of symmetry.

Compare your answer with the following:

The equation of the axis of symmetry is found from:

$$x = -\frac{b}{2a}$$

where a and b are coefficients of $\mathbf{x}^2$ and $\mathbf{x}$ respectively.

For $\mathbf{y} = 2\mathbf{x} - \frac{1}{2}\mathbf{x}^2$, $\mathbf{b} = 2$ and $\mathbf{a} = -\frac{1}{2}$.

By substituting b = 2 and $\mathbf{a} = -\frac{1}{2}$ into $x = -\frac{b}{2a}$, we get:

$$\mathbf{x} = -\frac{(2)}{2\left(-\frac{1}{2}\right)}$$

$$\mathbf{x} = 2$$

d) Find the coordinates of S.

Compare your answer with the following:

S has an x coordinate equal to the equation of the axis of symmetry; that is $\mathbf{x} = 2$.

Find the corresponding y by substituting it $\mathbf{x} = 2$ in $\mathbf{y} = 2\mathbf{x} - \frac{1}{2}\mathbf{x}^2$:

$$\mathbf{y} = 2(2) - \frac{1}{2}(2)^2$$

$y = 4 - 2$

$y = 2$

T is (2,2)

e) Reflection of R in the axis of symmetry coincides with U. Find the coordinates of U.

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Compare your answer with the following:

R and U have the same y coordinate; that is $y = -16$.

To find the corresponding x coordinate of U,

substitute $y = -16$ into $y = 2x - \frac{1}{2}x^2$:

$-16 = 2x - \frac{1}{2}x^2$

Multiply by 2 on both sides:

$-16(2) = 2x(2) - \frac{1}{2}x^2(2)$

$-32 = 4x - x^2$

Subtract $4x$ from each side:

$-32 - 4x = 4x - x^2 - 4x$

$-32 - 4x = -x^2$

Add x^2 to both sides:

$x^2 - 32 - 4x = -x^2 + x^2$

$x^2 - 32 - 4x = 0$

$x^2 - 4x - 32 = 0$

Factorise the quadratic expression on the left:

$(x - 8)(x + 4) = 0$

By using the zero factor property:

$x - 8 = 0 \ldots \textbf{or} \ldots x + 4 = 0$

$\mathbf{x} - 8 + 8 = 0 + 8 \ldots \mathbf{or} \ldots \mathbf{x} + 4 - 4 = 0 - 4$

$\mathbf{x} = 8 \ldots \mathbf{or} \ldots \mathbf{x} = -4$

The x coordinate of U is 8 since -4 is the x coordinate of R.

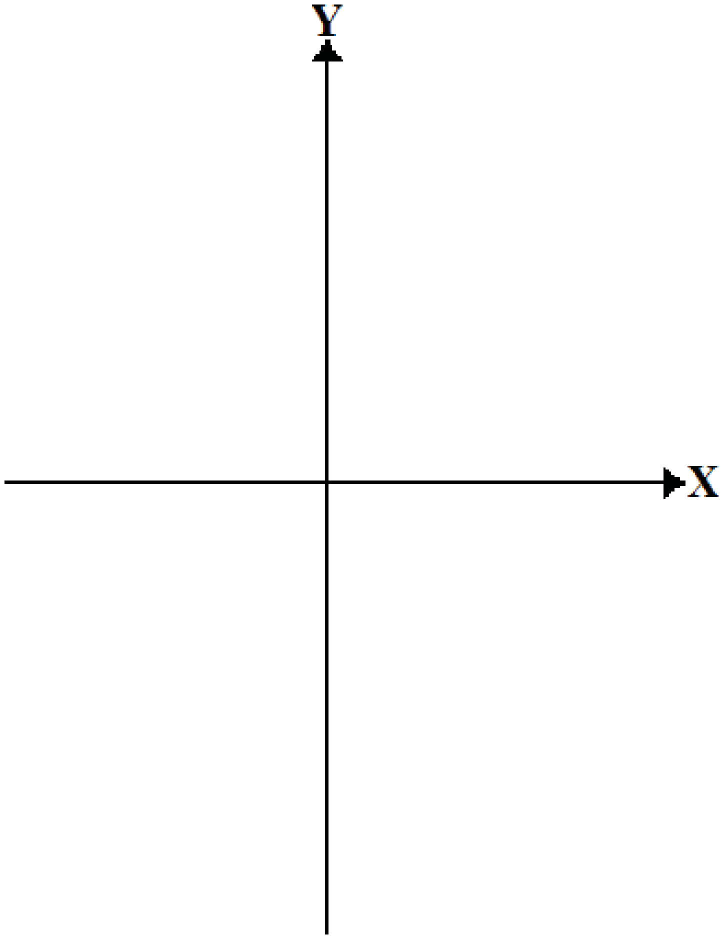
U is (8,-16).

Activity 5

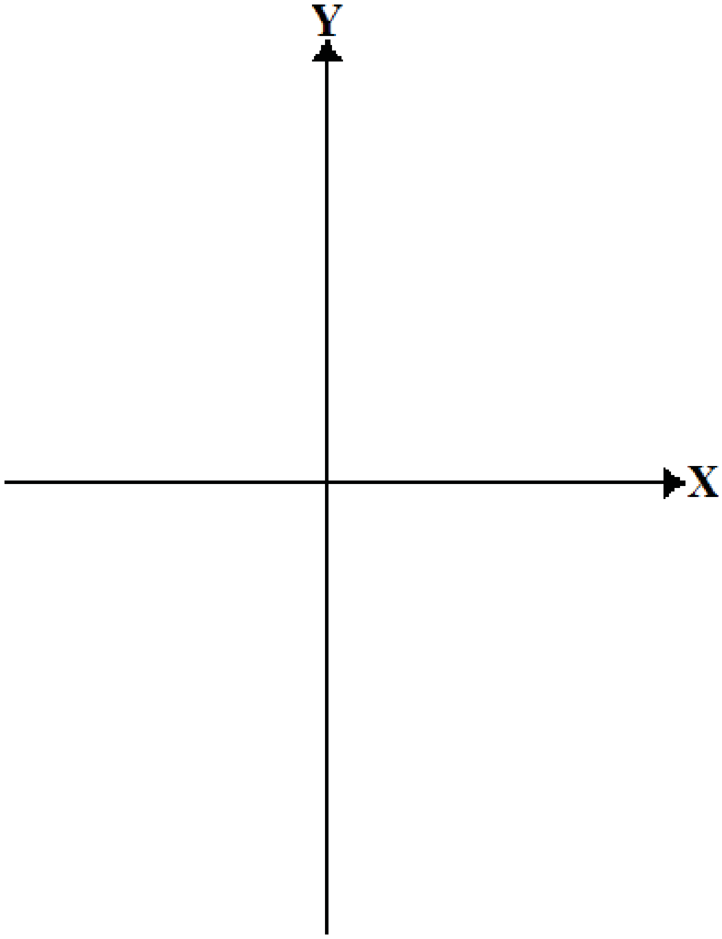
Activity 5

1. Sketch the graphs of the following parabolas.

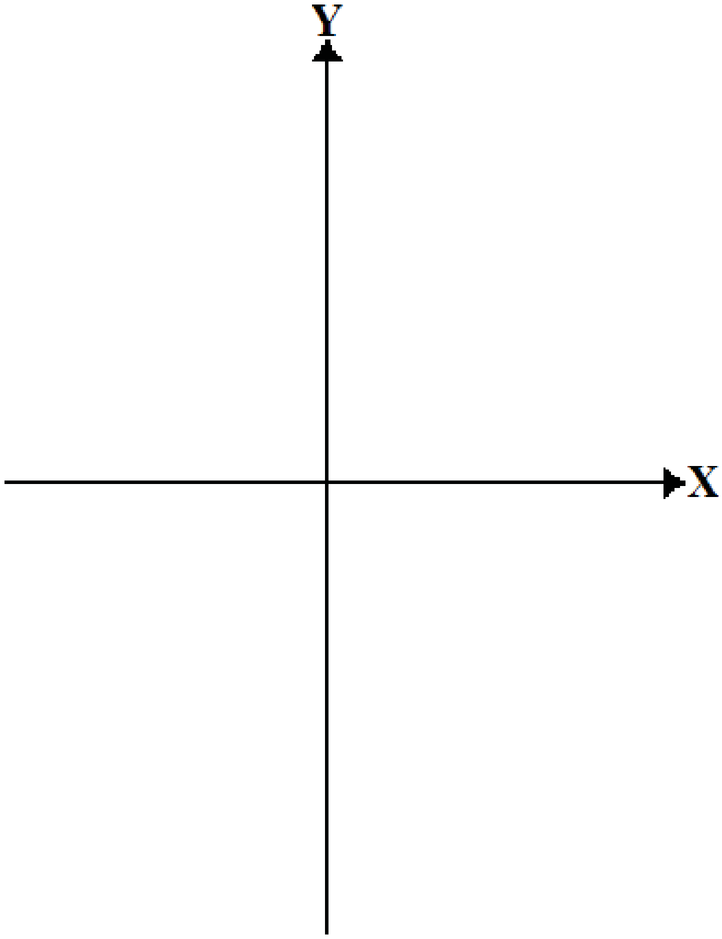
   a) $\mathbf{y} = (\mathbf{x} - 1)(\mathbf{x} - 5)$



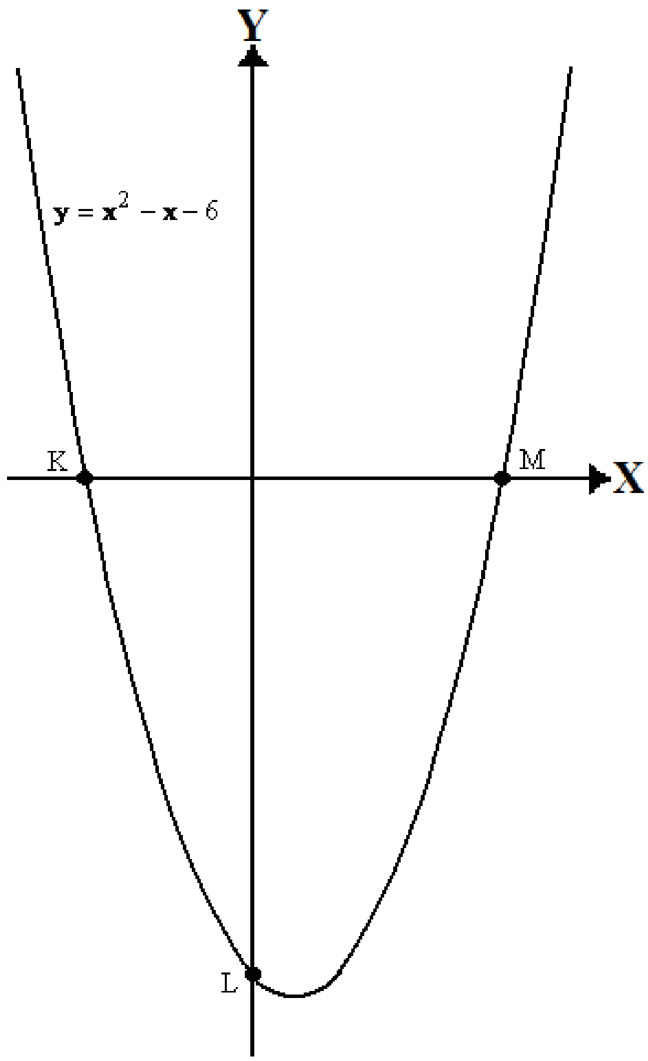
b) $y = x^2 - 4$



c) $y = -x^2 + 3x + 4$



2. Sketch graph of $y = x^2 - x - 6$ is shown in figure 34.

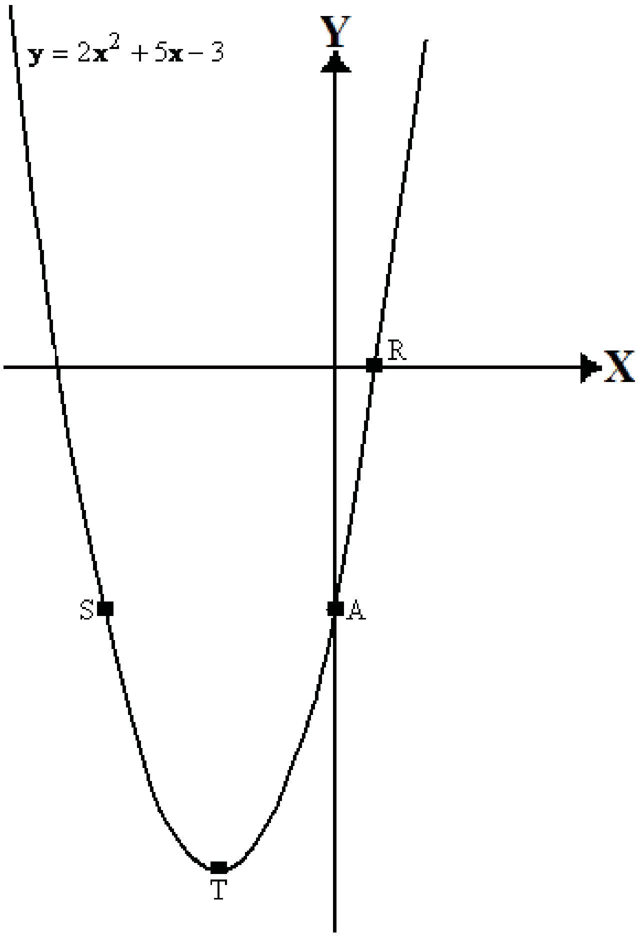


a) Write down the coordinates of point L.

b) Find the coordinates of K and M.

c) Find the equation of line of symmetry of $y = x^2 - x - 6$.

3. Sketch graph of $y = 2x^2 + 5x - 3$ is shown in figure 35.



a) What are the coordinates of A?

b) Given that S is on the horizontal line with A, find the coordinates of S.

c) Find the coordinates of R.

d) Given that T is at the minimum value of y, find the coordinates of T.

Check your performance against the given solutions at the end of this subunit. If you are satisfied with your performance continue to the assignment, after going through the summary. Otherwise review the parts of the subunit which you did not do well.

Note it!

Remember:

When sketching graphs of parabolas, important points in the sketch graph are **critical points** (points at which absolute maximum and absolute minimum exists) and the **intercepts** (y and x- intercepts).

You have now completed the last subunit of this unit on quadratics and other non-linear graphs. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Solutions to ACTIVITY 5:

1. Sketch graphs.

   a) $\mathbf{y} = (\mathbf{x}-1)(\mathbf{x}-5)$

When x =0 is put in $\mathbf{y} = (\mathbf{x}-1)(\mathbf{x}-5)$, y = 5. Then, the parabola intersects y-axis at (0,5).

When substituting y =0 in $\mathbf{y} = (\mathbf{x}-1)(\mathbf{x}-5)$, and solving for x, x = 1 and x =5 are solutions. Therefore, (1,0) and (5,0) are points where the parabola intersects x-axis.

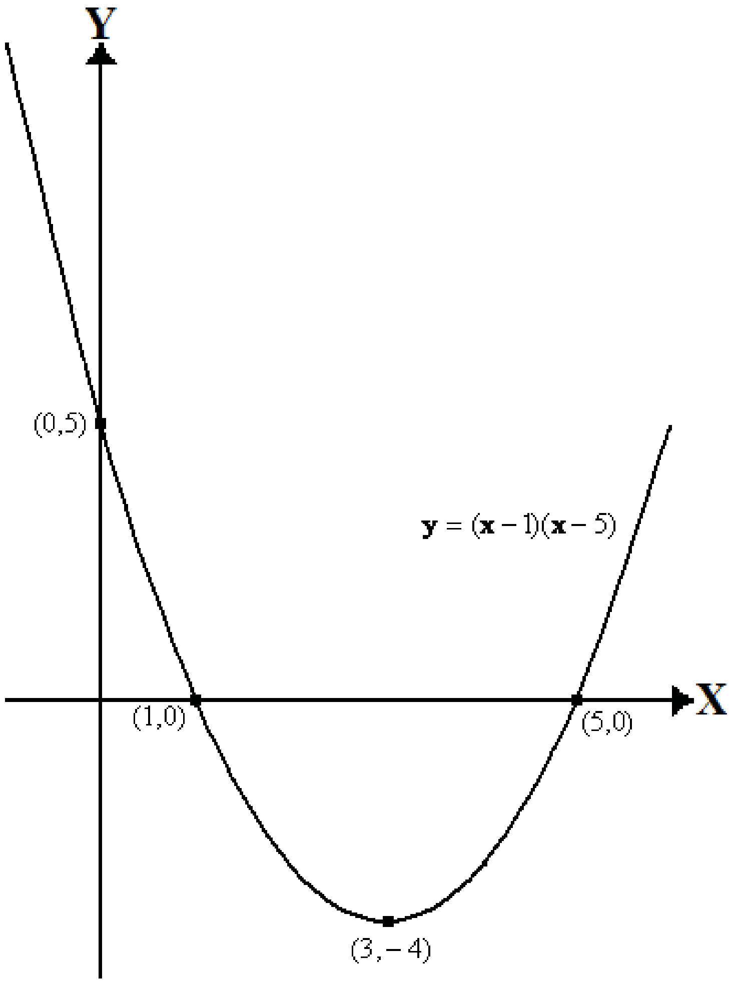
$\mathbf{y} = (\mathbf{x}-1)(\mathbf{x}-5)$ can be written as $\mathbf{y} = \mathbf{x}^2 - 6\mathbf{x} + 5$. Now, a =1, b = -6 and c = 5.

The equation of the line of symmetry is $\mathbf{x} = -\frac{\mathbf{b}}{2\mathbf{a}} = -\frac{(-6)}{(2\times 1)} = \frac{6}{2} = 3$.

Substitute $\mathbf{x} = 3$ in $\mathbf{y} = \mathbf{x}^2 - 6\mathbf{x} + 5$ to get the corresponding $\mathbf{y}$.

$\mathbf{y} = (3)^2 - 6(3) + 5 = 9 - 18 + 5 = -4$.

The point of absolute minimum is $(3, -4)$. Now, the sketch graph:



b) $\mathbf{y} = \mathbf{x}^2 - 4$

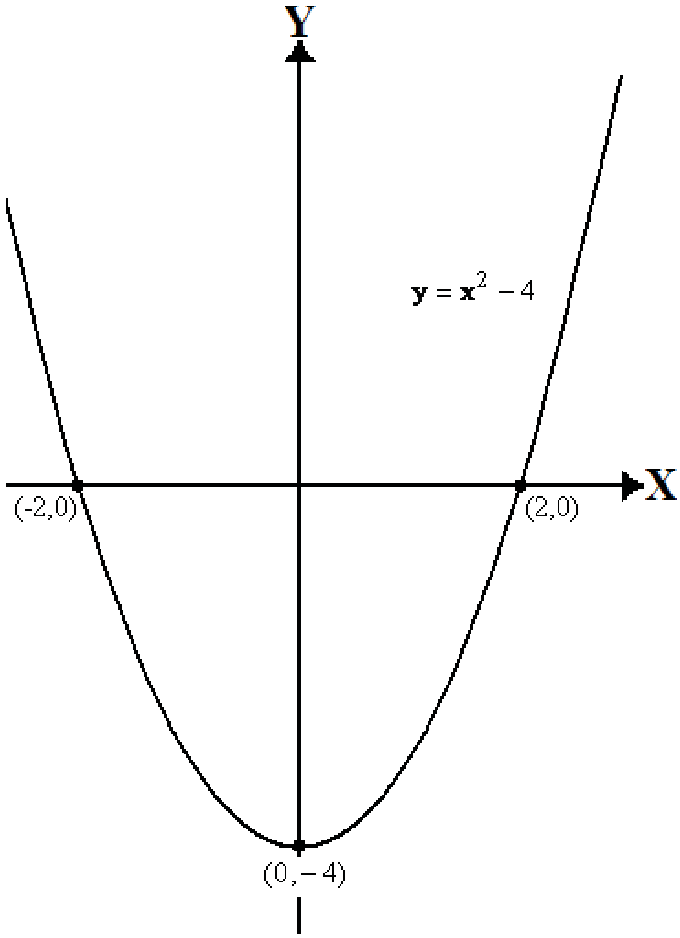
When x =0 is put in $\mathbf{y} = \mathbf{x}^2 - 4$, y = -4. Then, the parabola intersects y-axis at (0,-4).

When substituting y =0 in $\mathbf{y} = \mathbf{x}^2 - 4$, and solving for x in the difference of two squares on the right of $0 = \mathbf{x}^2 - 4$, x = 2 and x = -2 are solutions. Therefore, (2,0) and (-2,0) are points where the parabola intersects x-axis.

Now, from $\mathbf{y} = \mathbf{x}^2 - 4$ a =1, b = 0 and c = -4. The equation of the line of symmetry is $\mathbf{x} = -\frac{\mathbf{b}}{2\mathbf{a}} = -\frac{(0)}{(2 \times 1)} = \frac{0}{2} = 0$. Substitute $\mathbf{x} = 0$ in $\mathbf{y} = \mathbf{x}^2 - 4$ to get the corresponding $\mathbf{y}$.

$\mathbf{y} = (0)^2 - 4 = -4$.

The point of absolute minimum is $(0,-4)$. Now, the sketch graph:



c) $y = -x^2 + 3x + 4$

When putting x =0 in $y = -x^2 + 3x + 4$, y = 4. Then, the parabola intersects y-axis at (0,4).

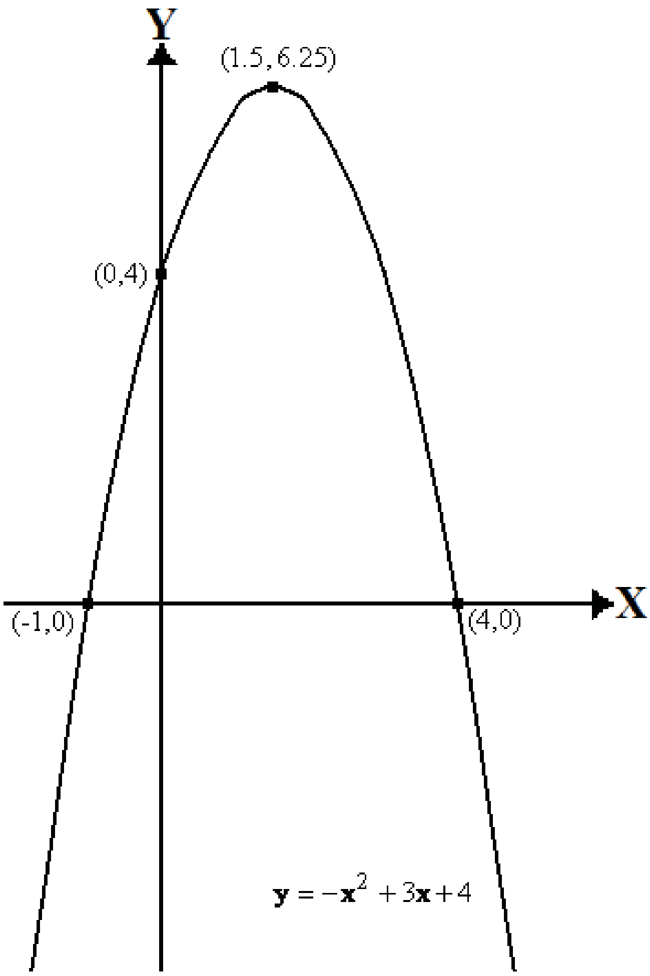
When substituting y =0 in $y = -x^2 + 3x + 4$, and solving for x, x = -1 and x = 4 are solutions. Therefore, (-1,0) and (4,0) are points where the parabola intersects x-axis.

Now, a = -1, b = 3 and c = 4 in $\mathbf{y}=-\mathbf{x}^2+3\mathbf{x}+4$. The equation of the line of symmetry is $\mathbf{x}=-\frac{\mathbf{b}}{2\mathbf{a}}=-\frac{(3)}{(2\times -1)}=\frac{3}{2}=1.5$. Substitute $\mathbf{x}=1.5$ in

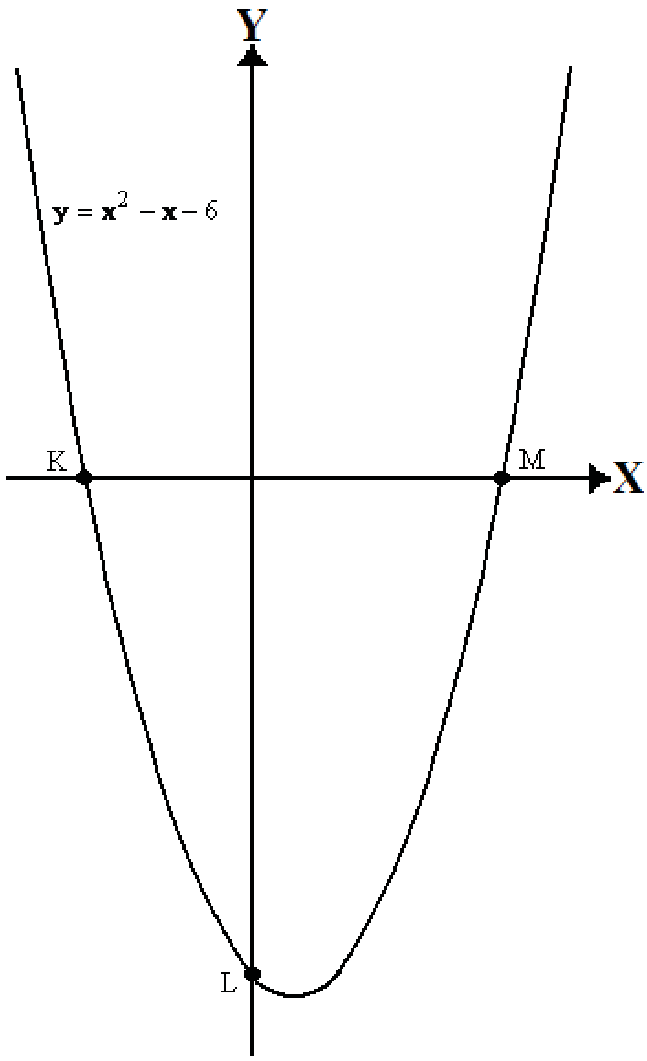
$\mathbf{y}=-\mathbf{x}^2+3\mathbf{x}+4$ to get the corresponding $\mathbf{y}$.

$\mathbf{y}=-(1.5)^2+3(1.5)+4=-2.25+4.5+4=6.25$.

The point of absolute maximum is $(1.5, 6.25)$. Now, the sketch graph:



2. Sketch graph of $\mathbf{y}=\mathbf{x}^2-\mathbf{x}-6$.



a) Coordinates of point L.

The x coordinate of L is 0, as it is on the y-axis. Substituting 0 in $y = x^2 - x - 6$ results in -6 (this is constant term in $y = x^2 - x - 6$).

L(0,-6).

b) Coordinates of K and M.

As K and M are on the x-axis, their y coordinates are equal to 0. Substituting 0 in $y = x^2 - x - 6$ and solving for x, -2 and 3 are the solutions. From the sketch graph, -2 is the x coordinate for K, as it is on the left side of the origin.

K(-2,0) and M(3,0).

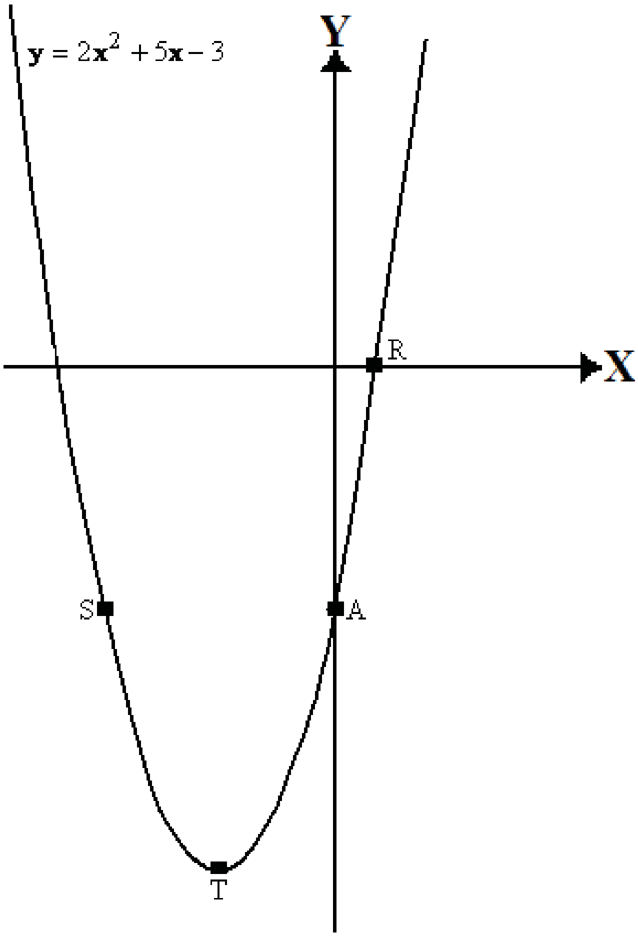
c) Line of symmetry of $y = x^2 - x - 6$.

Formula for the line of symmetry of a parabola is $\mathbf{x} = -\frac{\mathbf{b}}{2\mathbf{a}}$. In $\mathbf{y} = \mathbf{x}^2 - \mathbf{x} - 6$, a = 1, b = -1and c = -6.

So, $\mathbf{x} = -\frac{\mathbf{b}}{2\mathbf{a}} = -\frac{(-1)}{(2 \times 1)} = \frac{1}{2}$.

The line of symmetry is $\mathbf{x} = \frac{1}{2}$.

3. Sketch graph of $\mathbf{y} = 2\mathbf{x}^2 + 5\mathbf{x} - 3$.



a) Coordinates of A.

The x coordinate of A is 0, as it is on the y-axis. Substituting 0 in $\mathbf{y} = 2\mathbf{x}^2 + 5\mathbf{x} - 3$ results in -3 (this is constant term in $\mathbf{y} = 2\mathbf{x}^2 + 5\mathbf{x} - 3$).

A(0,-3).

b) Coordinates of S.

The y coordinate of S is also -3, as it is on the horizontal line as A.

Substitute -3 for y in $\mathbf{y} = 2\mathbf{x}^2 + 5\mathbf{x} - 3$ to get x coordinate:

$$-3 = 2\mathbf{x}^2 + 5\mathbf{x} - 3$$

$$-3 + 3 = 2\mathbf{x}^2 + 5\mathbf{x} - 3 + 3$$

$$0 = 2\mathbf{x}^2 + 5\mathbf{x}$$

$$0 = \mathbf{x}(2\mathbf{x} + 5)$$

$$\mathbf{x} = 0 \;\; or \;\; 2\mathbf{x} + 5 = 0$$

Solving $2\mathbf{x} + 5 = 0$, gives the x coordinate of S.

$$2\mathbf{x} + 5 - 5 = 0 - 5$$

$$2\mathbf{x} = -5$$

$$\frac{2\mathbf{x}}{2} = \frac{-5}{2}$$

$$\mathbf{x} = -2.5$$

Therefore, S (-3,-2.5).

c) Coordinates of R.

The y coordinate of R is 0, as it is on the x-axis.

Substitute 0 for y in $\mathbf{y} = 2\mathbf{x}^2 + 5\mathbf{x} - 3$ to get x coordinate:

$$0 = 2\mathbf{x}^2 + 5\mathbf{x} - 3$$

$$0 = (2\mathbf{x} - 1)(\mathbf{x} + 3)$$

$$2\mathbf{x} - 1 = 0 \;\; or \;\; \mathbf{x} + 3 = 0$$

0.5 and -3 are the solutions. The x coordinate of R is positive, and therefore, R(0.5,0).

d) Coordinates of T.

In $\mathbf{y} = 2\mathbf{x}^2 + 5\mathbf{x} - 3$, a = 2, b = 5 and c = -3; the x coordinate of T is equal to $\left(-\frac{\mathbf{b}}{2\mathbf{a}}\right)$, as it is the same equation of line of symmetry:

$$\left(-\frac{\mathbf{b}}{2\mathbf{a}}\right) = -\frac{(5)}{(2 \times 2)} = -\frac{5}{4} = -1.25.$$

Substitute -1.25 for x in $\mathbf{y} = 2\mathbf{x}^2 + 5\mathbf{x} - 3$ to get y coordinate:

$$\mathbf{y} = 2(-1.25)^2 + 5(-1.25) - 3 = 3.125 - 6.25 - 3 = -6.125$$

The coordinates of T are (-1.25,-6.125).

Unit Summary

Summary

In this unit you learned to complete tables of variables, with the help of the equations of those variables.

From a completed table, the graph can be drawn after plotting the points based on data in the completed table.

The drawn graphs can be used:

1. to find other corresponding values of x and y which satisfy the equation of the graph. They will be estimates based on how well the graph is drawn.
2. to solve some equations and inequalities. The important thing in this is to know that those equations and inequalities will always be related to the equation of your graph.

When you sketch graphs of parabolas are drawn, important points like **critical points** and the **intercepts**, are shown. Critical points include a point at which absolute maximum or absolute minimum exists. Intercepts are y and x- intercepts.

You have completed the material for this unit on quadratics and other non-linear graphs. You should now spend some time reviewing the content. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

When you work on this assignment, please observe the time allocated and show your work or reason for each answer.

TOTAL MARKS: 30

TIME: 40 minutes

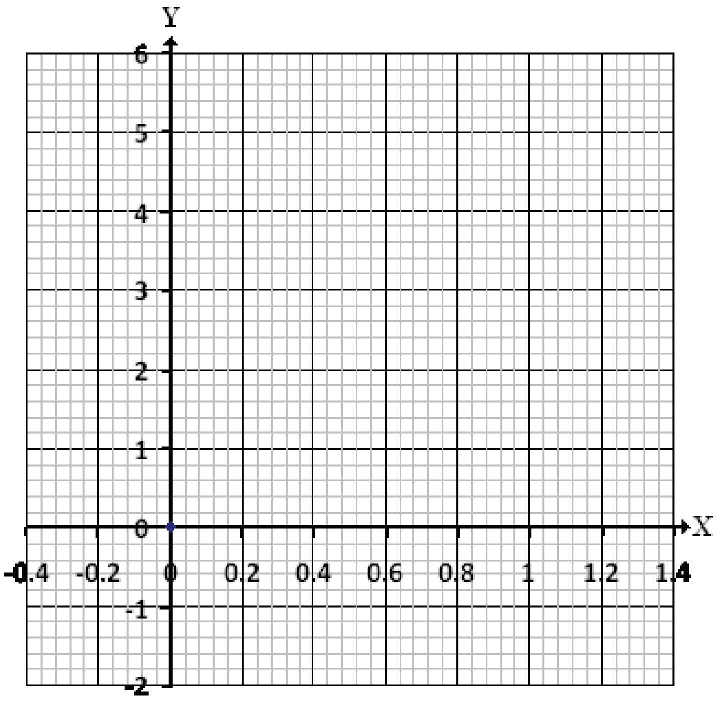
1. The height of a box, y in metres, falling freely from 5 metres above the ground is related to the time, x in seconds, of fall by $y = 5 - 5x^2$.

   i. Calculating y to 1decimal place, complete the table below for the values of x from 0 to 1.

x	0.0	0.2	0.4	0.6	0.8	1.0
y						

(2 marks)

   ii. Draw the graph of height against time for $y = 5 - 5x^2$,using the information from your completed table.



(3 marks)

iii. Use your graph to find the time when the box is half-way in its fall.

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(2 marks)

iv. According to your graph, how long does it take the box to hit the ground?

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(2 marks)

v. Use your graph to solve the equation $x = 5 - 5x^2$.

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(5 marks)

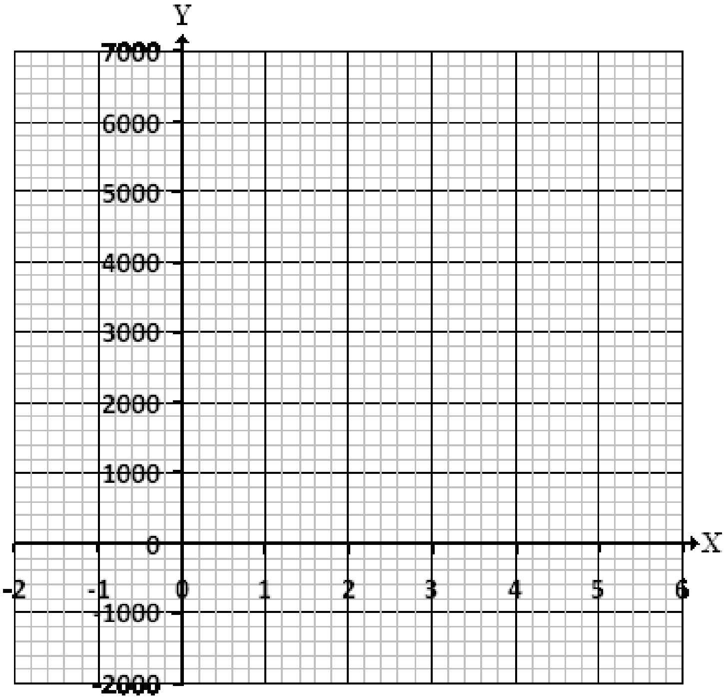
2. The movement energy, y in joules, of a car accelerating constantly is connected to its velocity, x in metres per second, by the equation $y = 250x^2$.

   i. Form a table for integer values of x from 0 to 5 and complete it by finding corresponding values of movement energy, which is y.

x						
y						

(2 marks)

ii. Draw the graph of $y = 250x^2$, using the information from your completed table.



(3 marks)

iii. What is the movement energy of the car, when it is travelling at 2.5 metres per second, from your graph?

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(2 marks)

iv. What is the approximate velocity of the car, when its movement energy is 3500 joules?

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(2 marks)

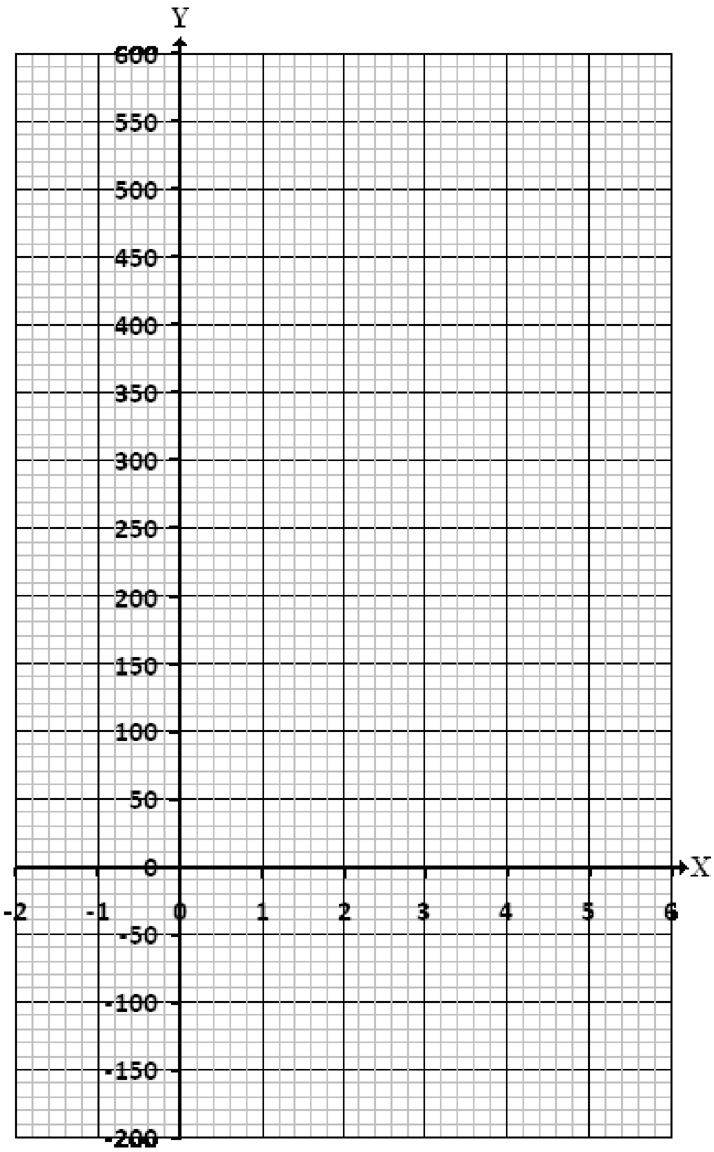
3. The mass, y in grams, of a cube of glass is related to the side, x in centimetres, of one of the cube faces by the equation $y = 2.5x^3$.

   i. Some values of y and x are shown in the table below. Complete the table for the values of y and x.

x	1	2	3	4	5	6
y	2.5		67.5			

(2 marks)

   ii. Draw the graph of y against x, with the data from the table.



(3 marks)

iii. Use your graph to find the side x of a glass cube, when the mass is 80 grams.

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(2 marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

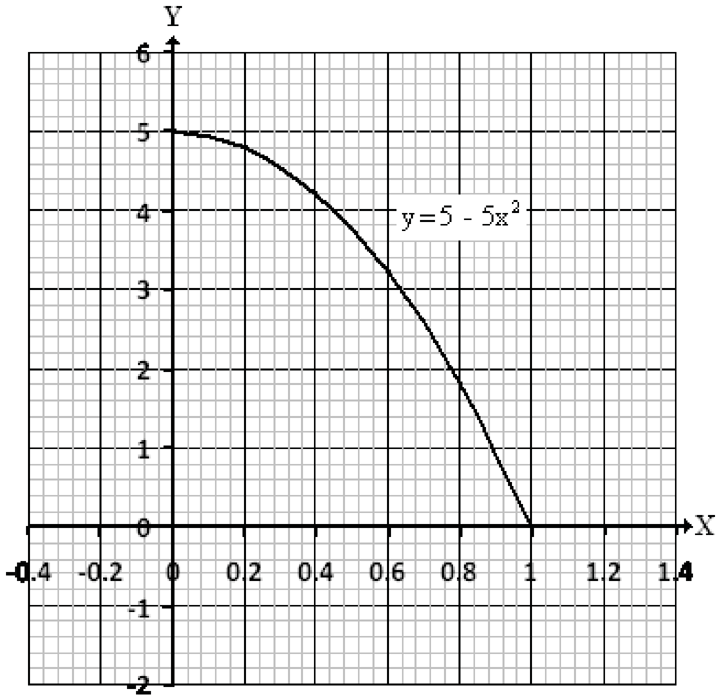
Solutions to the ASSIGNMENT:

1. The height of a box, y in metres, falling freely from 5 metres above the ground is related to the time, x in seconds, of fall by $y = 5 - 5x^2$.

   i. Completed table for the values of x from 0 to 1.

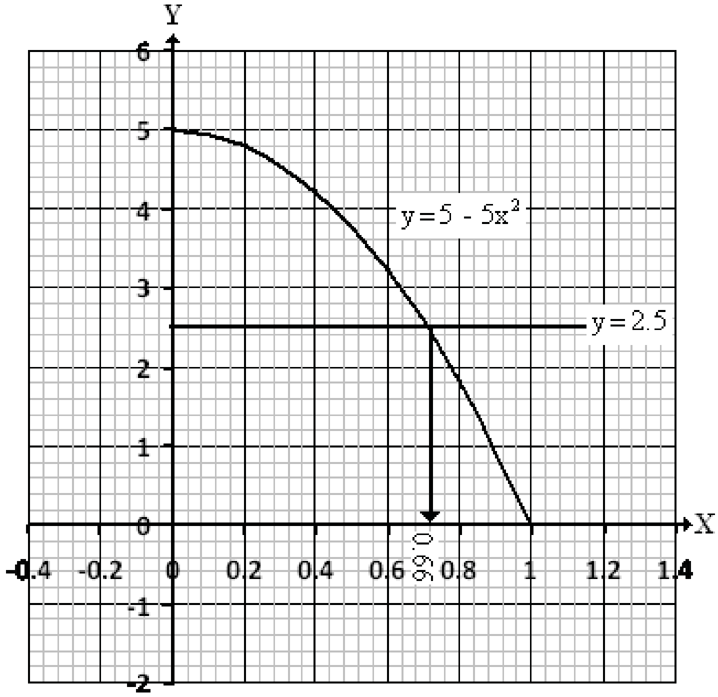
x	0.0	0.2	0.4	0.6	0.8	1.0
y	5.0	4.8	4.2	3.2	1.8	0.0

   ii. Graph of height against time for $y = 5 - 5x^2$,using the information from the completed table.



iii. Finding the time when the box is half-way in its fall:

Draw $y = 2.5$ and read the value of x at point of intersection with $y = 5 - 5x^2$.

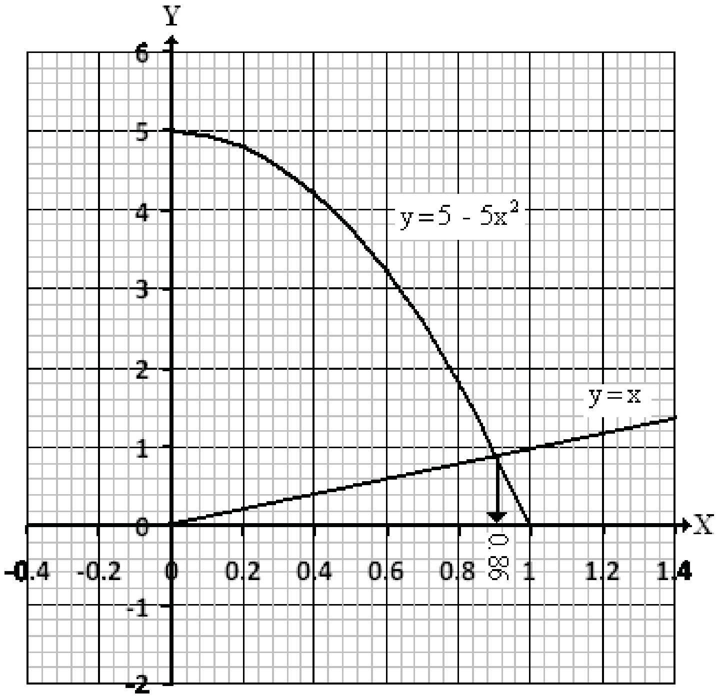


The time is approximately 0.66 seconds.

iv. The box hits the ground when y = 0. The line y = 0 intersects $y = 5 - 5x^2$ at the point where x =1. So, the box hits the ground in 1second.

v. Solving the equation $x = 5 - 5x^2$.

Draw $y = x$ and read the value of x at point of intersection with $y = 5 - 5x^2$.



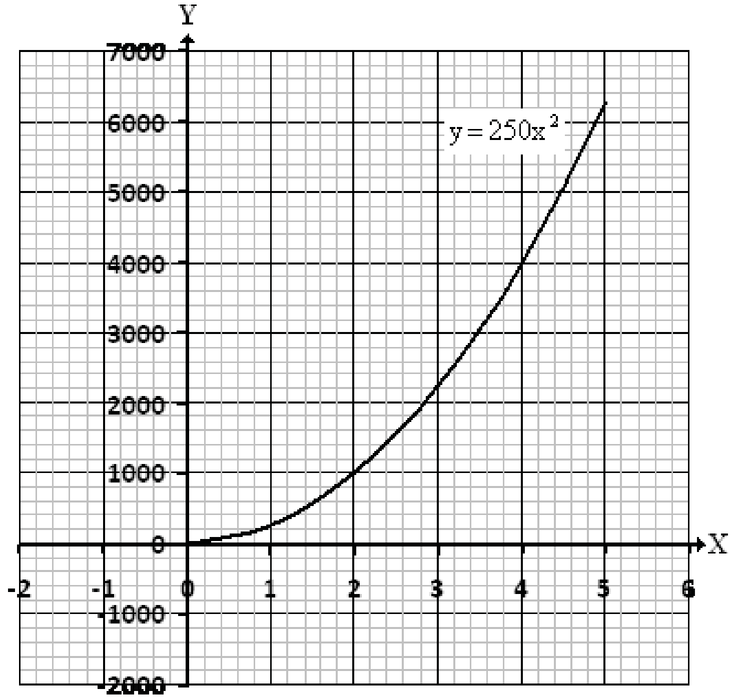
As seen from the diagram, x is approximately 0.86.

2. The movement energy, y in joules, of a car accelerating constantly is connected to its velocity, x in metres per second, by the equation $y = 250x^2$.

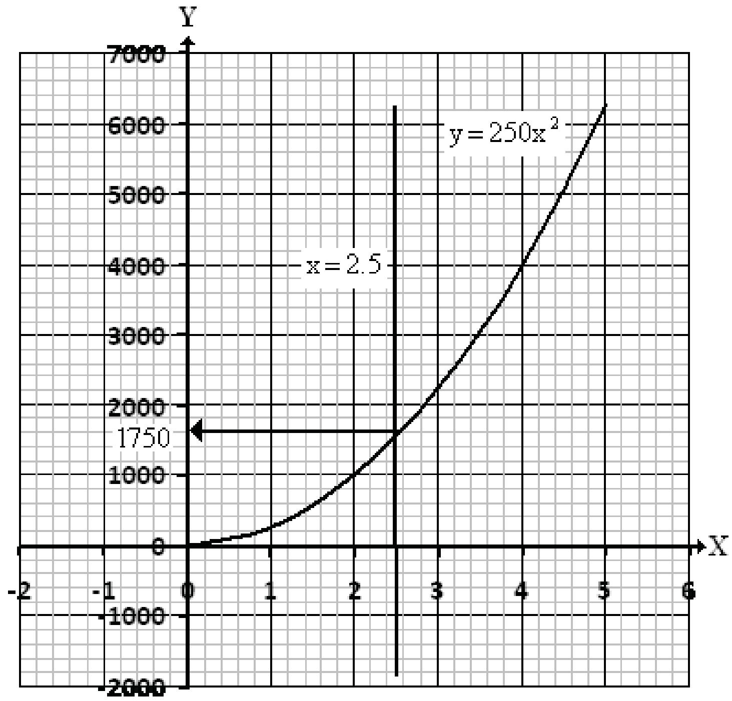
   i. The table for integer values of x from 0 to 5.

x	0	1	2	3	4	5
y	0	250	1000	2250	4000	6250

   ii. The graph of $y = 250x^2$, using the information from the completed table.



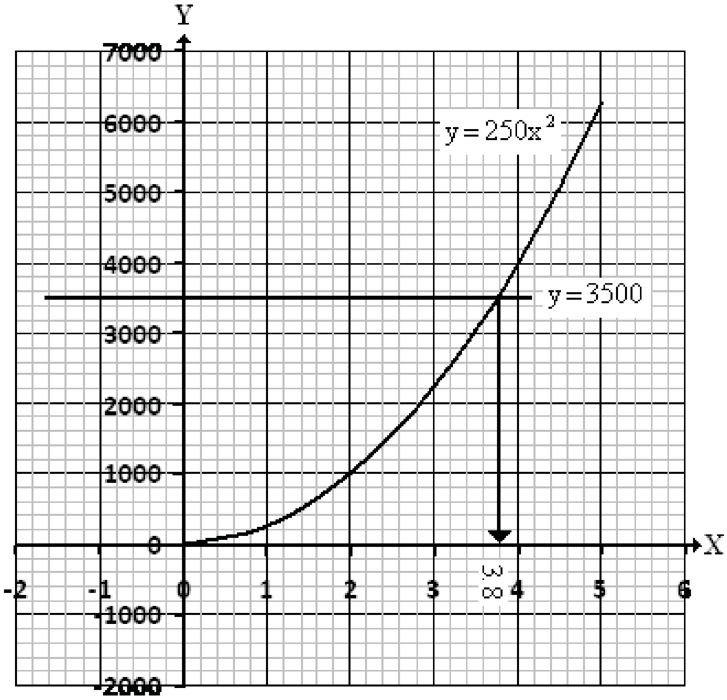
iii. Draw $x = 2.5$ and read the value of y at point of intersection with $y = 250x^2$.



It is approximately 1750 joules.

iv. The approximate velocity of the car, when its movement energy is 3500 joules:

Draw $y = 3500$ and read the value of x at point of intersection with $y = 250x^2$.



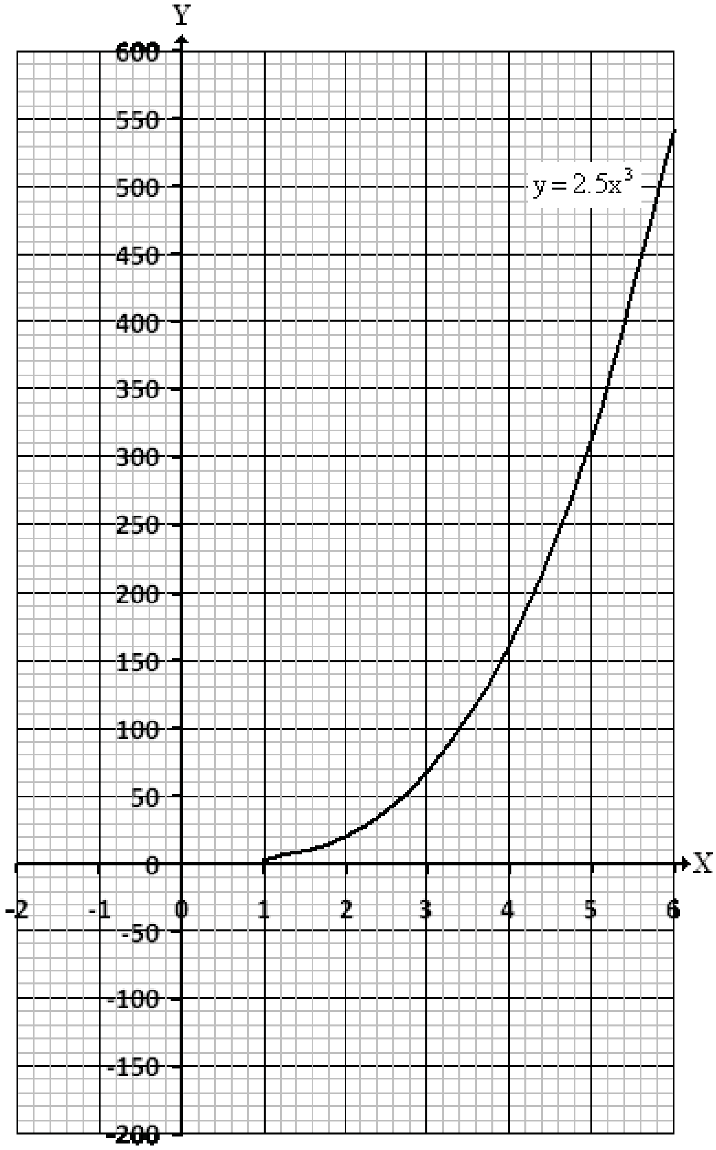
The velocity is approximately 3.8 metres per second.

3. The mass, y in grams, of a cube of glass is related to the side, x in centimetres, of one of the cube faces by the equation $y = 2.5x^3$.

i. The values of y and x are shown in the table below.

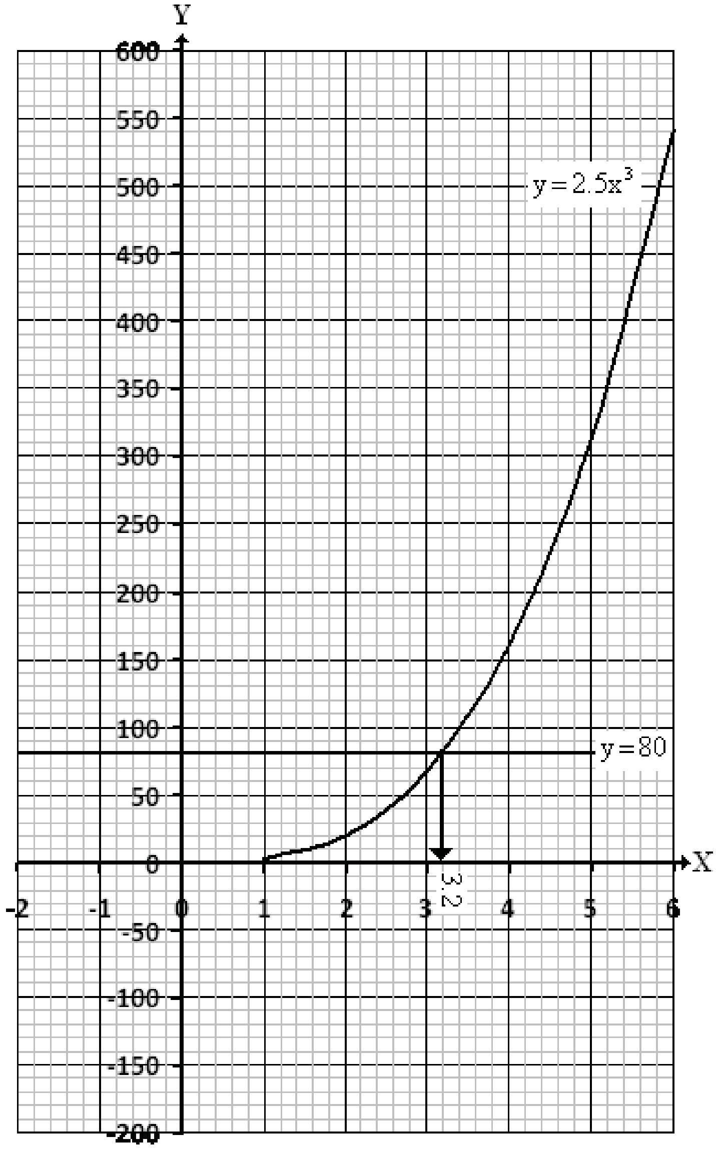
x	1	2	3	4	5	6
y	2.5	20	67.5	160	312.5	540

ii. The graph of y against x.



iii. Finding the side x of a glass cube, when the mass is 80 grams:

Draw $y = 80$ and read the value of x at point of intersection with $y = 2.5x^3$.



When the mass is 80 grams, the side x of a glass cube is 3.2 centimetres.

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

\

Assessment

Assessment

When you work on this assessment, please observe the time allocated and show your work or reason for each answer.

TOTAL MARKS: 25

TIME: 30 minutes

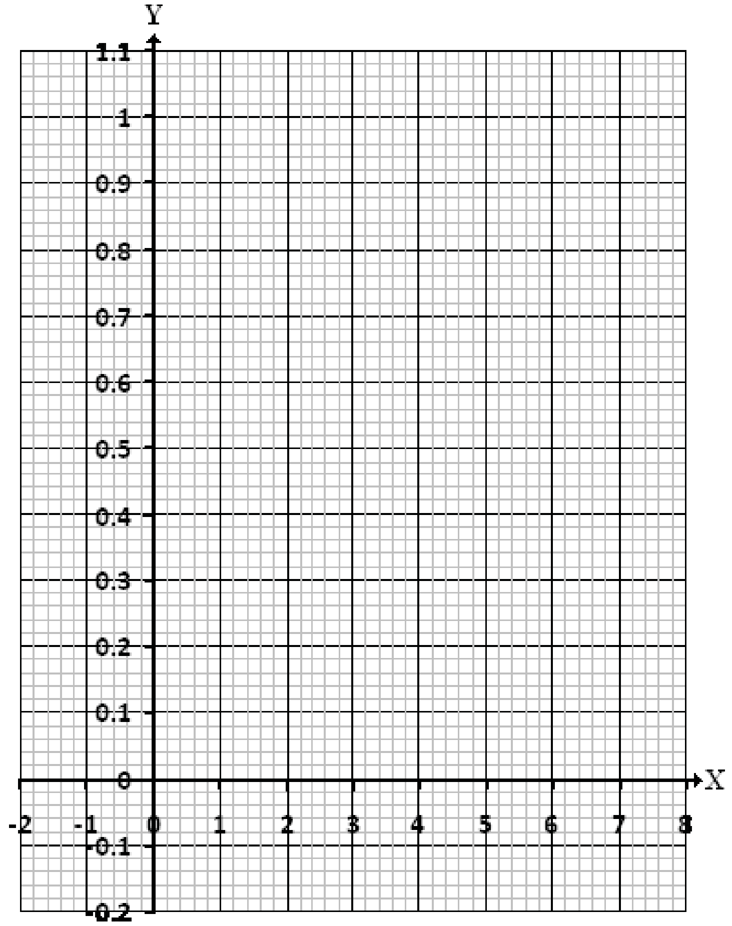
1. When cuboids of the same weight and with square cross sections are placed on the table, the pressure, y in pascals, exerted by each box is related to the side, x in metres, of the square face by $y = \frac{4}{x^2}$.

   i. Complete the table, writing the non- integer values of y to two decimal places, for the values of y and x below.

x	2	3	4	5	6	7
y	1		0.25			

(2 marks)

   ii. Draw the graph of $y = \frac{4}{x^2}$ with the data from the table.



(3 marks)

iii. Use your graph to estimate the pressure exerted by a box of a square face with the side 4.5 metres.

(2 marks)

iv. Use your graph to solve the equation $\frac{1}{10}x = \frac{4}{x^2}$.

(6 marks)

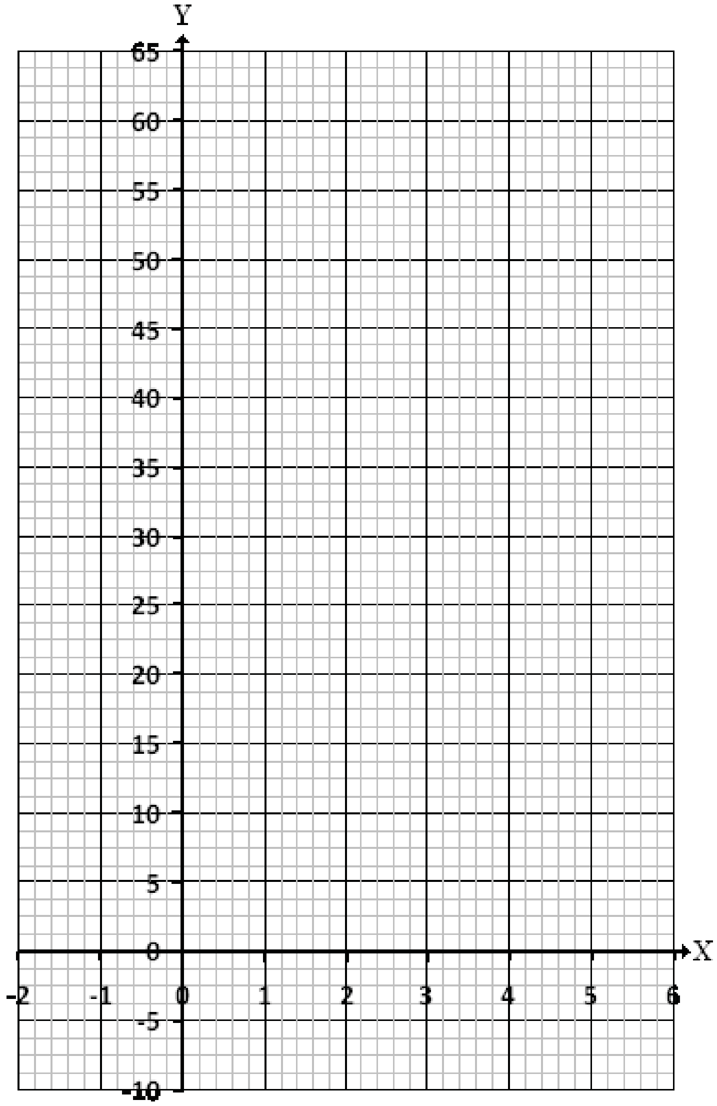
2. A simple interest, y in Maloti, on depositing M100.00 in the bank for x years, where the rate of yearly interest is 10% is given by the equation $\mathbf{y} = 100(1.1^{x} - 1)$.

   i. Complete the table below .

x	0	1	2	3	4	5
y			21		46.41	

(2 marks)

   ii. Draw the graph of $\mathbf{y} = 100(1.1^{x} - 1)$, using the information from your completed table.



(4 marks)

iii. Use your graph to find the interest earned after 4.5 years.

____

____

____

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(2 marks)

iv. According to your graph, how long does it take to accumulate M25, in interest?

(2 marks)

v. Solve the equation $30 = 100(1.1^x - 1)$ using your graph.

(2 marks)

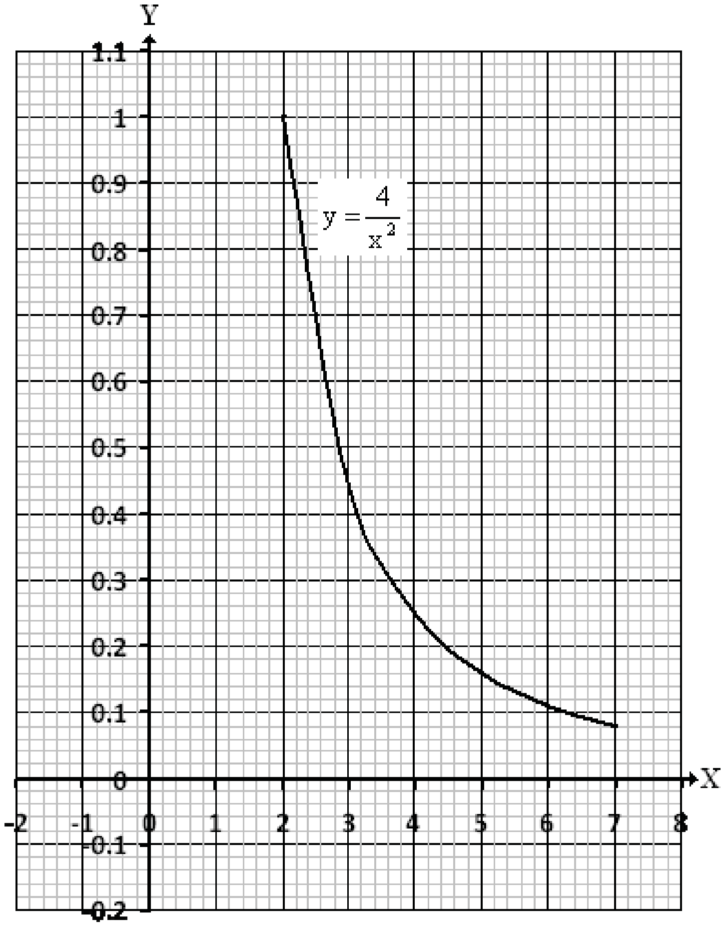
Solutions to the ASSESSMENT:

1. When cuboids of the same weight and with square cross sections are placed on the table, the pressure, y in pascals, exerted by each box is related to the side, x in metres, of the square face by $y = \frac{4}{x^2}$.

   i. The completed table.

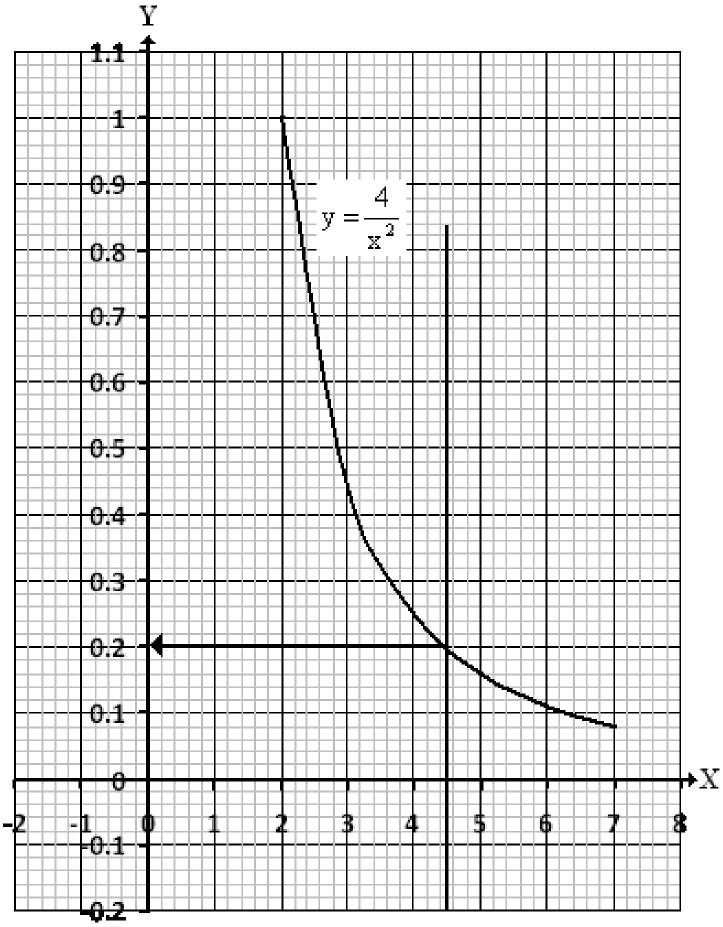
x	2	3	4	5	6	7
y	1	0.44	0.25	0.16	0.11	0.08

   ii. The graph of $y = \frac{4}{x^2}$.



iii. The pressure exerted by a box of a square face with the side 4.5 metres:

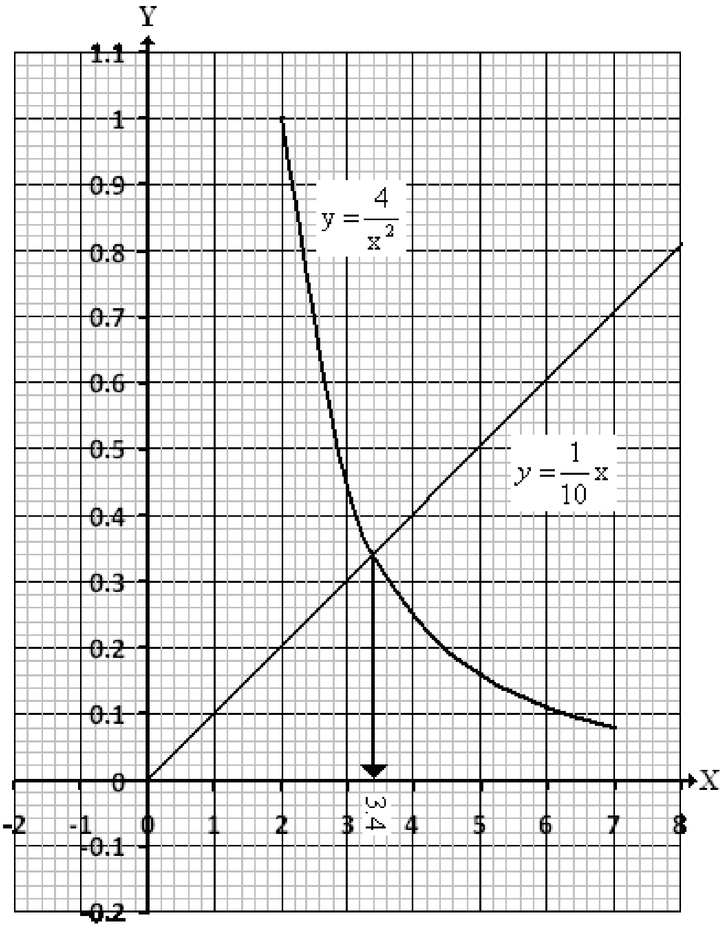
Draw $x = 4.5$ and read the value of y at point of intersection with $y = \frac{4}{x^2}$.



It is 0.2 pascals.

iv. Solving the equation $\frac{1}{10}x = \frac{4}{x^2}$.

Draw $y = \frac{1}{10}x$ and read the value of x at point of intersection with $y = \frac{4}{x^2}$.



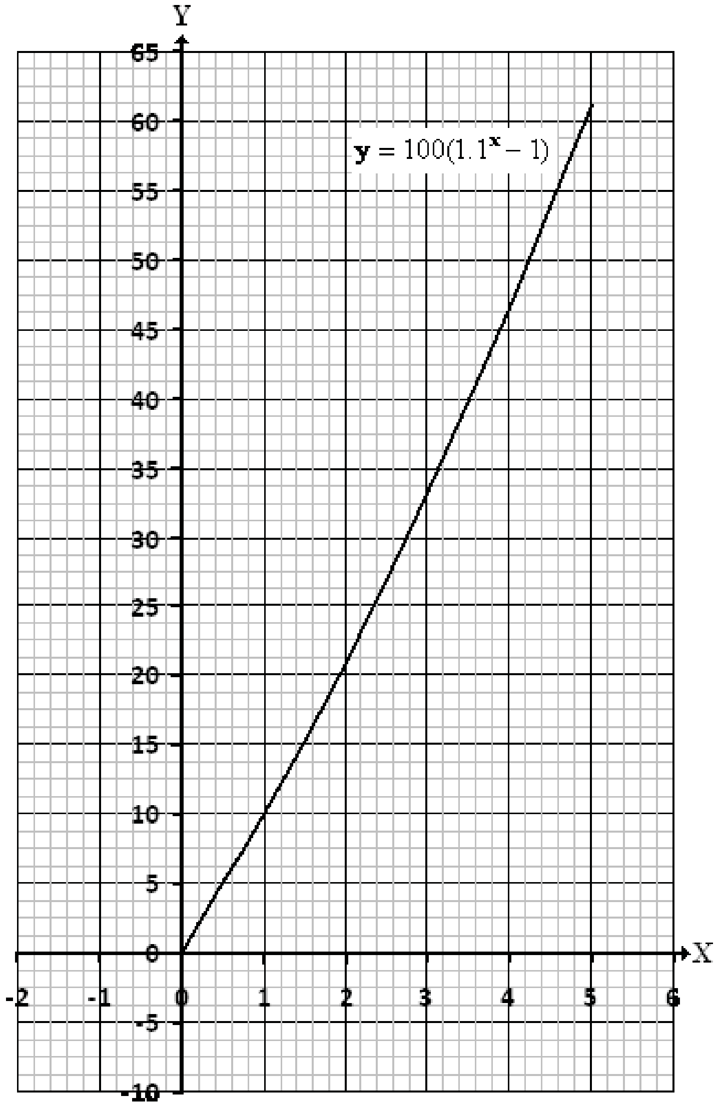
From the diagram, x is approximately 3.4.

2. A simple interest, y in Maloti, on depositing M100.00 in the bank for x years, where the rate of yearly interest is 10% is given by the equation $\mathbf{y} = 100(1.1^x - 1)$.

   i. The complete table.

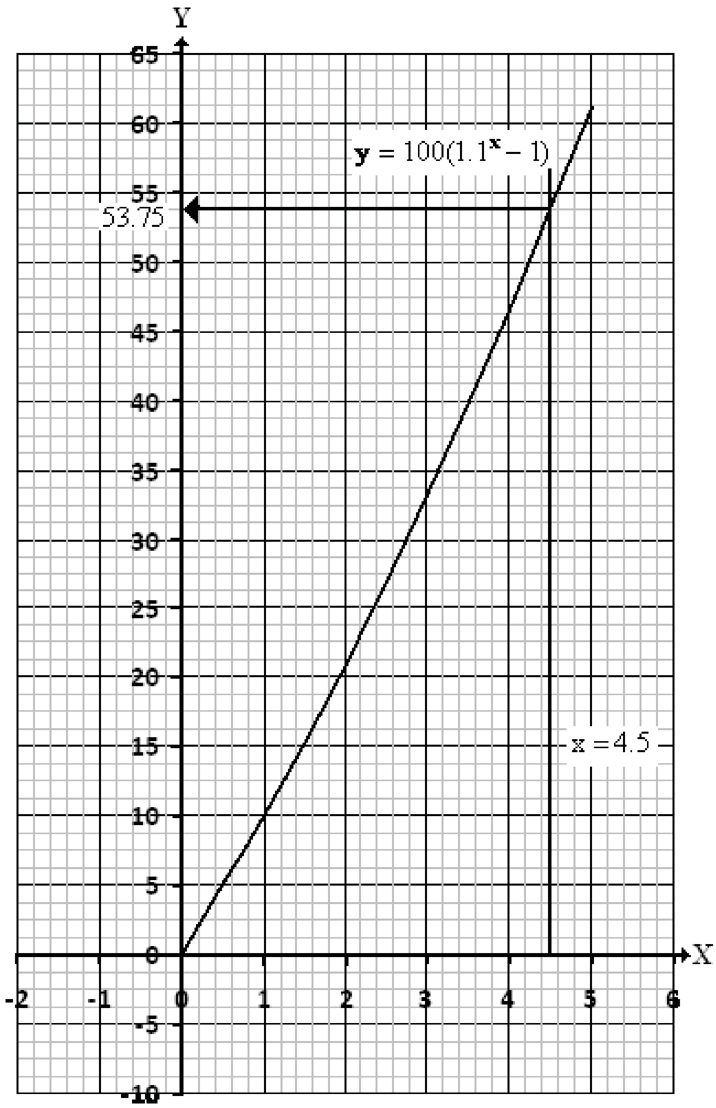
x	0	1	2	3	4	5
y	0	10	21	33.10	46.41	61.05

   ii. The graph of $\mathbf{y} = 100(1.1^x - 1)$, using the information from the completed table.



iii. The interest earned after 4.5 years:

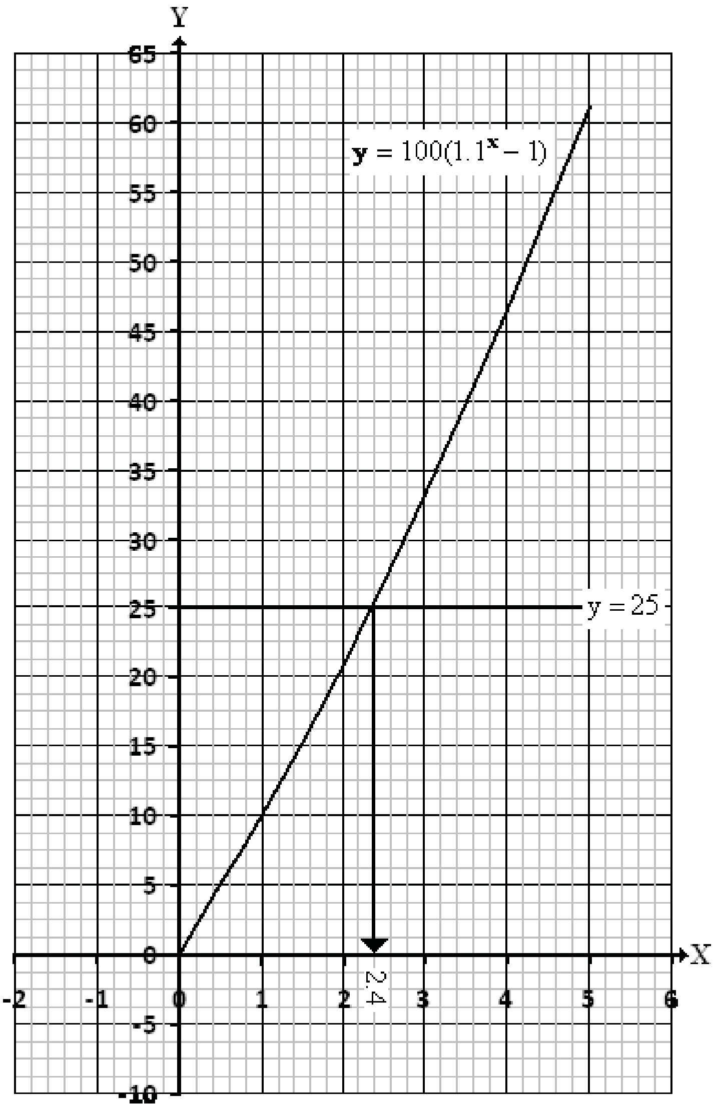
Draw $x = 4.5$ and read the value of y at point of intersection with $y = 100(1.1^x - 1)$.



The interest earned after 4.5 years is M53.75.

iv. The time it takes to accumulate M25, in interest:

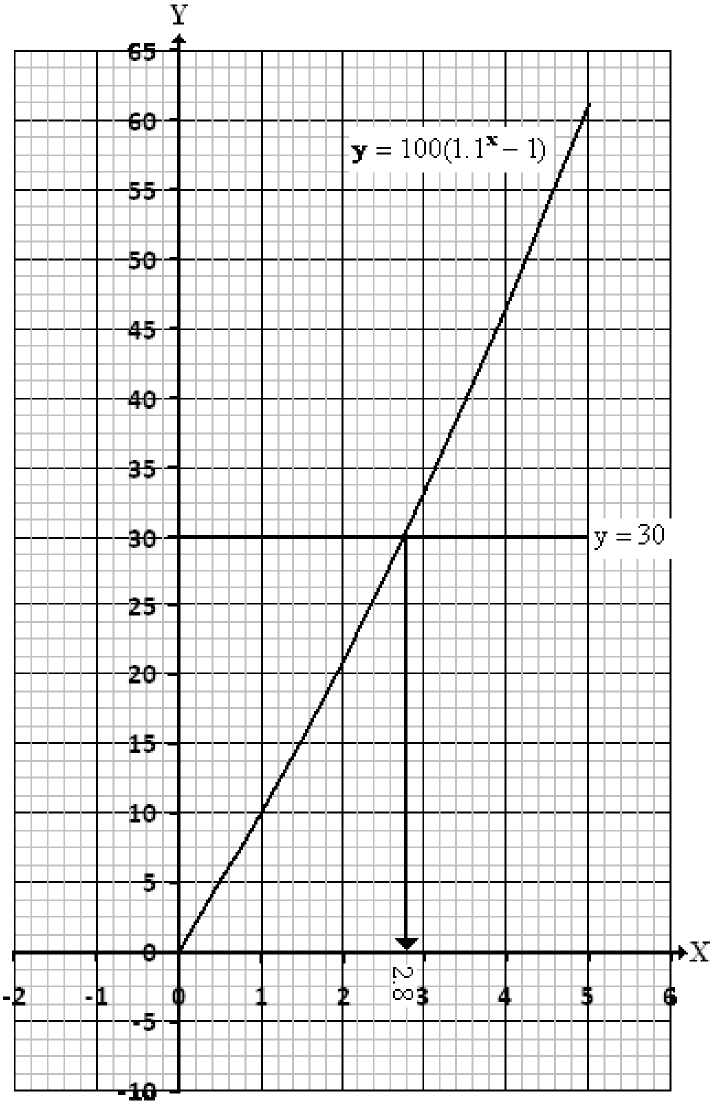
Draw $y = 25$ and read the value of x at point of intersection with $\mathbf{y} = 100(1.1^x - 1)$.



It takes 2.4 years to accumulate M25, in interest.

v. Solving the equation $30 = 100(1.1^x - 1)$ using your graph:

Draw $y = 30$ and read the value of x at point of intersection with $\mathbf{y} = 100(1.1^x - 1)$.



The value of x is 2.8, which means it takes 2.8 years to accumulate M30, in interest.

Unit Contents

Unit 22

Unit 22

Statistics

Introduction

Statistics is the branch in mathematics that is concerned with the collection, interpretation and analysis of data.

Governments need to know a lot about their countries in order to plan ahead. They need to know the rate of HIV/ AIDS infections in order to strategize on how to combat the pandemic. They need to know the population growth rate in order to determine how many schools, hospitals, health centres, etc they need to train.

A government also has to plan how it will spend the monies it receives. They need to know how much money they are going to collect, say, from the sale of the country's product to other countries and how much money they will collect from income tax paid by its citizens.

For all these needs, data has to be collected. One way to collect data is by survey people. In your junior mathematics study, you learned some ways of displaying statistical data/information. This unit is going to widen the knowledge of statistics you have gained so far.

This unit consists of 96 pages and it is 4% of the whole course. As reference, you will need to devote 30 hours to work on this unit, 20 hours for formal study and 10 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Four Lessons:

Lesson 1 Describing Sets Using Set Builder Notation and Listing
Lesson 2 Expressing Relationships in Sets Using Set Notation
Lesson 3 Interpreting Information That Is Represented with Venn Diagrams
Lesson 4 Drawing Venn Diagrams in Problem Solving

Upon completion of this unit you will be able to:

utcomes

- *Construct* bar charts, pie charts, pictograms, simple frequency distributions and frequency polygons to represent statistical data.
- *Calculate* the mean, median and mode for individual data and distinguish between the purposes for which they are used.
- *Interpret and draw* simple inferences from tables and statistical diagrams.
- *Construct* cumulative frequency diagrams.
- *Estimate* medians, percentiles, quartiles and inter-quartile ranges from cumulative frequency diagrams.
- *Calculate* the mean and median for grouped data.
- *Identify* the modal class from grouped frequency distribution.
- *Construct histograms* with equal and unequal intervals using frequency density.

minology

Statistics: Involves the collection, display and analysis of information.

Data: Is the complete set of individual pieces of information which is being used in any of the processes connected with statistics.

Frequency: The number of times each piece of data is found.

Class interval: Is the width of a class as measured by the difference between the class limits.

Mode: The most frequently appearing piece of data.

Median: Is the numerical value of the piece of data in the middle of the set after arranging the set in order of size.

Mean: Is the numerical value found by adding together all the separate values of data and dividing by how many pieces of data there are.

Range: Is the numerical difference between the smallest and the greatest values in a data.

line Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

esson 1 Review

Frequency Distribution Table, Bar Charts, Pie Charts Pictograms and Frequency Polygons

Introduction

By the end of this subunit, you should be able to:

-Distribute statistical data on a frequency distribution table.
-Display statistical data on a bar chart.
-Display statistical data on a pie chart.
-Use the pictogram to display statistical data
-Display statistical data on a frequency polygon.

This subunit is about 12 pages in length.

Frequency Distribution table

You probably have come across the frequency distribution table before. We will, however, start our discussion on statistics with it so that you may understand the other coming subtopics.

Consider the following data:

It is collected from 30 students' scores in a mathematics exam.

20	60	80	70	80
20	50	30	60	80
25	60	70	90	60
80	20	60	60	60
60	70	80	90	20
60	70	80	20	60

Think about whether you can quickly say how many scored 20, 50, or any other score.

Think about the challenge if there were 2,000 pieces of data.

Since the data is scattered, it is not obvious how many students scored 20 or 50. It would take time to count the scores one by one. To make this data look more readable what do you think we should do?

Well, there are quite a number of things you could do to make the data look more readable. One of them is to tabulate it.

When tabulating the data you could make use of the tally marks as they help in making counting easier.

With tally marks to represent the number five for instance, we put four marks in line and cross them with the fifth mark. i.e. 𝍸 represents five.

Now that we know how to use tally marks, we can represent the 30 students' scores on a frequency distribution table as shown in table 1.1.

Score	**Tally Marks**	**Number of Students**
20	𝍸	5
25	|	1
30	|	1
50	|	1
60	𝍸 𝍸	10
70	||||	4
80	𝍸 |	6
90	||	2
Total		**30**

Table 1.1 – Frequency distribution table of test scores

Note how the data in the table is easier to read than the rows and columns of the original data. We can easily see from the table that 5 students scored 20, and only 1 student scored 50.

The column named <u>number of students</u> is the frequency column, it shows how frequently a score appears, it shows that 20 appears 5 times, 25 once, 30 once and so on, hence the table is called the **frequency distribution table.** The frequencies add up to the total number of students which is 30.

Now work on activity 1.1

Activity

Activity 1.1

In a class of 40 students, a class teacher was making investigations on which subject was liked the most between, Biology, Mathematics and Sesotho. Each student wrote the subject they liked the most and these were the outcomes.

B	S	M	M	S	S	B	S
M	B	B	B	S	M	S	M
S	B	M	M	S	S	M	B
B	S	M	B	B	S	S	M
B	S	M	S	M	M	B	B

Whereby: B represents Biology, S represents Sesotho and M represents mathematics.

Distribute the above data on a frequency distribution table making use of tally marks.

The headings of the table have been provided for you in the table 1.2.

Subject	Biology	Mathematics	Sesotho
Tally Marks			
No. Of students			

Table 1.2 – Frequency distribution table of students' favourite subjects

After completing the questions, compare your answers to the correct answers at the end of this subunit. Take the time needed to understand each answer before continuing.

To make the statistical data visual and easier to make comparisons, the data can also be represented in the form of Bar charts, pie charts or pictograms.

Bar Charts

In bar charts, the numerical data is shown on a graph with two axis, the vertical and the horizontal axis. The frequency is shown on the vertical axis while the data is shown on the horizontal axis. Hopefully that sounds familiar.

Bar charts are best to use to compare things between differed groups or used to display growing database so as to track changes.

Now consider the following example on bar charts.

<u>Example 1</u>

35 students were asked to mention their favourite cars and the results are shown in table 1.3, which is a frequency distribution table.

Cars	Astra	Sentra	Pajero	Jetta	Velocity
Frequency	1	9	8	7	10

Table 1.3-Frequency distribution table of student's favourite cars

The bar chart shown in figure 1.1 illustrates the frequencies of the data. The frequencies are represented by the heights of the bars. So from the bar graph we can easily see that 1 student's favourite car was the Astra, for 9 students it was the Sentra, and so on.

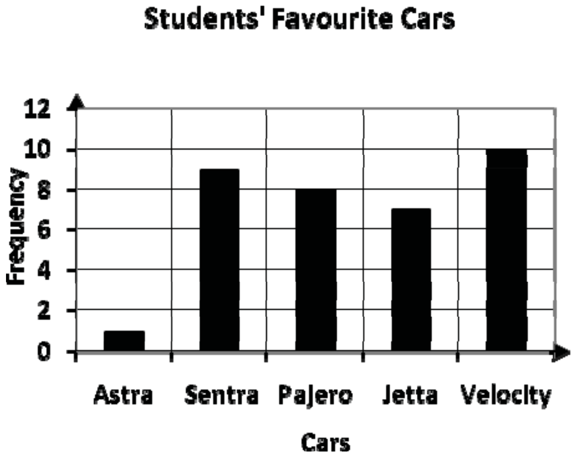


Figure 1 Bar chart showing students' favourite cars

When drawing the bar chart, **always** make sure that:

- The bars are the same width.
- The spaces between the bars are the same widths.
- The axes are labelled.
- The graph has a title.

Now try this:

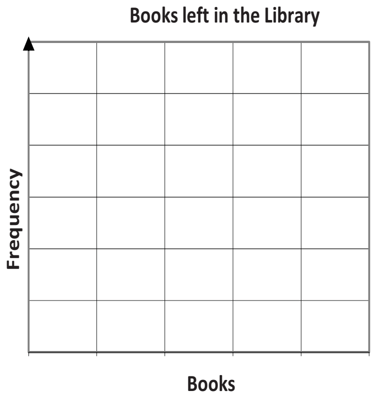
Activity 1.2

Table 1.4 shows the books that were left in a library after all the other books were stolen from the library.

Books	Prism	I'm David	Bona	Bible	Basic English
No of Books (Frequency)	10	9	1	15	20

Table 1.4 – Books left in a library

From Table 1.4, draw a bar chart to show the data of the books:



Compare your graph to the one at the end of the subunit. If you did not get it correct, study example 1 to see what needs to be done.

Pie Charts

If you have heard of the bar chart before you most probably have heard of the pie chart also. You may recall that in a pie chart, data is displayed as sectors of a circle. Pie charts are best to use when trying to compare parts of a whole.

Let us consider example 1

<u>Example 1:</u>

100 students were asked for their favourite subjects and the results are shown in table 1.5.

Subject	Mathematics	English	Sesotho	Geography	Physics
Number of students(frequency)	25	20	15	10	30

Table 1.5-Frequency distribution table of student's favourite subjects

Now let us display the data on a pie chart.

First we calculate the angles of the sectors of the circle. To calculate the sector angle we put the frequency over the total number of elements and multiply that fraction by 360º. Table 1.6 shows three of the sector angles.

Fill in the missing sector angles on table 1.5.

Subject	Mathematics	English	Sesotho	Geography	Physics

Frequency	25	20	15	10	30
Sector Angle	$\frac{25}{100} \times 360^0 = 90^0$	$\frac{20}{100} \times 360^0 = 72^0$	$\frac{15}{100} \times 360^0 = 54^0$		

Table 1.6-Frequency distribution table of student's favourite subjects with sector angles

Compare your answers with the following.

Solution:

The missing sector angles on table 1.6 are:

Geography:

$$\frac{10}{100} \times 360^0 = 36^0$$

Physics:

$$\frac{30}{100} \times 360^0 = 108^0$$

Now that we know the angles we can draw the pie chart, as shown in figure 2.

With the pie charts, if you choose to use shading, shade each section differently. It is also acceptable to draw the sectors without shading.

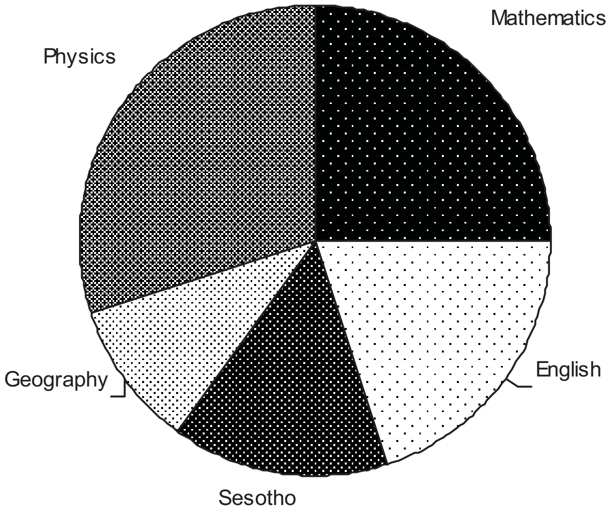


Figure 2 – Pie chart of student's favourite subjects

From the pie chart the largest angle is the angle that represents Physics since it has the highest Frequency. The smallest angle represents Geography as it has the lowest frequency.

Now work on activity 1.3.

Activity 1.3

1. Frequency table 1.7 below shows the countries of the 20 people who attended a seminar in China on distance learning.

Country	Lesotho	Botswana	Swaziland	Nigeria	Namibia	Malawi	Angola
Frequency	2	2	3	3	6	2	2

Table 1.7 - Frequency distribution table of participant's countries

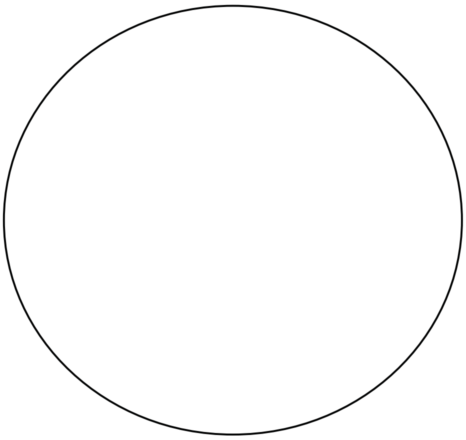
(i) Complete the following frequency table for the same data.

Country	Lesotho	Botswana	Swaziland	Nigeria	Namibia	Malawi	Angola
Frequency	2	2	3	3	6	2	2
Sector	2/20 x	2/20 x					

angle	$360^0 =$ 36^0	$360^0 = 36^0$					

Table 1.8- Frequency distribution table of participant's countries with sector angles

(ii) Display the information on Table 1.8 on a pie chart.



Compare your answers those given at the end of the subunit. If your table results and pie chart do not match the provided answers, spend the time needed to be sure that you understand the concepts.

Pictograms

You have now learned about bar charts and pie charts. A third way of displaying statistical data is making use of pictograms. With pictograms, statistical data is displayed in the form of pictures. Just like bar charts, pictograms are best to use to compare things between differed groups or used to display growing database so as to track changes.

Example 1

A group of 90 children were asked to mention their favourite colours and table 1.9 shows the results.

Colour	Red	Pink	Black	Yellow	White
No. Of Children	10	15	30	25	10

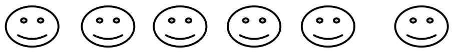
Table 1.9-frequency distribution table of children's favourite colours.

The results of table 1.9 are shown in the form of a pictogram below.

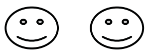
The key is:  = 5 students. Fill in the yellow row:

Red: 

Pink: 

Black: 

Yellow:

White: 

You should have drawn five faces in the yellow row.

That is how to display statistical data in the form of a pictogram.

Now try activity 1.4.

Activity 1.4

Pulane counted animals passing near her home on one day and her results are shown in table 110. Use table 1.10 to make a pictogram that represents the data that Pulane got on that day and in your pictogram let each symbol represent 6 animals.

Animals	Cows	Horses	Donkeys	Dogs	Cats
Frequency	12	24	6	30	3

Table 1.10-Frequency distribution of animals passing on one day

Key:

Cows

Horses

Donkeys

Dogs

Cats:

If you have drawn the pictogram, compare it with the one given at the end of the subunit and see how well you did. If needed, clarify your misunderstandings before continuing.

Frequency Polygons:

A frequency polygon is another way of displaying statistical data.

Let us consider table 1.11 and learn how to draw the frequency polygon.

Temperature	-2	0	4	11	15	24
Frequency	1	3	5	7	6	3

Table 1.11-Frequency distribution table of high temperatures on 25 days during the year.

To draw the frequency polygon for the distribution in table 1.11, we use ordered pairs, i.e. (variable, frequency).

The ordered pairs that we are coming to use to draw the frequency polygon are : (-2,1) (0,3) (4,5) (11,7) (15,6) (24,3). In a frequency polygon the points are plotted and the joined with straight lines. From the above points, the following frequency polygon can be drawn:

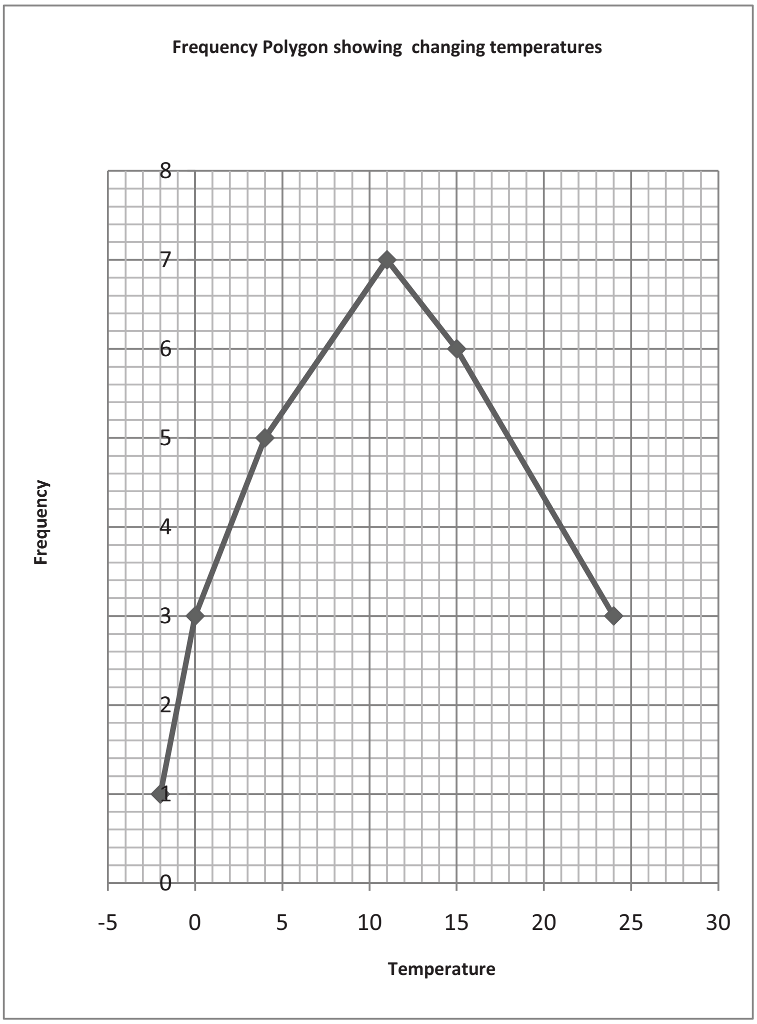


Figure 3- Frequency Polygon showing changing temperatures in 5 days.

Now try activity 1.5

Activity 1.5

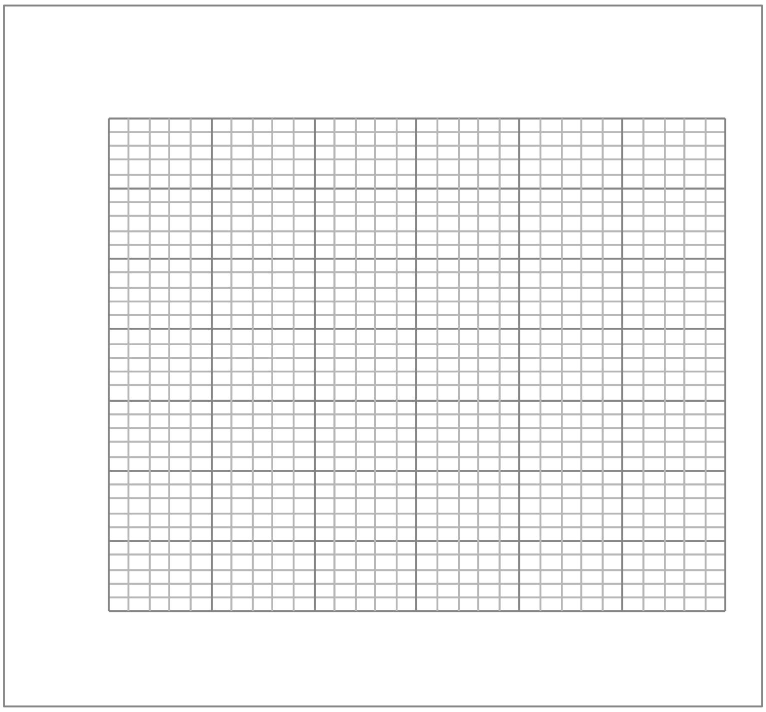
Table 1.12 shows the prices of the items sold in a restaurant over the past two weeks.

ctivity

Prices (in rands)	8	9	10	16	12	13
Number of items sold	4	7	8	15	12	6

Table 1.12-Frequency distribution table of items sold in a restaurant.

Draw the frequency Polygon for the information shown on table 1.12.



Compare your answers with those provided at the end of the subunit. Continue on if you can accurately draw frequency polygons.

Key Points to Remember

The key points to remember in this subunit on frequency distribution table, bar charts, pie charts pictograms and frequency polygons are:

-Frequency distribution table helps to make statistical data look more readable.

-In bar charts, the numerical data is shown on a graph with two axis, the vertical and the horizontal axis. The frequency is shown on the vertical axis while the data is shown on the horizontal axis.

-When drawing the bar chart, we **always** make sure that:

- The bars are the same width.
- The spaces between the bars are the same widths.
- The axes are labelled.
- The graph has a title.

-When drawing a pie chart, first we calculate the angles of the sectors of the circle. To calculate the sector angle we put the frequency over the total number of elements and multiply that fraction with 360º.

- With pictograms, statistical data is displayed in the form of pictures.

-Frequency polygons are used mostly for constantly changing data.

Solutions to activities:

Solution to activity 1.1

Subject	**Tally marks**	**Frequency**
Biology	𝍸 𝍸 |||	13
Sesotho	𝍸 𝍸 ||||	14
Mathematics	𝍸 𝍸 |||	13

Table1.13-Frequency distribution of student's favourite subjects

Solution to activity 1.2

To be correct, in your bar graph:

-The bars should have the same width.

-The spaces between the bars should be the same widths.

-The axes should be labelled.

-The graph must have a title.

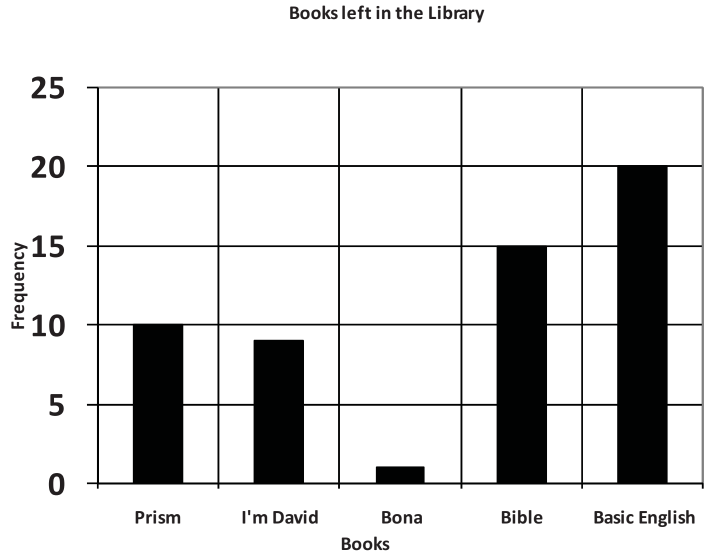


Figure 4 -Bar chart showing books left in the library

Solution to activity 1.3

i) First we calculate the sector angle and then draw the pie chart.

Country	Frequency	Sector Angle
Lesotho	2	$\frac{2}{20} \times 360^0 = 36^0$
Botswana	2	$\frac{2}{20} \times 360^0 = 36^0$
Swaziland	3	$\frac{3}{20} \times 360^0 = 54^0$
Nigeria	3	$\frac{3}{20} \times 360^0 = 54^0$
Namibia	6	$\frac{6}{20} \times 360^0 = 108^0$
Malawi	2	$\frac{2}{20} \times 360^0 = 36^0$
Angola	2	$\frac{2}{20} \times 360^0 = 36^0$

Total	20	360^0

Table 1.14-Frequency distribution table of participant's countries

ii)

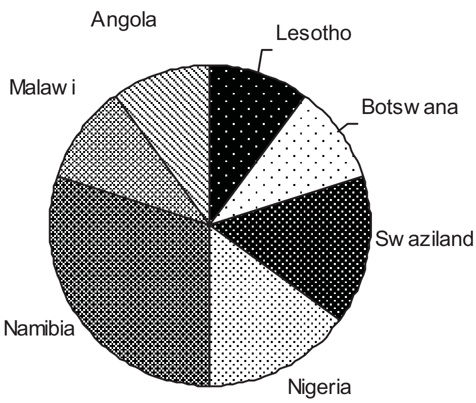


Figure 5- Bar chart showing Participants' Countries.

Now if you use your protractor to measure the angles you will see that the angle of the largest sector is 108° which represents the angle of mark 18 and the smallest sector angle is 36°.

That is the way of displaying data on a pie chart.

Solution to activity 1.4

<u>The pictogram to show animals that passed near Lineo's home one day.</u>

Key:  = 6 animals

Then

Cows: ¤ ¤

Horses: ¤ ¤ ¤ ¤

Donkeys: ¤

Dogs: ¤ ¤ ¤ ¤ ¤

Cats: 

Since there are only 3 cats, the symbol that represents 3 cats is half the other symbols as 3 is half of six.

Solution to activity 1.5

The ordered pairs for the distribution are: (8, 4), (9, 7), (10, 8), (16, 15), (12, 12), (13, 6)

Figure 6- Frequency polygon showing prices of items sold in a restaurant.

Lesson 2 Mean, Median and Mode

Introduction:

Earlier on we said statistical data needs to be collected analysed and displayed. We have shown different ways of displaying statistical data and now we are to address how to analyse it.

By the end of this Subunit on mean, median and mode you should be able to:

Calculate the mean, get the mode and median from data which is not tabulated and tabulated data

This subunit has about 10 pages.

Mean:

Mean is the statistical name for average. It is a way of analysing statistical data. For instance, in a class a teacher may use the mean to see the general performance of the class from a test.

Example 1

Let us consider the following data and calculate its mean.

4, 10, 10, 15, 20, 12, 16, 10, 18, 21

To calculate the mean we add all data together i.e.

4+10+10+15+20+12+16+10+18+21 = 130

Then we divide the sum by 10 which is the total number of elements of data, i.e.

$$\frac{130}{10} = 13$$

So, 13 is the mean of the data.

Now let us do example 2 on mean.

Example 2

Calculate the mean of the following numbers and round it to two decimal places.

5.1, 6.4, 0.6, 2.3, 5.5, 3.9, 1.7

To calculate the mean, the first step is to add up all of the data elements. Do this now.

____

____

____

Compare your answer with the following:

5.1+6.4+0.6+2.3+5.5+3.9+1.7=25.5

The second step is to divide the sum we just got by the number of data elements and round to two decimal places.

____

____

____

Compare your answer with the following:

$$\frac{25.5}{7} = 3.64$$

3.64 is the mean rounded to two decimal places.

Median:

It is the number in the middle after all data is arranged in order from the smallest to the largest. That is the median is found by putting all scores or data in order and the one in the middle is the median.

Example 1

Find the median of the following distribution:

1 2 4 3 4 5 2 1 6

Solution to example 1

Step 1:

Put the numbers in order starting with the smallest.

___ ___ ___ ___ ___ ___ ___ ___ ___

Compare your answer with the following.

1 1 2 2 3 4 4 5 6

What is the number in the middle? _____

The number in the middle is 3, so 3 is the median.

Now let us do example 2.

Example 2

Find the median of the following distribution.

1 5 4 3 1 4 5 7 5 6

Solution to example 2

First put the numbers in order stating with the smallest.

___ ___ ___ ___ ___ ___ ___ ___ ___ ___

Compare your answer with the following.

1 1 3 4 4 5 5 5 6 7

And now what is the number in the middle? _____

Compare your answer with the following.

There are two numbers in the middle 4 and 5.

The median is one number not two. So what do you think we should do?

___

___

Compare your idea with the one below:

We add these two numbers together and divide them by two to get one number which is the median.

$4 + 5 = 9$

$\frac{9}{2} = 4.5$

So the median is 4.5.

If you clearly understand this continue to the next subtopic which discusses the mode. If not, revisit examples 1 and 2.

Mode:

Now that we know how to calculate mean and median, let us learn how to find the mode.

By definition, mode is the most frequently appearing piece of data.

To illustrate what mode is, let us consider the following quiz scores.

1	2	6	8	7	0	7	2	7	5	7
5	7	7	7	7	7	7	7	7	1	2

How many times do the following numbers appear?

0: ___ 1: ___ 2: ___ 3: ___ 4: ___

5: ___ 6: ___ 7: ___ 8: ___ 9: ___

Which number appears the most number of times? ____

Compare your answer with the one below:

7 is the mode or the modal number as it appears more often than any of the other numbers.

Now that we know what mode is, let us do activity 1.6 to give you practice calculating the mean, median, and mode.

Activity

Activity 1.6

In each of the following distributions, find the mean, the median and the mode.

a) Maximum temperatures of nine consecutive days:
25 27 28 29 30 26 27 24 27

Mean: ___

Median:

Mode:

b) Ages of 8 men:
50 55 51 56 59 55 53 54

Mean:

Median:

Mode:

c) Number of people present on different days:
1 9 10 13 1 19 15 9 9 5

Mean:

Median:

Mode:

Compare your answers to those given at the end of the subunit. If you can comfortably find the mean, median, and mode from a distribution of numbers continue on. Otherwise review this material before continuing.

son 3 Mean Median and Mode from a Frequency Table

Up to this far we have been dealing with mean, median and mode from data that is not tabulated. Now we will use data from a frequency table to find the mean, the median and the mode.

Let us consider the following data regarding the number of tyres a mechanic repaired in each preceding day:

1 1 5 2 3 2 1 5 3 2

6 1 5 2 2 1 1

Table 1.15 shows the distribution of the above numbers.

Number	1	2	3	5	6
Frequency	6	5	2	3	1

Table 1.15- Frequency distribution table of tyres repaired by a mechanic

Median

When we have arranged the data on table 1.15 in order,

On the 1st ,2nd ,3rd, 4th, 5th and 6th positions we have the number 1.

On the positions :7 ,8 ,9,10 and 11 we have the number 2.

On the 12th and 13th position we have the number 3

the 14th, 15th and 16th numbers are 5

And the 17th number is 6.

So, there are 17 numbers all in all and the median number is the 9th number as there are 8 numbers on its left and 8 numbers on its right. **So the median number is the ninth number which is 2**, since we said that the 7th, 8th, 9th, 10th and 11th numbers are 2.

So we can see that there are 17 values of the variable, and the median position is the 9th position. The 9th position can be found using the following calculation.

$$9 = \frac{17+1}{2}$$

Now the general formula for the position of the median is :

$median\_position = \frac{1}{2}(n+1)$, where n is the total number of variables or sum of all frequencies.

Mode

As stated earlier, the modal number is the number which occurs most often. So, which number is the most common in table 1.15?

Compare your answer with the one below:

1 is the modal number.

Mean

When finding the median we said there are 17 numbers all in all, so when calculating the mean we should calculate the mean of all the 17 numbers.

$$\text{So mean} = \frac{1+1+1+1+1+1+2+2+2+2+2+3+3+5+5+5+6}{17} = \frac{43}{17} = 2.53$$

But the calculations can be very long with large frequencies, so we can use multiplication as shown in the following.

$$\text{Mean} = \frac{(1\times 6)+(2\times 5)+(3\times 2)+(5\times 3)+(6\times 1)}{17} = \frac{43}{17} = 2.53$$

So, the second method is the shorter way of calculating the mean from a frequency table.

$$\text{Generally mean} = \frac{sum\_of\,(frequency \times variable)}{sum\_of\_frequencies}$$

Now apply these ideas in activity 1.7 for calculating the mean, median and mode from a frequency table.

Activity 1.7

1. 20 pupils sat for a mathematicss test and the results of the test are shown on table 1.16.

Mark	50	60	65	77	84
Number of pupils	12	10	15	6	3

Table 1.16- Frequency distribution table of students' scores in a mathematics test

Find the

a) mean mark

b) modal mark

c) median

2. Table 1.17 shows the number of animals owned by farmers.

Number of animals	20	30	35	46	50	55	**Total**
Number of farmers	2	4	12	15	10	5	48

Table 1.17-Frequency distribution table showing animals owned by different farmers.

From table 1.17 find the

a) mean number of animals

b) modal number of animals

c) median

Compare your answers with those given at the end of the subunit. If you had at least 5 out of 6 correct, continue to the next subunit. If not, review the content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on mean, median and mode are:

-Mean is the statistical name for average

-To calculate the mean we add all data together then we divide the sum by the total number of elements of data.

- Median is the number in the middle after all data is arranged in order from the smallest to the largest.

-Mode is the most frequently appearing piece of data.

The general formula for the position of the median is

$median\ position = \frac{1}{2}(n+1)$, where n is the total number of variables or sum of all frequencies.

$$\text{Mean} = \frac{sum\_of\,(frequency \times variable)}{sum\_of\_frequencies}$$

Solution to activity 1.6

a) $Mean = \frac{25+27+28+29+30+26+27+24+27}{9} = 27$

Median: 24 25 26 27 <u>27</u> 27 28 29 30

Median = 27 because it appears in the middle after arranging all the numbers in order.

Mode=27 because it appears more often.

b) $Mean = \frac{50+55+51+56+59+55+53+54}{8} = 54.1$ correct to 1 significant figure.

Median: 50 51 53 <u>54</u> <u>55</u> 55 56 59

$Median = \frac{54+55}{2} = 54.5$

Mode=55

c) $Mean = \frac{1+9+10+13+1+19+15+9+9+5}{10} = 9.1$

Median: 1 1 5 9 <u>9</u> <u>9</u> 10 13 15 19

$Median = \frac{9+9}{2} = 9$

Mode= 9

Solutions to Activity 1.7

1. In this case of a frequency table we have 12 students who have scored 50 marks, that says all in all there are twelve 50's. We have ten 60's, fifteen 65's and so on. Now when calculating the mean mark, all the twelve 50's which is (12×50) will need to be included, all the ten 60's and so on. In other words the whole data has to be included when calculating the mean mark.

We can redraw the frequency table and in it include the column of frequency × mark.

Mark	50	60	65	77	84	Total
Frequency	12	10	15	6	3	46
Frequency × mean	600	600	975	462	252	2889

Table 1.16-Frequecy Table showing students' marks and the frequency mean

So this is how we calculate the mean: we multiply each piece of data by its frequency like we have done in the frequency × mark column and the we add all the products together, and then divide the whole sum by the sum of the frequencies, i.e.

$$mean = \frac{2889}{46} = 62.8$$

The general formula of finding the mean from a frequency table is:

$$\frac{\text{Sum of (frequency × variable)}}{\text{Sum of all frequencies}}$$

To find the mode:

a) The modal mark is the mark that appears most frequently, so we can see from table 1.12 that 15 is the largest frequency, so the corresponding mark, 65, is the modal mark.

b) The median: There are 46 marks all in all because there are 12, 50's, 10, 60's, 15, 65's, 6, 77's and 3, 84's. So the median position is at the $\frac{1}{2}(46+1)=23.5$ position. So from position 1 to position 12, we have marks of 50, and from position 13 to position 22 we have marks of 60. Now from position 23 to position 37 we have marks 65. Now the median mark lies between position 23 and 24, which is at position 23.5. So to get the median mark we add the marks at position 23 and position 24 and divide their sum by 2. So the marks at position 23 and 24 are 65's.

$$\therefore median = \frac{65+65}{2} = \frac{130}{2} = 65$$

So the median mark is 65.

2.

a) Mean

First let us include the Frequency ×Number of Animals column to the given table 1.17

Number of Animals	20	30	35	46	50	55	Total
Number of Famers (Frequency)	2	4	12	15	10	5	48
Frequency × Number of animals	40	120	420	960	500	275	2045

Table 1.17-Frequency distribution table showing number of animals owned by different farmers.

So the mean = $\frac{\text{Sum of all the animals}}{\text{Sum of all farmers (Frequency)}}$

$$\therefore mean = \frac{2045}{48} = 42.6$$

b) The modal number of animals is 46

c) There are 48 variables all in all, so the median position is at position:

$$\frac{1}{2}(48+1) = 24.5$$

Which means that the median is between positions 24 and 25. So to get the median we add the numbers of animals at positions 24 and 25 and then divide them by 2. So the median number of animals is 46 animals.

son 4 Grouped Data/Grouped Class Intervals

Introduction:

Up to this point, we have been dealing with ungrouped data. Sometimes data is grouped, especially large data to make it more manageable. Grouped data is made out of ungrouped data or sometimes when data is being collected; it is recorded in groups or class intervals.

By the end of this subtopic you should be able to:

-Find the mean, median and mode from grouped data.

This subunit is about 6 pages. Let us consider the following data for instance,

40 students sat for a chemistry test marked out of 100 and the following results were obtained.

01	02	04	06	07	08	10	31	34	25
11	13	14	18	18	19	20	36	61	50
21	22	22	25	27	28	29	55	59	66
77	40	85	90	95	97	83	84	51	60

Example 1.

The data above can be grouped or arranged into class intervals, let us consider at table 1.18 and see how data can be grouped.

Marks(Class)	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99	Total
Frequency (*f*)	6	7	9	3	1	4	3	1	3	3	40

Table 1.18- Frequency distribution table of grouped data of student's marks in a chemistry test.

In table 1.18 data has been grouped into groups of 10. It can be grouped anyhow, in groups of 3, 5 or any number desired.

Mean Median and Mode from Grouped data

Now let us see how we can find the mean, the median and the mode from the grouped data.

Finding the Mode:

Mode as we said is the data that appears most frequently. So from table 1.15, which group has the highest frequency?

Compare your answer with the one given below.

It is the 20-29 marks group.

Finding the Median:

From the given data on table 1.15, there are 40 values of the variable, now from the table alone we do not know what individual values of marks are. But if we would list the marks in order, the median position would be in Position $\frac{40+1}{2} = 20.5$ that is between positions 20 and 21. So we add the marks at these two positions and divide their sum by 2 to get the median.

From position 1 to 6 we have marks 0—9

From position 7 to 13 we have marks 10—19

From Position 14 to 21 we have marks 20—29

Now we can see that the median falls in the group of 20—29 marks since 20.5 falls in that range. Now, using the information from the table alone it is not easy to tell what the marks on positions 20 and 21 are. So with grouped data it is very difficult to tell what the exact median is. We can only tell the median group. In the next subunit we will use the cumulative frequency diagram to estimate the median.

Finding the Mean in Grouped Data

You have seen how to find the mode and median in grouped data. Now we are coming to learn how to find the mean in grouped data.

Like we said earlier, with grouped data, the individual marks are not known, we can only tell that the first six marks are in group 1, which is all of the marks between 0 and 9. So now how do we find the mean when we do not know what our exact data is?

This is how we do it:

First we make assumptions, let us take the class of 0 to 9 marks. From table 1.18, we only know that we have 6 marks in this group. If we assume that all the 6 marks were 0, their sum would be 0 and their mean would be 0. Now if we assume that all the six marks were 9, their sum would 54 and their mean would be 9. The estimated class mean would be (0+54)/2 which is 27. We can get the same class mean if we take the mid-value of the class of 0—9 marks and multiply it by 6 because we know that there are 6 marks in this class. To get the mid value of a class we add lowest value and the highest value of that class and divide their sum by two. The mid value of the class of 0—9 marks will be given by:

Mid-value $= \frac{0+9}{2} = 4.5$.

So the estimated class mean = $4.5 \times 6 = 27$. This is how we estimate the class mean.

Now to get the overall mean of the grouped data, we add the class means which are also called frequency means (f_m) and divide their sum by the sum of all frequencies.

So the Estimated mean in grouped data is given by:

$$Estimated\ mean = \frac{Sum\ of\ f_m\ values}{sum\ of\ f\ values}$$

Now to get the estimated mean of the data in table 1.18, let us first redraw it and include the columns on frequency means and class mid-values as we are going to need the class mid-values when calculating the frequency means as shown in table 1.19.

Marks(Class)	0-9	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99	Total
Frequency (*f*)	6	7	9	3	1	4	3	1	3	3	40
Class mid-value	4.5	14.5	24.5	34.5	44.5	54.5	64.5	74.5	84.5	94.5	
f_m	4.5×6 =27	14.5×7 =101.5	24.5×9 =220.5	34.5×3 =103.5	44.5×1 =44.5	54.5×4 =218	64.5×3 =193.5	74.5×1 =74.5	84.5×3 =253.5	94.5×3 =283.5	**1508**

Table 1.19-- Frequency distribution table of grouped data of student's marks in a chemistry test.

So the

$$Estimated\ mean = \frac{Sum\ of\ f_m\ values}{sum\ of\ f\ values}$$

$$Estimated\ \ mean = \frac{1508}{40} = 37.7$$

So in grouped data:

The Modal class is the class with the highest frequency.

The Median class is the class in which the middle value of the variable falls.

Example 2

For following distribution:

a) Determine the modal class
b) Determine the median class
c) Calculate the mean

Class	20-29	30-39	40-49	50-59	60-69
Frequency	2	4	9	16	8

Table –1.20-Frequency distribution table of students' marks in a test

Solutions

a) The modal class is the class of 50-59 marks.

b) The median position is at position $\frac{39+1}{2} = 20$, so the median which is in position 20 falls on group 50-59.

c) In finding the median we will have to find first the class mid-value and the frequency mean.

Class	20-29	30-39	40-49	50-59	60-69	Total
Frequency	2	4	9	16	8	39
Class mid-value	24.5	34.5	44.5	54.5	64.5	
f_m	49	138	400.5	872	516	1975.5

Table-1.21- Frequency distribution table of students' marks in a test

So the estimated mean:

$$estimated\ mean = \frac{1975.5}{39} = 50.65$$

Now consider activity 1.8 and answer all the questions.

Activity

Activity 1.8

1. For the following distributions on tables 1.22, and 1.23:

a) Determine the modal class.

b) determine the median class

c) calculate the mean

i)

Class	16-20	21-25	26-30	31-35	36-40
Frequency	3	6	5	4	2

Table 1.22- The Frequency distribution table of the marks of students in a test marked out of 50

ii)

Class	11-20	21-30	31-40	41-50	51-60
Frequency	6	8	7	6	3

Table 1.23- The frequency distribution table of the ages of people in a church choir

No. 2

Determine the mean, median class and modal group for the following distribution.

Group	3-5	6-8	9-11	12-14	15-17
Frequency	2	6	7	5	5

Table 1.24 – Frequency distribution table of the age of people in Sunday class

Compare your answers with those given at the end of the subunit. Be sure that you understand each calculation before continuing.

Key Points to Remember

In this subunit the key points to remember are that in grouped data:

-The Modal class is the class with the highest frequency

-The Median class is the in which the middle value of the variables fall.

-The $\textit{Estimated mean} = \frac{\text{Sum of } f \times \text{values}}{\text{sum of } f \text{ values}}$

Solutions to Activity 1.8

No.1 i)

a) The modal class is class 21-25

b) The median is in position $\frac{20+1}{2}=10.5$

From position 10 to position 14 we have the class of 26-30, so class 26-30 is the median class.

c) To calculate the mean, let us first redraw table 1.20 and include the class mid-values and the frequency means.

Class	16-20	21-25	26-30	31-35	36-40	Total
Class mid-value	18	23	28	33	38	
Frequency	3	6	5	4	2	20
fx	54	138	140	132	76	540

Table 1.25- The Frequency distribution table of the marks of students in a test marked out of 50

Therefore the estimated mean = $\frac{540}{20}=27$

ii) a) The modal class is class 21-25

b) The median lies in position $\frac{30+1}{2}=15.5$

So the median lies between position 15 and position 16. So position 15 and 16 are in class 31-35. So the median class is class 31-35.

c) Now we can find the mean by finding the class mid values for each class and then getting the frequency mean for each class.

Class	11-20	21-30	31-40	41-50	51-60	**Total**
Class mid-value	15.5	25.5	35.5	45.5	55.5	
Frequency	6	8	7	6	3	30
fx	93	204	248.5	273	166.5	985

Table 1.26 - The frequency distribution table of the ages of people in a church choir

Therefore the estimated mean = $\frac{985}{30}=32.83$

2.

i) The Modal class is the class of 9-11

ii) The median position is at position $\frac{25+1}{2}=13$

iii) The estimated mean

Group	3-5	6-8	9-11	12-14	15-17	Total
Group mid-value(m)	4	7	10	13	16	
Frequency(f)	2	6	7	5	5	25
f_m	8	42	70	65	80	265

Table 1.27-Frequency distribution table of the age of people in Sunday class

$$estimated\ median=\frac{265}{25}=10.6$$

on 5 Cumulative Frequency Diagrams, Median, Quartiles and Percentiles

Introduction:

Earlier on we said that it is difficult to say exactly what the median is in grouped data, and we said we can only tell the median group. However in this subunit we will learn how to estimate the median, the quartiles and the percentiles using the cumulative frequency diagram.

By the end of this subtopic you should be able to:

-Draw the cumulative frequency diagram and use it to find the median, the upper quartile, the lower quartile, the percentiles and the inter quartile range.

This subunit has about 9 pages.

To understand this subunit, first work on activity 1.9. It introduces you to the cumulative frequency that you will need in order to be able to estimate the median, the quartiles and the percentiles using the cumulative frequency diagram.

Activity

Activity 1.9

Table 1.28 shows the marks of students in a test.

From In table 1.28

a) Complete the cumulative frequency column.

b) Find the median position.

c) Looking at the cumulative frequency column, in which group does the median fall? ______________

Marks	0-4	5-9	10-14	15-19	20-24	25-29
Frequency	4	10	15	10	6	5
Cumulative Frequency	4	14	29			50

Table 1.28- Cumulative frequency table of the marks of students in a test.

Compare your answers with those given at the end of the subunit. If you got all of the answers correct you can move on to the quartiles. If not, go back and review the subunit on grouped data.

The Quartiles

Into how many equal parts does the median divide the total frequency?

____

____

Compare your answer with the following.

It divides it into two equal parts.

Note: Sometimes it is necessary to divide the total frequency into four equal parts. Now the values of the variables at the points of division are called **quartiles**.

The lower quartile (L.Q.) has a quarter of the variable values below it. The lower quartile position is given by $\frac{1}{4}(n+1)$, or $\frac{1}{4}n$ for large *n (you can consider n as large from 100 upwards), where n is the sum of frequencies.*

The median has half of the variable values below it and the median position is given by $\frac{1}{2}n+1$ or $\frac{1}{2}n$ for large *n*.

The upper quartile has three quarters of the variable below it, whereby the upper quartile position is given by $\frac{3}{4}(n+1)$ or $\frac{3}{4}n$ for large *n*.

Example 1:

The following diagram is drawn from the data on table 1.28 and is called the cumulative frequency diagram. It shows an example of the upper quartile (UP), the median (M) and the lower quartile (LQ).

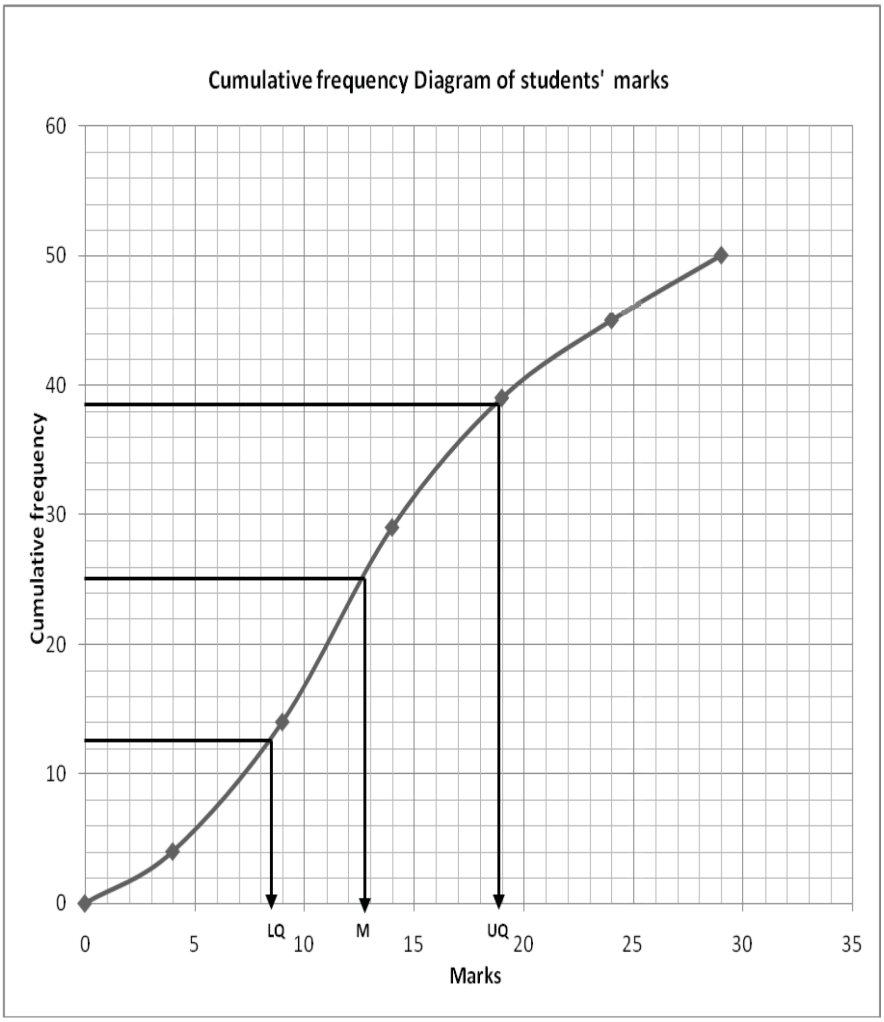


Figure 7-The cumulative frequency diagram showing students' marks from a test.

To find the median the cumulative frequency diagram has been used, first the median position has been calculated by $\frac{1}{2}(n+1) = \frac{1}{2}(50+1) = 25.5$

So after finding the median position,

- We draw a horizontal line that joins the point (0, 25.5) and the graph.

-At the point where the horizontal line joins the graph we draw a vertical line.

The mark coordinates of vertical line give us the median.

So the median mark is about 12.7. This is shown on the graph above.

The lower Quartile position is found by:

$$\frac{1}{4}(n+1) = \frac{1}{4}(50+1) = 12.75 \approx 12.8$$

By the same procedure of finding the median, the lower quartile mark is estimated to be 8.4.

The upper quartile position is given by:

$$\frac{3}{4}(50+1) = 38.25$$

Then the upper quartile position is estimated to be 18.8.

The cumulative frequency diagram is a graph of which its ordered pairs consist of (upper class value, cumulative frequency). With table 1.24 the ordered pairs are (4, 4), (9, 14), (14, 29), (19, 39), (24, 45), (29, 50) It is important to understand why we use the upper class value in ordered pairs. The upper class value accommodates all the marks of the class. Let us explain it with an example, (4, 4) means 4 pupils scored 4 marks or below, (9,14) means 14 students scored 9 marks or below, It includes even those 4 who scored 4 marks and below. (14, 29) means that 29 pupils scored 14 marks or less, (29, 50) means that 50 pupils scored 29 marks or less and so on.

Example 2

For the following distribution on the ages of people in church:

a) Draw the cumulative frequency diagram
b) Use your cumulative frequency diagram to estimate the median and the quartiles.

Class	1-5	6-10	11-15	16-20	21-25
Frequency	5	12	16	9	5

Table 1.29-The cumulative frequency table showing peoples' ages in a church

Solution:

a) To draw the cumulative frequency diagram we will need the coordinates of the curve being (upper class value, corresponding cumulative frequency).

Class	1-5	6-10	11-15	16-20	21-25
Frequency	5	12	16	9	5
Cumulative Frequency	5	17	33	42	47

Table 1.30- The cumulative frequency table showing peoples' ages in a church

The coordinates of the graph are: (5, 5) (10, 17) (15, 33) (20, 42) (25, 47).

The graph is as follows:

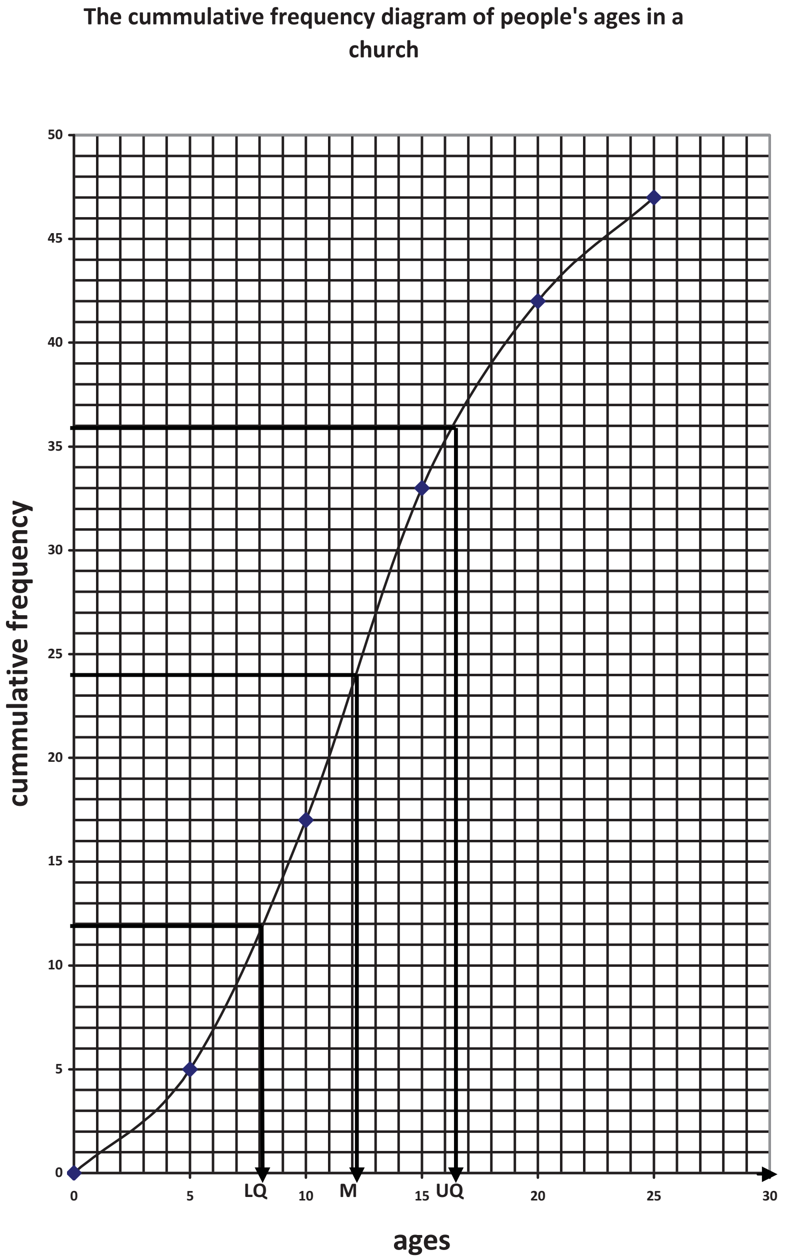


Figure 8- The cumulative frequency table showing peoples' ages in a church

b) -The median position is given by $\frac{47+1}{2} = 24$

From the graph the median is 12.2 years.

- The upper quartile position is $= \frac{3}{4}(47+1) = 36$

From the graph the upper quartile position is estimated to be 16.5

- The lower quartile position is $\frac{1}{4}(47+1)=12$

  So the lower quartile position is estimated from the graph to be 8 years.

Now work on activity 1.10 to get some practice on the cumulative frequency diagram, the median and the quartiles.

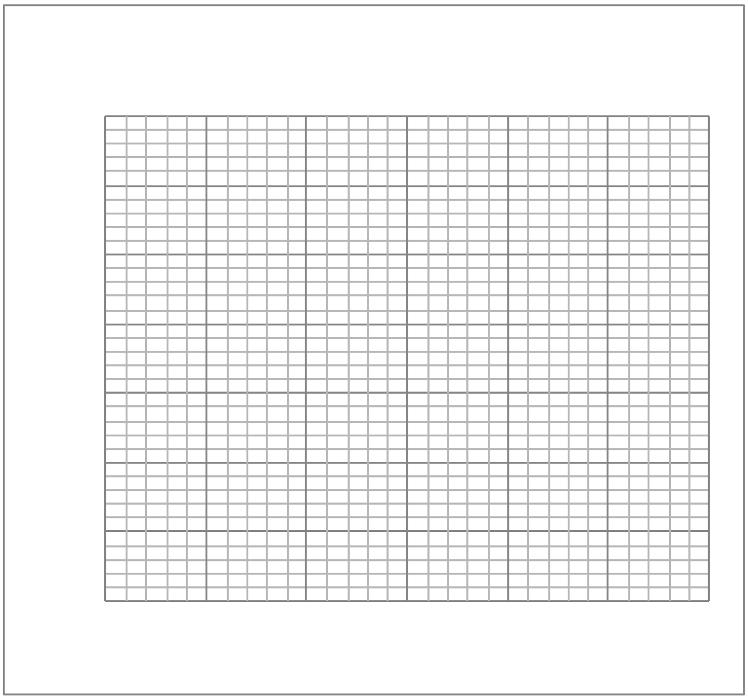
Activity 1.10.

The following table shows the masses of 49 people in a weight loss challenge club.

Group	50-54	55-59	60-64	65-69	70-74
Cumulative Frequency	4	13	33	46	49

Table 1.31-The cumulative frequency table showing the masses of 49 people in a weight loss challenge club

a) Draw the cumulative frequency diagram to show the information.



b) From your graph estimate the median and the quartiles.

Compare your answers with those given at the end of the subtopic. If your answers to the median or the quartiles differ slightly, your answers are still correct as it is an estimate from the graph.

Inter quartile range

Let us suppose that we have a student who sat for a Physics test with other students and obtained 60%. The mark as it is has limited meaning, but if we know what other students have obtained, we can rate the performance. Let us suppose we now have the following scenario:

Mpho got 60% in a Physics test, the highest mark of the test is 65% and the lowest 10%

In a Chemistry test Mpho got 60% also, the highest mark of the test is 90% and the lowest 59%.

Has Mpho performed better on the Physics test or on the Chemistry test?

Compare your answer with the following.

Since on the chemistry test she is almost the lowest of the class and on the Physics test she is not far from the highest she has performed better on the Physics test than on the Chemistry test.

To rate the performance of the class as a whole we can take the **range** of the class which is given by the highest mark minus the lowest mark. That is we can assess the value of the mark. However, with the range we are looking only at the extreme data and not considering the data in between. So the range is not the best method of assessing the value of data as it looks only at the extremes.

The alternative method to the range is the **inter quartile range**. The **inter quartile range** is defined as: **upper quartile - lower quartile**. The inter quartile range depends on the middle half of the values and is not affected by the extreme values.

That is it with the inter quartile range. Now let us move on to the percentiles.

The Percentiles

The word percent means over 100.

We said the quartiles divide the total frequency into four equal parts. Now there are what we call percentiles and they divide the total frequency into 100 equal parts hence the name percentile. The value of the variable at each point of the division is called the percentile. Thus the s^{th} percentile is given by $\frac{s}{100}(n+1)$ or $\frac{s}{100}n$ for large *n*.

Example 1:

The contents of table 1.24 are relisted below,

Marks	0-4	5-9	10-14	15-19	20-24	25-29
Frequency	4	10	15	10	6	5

Cumulative Frequency	4	14	29	39	45	50

Table 1.32 frequency distribution table showing students' marks:

-The Cumulative frequency diagram is also drawn for you in example 1 on page 29.

-Calculate the inter quartile range.

-Estimate the 90th percentile from your graph.

Solution

Inter quartile range = upper quartile - lower quartile

- From activity 1.9 where the quartiles were introduced, we found that

the upper quartile = 18.8

and the lower quartile = 8.4,

so the estimated inter quartile range= 18.8-8.4=10.4.

-The 90th percentile is given by $\frac{90}{100}(50+1) = 45.9$

From the graph the 90th percentile is estimated to be 25 marks.

The graph follows.

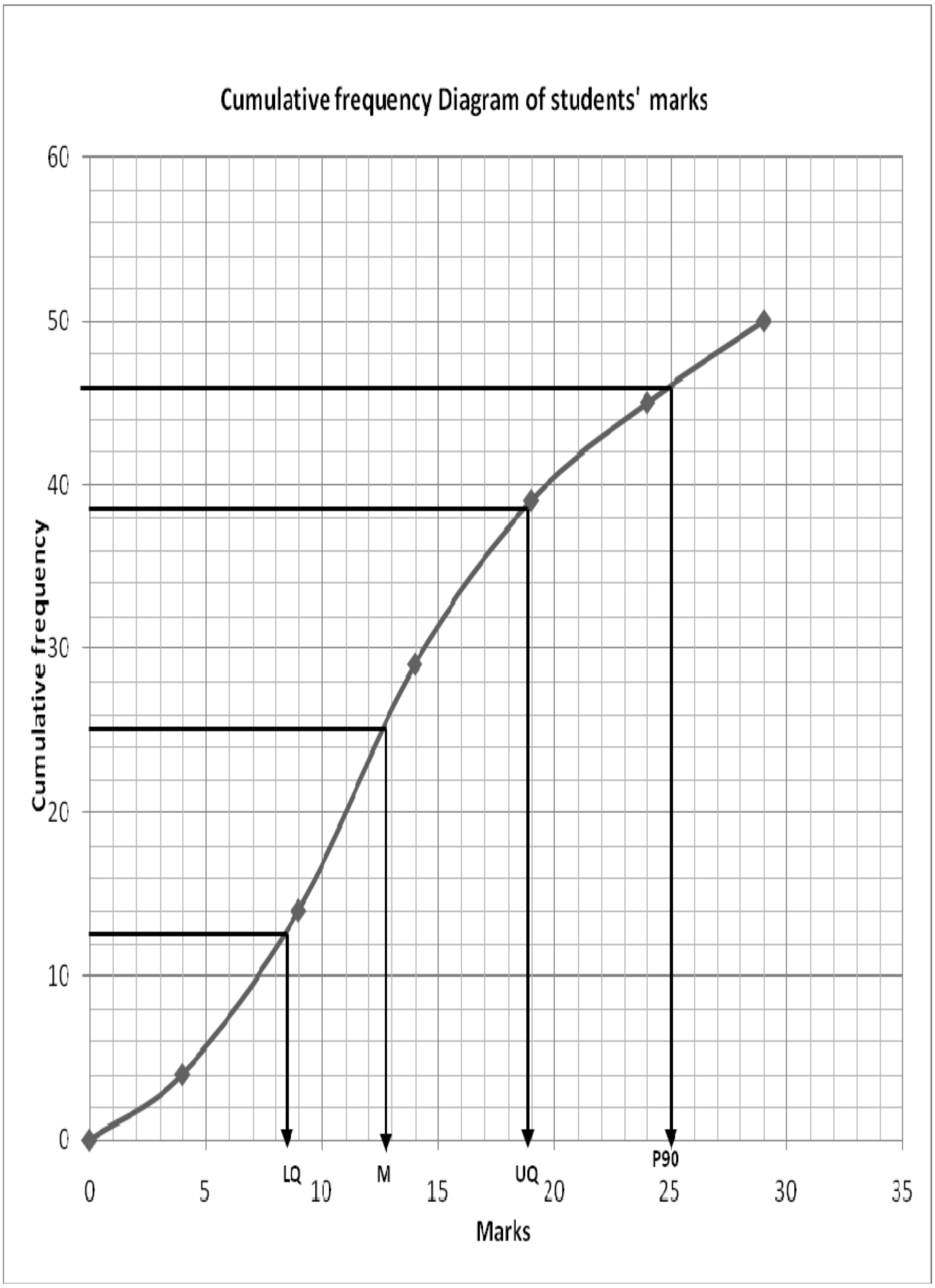


Figure 9-The Cumulative frequency diagram showing students' marks.

Example 2

The following cumulative frequency diagram has been drawn for you. Answer the questions that follow.

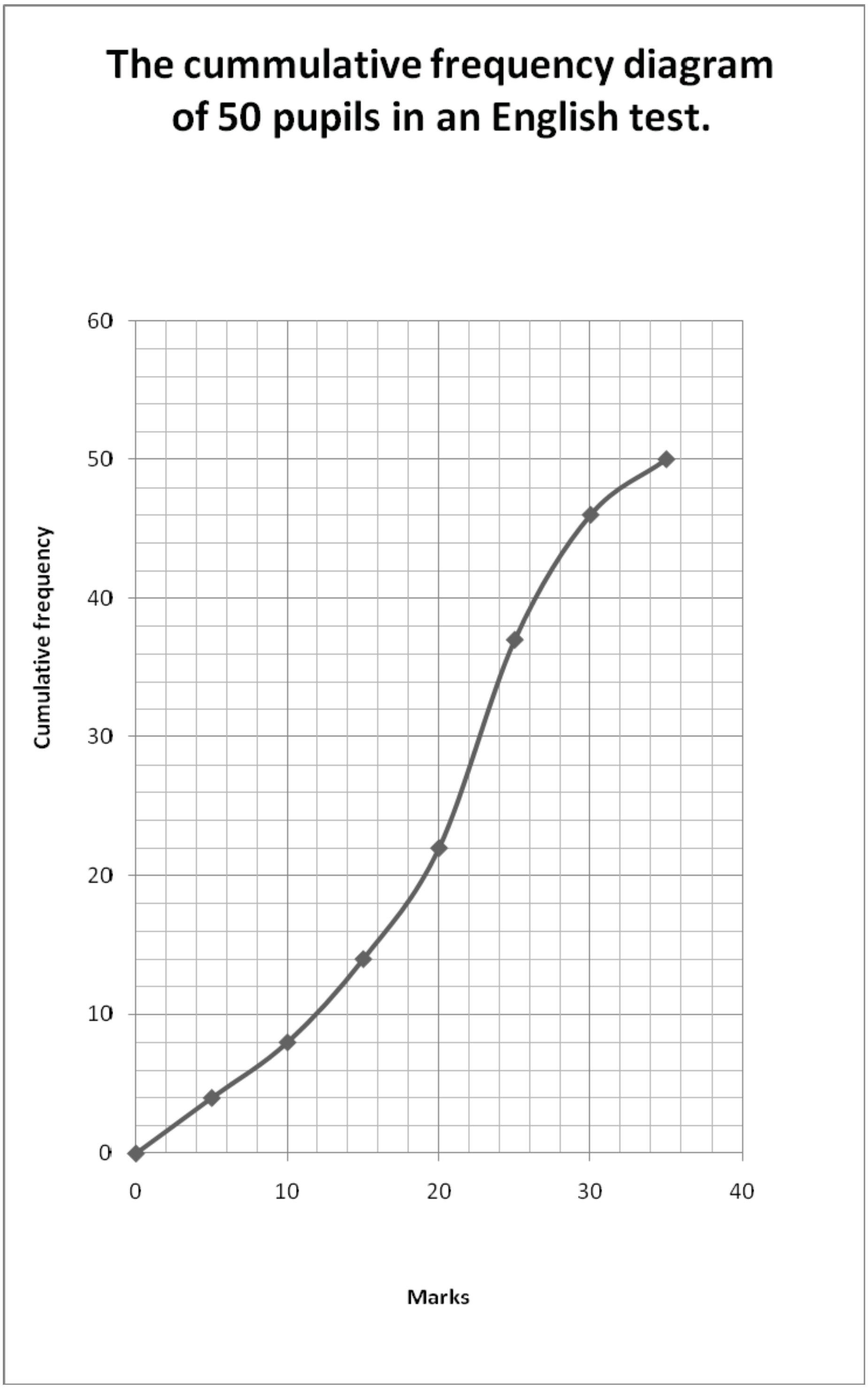


Figure 10 – Cumulative frequency diagram of 50 pupils' marks in an English test

Estimate:

a) The median

b) The lower quartile

c) The upper quartile

d) The inter quartile range

e) The 20^{th} percentile

Solution to example 2

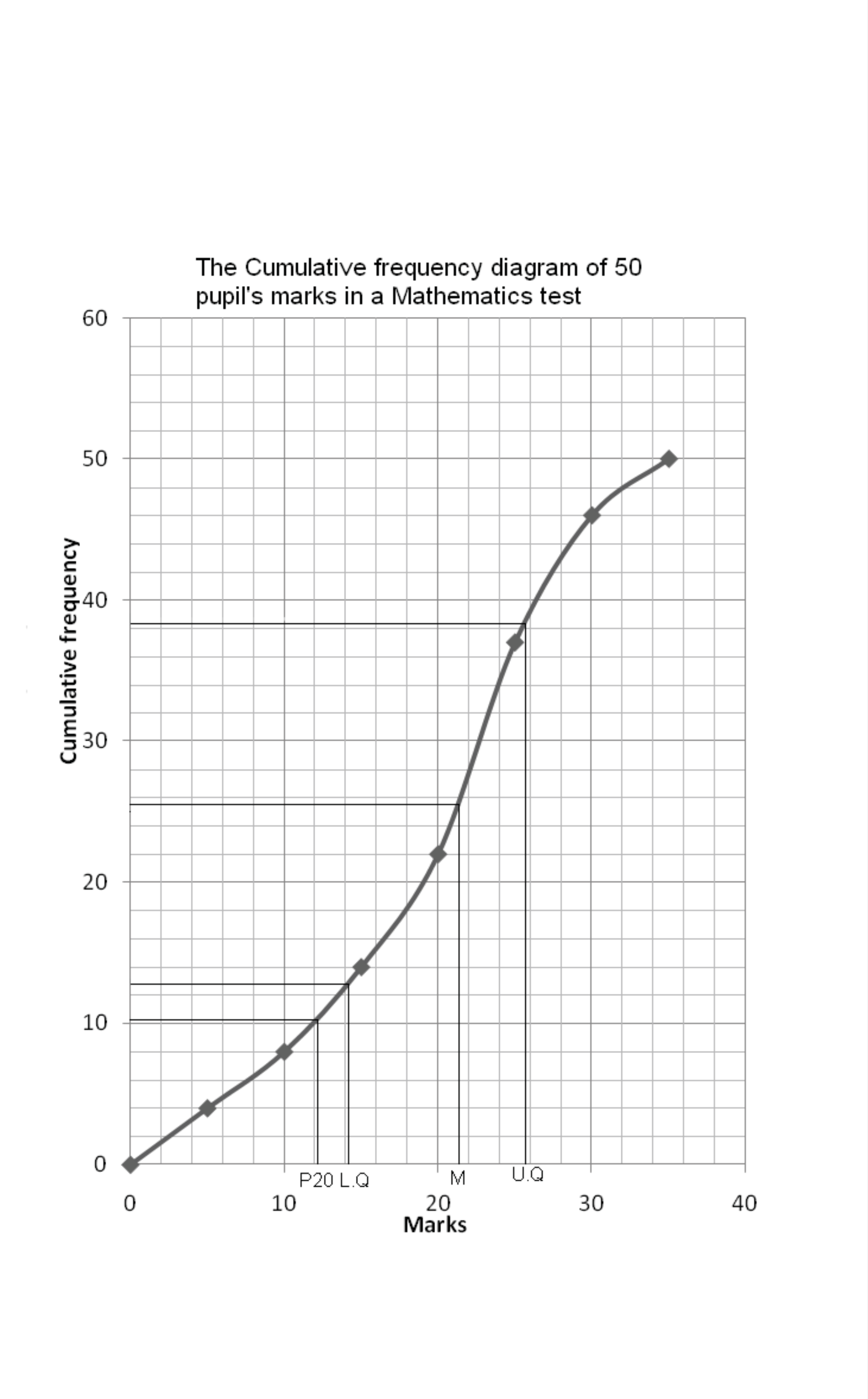


Figure 21- The cumulative frequency diagram of 50 pupils' marks in a mathematics test

a) The median position is $\frac{1}{2}(50+1) = 25.5$

So from the graph the median is estimated to be 21.4 marks.

b) The lower quartile position is $\frac{1}{4}(50+1) = 12.75$

So the lower quartile is estimated from the graph to be 14.1 marks.

c) The upper quartile position is $\frac{3}{4}(50+1) = 38.25$

From the graph the upper quartile is estimated to be 25.7

d) Inter quartile range = 25.7-14.1=11.6 marks.

e) 20^{th} percentile position is $\frac{20}{100}(50+1) = 10.2$

From the graph the 20^{th} percentile is estimated to be 12.2 marks.

Now work on activity 1.11 which is on the cumulative frequency diagram, the median, the quartiles and the percentiles.

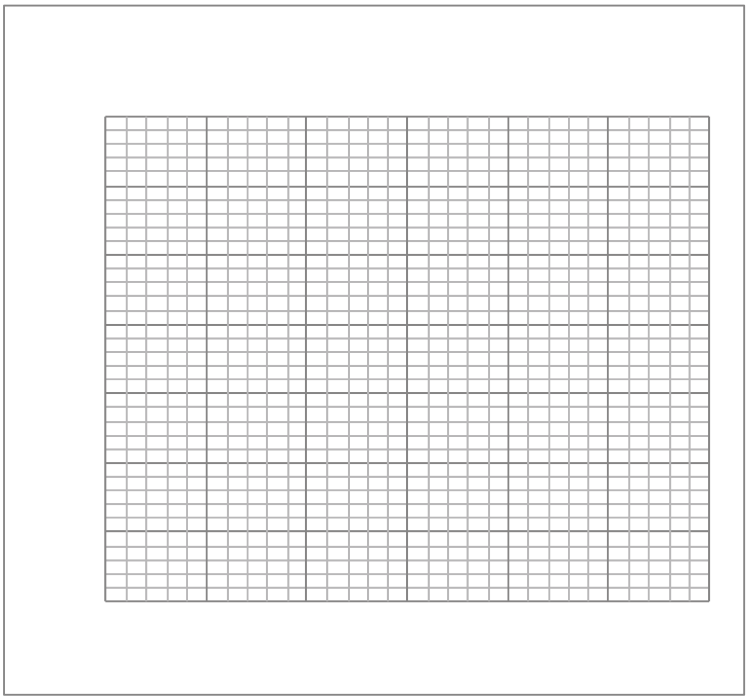
Activity 1.11

1) The following cumulative frequency table gives data about the number of letters in the 59 words of a written summary.

No. Of letters	1-5	6-10	11-15	16-20	21-25	26-30	31-40
Number of words	4	15	36	48	57	58	59

Table 1.33- cumulative frequency table of the number of letters in 59 words of a summary.

a) Draw a cumulative frequency diagram to show the information.



b) Use your cumulative frequency diagram to estimate

- The median number of letters per word

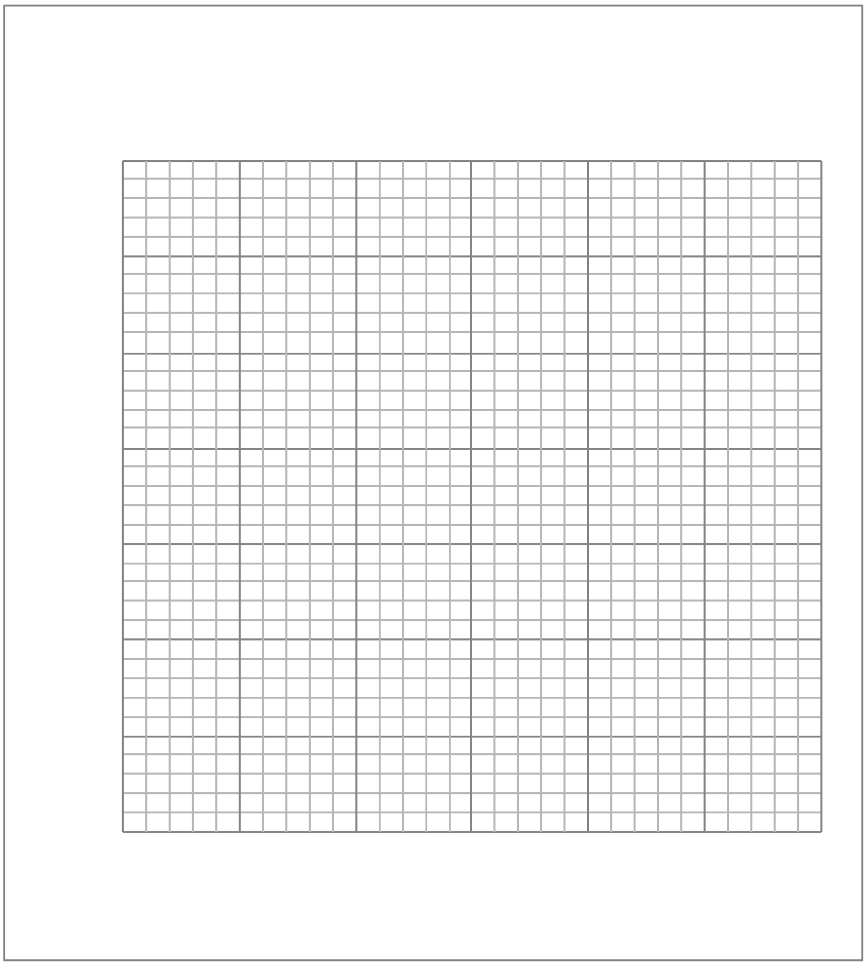
- The inter quartile range and the 80th percentile

2. Table 1.34 shows the percentage marks of 60 pupils in a mathematics test.

Class interval	0-9	10-19	20-29	30-39	40-49	50-59	60-69
Frequency	4	5	5	10	9	15	12
Cumulative frequency	4	9	14	24	33	48	60

Table 1.34-Frequency distribution table of students' percentage marks

a) Draw the cumulative frequency diagram of the data shown on table 1.25.



b) Use your diagram to estimate the median

c) Find the 60^{th} percentile

d) Estimate the mean of the data

e) If the pass mark was 40%, how many pupils passed?

Compare your answers with those at the end of this subunit. Continue on if you scored at least 80%. If not, review the above content and work through the activity again.

Key Points to Remember:

The key points to remember in this subunit on cumulative frequency diagrams, median, quartiles and percentiles are:

-The lower quartile (L.Q.) has a quarter of the variable values below it. The lower quartile position is given by $\frac{1}{4}(n+1)$, or $\frac{1}{4}n$ for large *n, where n is the sum of frequencies.*

-The median which is the middle quartile has half of the variable values below it and the median position is given by $\frac{1}{2}n+1$ or $\frac{1}{2}n$ for large *n.*

-The upper quartile has three quarters of the variable below it, whereby the upper quartile position is given by $\frac{3}{4}(n+1)$ or $\frac{3}{4}n$ for large *n*.

-Inter quartile range = upper quartile- lower quartile**.**

Percentiles divide the total frequency into 100 equal parts,

Thus the sth percentile is given by $\frac{s}{100}(n+1)$ or $\frac{s}{100}n$ for large *n*

-The cumulative frequency diagram is used to estimate the median and the quartiles.

-The cumulative frequency diagram is a graph of which its ordered pairs consist of (upper class value, cumulative frequency).

<u>Solutions to activities:</u>

Solution to activity 1.9

a)

Marks	0-4	5-9	10-14	15-19	20-24	25-29
Frequency	4	10	15	10	6	5
Cumulative Frequency	4	14	29	39	45	50

Table 1.35- The cumulative frequency table of students' marks

b) Median position = $\frac{1}{2}(50+1) = 25.5$

c) The median is between positions 25 and 26. Class 10-14 occupies positions 15 to 29. Therefore positions 25, 25.5 and 26 are all on class 10-14. The median class is the 10-14 class.

Solutions to activity 1.10

a)

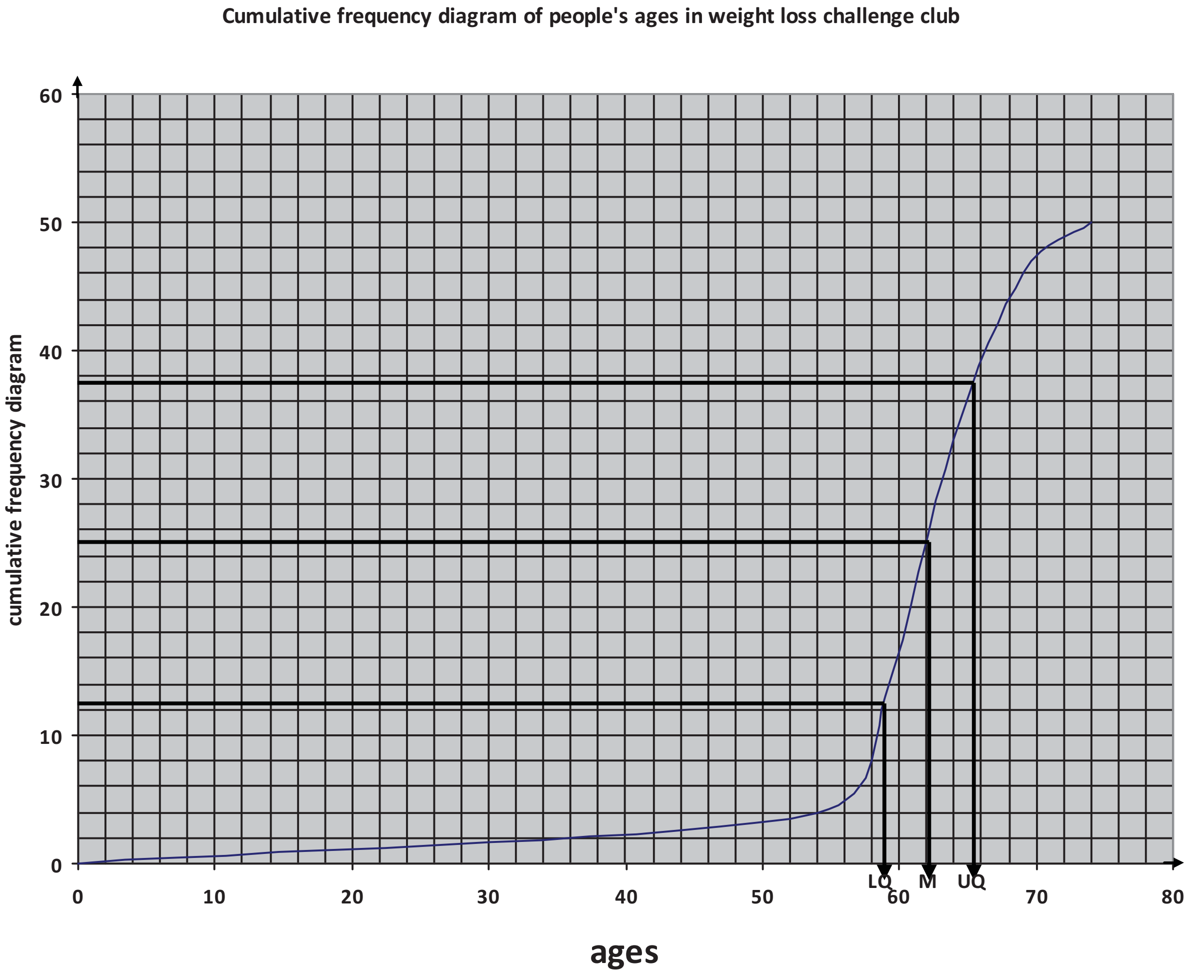


Figure 12-Cumulative frequency diagram of peoples' ages in a weight loss challenge

b)

- The median position is given by $\frac{1}{2}(49+1)=25$

  Now from the graph the median is 62 years.

- The upper quartile position is given by $\frac{3}{4}(49+1)=37.5$

So an estimated value of the upper quartile is 65.5 years.

- The lower quartile position is $\frac{1}{4}(49+1)=12.5$

  An estimated value of the lower quartile from the graph is 59 years.

Solutions to activity 1.11

1.

a)

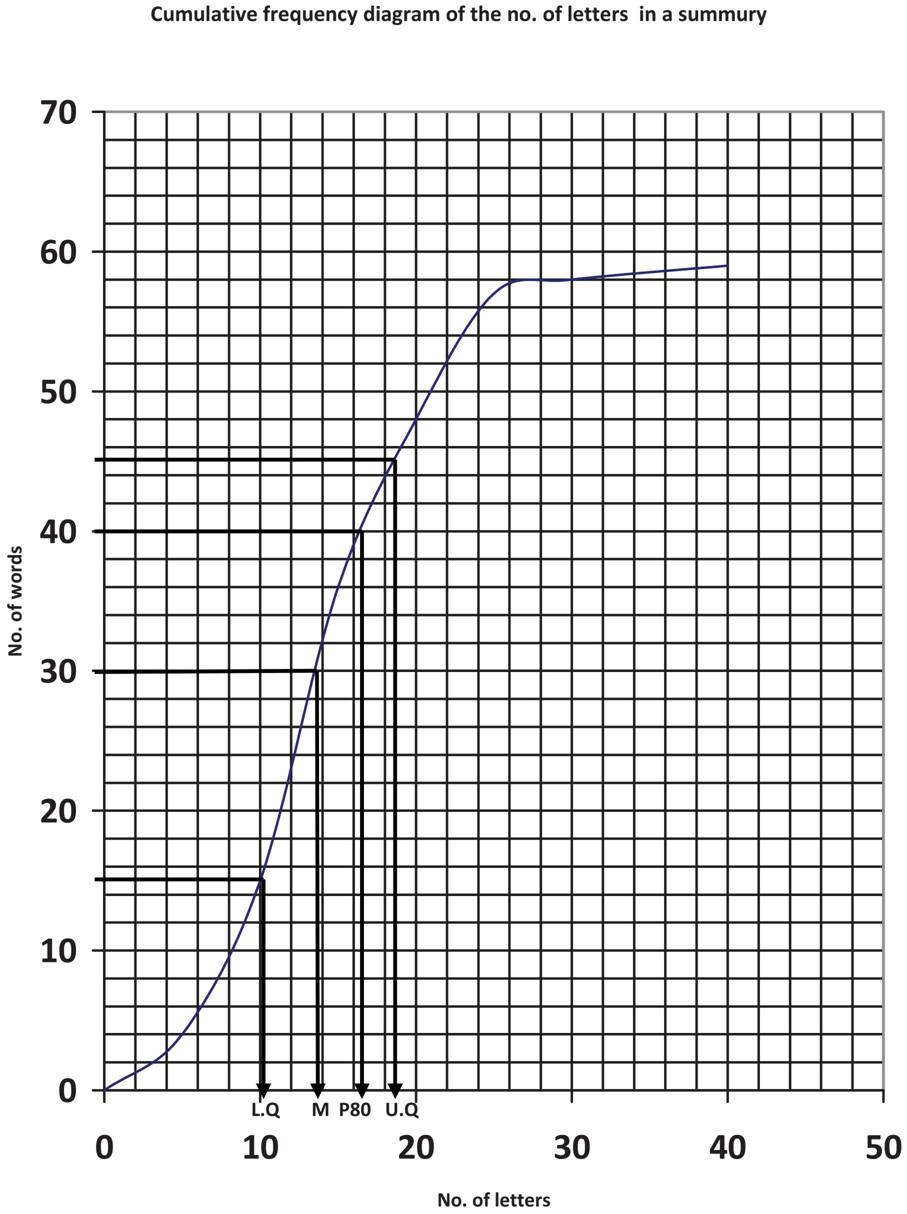


Figure 13- The cumulative frequency diagram of the no. of letters in a 59 word summary

b)

- The median position is at $\frac{1}{2}(59+1)=30$

From the graph the median number of letters per word is estimated to be 13.5 letters.

-To find the inter quartile range we first need to have the lower and the upper quartiles.

-The lower quartile position is at $\frac{1}{4}(59+1)=15$

Now from the graph the lower quartile is estimated to be 10 letters.

-The upper quartile position is at $\frac{3}{4}(59+1)=45$

And from the graph the upper quartile is estimated to be 18.4 letters

So the inter-quartile range= 18.4-10=8.4 letters

- The 80^{th} percentile position is at $\frac{80}{100}(59+1)=40$

From the graph the estimated value of the 80^{th} percentile= 18.4 letters.

2.

a)

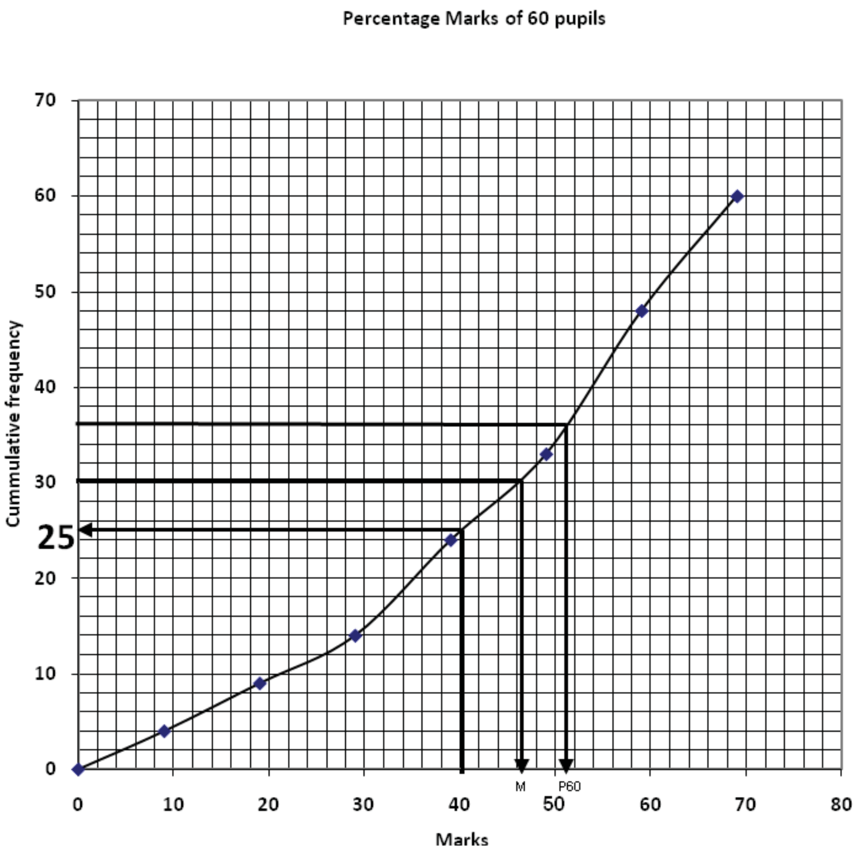


Figure 3- Cumulative frequency diagram of percentage marks of 60 pupils in a test

b) Median position $= \frac{1}{2}(60+1) = 30.5$

Now, looking at the graph the median has been estimated to be 46 marks.

c) The 60^{th} percentile position $= \frac{60}{100}(60+1) = 36.6$

So the 60^{th} percentile from the graph is estimated as 51.5 marks

d) To calculate the mean let us first redraw table 1.34 but including the column on frequency mean.

Class interval	0-9	10-19	20-29	30-39	40-49	50-59	60-69	**Total**
Frequency	4	5	5	10	9	15	12	**60**
Mid-value	4.5	14.5	24.5	34.5	44.5	54.5	64.5	
fx	18	72.5	122.5	345	400.5	817.5	774	**2550**

Table 1.36-frequency distribution table of students' marks

So the estimated mean= $\frac{2550}{60} = 42.5$

e) From the graph 25 students passed. The frequency 25 is on the pass mark according to the graph.

sson 6 Histograms and Frequency Polygons of Grouped Data

In the first subunit we dealt with the frequency polygon and other methods of displaying the ungrouped data. We will now discuss the other two methods that can also be used to display; the frequency polygon and the histogram.

By the end of this subunit you should be able to:

-Draw the frequency polygon for grouped data.

-Draw the histogram for any given data.

This subunit is about 15 pages long.

The Frequency Polygon in Grouped Data

We have seen how to draw the frequency polygon of ungrouped data before, but now we have learned that sometimes data is grouped. The following examples will show you step by step how to draw a frequency polygon of grouped data.

The Frequency polygon of grouped data has just a little difference from that of ungrouped data. The only difference is that with grouped data, the variable is the class mid value which is the average of the class.

Consider the following example on frequency polygon of grouped data.

Example 1

The following data on table 1.36 shows the sales of liters of milk bought by29 people last week.

liters sold	1-5	6-10	11-15	16-20	21-25	26-30
Number of People	1	3	6	10	7	2

Table 1.37-cummulative frequency table of litters of milk sold last week

Represent the distribution on table 1.37 on a frequency polygon

Solution to example 1

Earlier on we said the coordinates of a frequency polygon of ungrouped data are (variable, frequency). With grouped data the variable is taken as the class-mid value because it is the average of the class.

So to find the coordinates we first need to find the class mid values.

The following table shows the class mid-values

litters sold	1-5	6-10	11-15	16-20	21-25	26-30

Mid-value	3	8	13	18	23	28
Number of People	1	3	6	10	7	2

Table 1.38-cumulative frequency table litters of milk sold last week

Now the point that we are to plot from table 1.37 are (class mid-value, frequency), which are: (3,1) (8,3) (13,6) (18,10) (25,7) (28,4).

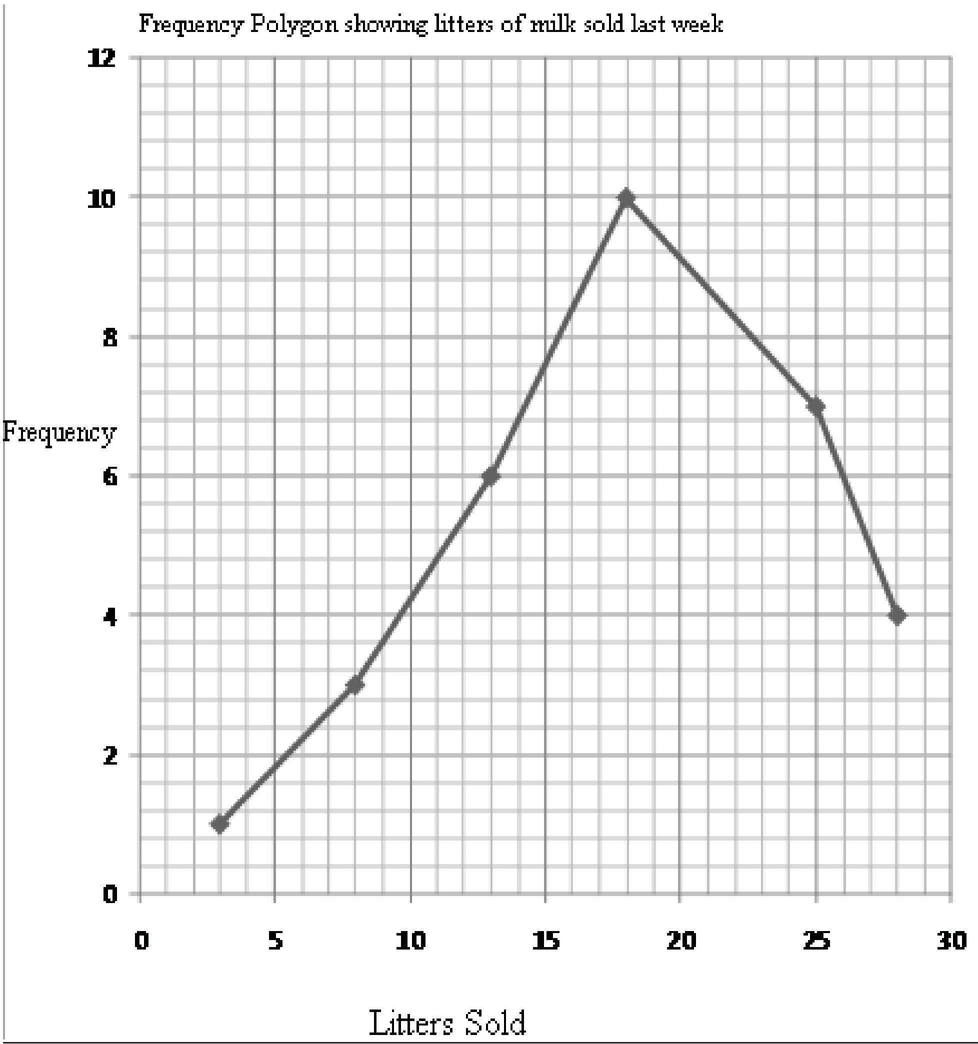


Figure 15

If you do not understand how to draw the frequency polygon of grouped data, revisit example 1. The only trick is to get the class mid values. Now work on activity 1.13.

Activity

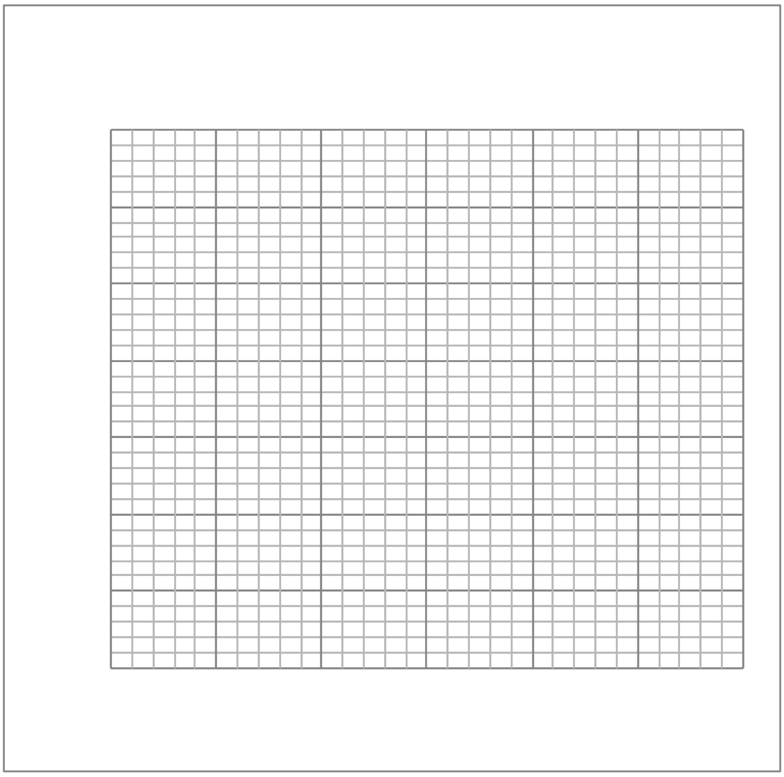
Activity 1.12

1. Table 1.39 shows litters of milk produced by 70 cows in a week

Litters of Milk	0-9	10-19	20-29	30-39	40-49	50-59
Number of Cows	2	4	8	15	13	7

Table 1.39- Frequency distribution table showing litters of milk produced by 70 cows in a week.

Draw the frequency polygon to represent the distribution shown on table 1.39

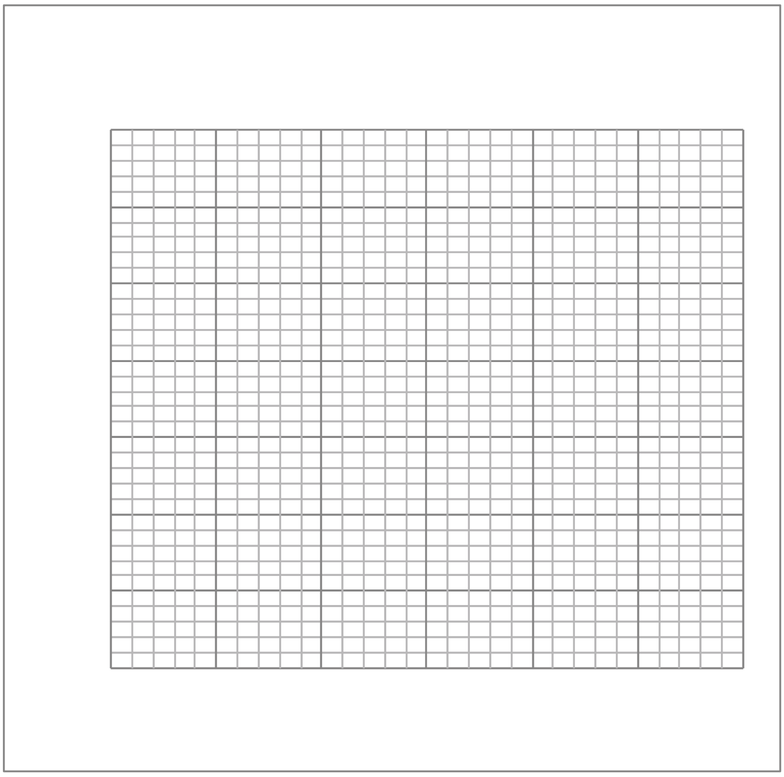


2. The following table shows the number of chicks hatched in 2 months.

Number of Chicks	1-10	11-20	21-30	31-40	41-50	51-60
Number of Hens	12	16	45	36	25	10

Table 1.40- Frequency distribution table showing the number of chicks hatched in 2 months

Draw a frequency polygon to show the information.



When you are done answering the questions, compare your answers with those given at the end of the subunit. Remember to go back and check the concepts if you feel like you missed something.

Histograms

Activity

Activity 1.13

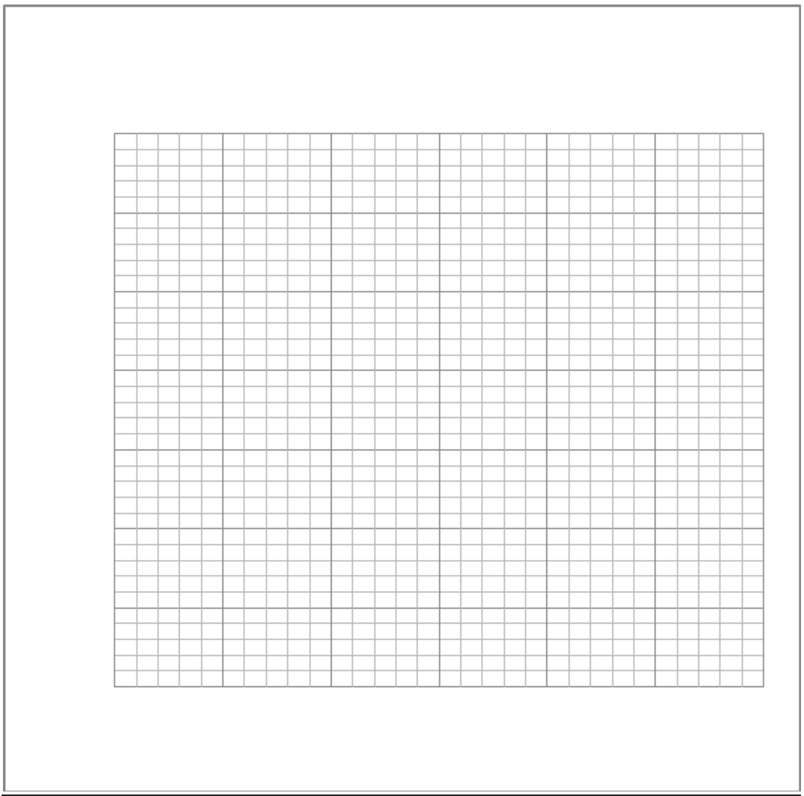
This activity addresses how to draw the histogram.

Table 1.41 shows the favourite numbers of 50 students.

Represent the information shown on table 1.41 on a bar chart.

Number	0	1	2	3	4	5	6	7
Frequency	8	10	9	10	3	2	5	3

Table 1.41-Frequency distribution table showing students' favourite numbers



Compare your answers with those given at the end of the subunit. Be sure you understand how to draw the bar chart before continuing.

The information shown on table 1.41 could also be represented as shown on table 1.42

Number	0	1	2	3-4	5-7
Frequency	8	10	9	13	10

Table 1.42- Frequency distribution table showing students' favourite numbers

Note that the variables are not the same width; some contain 1 number, others two numbers like 3-4 and others 3 numbers like 5-7.

So in table 1.42 the data is grouped and the groups have different widths.

With a bar chart, the groups or classes have the same widths; hence the bars have the same widths. **As a result the heights of the bars give the frequencies of the groups/variables.**

On the other hand **with a histogram,** the classes have different widths and the bars also have different widths, so the bars are now called the rectangles. Unlike the bar chart, **with the histogram, the frequency of the variables is given by the area of the rectangles.** The height of the rectangles is not the frequency; instead it is the **frequency density.**

Example 1

Draw the histogram to represent the information shown on table

1.43.

Number	0	1	2	3-4	5-7
Frequency	8	10	9	13	10

Table 1.43- frequency distribution table showing 50 students' favourite numbers

Solution to example 1

Before drawing the histogram, we are first going to expand table 1.43 by adding two extra rows. Since we said in a histogram our rectangles have different widths, one row is on the widths. The other one shows the heights of the bars.

To get the width we can decide to have a unit or a standard width and then build other units on the standard one. For instance, in this example we are taking our unit as 1 number.

Then to get the height we divide the frequency by the width since the frequency is given by the area of the rectangle which is the width multiplied by the height.

Class	0	1	2	3-4	5-7
Width	1 unit	1 unit	1 unit	2 units	3 units
Frequency	8	10	9	13	10
Height	$8/1 = 8$	$10/1 = 10$	$9/1 = 9$	$13/2 = 6.5$	$10/3 = 3.3$

Table 1.44- frequency distribution table showing 50 students' favourite numbers

Then from this new table we can draw the histogram.

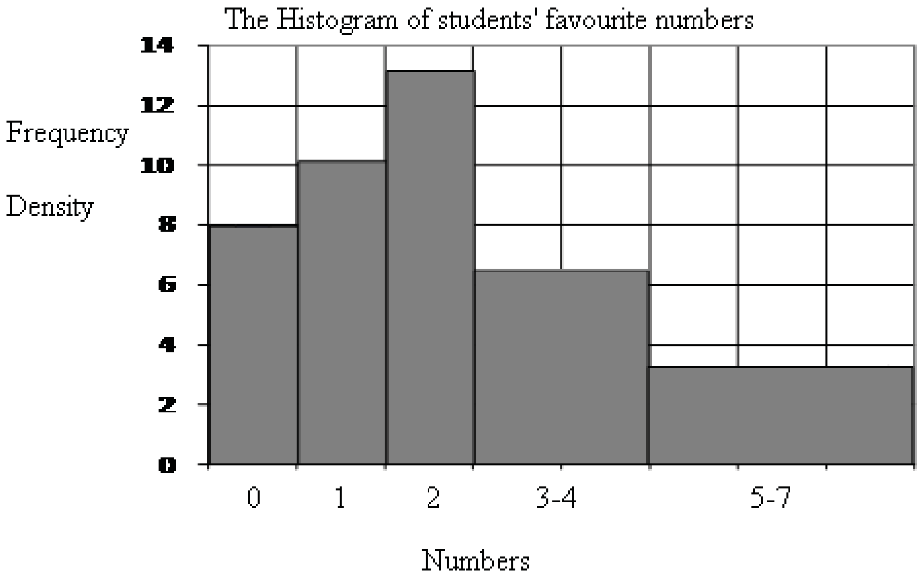


Figure16-Histogram showing 50 students' favourite numbers

Example 2

8000 electric light bulbs were tested to see how long (in hours) they would last. The results are shown in the following table:

Number of hours	0-400	400-1200	1200-2000	2000-2400
Number of bulbs	1200	4000	2000	800

Table 1.45-Frequency distribution table showing the tests of 8000 electric bulbs

Draw the histogram to represent the distribution.

Solution to example 2

First let us put the rows on the width and frequency density (height).

Let 1 unit = 400 hours.

Frequency Density = $\frac{Frequency}{Width}$

Number of hours	0-400	400-1200	1200-2000	2000-2400
Width	1 unit	2 units	2 units	1 unit
Number of bulbs(Frequency)	1200	4000	2000	800

Frequency Density (Height)	$\frac{1200}{1} = 1200$	$\frac{4000}{2} = 2000$	$\frac{2000}{2} = 1000$	$\frac{800}{1} = 800$

Table 1.46- Frequency distribution table showing the tests of 8000 electric bulbs

Now let the histogram is draw below.

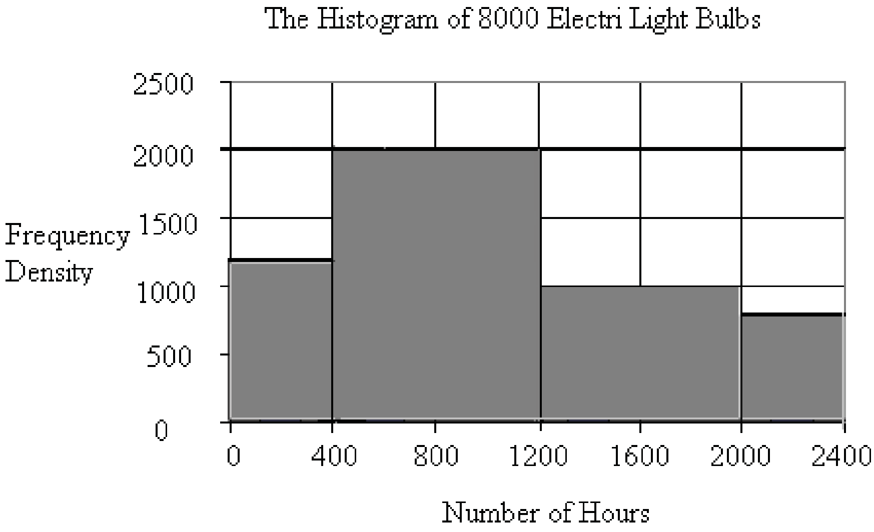


Figure 17- The Histogram showing the tests of 8000 electric bulbs

Now try activity 1.14 on histograms.

Activity 1.14

1. From the Histogram in figure 18 below, state the frequency for each class of the ages of the students.

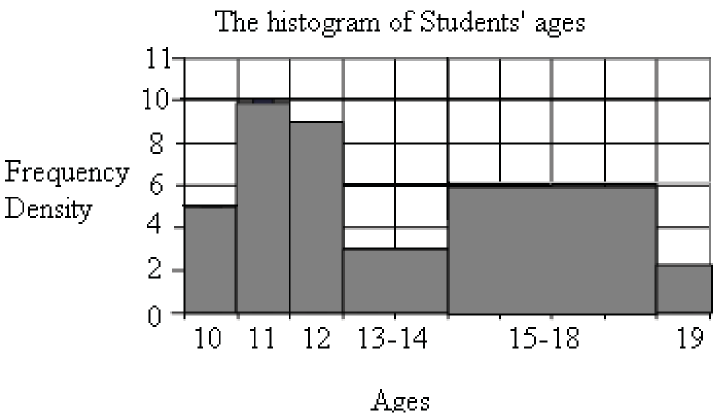


Figure 18- The histogram showing students'ages

____

____

____

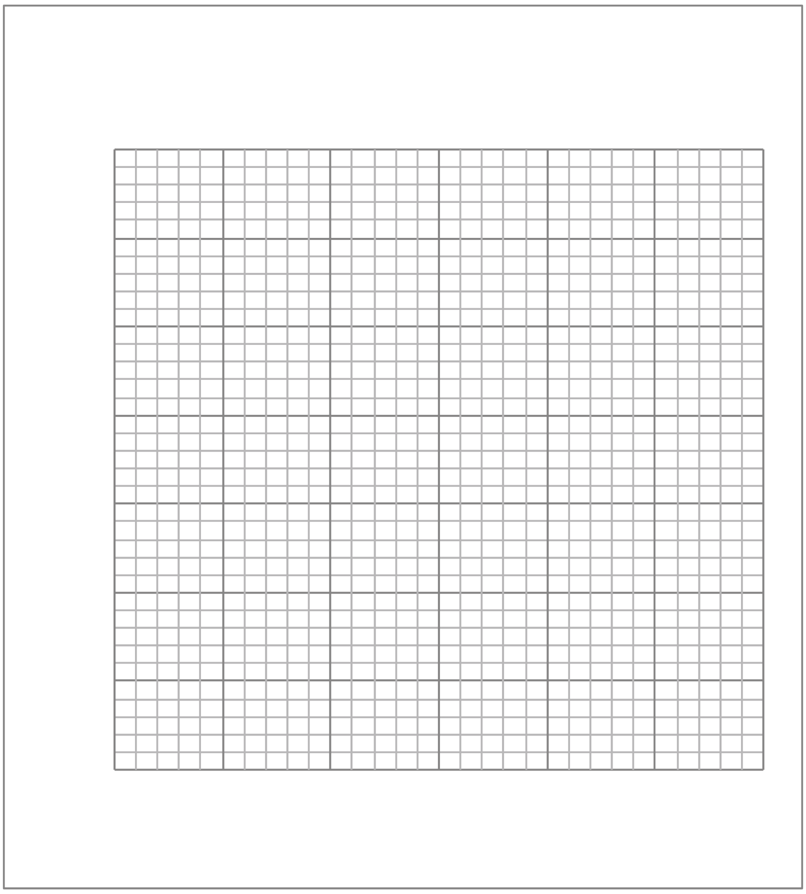
____

2. Table 1.47 shows the percentage marks of 39 students.

   Represent the information shown on table 1.47 on a histogram making 5 marks to represent 1 unit.

Marks	50-54	55-59	60-64	65-74	75-89
Frequency	3	10	4	10	12

Table 1.47-Frequency distribution table showing percentage marks of 39 students



Compare your answers with those given at the end of the subunit. If you got all of the answers correct, feel free to move onto the next subtopic, if not, review this subunit again.

Key Points to Remember

The key points to remember in this subunit on histograms and frequency polygons of grouped data are:
- With grouped data the frequency polygon coordinates consist of the class (class mid-value, frequency)
- With the histogram, the frequency of the variables is given by the area of the rectangles.
-The height of the rectangles is not the frequency; instead it is the **frequency density.**

You have now completed the last subunit of this unit. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Solution to activity 1.12

1. First let us draw the table to show the mid-values and then plot the mid-values against the corresponding frequencies (Number of cow).

Litters of milk (class)	0-9	10-19	20-29	30-39	40-49	50-59
Mid-value	4.5	14.5	24.5	34.5	44.5	54.5
Number of Cows	2	4	8	15	12	7

Table 1.48-Frequency distribution table showing liters of milk produced by 70 cows

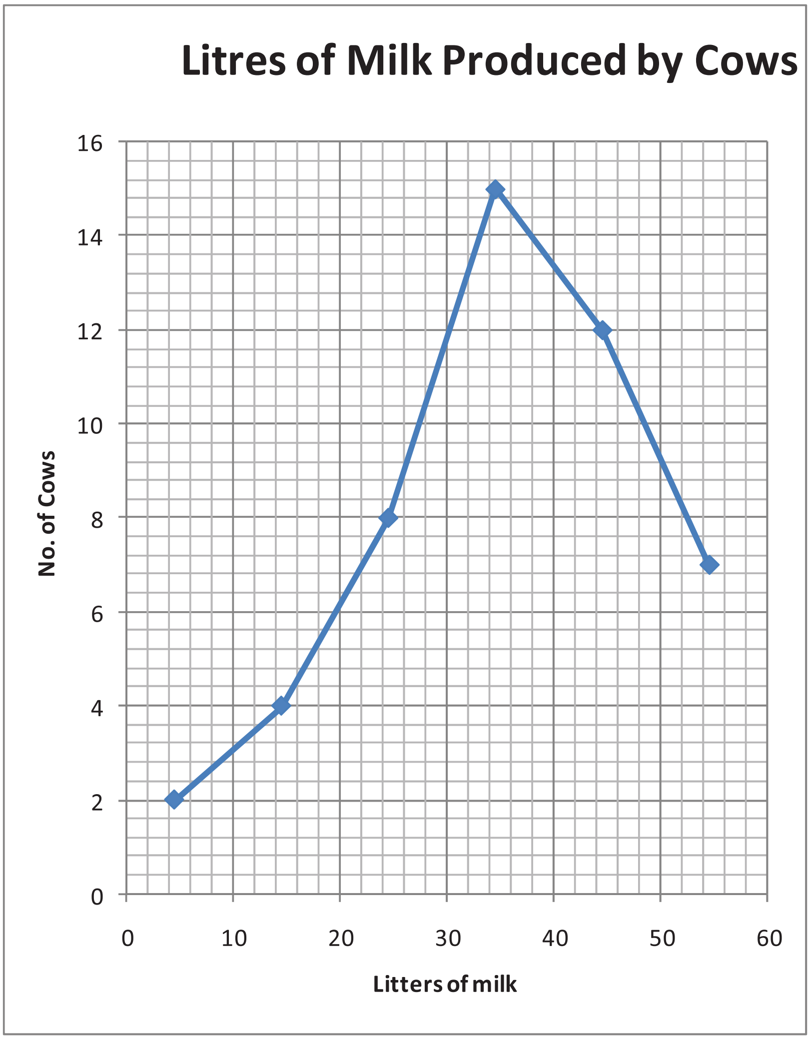


Figure 19

2. The mid-values are shown in the following table.

Number of Chicks	1-10	11-20	21-30	31-40	41-50	51-60
Mid-value	5.5	15.5	25.5	35.5	45.5	55.5
Number of Hens	12	16	45	36	25	10

Table 1.49-Freequency distribution table showing number of chicks hatched.

The coordinates are (5.5, 12) (15.5, 16) (25.5, 45) (35.5, 36) (45.5, 25) (55.5, 10).

The frequency polygon is show below.

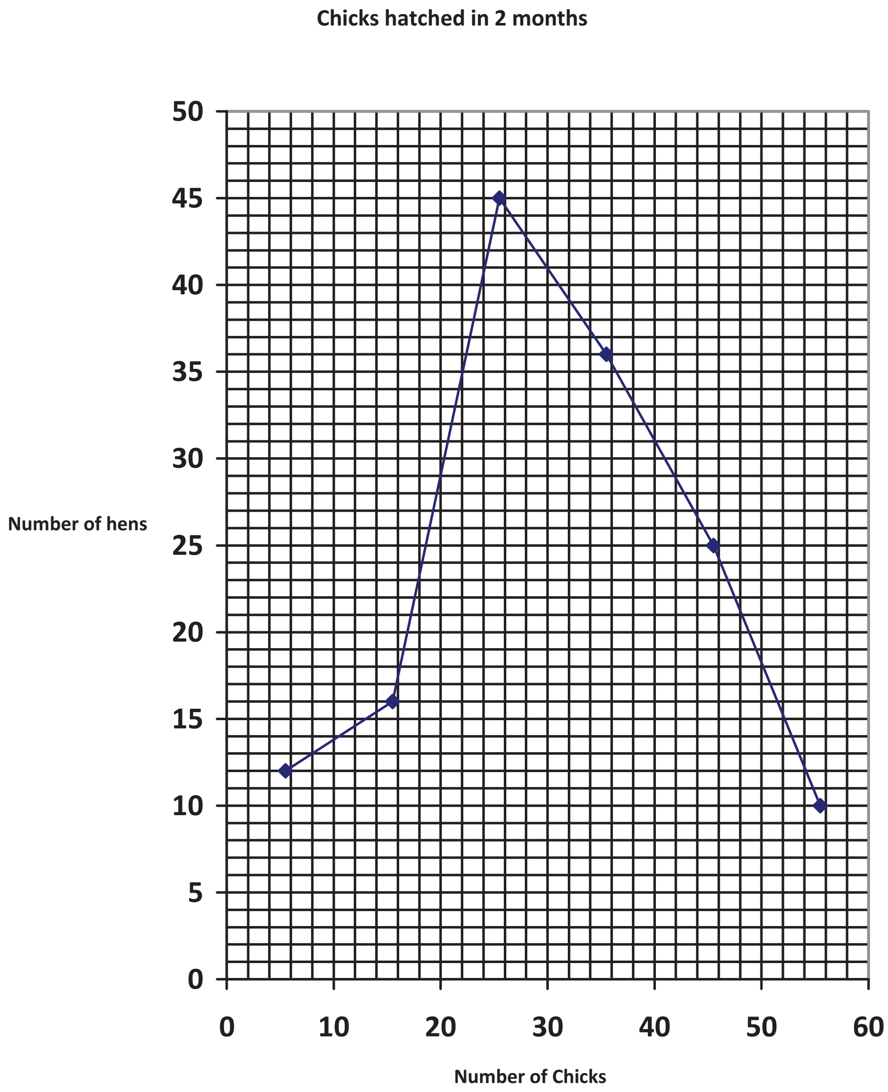


Figure 20-Frequecy polygon showing chicks hatched in 2 months.

Solution to activity 1.13

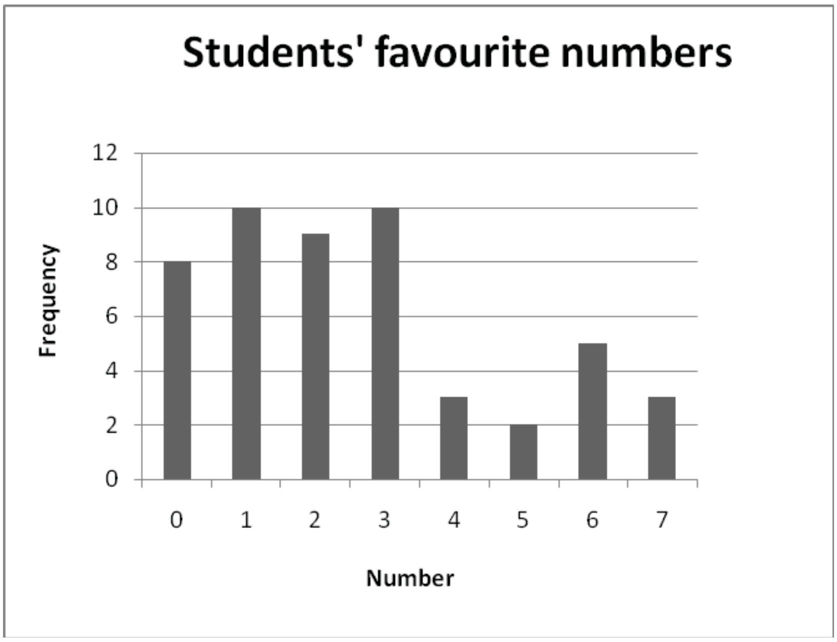


Figure 21-Bar chart showing students' favourite numbers

Solution to activity1.14:

1.

Class	10	11	12	13-14	15-18	19
Width	1 unit	1 unit	1 unit	2 units	3 units	1 unit
Height	5	10	9	3	6	2
Frequency(width $\times$ Height)	5	10	9	6	18	2

Table 1.50- The histogram showing students' ages

The frequencies are on the last column.

2. Let 5 marks make 1 unit.

Class	50-54	55-59	60-64	65-74	75-89
Width	1 unit	1 unit	1 unit	2 units	3 units
Frequency	3	10	4	10	12

Height/ Frequency Density	$\frac{3}{1}=3$	$\frac{10}{1}=10$	$\frac{4}{1}=4$	$\frac{10}{2}=5$	$\frac{12}{3}=4$

Table 1.51

Now the histogram is as show in the following figure.

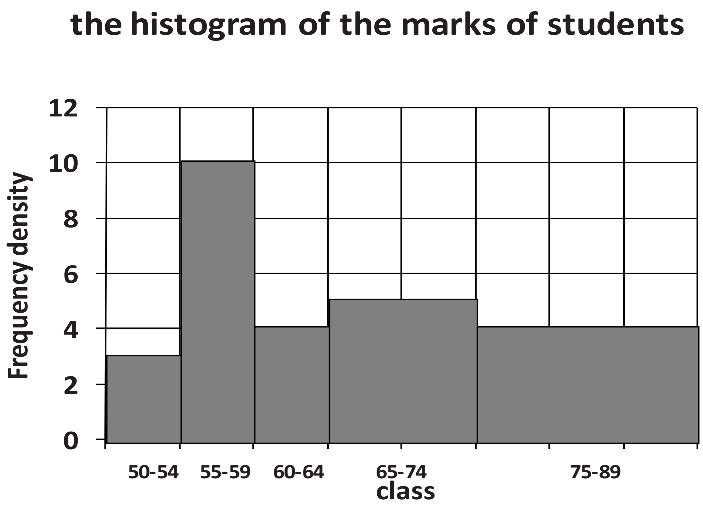


Figure 22-Histogram showing students' marks

it Summary

ummary

In this unit you learned that :

- Statistical data can be displayed in the form of

  i. **Frequency distribution table**

  ii. **Bar Charts** where the heights of the bars represent the frequencies.

  iii. **Pie Charts** whereby the data is represented as sector angles and each sector represents the frequency, the formula for the sector angle is given as:

  $$\frac{x}{sum\_of\_frequencies} \times 360^0$$

  iv. **Pictograms** in which data is displayed with pictures or symbols

  v. **Frequency Polygon.** The coordinates of the frequencies are (variable, frequency).

- **Mode** is the data which appears most frequently.
- **Modal group/ class** is the class in grouped data which appears most frequently.
- **Median** is found as data in the middle after arranging all data in order starting with the smallest.
- With tabulated data **Median position** is found by $\frac{1}{2}(n+1)$ or $\frac{1}{2}n$ (for large data) where n is the total frequency.
- Mean is found by adding all data and dividing the sum by the number of data
- With tabulated data mean $= \frac{sum\_of(frequecy \times variable)}{sum\_of\_frequency}$, but with grouped data the mid-value of each class is taken as the variable.
- The cumulative frequency diagram is used to estimate; the median, the upper quartile, the lower quartile and the inter quartile range, whereby;

  - Upper quartile position $= \frac{3}{4}(n+1)$ or $\frac{3}{4}n$ for large n, where n is the sum of frequencies.
  - Lower quartile position $= \frac{1}{4}(n+1)$ or $\frac{1}{4}n$ for

large n.

- x^{th} percentile position $= \frac{x}{100} \times (n+1)$ or $\frac{x}{100} n$ for large n.
- Inter quartile range = upper quartile – lower quartile.

➢ With grouped data the frequency polygon coordinates are given as: (class mid-value, frequency)

➢ With a Histogram the classes have different widths and the area of each rectangle represents the frequency.

You have completed the material for this unit on Statistics. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

signment

Assignment

Instructions

i. This assignment contains 70 marks.

ii. You should be able to complete it in 2 hours.

iii. Use calculators where necessary

iv. Answer all questions

Questions

No 1. The following data shows the marks of 40 pupils in a mathematical exam.

Display the data on a frequency distribution table making use of tally marks. (3 marks)

10 30 20 30 40 50 40 50 30 40 60 70 80 90 40 50 60 70 80 90
50 60 20 50 50 70 60 70 10 20 50 40 30 60 20 10 40 80 60

No 2.

Display the data given on question No 1 above in the form of a:

a) Bar chart (4 marks)

b) Frequency polygon (3 marks)

c) Pie chart (5 marks)

d) Pictogram (3 marks)

No 3.

Find the mode, mean and median for each of the following distributions.

a) 1 2 4 5 1 7 3 (3 marks)

b) 3 7 16 4 9 12 4 16 7 16 1 (3 marks)

c) 3 10 11 6 7 3 4 7 7 3 7 7 (3 marks)

d) 6 3 5 9 9 2 4 9 1 9 (3 marks)

No 4.

Table 1.40 shows the marks of 30 pupils in a test marked out of 20.

Mark	2	3	4	7	10	15	19	20
f	3	2	5	6	7	3	2	2

Table 1.52- the marks of 30 pupils in a test marked out of 20.

From table 1.52 find

a. The mean (4 marks)

b. The median (3 marks)

c. And the mode (1 mark)

No 5. Table 1.41shows the classes of masses of 50 objects measured in kilograms.

Mass-kg	30-39	40-49	50-59	60-69	70-79	80-89	90-99
Number of objects	13	10	7	10	5	3	2

Table 1.53- the classes of masses of 50 objects measured in kilograms

Use table 1.41 to find the

a. Median class (2 marks)

b. Modal class (1 marks)

c. Estimated mean (4 marks)

No 6. Table 1.54 shows the marks of 45 pupils in a mathematics quiz.

Marks(class)	11-15	16-20	21-25	26-30	31-35	36-40
Frequency	3	5	8	14	10	5

Table 1.54- the marks of 45 pupils in a mathematics quiz

a) Draw the cumulative frequency diagram of the given data. (4 marks)

b) Use the cumulative frequency diagram to estimate the median and the inter quartile range. (6 marks)

c) Which group(s) are above average? (4 marks)

d) Find the probability that a student chosen at random scored below marks (3 marks)

e) Display the data on table 1.42 on a frequency polygon. (4 marks)

No 7.

Table 1.43 shows the salaries of nurses per hour in rands depending on their qualifications.

Salary	10-19	20-29	30-49	50-89	90-99
Frequency	6	11	13	20	5

Table 1.55 - the salaries of nurses per hour in rands

Display the data shown on table 1.55 on a Histogram. (4 marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Solutions to the assignment

1. We make three columns, on marks, tally marks and on frequency.

Marks	Tally Marks	Frequency
10	|||	3
20	||||	4
30	卌	5
40	卌 |	6
50	卌 ||	7
60	卌 |	6
70	||||	4
80	|||	3
90	||	2

Table 1.56- The frequency distribution table showing marks of students in a mathematics test.

2.

a. Following is the bar chart of 40 pupils in a mathematics test as shown on table 1.44 on question 1. Remember we said the spacing between the bars and the widths of the bars should be the same and the bar chart as well as the axes must be labelled.

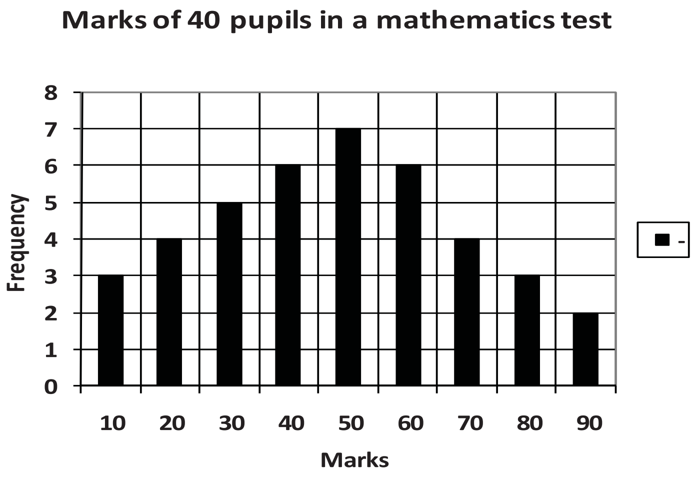


Figure 23- The bar chart showing 40 pupils' marks in a mathematics test

b. Following is the frequency polygon of the marks of 40 pupils as shown on table 1.56 in question 1. Remember with the frequency polygon we plot the marks against the corresponding frequency and the join the points with straight lines.

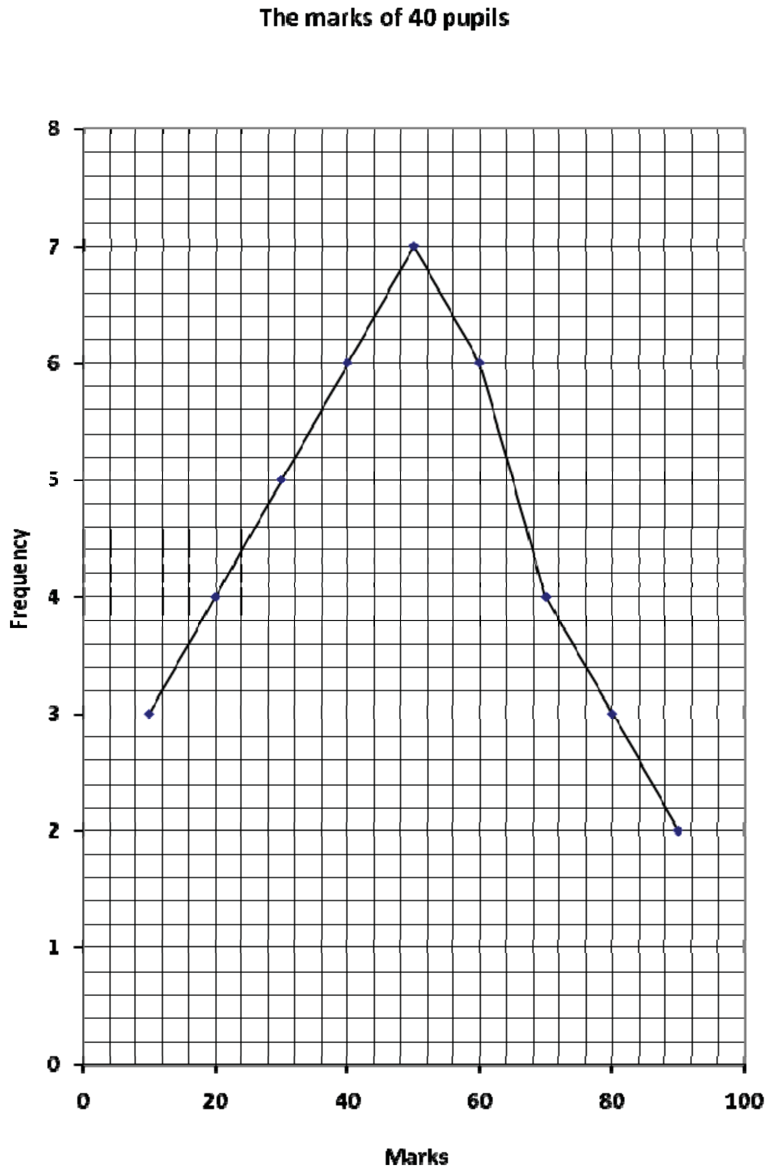


Figure 24- The frequency polygon showing pupils' marks in a mathematics test

c. Remember that with the pie chart we first need to know the angles representing each mark. Let us use table 1.57 to show the sector angles. One other thing that we need to put in our mind the formula we use to calculate the sector angles. The sector angle of each mark will be given by:

$$\frac{mark\_frequency}{sum\_of\_frequencies} \times 360^0$$

Mark	Frequency	Sector angle

10	3	$\frac{3}{40}\times 360^0 = 27^0$
20	4	$\frac{4}{40}\times 360^0 = 36^0$
30	5	$\frac{5}{40}\times 360^0 = 45^0$
40	6	$\frac{6}{40}\times 360^0 = 54^0$
50	7	$\frac{7}{40}\times 360^0 = 63^0$
60	6	$\frac{6}{40}\times 360^0 = 54^0$
70	4	$\frac{4}{40}\times 360^0 = 36^0$
80	3	$\frac{3}{40}\times 360^0 = 27^0$
90	2	$\frac{2}{40}\times 360^0 = 18^0$

Table 1.57- The frequency distribution table showing marks of students in a mathematics test

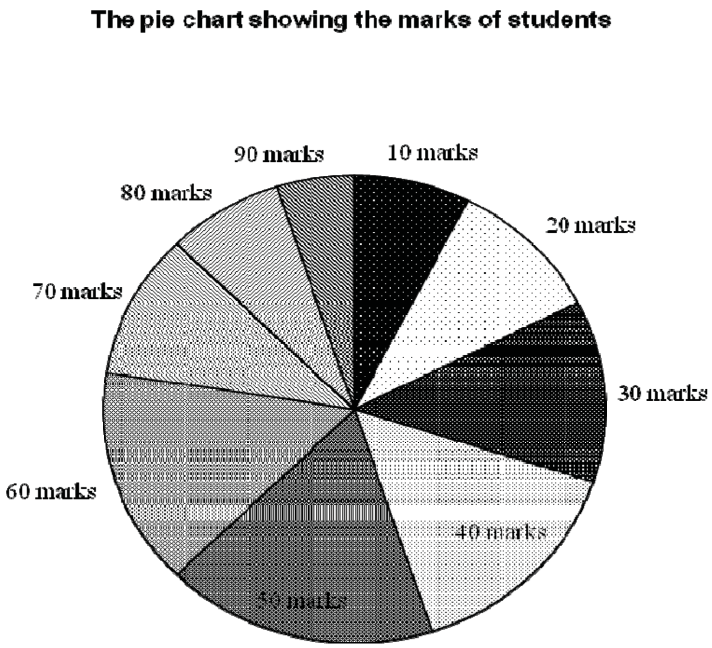


Figure 25- The pie chart table showing marks of students in a mathematics test

d.. Remember with the pictogram we display statistical data with pictures or symbols.

Key: let ☆ represent 2 students.

Mark	**Frequency**
10	½☆ ☆
20	☆ ☆
30	½☆ ☆ ☆
40	☆ ☆ ☆
50	½☆ ☆ ☆ ☆

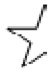
60	☆ ☆ ☆
70	☆ ☆
80	 ☆
90	☆

Table 1.58- The Pictogram showing the marks of students in a mathematics test

3.

a) 1 2 4 5 1 7 3

$Mode = 1$ (Remember mode is a number which appears most frequently)

$Mean = \frac{1+2+4+5+1+7+3}{7} = \frac{23}{7} = 3.3$ (When finding the mean we add all the numbers and divide them by their number)

Median: 1 1 2 <u>3</u> 4 5 7 (to find the median we first arrange the numbers in order starting with the smallest)

Median = 3.

b) 3 7 16 4 9 12 4 16 7 16 1

Mode= 16

$Mean = \frac{3+7+16+4+9+12+4+16+7+16+1}{11} = \frac{95}{11} = 8.6$

Median=1

c) 3 10 11 6 7 3 4 7 7 3 7 7

Mode = 7

$Mean = \frac{3+10+11+6+7+3+4+7+7+3+7+7}{12} = \frac{75}{12} = 6.25$

Median: 3 3 3 4 6 <u>7 7</u> 7 7 7 10 11 (we have two numbers 7 and 7 in the middle so to find the median we divide their sum by 2)

$Median = \frac{7+7}{2} = \frac{14}{2} = 7$

d) 6 3 5 9 9 2 4 9 1 9

Mode = 9

$$Mean = \frac{6+3+5+9+9+2+4+9+1+9}{10} = \frac{57}{10} = 5.7$$

Median: 1 2 3 4 5 6 9 9 9 9 (we have two number in the middle, 5 and 6)

So $median = \frac{5+6}{2} = 5.5$

4.

a) To find the mean, we are going to add a column on the frequency × mark so that we get the mean as the sum of frequency means divided by sum of frequencies

Mark	2	3	4	7	10	15	19	20	**Total**
f	3	2	5	6	7	3	2	2	**30**
***f*×mark**	6	6	20	42	70	45	38	40	**267**

Table 1.59- the marks of 30 pupils in a test marked out of 20.

So the $mean = \frac{267}{30} = 8.9$

b) The median position $= \frac{30+1}{2} = 15.5$. So the median lies between position 15 and position 16. Thus to find the median we add the marks in those positions and divide them by 2. At position 15 we have mark 7 and at position 16 we also have mark. So the $median = \frac{7+7}{2} = 7\ marks$

c) The mode is 10 marks because it has the highest frequency.

No. 5

a) The median position $= \frac{50+1}{2} = 25.5$. The median lies between positions 25 and 26. So the median class is class 50-59.

b) The modal class is class 30-39

c) To find the mean we first need to get the class mid-values and the frequency means.

Mark	30-39	40-49	50-59	60-69	70-79	80-89	90-99	**Total**
Mid-value	34.5	44.5	54.5	64.5	74.5	84.5	94.5	
No. of objects	13	10	7	10	5	3	2	**50**

f_m	448.5	445	381.5	645	372.5	253.5	189	**2735**

Table 1.60- the classes of masses of 50 objects measured in kilograms

So the estimated mean $\text{Estimated mean} = \frac{\text{Sum of } f_m \text{ values}}{\text{sum of } f \text{ values}}$

$= \frac{2735}{50} = 54.7 \text{ kg}.$

No 6.

a) Remember that when drawing the cumulative frequency diagram we first need to have the cumulative frequencies since the coordinates of the cumulative frequency curve are (upper class value, cumulative frequency). So table 1.61 has the column on the cumulative frequency.

Class	11-15	16-20	21-25	26-30	31-35	36-40	Total
f	3	5	8	14	10	5	45
Cumulative f	3	8	16	30	40	45	45

Table 1.61- the marks of 45 pupils in a mathematics quiz

Now we can draw our cumulative frequency diagram. We can also add the point (0, 0) to have our graph staring at the origin.

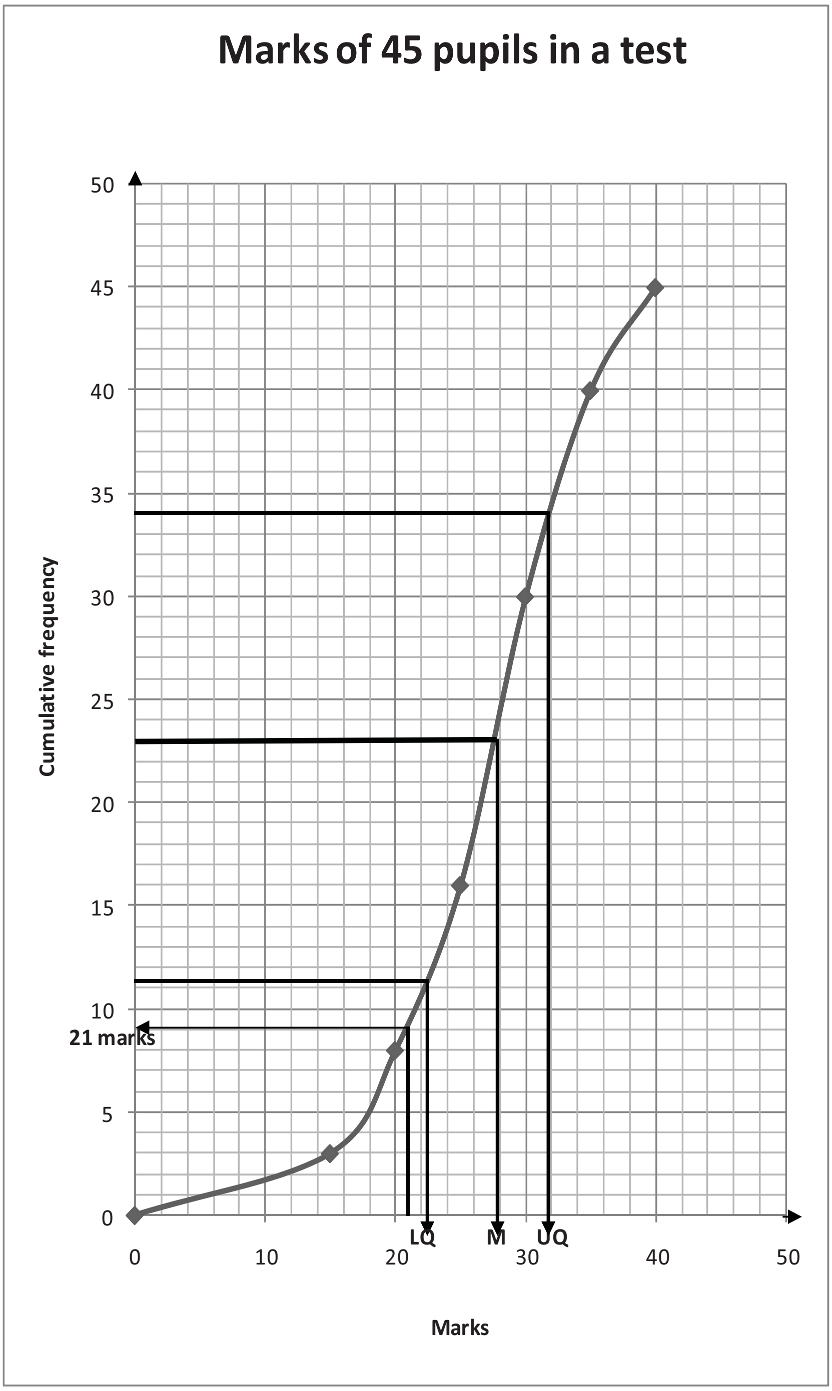


Figure 26- cumulative frequency diagram showing marks of 45 pupils' in a test

b) To find the median we first need to find the median position.

i. Median position $= \frac{1}{2}(45+1) = 23$

We first draw a straight line from the cumulative frequency axis at point 23 and at the point where it meets the graph we draw another line to the marks axis which gives us the value of the median mark.
Now the median mark is 28 marks.

ii. The inter quartile range is given by upper quartile – lower quartile. So that says we first need to find both the lower quartile and the upper quartile first.

Lower quartile position $= \frac{1}{4}(45+1) = 11.5$, so the lower quartile position from the graph is 22.2 marks.

The upper quartile position $= \frac{3}{4}(45+1) = 34.5$, and from the graph the upper quartile is 31.8 marks.
So the inter quartile range is 31.8 marks - 22.2 marks = 9.6 marks.

c) To find the group/s which are above average we first need to find the average mark or the mean mark.

Mark	11-15	16-20	21-25	26-30	31-35	36-40	**Total**
Mid-value	13	18	23	28	33	38	
f	3	5	8	14	10	5	**45**
f_m	39	90	69	392	330	114	**1034**

Table 62- the marks of 45 pupils in a mathematics quiz

So the mean $= \frac{sum\_of\_f_m\_values}{sum\_of\_f\_values} = \frac{1034}{45} = 23 marks$

Now looking at our data groups 26-30, 31-35, and 36- 40 are above average.

d) We know that all in all there are 45 students in this class and from the graph, from student nine upwards they scored 21 and above, that says 8 students scored below 21 marks. So the probability that a student chosen at random scored below 21 marks is $\frac{8}{45}$. (Refer to the unit on probability).

e) Remember that the frequency polygon of grouped data is drawn with coordinates that consist of (mid-value, the corresponding frequency).

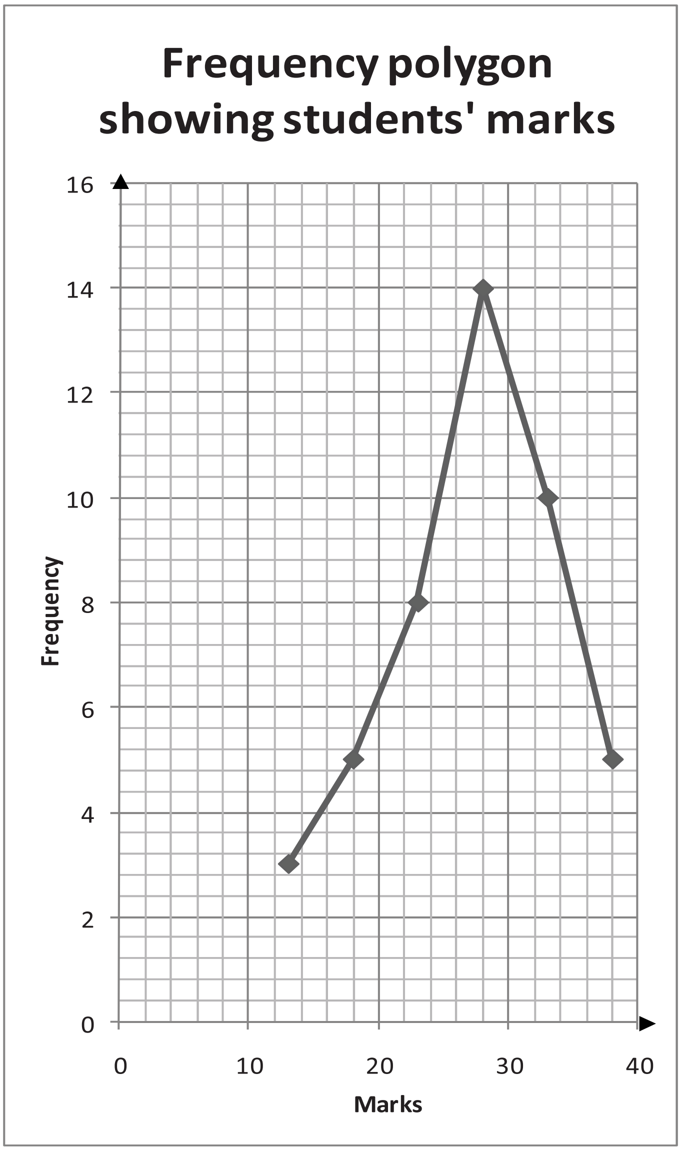


Figure 27-Frequency Polygon showing students' marks

No 7.

With the histogram remember that you need to know the width and the height of each, and to get the height we divide the frequency by the width.
Let one unit be 10 marks.

Salary	10-19	20-29	30-49	50-89	90-99
Width	1 unit	1 unit	2 unit	4 units	1 unit

Frequency	6	11	13	20	5
Height (Frequency density)	6	11	6.5	5	5

Table 1.62- Frequency distribution table showing nurses' salary

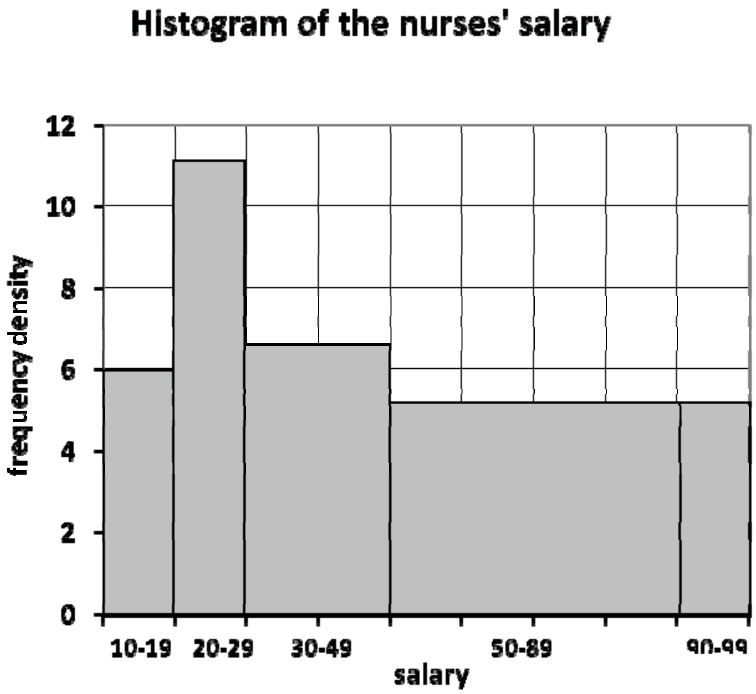


Figure 28-Histogram showing nurses' salaries

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

ssessment

sessment

Instructions:

- ✓ This assessment contains 40 marks
- ✓ You should be able to complete this assessment in no more than one hour.
- ✓ Calculators can be used
- ✓ Answer all questions

No. 1 Table 1.63 shows the percentage marks obtained by pupils in an English exam.

Mark	**Number of pupils(frequency)**
50	12
60	10
65	15
77	5
84	3
Total	46

Table 1.63- Percentage marks obtained by pupils in an English language exam.

a) Find the mode and estimate the mean of the data on table 1.63 (4 marks)

____

____

b) On a separate graph paper Draw the cumulative frequency diagram of the data shown on table 1.63 (3 marks)

____

____

c) Use your diagram to estimate the median and the quartiles. (3 marks)

____

____

d) Find the 90^{th} percentile (1 mark)

____

____

e) If 30% of the pupils passed, what would have been the pass mark? (3 marks)

____

____

f) Display the data on table 1.63 on both a bar chart and a frequency polygon. (6 marks)

No. 2

One day a farmer collected 360 eggs from his chickens.

Table 1.64 shows the distribution of the masses of the eggs.

Mass (m grams)	$34<m\leq 42$	$42<m\leq 46$	$46<m\leq 48$	$48<m\leq 50$	$50<m\leq 54$	$54<m\leq 58$	$58<m\leq 66$
Frequency	40	60	50	54	70	56	30

Table 1.64-Frequency distribution table showing masses of eggs

a) When a histogram is drawn to illustrate this information, the rectangle representing the eggs with masses in the interval $42<m\leq 46$ has width 2 cm and height 3cm.

Find the width and the height of the rectangle representing the eggs with masses in the interval $46<m\leq 48$. (2 marks)

b) Copy and complete the cumulative frequency table 1.65 which follows. (2 marks)

Mass (in grams)	≤ 34	≤ 42	≤ 46	≤ 48	≤ 50	≤ 54	≤ 58	≤ 66
Cumulative Frequency	0	40	100					360

Table 1.65- Frequency distribution table showing masses of eggs

c) Draw a horizontal m-axis for $0\leq m\leq 70$ and the vertical y- axis for values from 0 to 400.

On your axis, draw a smooth cumulative frequency curve to illustrate this information. (2 marks)

d) Use your graph to find

i) The median mass of the eggs, (3 marks)

ii) The inter quartile range. (4 marks)

e) The farmer classifies 300 of the eggs to be 'Small Eggs' and 60 of the eggs to be 'Large Eggs'

i) Use your graph to find the least mass of a 'Large Egg" (2 marks)

3.

0ne hundred and sixty students took an examination.

Table 1.66 shows the marks needed for each grade.

Grade A	70< Mark
Grade B	60<Mark≤70
Grade C	50<mark≤60
Grade D	20<mark≤50
Grade U	mark≤20

Table 1. 66-Table showing marks needed for each grade.

The cumulative frequency curve shows the distribution of their marks.

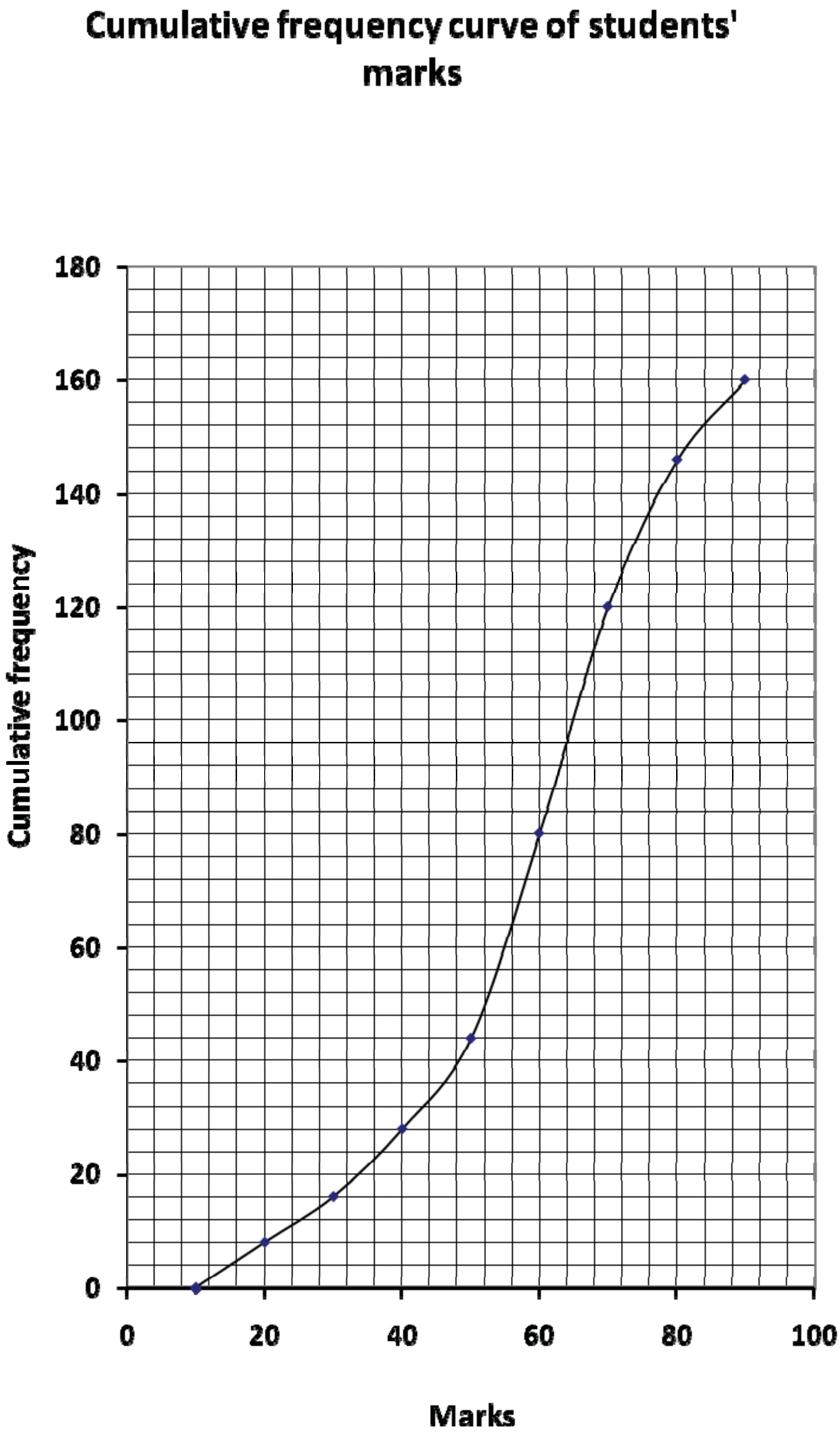


Figure 29-Cumulative frequency diagram showing students’ marks

i) Use the graph to estimate the number of students who were awarded a grade C. (2 marks)

____

____

ii) A pie chart was drawn to illustrate the grades awarded to the students who were awarded a grade of C.

Calculate the angle of the sector which represented the number of students who were awarded a grade of C. (4 marks)

____

____

Solutions to the assessment

a) The mode of the distribution is 65 marks, as it appears 15 times.

To estimate the mean let us include the column on frequency $\times$ mark as shown on table 1.67.

Mark	**Number of pupils(frequency)**	**Frequency $\times$ Mark**
50	12	600
60	10	600
65	15	975
77	5	385
84	3	252
Total	**45**	**2812**

Table 1.67- Percentage marks obtained by pupils in an English language exam

So the mean= $\frac{sum\_of(frequency \times variable)}{sum\_of\_frequencies} = \frac{2812}{45} = 62.5$ marks.

b) To draw the cumulative frequency diagram, first we will draw a table with cumulative frequencies.

Mark	50	60	65	77	84
Cumulative frequency	12	22	37	42	45

Table 1.68-Cumulative frequency table showing Percentage marks obtained by pupils in an English language exam

Now following is the cumulative frequency diagram.

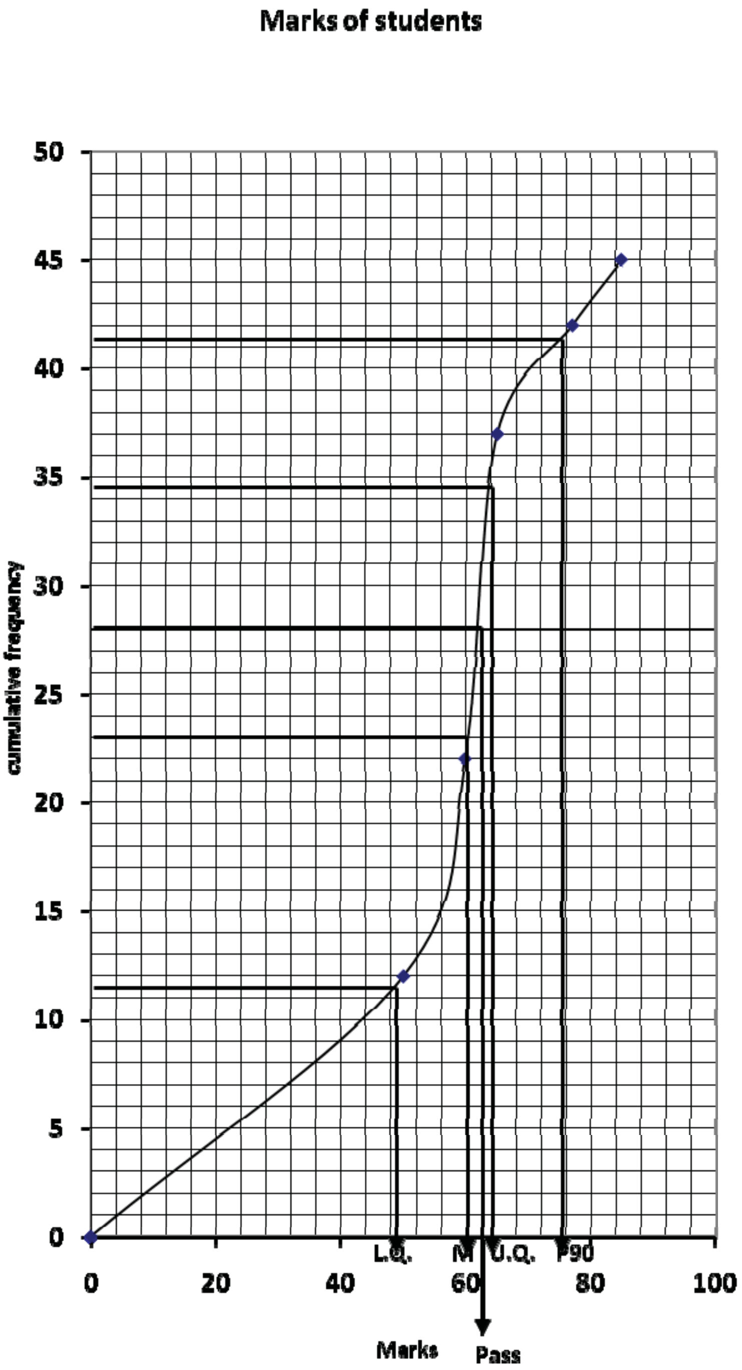


Figure 30-Cumulative frequency diagram showing Percentage marks obtained by pupils in an English language exam

c) **1**. Median position = $\frac{1}{2}(45+1) = 23$, now from the graph the median = 60 marks

2. Lower quartile position = $\frac{1}{4}(45+1) = 11.5$, and the lower quartile = 49 marks.

3. Upper quartile position = $\frac{3}{4}(45+1) = 34.5$, so from the graph the upper quartile = 64 marks

d) The 90th percentile position = $\frac{90}{100}(45+1) = 41.4$, and from the graph the 90th percentile = 75 marks.

e) 40% of the pupils is $\frac{40}{100} \times 45 = 18$ pupils. So if 18 pupils passed, the pass mark should have been 62 marks when looking at the graph.

f) Bar chart

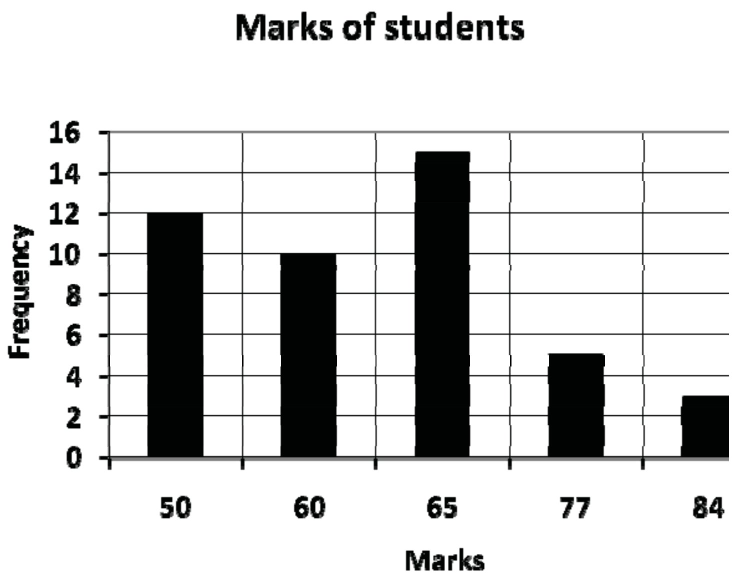


Figure 31-Bar chart showing students' marks.

The frequency polygon follows.

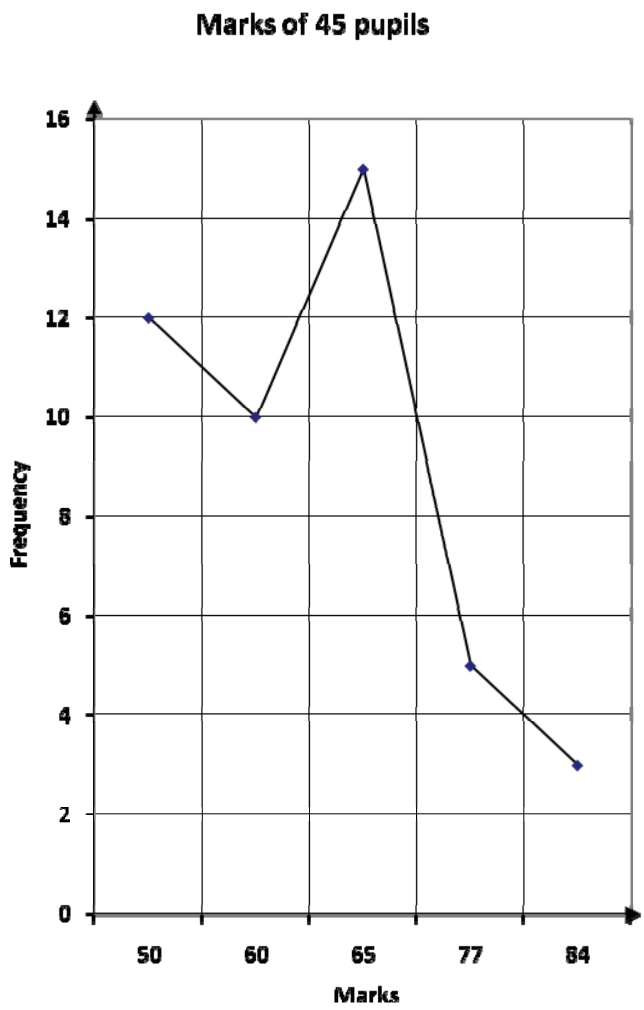


Figure 32- Frequency polygon showing the marks of 45 pupils.

No 2.

a) The width would be 1cm and the height would be 5 cm.

b)

Mass (in grams)	≤34	≤42	≤46	≤48	≤50	≤54	≤58	≤66
Cumulative Frequency	0	40	100	150	204	274	330	360

Table 1.69- Frequency distribution table showing masses of eggs

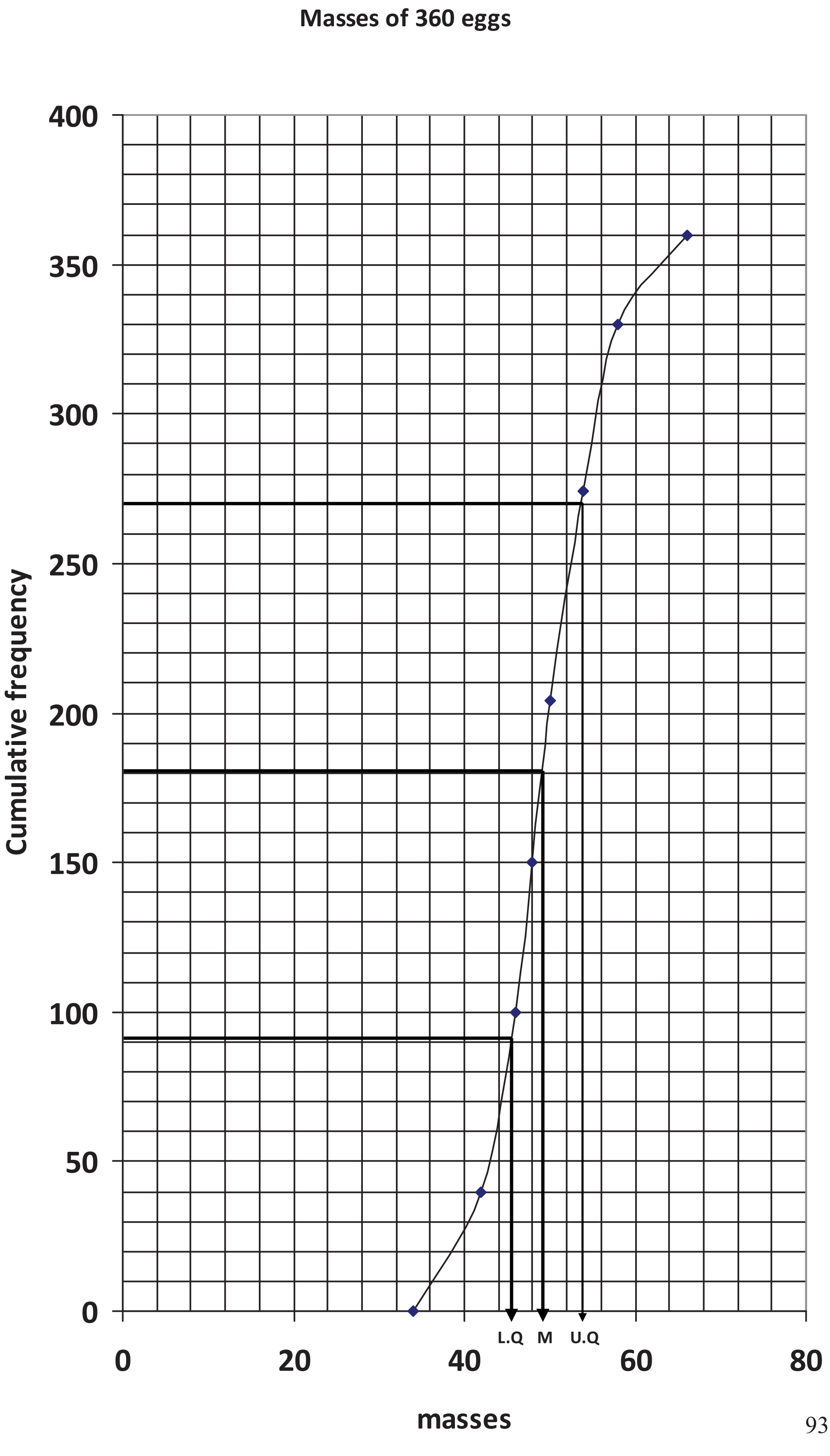

Masses of 360 eggs
Cumulative frequency
400
350
300
250
200
150
100
50
0
L.Q
M
U.Q
0
20
40
60
80
masses

Figure 33-Cumulative frequency diagram of 360 eggs

i. The median position = $\frac{1}{2}(360) = 180$ (we use $\frac{1}{2}n$ to find the median position now because our n is large).

So from the graph the median = 49 grams.

ii. To find the inter quartile range we need to know the lower quartile and the upper quartile.

The lower quartile position = $\frac{1}{4}(360) = 90$, and from the graph the lower quartile = 45.6 grams.

The upper quartile position = $\frac{3}{4}(360) = 270$, now from the graph the upper quartile = 54 grams

Now the inter quartile range = 54 grams – 45.6 grams = 8.4 grams.

d) From egg number 301 we have large eggs since there are 300 small eggs. From the graph the least mass of a large egg is 56 grams.

No.3

a) 36 students were awarded a grade C, because it is from students who scored just above 50 to 60 marks. And when we look at the graph, students who scored 50 and below are 44, which says from student no. 45 to students no. 80 are on grade C which is 36 students all in all.

b) The sector angle for Grade C students = $\frac{36}{160} \times 360^0 = 81^0$ because there are 36 students in grade

c)

Unit Contents

Unit 23

Unit 23

Probability

Introduction

The concept of probability plays a very important part in everyday life. It has interesting applications in a number of fields such as the business world, the sciences, higher mathematics and production industry. Insurance companies base their premiums on the probability of events occurring. For example, to decide on a premium for fire insurance on a house, an insurance company must establish the likelihood of the house being burnt. A house with a thatched roof will require a higher premium since the chances of its roof catching fire are higher.

In our everyday conversations we could hear expressions such as;

- "I am 100% sure that he will arrive today."
- The chances that LDF football team will win the game against Masheshena are fifty-fifty."
- "By the look of things, he has a very slight chance of surviving from this accident."
- "It is impossible that one can agree to kill his own mother."

What probabilities are implied in these statements? Well, let us find out!

This unit consists of 60 pages. It covers approximately 3% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 30 hours to work on this unit, 20 hours for formal study and 10 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Four Lessons:

Lesson 1 Theoretical Probability
Lesson 2 The Possibility Space
Lesson 3 Tree Diagrams and Independent Events
Lesson 4 Conditional Probability

Spend a few moments reading the following learning outcomes. They are a guide to what you should focus on while studying this unit.

Upon completion of this unit you will be able to:

Outcomes

- *determine* the probability of an event and express it as a fraction or decimal.
- *predict* the number of elements of a sample space for combined events.
- *distinguish* between mutually exclusive and independent events.
- *calculate* the probability of simple combined events, using possibility space and tree diagrams where appropriate.
- *solve* real-life problem situations involving probabilities of combined events.

Terminology

Probability: The numerical measure of the likelihood of an event to occur or not to occur.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Preview

Activity

ACTIVITY 1

The following activity is intended to help you remember the basic concepts in probability**.**

Indicate the likelihood of each of the following events on the likelihood scale shown below.

Write the letter of the event below the correct probability.

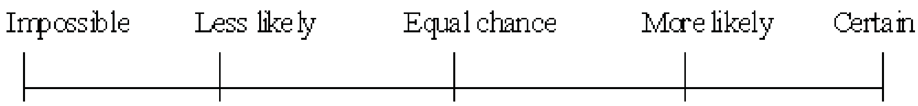
a) It will snow in Maseru on Christmas day.

b) The sun will not set tomorrow.

c) You will get an even number when you roll a six-sided die.

d) A baby will be born today.

e) Lesotho is a country.



Compare your answers with those at the end of this subunit. Be sure that you understand each answer before continuing.

Probability Scale

You have likely come across a word scale before.

The probability or possibility of an event occurring or not occurring is normally represented on a probability scale. The scale is shown in figure 1.

Probability Scale: $0 \leq \mathbf{P} \leq 1$

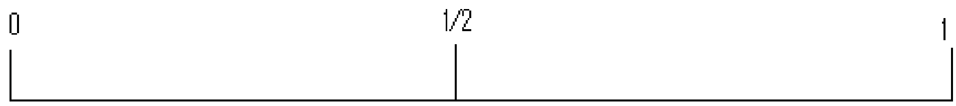


Figure 1 – Probability scale

Looking at the probability scale in figure 1 what do you think the P, 0, ½ and 1 mean?

P: ____

0: ____

½: ____

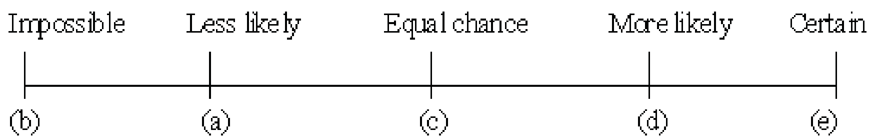
1: ____

Check your performance against the given solutions at the end of this subunit. Continue if you are satisfied with your ability to answer the questions.

Answers to subunit activities

Solution to activity 1

(b) It is impossible for the sun not to set on any day	(a) It is less likely to snow in Maseru on Christmas Day because Christmas day is in summer	(c) You can get an even number in 3 different ways (2,4,6) and an odd number in 3 different ways(1,3,5)	(d) It is highly possible that at least one baby is born every day on planet earth.	(e) It is certain, that Lesotho is country.



Solution to the Probability Scale Question

The **P** on the scale represents probability. Hopefully that is what you thought. The scale shows that the probability of an event to occur or not to occur ranges from 0 to 1, where:

0 probability means that an event con not by any means occur, it is impossible.

½ probability means equal chances, the event has the same chance to ocur as the chance not to occur.

1 probability means an event will certainly occur.

Lesson 1 Theoretical Probability

Introduction

By the end of this subunit, you should be able to calculate the probability of single events by using the theoretical probability formula.

This subunit is about 3 pages in length.

Probability Defined

In your junior secondary mathematics, you learned that experimental probability was calculated as:

Probability of an event $= \dfrac{Number\ of\ successes}{total\ number\ of\ attempts}$, these were based on the results of experiments.

Theoretical probability of an event is NOT based on the results of an experiment. It is calculated without doing any experiments and is expressed as a fraction:

Probability of an event occurring $= \dfrac{number\ of\ targeted\ outcomes}{total\ number\ of\ possible\ outcomes}$

Or we can say the probability of an event to occur is given as:

$$probability\ of\ an\ event\ occuring(P(S)) = \frac{number\ of\ ways\ in\ which\ an\ event\ can\ occur(n(S))}{total\ number\ of\ possible\ outcomes(n(\varepsilon))}$$

$$i.e.\ P(S) = \frac{n(S)}{n(\varepsilon)}$$

Single Events

In this section we are going to learn how to calculate probabilities of getting some outcomes for single events.

Now let us look at the following examples.

Example 1

Five pieces of fruit are put in a container; 2 apples and three oranges. If a fruit is picked at random from this container, find the probability that it is:

a) An apple

b) An orange

c) A peach

Solution to example 1

Our set of all possible outcomes is as follows:
$(\varepsilon) = \{apple, apple, orange, orange, orange\}$

a) There are five pieces of fruit, so the total number of possible outcomes is 5, i.e. $n(\varepsilon) = 5$, now the number of ways in which an event of choosing an apple can occur are 2 because there are three apples. So
$n(S) = n(apples) = 2$

$$\therefore P(apple) = \frac{n(apples)}{total\ no.\ of\ \ possible\ outcome} = \frac{2}{5}$$

b) $P(orange) = \dfrac{n(oranges)}{n(\varepsilon)} = \dfrac{3}{5}$

c) $P(peach) = \dfrac{n(peaches)}{n(\varepsilon)} = \dfrac{0}{5} = 0$, because there are no peaches in the container.

Think about the above concept. If it makes sense to you, continue on. If it is unclear, review the content to determine where you lost the concept.

Solutions are based on finding the sample space for the event and getting the number of target outcomes from the sample space. The sample space is the set of all the possible outcomes of an event, while the target outcomes are the outcomes we are interested in. In expressing probabilities values are normally left as fractions in their lowest terms or sometimes as decimals.

Example 2

A bag contains six identical marbles of different colours. Four of them are blue and two are green. If a marble is taken from the bag at random, what is the probability that it is:

a) Blue?

b) Green?

c) White?

<u>Solution to example 2</u>

There are six possible outcomes for this event. The sample space or set of all possible outcomes is as follows: $\varepsilon = \{b, b\ , b, b, g, g\}$ where b represents a blue marble and g a green marble.

a) $P(blue\ marble) = \dfrac{n(blue\ marbles)}{n(\varepsilon)} = \dfrac{4}{6} = \dfrac{2}{3}$

b) $P(green\ marble) = \dfrac{n(green\ marbles)}{n(\varepsilon)} = \dfrac{2}{6} = \dfrac{1}{3}$

c) $P(white\ marble) = \dfrac{n(white\ marbles)}{n(\varepsilon)} = \dfrac{0}{6} = 0$

Now try activity 2.

Activity

ACTIVITY 2

Answer the following questions:

1. In the name THOKOLOSIHALI, a letter is chosen at random, find the probability that:

   a. It is an ‘O’

   ____

   ____

   ____

   b. It is a vowel

   ____

   ____

   ____

   c. It is a consonant

   ____

   ____

   ____

2. From a pack of 52 playing cards, find the probability of drawing:

a. A king

b. A black king

c. A black card

d. A spade

3. A die is thrown. What is the probability that the number it shows is :
   a. a cubic number
   b. a square number
   c. is a multiple of 2

4. A back contains 10 sweets, 4 pink, 3red and 3 yellow. If a sweet is chosen at random from this back, find the probability that it is a:
   a. A yellow sweet
   b. Not a brown sweet
   c. Not a pink sweet

5. In the year 2004, there were 60 private candidates writing an examination at centre A. The probability that a candidate chosen at random from the centre will be a female is $\frac{4}{5}$. How many female candidates were at centre A?

Now compare your answers with the ones given at the end of the subunit.

If you had any difficulty in this activity, review the above notes to clarify your misunderstandings.

Combined Events, Possibility Space and Tree Diagrams

By the end of this subunit, you should be able to:

1. Calculate the combined probability of two or more events occurring.

2. Draw and use the possibility space to find the probability of combined events.

3. Draw and use tree diagrams to calculate the probability of combined events. This sub-unit on combined events, the possibility space and tree diagrams are about 11 pages long.

With combined events, two or more different events occurring are combined together to give one probability.

Consider the following events for instance:

A 5 cent coin and a 10 cent coin are tossed; find the probability that they both show heads.

When tossing these two coins, there are four possible combinations of outcomes. It is possible that the 5cent coins shows a head while a 10cent coin shows a tail, the combination can be shown as (H, T). So the four possible outcomes are (H, T), (H, H), (T, H), (T, T). Now we want to find the probability that the two coins both show heads.

Answering the following questions will help you understand how to find probabilities of combined events.

Questions:

a. How many possible outcomes (combinations) do we have?

Compare your answer to the following:

We have four possible outcomes all in all which are: (H, T), (H, H), (T, H), (T, T) ,i.e. $n(\varepsilon) = 4$

b. Now how many combinations of (H, H) do we have?

Compare your answer to the following:

We have only one way in which we can get (H, H), i.e. $n(S) = 1$.

c. And what is the probability that both coins show heads?

Compare your answer to the following:

The probability that both coins show heads is given by:

$$P(H,H) = \frac{n(S)}{n(\varepsilon)} = \frac{1}{4}.$$

Activity 3

Now work on activity 3 and practice working with combined events.

No. 1

Two dice are thrown; find the probability that they both show a 2.

No. 2

A coin is tossed twice; find the probability that in both occasions it shows a tail.

No. 3

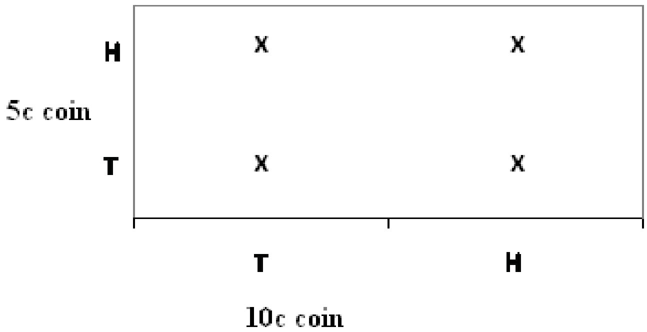
A coin is tossed and a die is thrown, find the probability of obtaining a head and an even number greater than 3.

Compare your answers with the ones given at the end of this subunit, just before the unit summary. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content and work through the activity again.

Lesson 2 The Possibility Space

We can show the same event of two coins on a **possibility space** and get the same probability. The possibility space is a diagram that is used to display possibilities when two or more events occur.

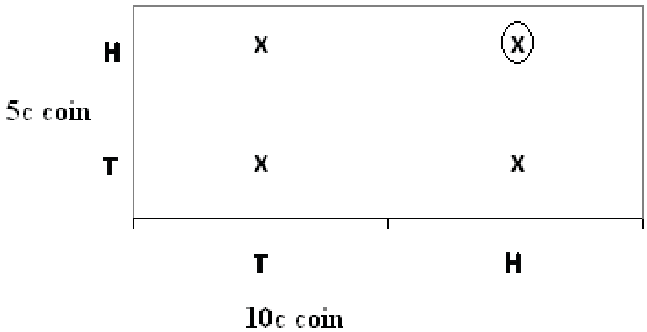
This is how the possibility space looks like:



Activity

The crosses show the possibilities, so we can see that the first cross is where the T's of the two coins cross and it represents (T, T), the second cross represents (T, H) and so on and on.

Now on the following diagram the cross on (H, H) has been circled which means that the there is one possibility of getting (H, H) out of 4 possibilities. Therefore the probability of getting (H, H) $= \frac{1}{4}$.



Now work on activity 4 to get some practice on combined events and the possibility space.

ACTIVITY 4

This activity is intended to help you determine the sample space and calculate probabilities for combined events using a possibility space.

1. Mpho has 5 coins in her pocket, two 20c coins and three 50c coins; she draws a coin from her back, puts it back and then she draws another one.

a. Draw the possibility space to represent these two events and the possible outcomes.

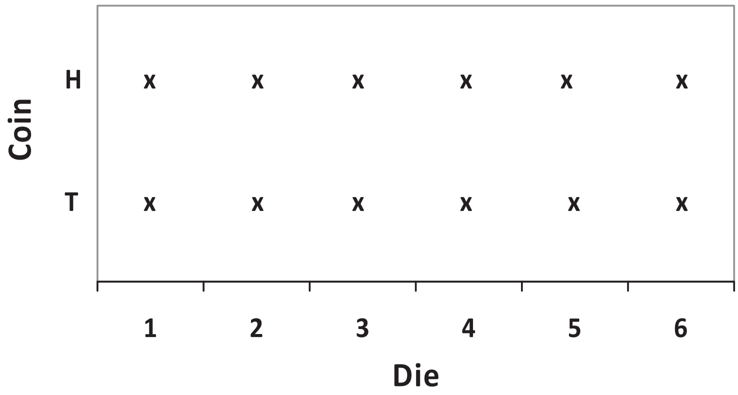
b. Find the probability of getting:

i. Two 20c coins

ii. A 20c coin and a 50c coin

iii. two 30c coins

2. An ordinary six-sided die and a coin are tossed together. The possible outcomes for these two events are shown on the following possibility space.



a) How many possible outcomes are there altogether?

b) Write down all the possible outcomes.

c) What is the probability of getting a 2 and a tail?

d) What is the probability of getting a head and an even number?

Compare your answers with those at the end of this subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content again.

Under combined events we are coming to discuss mutually exclusive events and independent events as they both are on combined events.

Activity

Mutually Exclusive Events

To understand mutually exclusive events carry out activity 4.

ACTIVITY 5

Toss an ordinary die.

a) How many possible outcomes are there?

b) What is the probability of getting a 4?

c) What is the probability of getting a 3?

d) What is the probability of getting either a 3 or a 4?

Check your performance against the given solutions at the end of this subunit. Continue if you are satisfied with your ability to answer the questions. If not, review this content again.

Now from question d) in activity 4 we can get the same answer when we calculate the probability of getting either a three or a 4 as:

$$P\ (3\ \textbf{or}\ 4) = P\ (3) + P\ (4) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}.$$

This still makes sense because we would be successful when we get a 3 and also when we get a 4. However we should note that these two outcomes cannot both appear at the same time. We can get either a 3 or a 4.

We say these two events are **mutually exclusive.** We can therefore conclude that:

If A and B are mutually exclusive events, then

P(A or B) = P(A) + P(B)

Work on activity 5 to see how well you understand this concept.

ACTIVITY 6

1. A bag contains three red balls, six white balls and eleven blue balls. A ball is drawn from the bag at random, find the probability that it is:

a) Blue ball

b) Either white or blue, i.e. find P (W or B)

c) Not blue

c) Neither red nor white, i.e. find P (not R and not W)

d) Red or not red

e) Neither black nor pink

2. A card is drawn from the pack of 52 playing cards. Find the probability that it is:

a. either a heart or a black queen.

b. Either an ace or a king

c. Neither a red jack nor a black card

3. A die is thrown. What is the probability that the number shown is:

a) A factor of 6 or a factor of 5?

b) Neither a square number nor a multiple of 3

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review this content.

Lesson 3 Tree Diagrams and Independent Events

Tree Diagrams

The same information that we display on a possibility space may also be displayed on a tree diagram. The tree diagram is a diagram with tree like branches which are used to display probabilities. Each branch of such a diagram represents a probability.

Try the following activity to learn how tree diagrams represent probabilities.

Activity

ACTIVITY 7

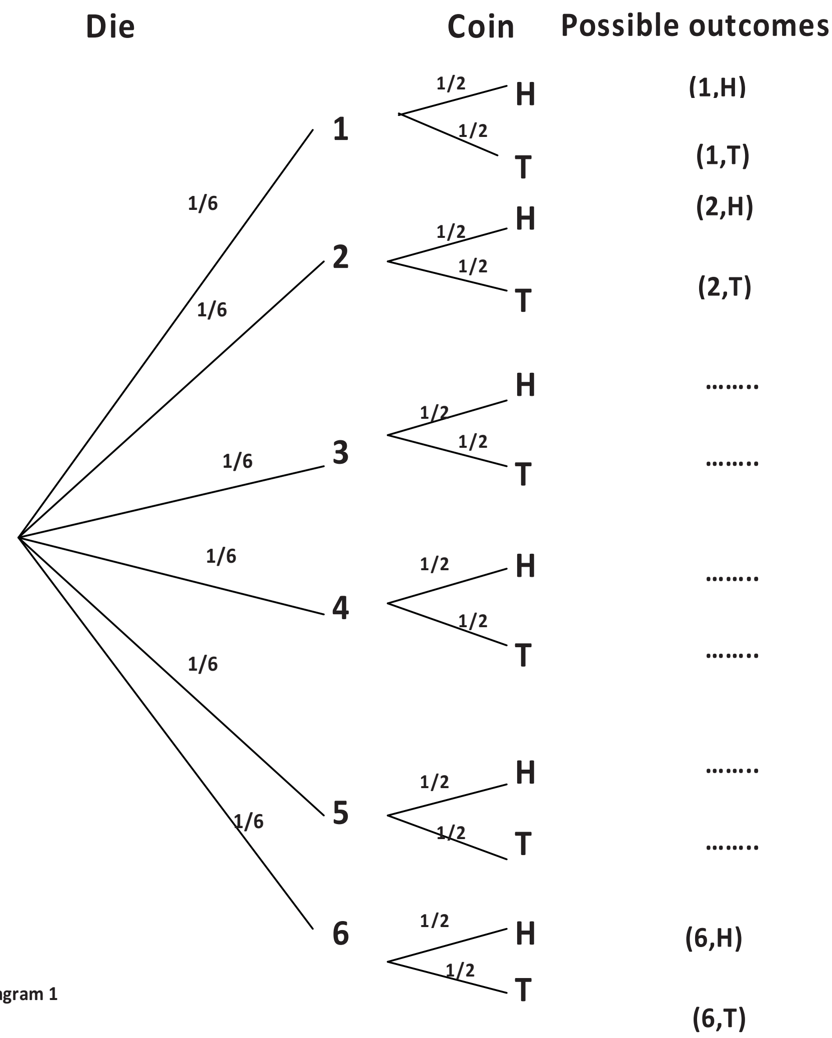
This activity is designed to help you to determine sample space and calculate the probabilities of events using tree diagrams.

Consider question 2 in Activity 3. An ordinary six-sided die and a coin are tossed together.

The tree diagram for this event is shown below. In this diagram, there are six branches to a die, these branches are labelled 1,2,3,4,5,6 respectively (which are the 6 possible outcomes on a die), on each branch we see a probability labelled. The probabilities are 1/6 throughout the branches of a die because the probability of a die resting as a 1 is 1/6, as a 2 is also 1/6 and so on and on.

When coming to a coin braches, we can see that they are duplicated six times. What is implied here is that, a die can rest as a1 and a coin maybe as a H or T, or a die may rest as a 2 and a coin as either a head or a tail, or a die as a 3 and a coin as a H or T and so on and on.

Now the possible outcomes are shown at the end of the branches.



Tree diagram 1

Take a moment to think about that you get the same possible outcomes in a tree diagram as you would get from a possibility space.

a. From tree diagram 1 complete the column on possible outcomes.

b. What is the probability of getting a 2 and a tail?

c. What is the probability of getting either a (2,T) or (3,H)?

____
____
____

d. What is the probability of getting a head and an even number?

____
____
____

Compare your answers with those at the end of this subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review the concept of tree diagrams again.

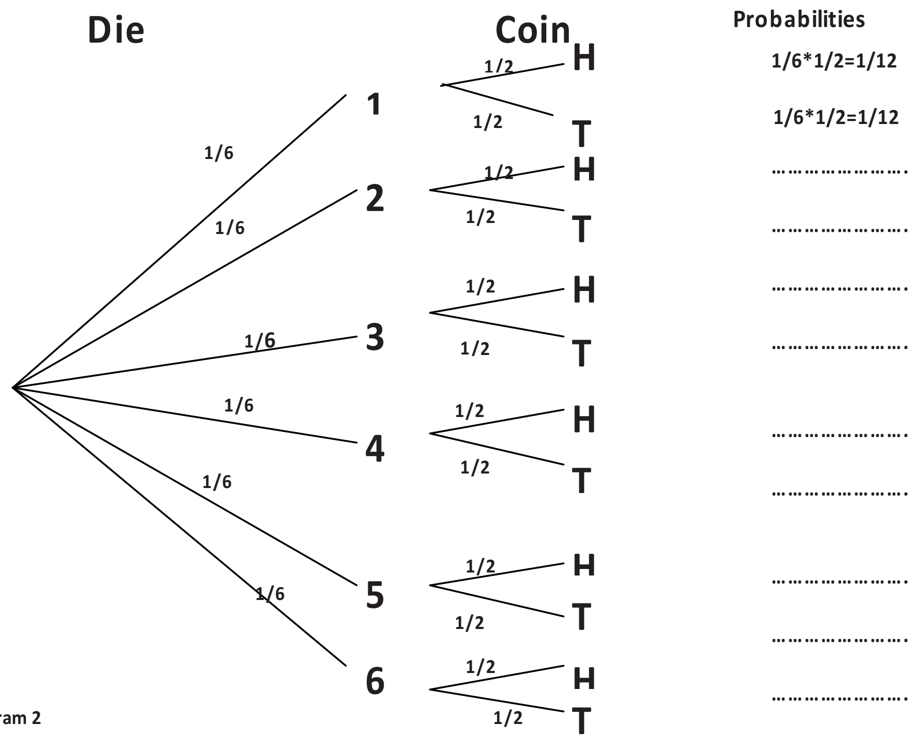
Things to note:

This shows that you can use a tree diagram or a possibility space to solve probability problems involving combined events. However at this level, you are advised to use possibility spaces for two combined events only and use tree diagrams with more than two combined events.

You should note that probabilities for each outcome are written on the branch pointing to that outcome.

Independent Events

Activity 8



Tree diagram 2

1.

i) From tree diagram 2, copy and complete the column on probabilities by multiplying the probabilities on the branches for each outcome,

____

____

Compare your answer to that of activity 4 b) where $P(2,T) = \frac{1}{12}$. What do you notice?

____

2. i) What is the probability of getting a Head and an even number?

____

____

Compare your answer to that of 1d in activity 4. What do you notice?

ii) What is the probability of getting a tail and a 3?

Compare your answers to those given at the end of the subunit. Note that it is important to understand these ideas. If you do not understand them, review this content.

The events in activity 8, regarding a die and a coin, are **independent events.** This means that the occurrence of one event has no effect on the other. For instance the probability of obtaining a 2 and a tail as we have seen in activity 4 is $P(2,T)=\frac{1}{12}$, because there is only one combination of P(2,T) out of 12 combinations. But in activity 5 we discovered that the probability of obtaining a tail and 2 can be given by: $P(2,T)=P(2)\times P(T)=\frac{1}{6}\times\frac{1}{2}=\frac{1}{12}$, now we can see that get the same answer either way, so the occurrence of a 2 in a die have no impact on the occurrence of a tail in a coin, these are called independent events.

With independent events: if A and B are independent events, $P(A \text{ and } B)=P(A)\times P(B)$ **.**

Activity

Activity 9

This activity gives more practice on: combined events, possibility space, mutually exclusive events, tree diagrams and independent events.

1. Two dice are thrown at the same time,

   a) Draw the possibility space to illustrate all the possible outcomes.

b) Find the probability that both dies show 2,

c) Find the probability that the sum of the numbers shown by the die is:

i) Equal to 10

ii) Greater than 5

iii) Less than 3

iv) Multiple of 4

2. Three sets are defined.

$$A = \{prime\ numbers\ less\ than\ 7\}$$
$$B = \{Even\ numbers\ less\ than\ 8\}$$
$$C = \{multiples\ of\ 3\ less\ than\ 9\}$$

A number is picked at random from each set,

a. Draw a tree diagram to represent all the possible outcomes.

b. Find the probability of picking a 2 from set A, a 4 from set B and a 6 from set C.

3. Three coin are thrown at the same time,

   a) Draw the tree diagram to represent all possible outcomes.

   b) Find the probability that at least two heads are shown.

   c) Find the probability that exactly two heads and one tail or exactly two tails and one head are shown.

Compare your answers with those at the end of this subunit. Continue on if you had at least 9 out of 11 correct. If not, review the above content and try the activity again.

<u>In this subunit you have learned that</u>:

$$- \textit{probability of an event occuring}(P(S)) = \frac{\textit{number of ways in which an event can occur}(n(S))}{\textit{total number of possible outcomes}(n(\varepsilon))}$$

- With combined events, two or more different events occurring are combined together to give one probability.

-If A and B are mutually exclusive events, then P(A or B) = P(A) + P(B)

-With independent events: if A and B are independent events, $P(A \text{ and } B) = P(A) \times P(B)$ **.**

Lesson 4 Conditional Probability

At the end of this sub unit you should be able to calculate the conditional probabilities of events.

This involves cases where outcomes in the previous event affect or determine those of the subsequent events.

Activity

The following activity is intended to help you handle such cases.

ACTIVITY 10

A bag contains 7 balls of equal size in different colours. There are 3 red balls and 4 blue balls.

A ball is taken from a bag at random.

a) What is the probability that this ball is red?

Compare your answer to the following:

$$P(red) = \frac{n(S)}{n(\varepsilon)} = \frac{red\ balls}{all\ balls} = \frac{3}{7}$$

b) The first ball is not returned to the bag. A second ball is taken from the same bag, also at random. How many balls are now left in the bag? And what is the new $n(\varepsilon)$?

Compare your answer to the following:

There are 6 balls left in the bag. And the new $n(\varepsilon)$ =6.

c) If the first ball was red, what is the probability that the second ball is:

i) Blue__________

ii) Red__________

Compare your answers to the following:

i) There are now six balls are in all, the red are now 2 because we are told the first was red, so they have reduced from 3 to 2. The blue are still 4.

$$\therefore P(blue) = \frac{n(S)}{n(\varepsilon)} = \frac{blue\ balls}{all\ remaining\ balls} = \frac{4}{6} = \frac{2}{3}$$

ii) in i) we said the red balls remaining in the bag are now 2 out of 6.

$$\therefore P(red) = \frac{n(S)}{n(\varepsilon)} = \frac{red\ balls}{all\ remaining\ balls} = \frac{2}{6} = \frac{1}{3}$$

d) If the first ball was blue, what is the probability that the second ball is:

i) Blue____
ii) Red____

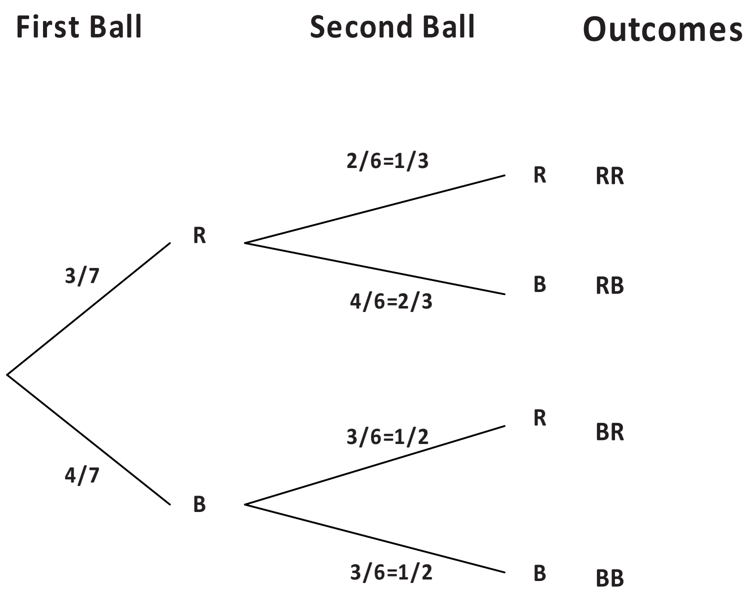
Compare your answers to the following:

i) In this case we are told that the first ball which was taken was blue, so that says we are now left with 6 balls of which the blue have reduced from 4 to 3, while the red still remain as 3. $\therefore P(blue) = \frac{n(blue)}{n(all)} = \frac{3}{6} = \frac{1}{2}$

ii) The red are still 3 because the once which were reduced in the first drawing were the blue. So $P(red) = \frac{n(S)}{n(\varepsilon)} = \frac{red}{all\ balls} = \frac{3}{6} = \frac{1}{2}$

e) Draw a tree diagram to show these two events. In your tree diagram, the first event should be labelled first ball, and the second event should be labelled second ball.

Compare your tree diagram to the following:



Tree diagram 4

f) Use the tree diagram to calculate the following probabilities for the combined events of the first ball and the second ball,
Find the probability that:

i) Both balls are red ______________________________

ii) One ball is red and the other is blue ______________________________

iii) The two balls have the same colour______________________________

Compare your answers with the ones given below:

i) $P(R \ and \ R) = P(1^{st} \ ball - red) \times P(2^{nd} \ ball - red) = \frac{3}{7} \times \frac{1}{3} = \frac{1}{7}$

ii)

$$P(R \text{ and } B) = P(R \text{ and } B) \text{ or } P(B \text{ and } R)$$

$$= P(1^{st} \text{ ball} - red) \times P(2^{nd} \text{ ball} - blue) \text{ or } P(1^{st} \text{ ball} - blue) \times P(2^{nd} \text{ ball} - red)$$

$$\frac{3}{7} \times \frac{2}{3} + \frac{4}{7} \times \frac{1}{2} = \frac{2}{7} + \frac{2}{7} = \frac{4}{7}$$

iii)

$$P(R \text{ and } R) \text{ or } P(B \text{ and } B) =$$

$$P(1^{st} \text{ ball} - red) \times P(2^{nd} \text{ ball} - red) + P(1^{st} \text{ ball} - blue) \times P(2^{nd} \text{ ball} - blue)$$

$$= \frac{3}{7} \times \frac{1}{3} + \frac{4}{7} \times \frac{1}{2} = \frac{1}{7} + \frac{2}{7} = \frac{3}{7}$$

It is important to note that from the above activity, the outcomes of the first event in this case actually affect the probabilities of the second event. This is because the first ball is not replaced before the second ball is picked.

Activity

Now work on the following activity to see how well you understand the concepts.

Activity 11

1. Three cards are taken from a pack of 52 playing cards, one after another without replacement.

With the use of tree diagrams or otherwise, find the probability that:

a) All the three are kings.

b) At least two are spades.

Compare your answers with those at the end of this subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review the above content.

You have now completed this subunit. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Solutions to Subunit Activities

Solutions to Activity 2

1. a) There are 3 'O's in the name thokolosihali and 13 possible outcomes so $P(O) = \frac{n(O)}{n(\varepsilon)} = \frac{3}{13}$

   b) There are 13 letters hence 13 possible outcomes, and there are 6 vowels so that says there are 6 ways in which a vowel can be drawn. So $P(vowel) = \frac{n(vowels)}{n(\varepsilon)} = \frac{6}{13}$

   c) There are 13 possible outcomes and 7 consonants, so $P(consonant) = \frac{n(consonants)}{n(\varepsilon)} = \frac{7}{13}$

2. a) there are 52 playing cards hence 52 possible outcomes, again there are in the 52 playing cards 4 kings, so probability of drawing a king

   $= P(king) = \frac{n(kings)}{n(\varepsilon)} = \frac{4}{52} = \frac{1}{13}$

   b) There are 2 black kings in 52 playing cards. So the probability of drawing a black king

   $= P(black\ king) = \frac{n(black\ kings)}{n(\varepsilon)} = \frac{2}{52} = \frac{1}{26}$

   c) There are 26black cards and 26 red cards in 52 playing cards, so the probability of drawing a black card =

   $P(black\ card) = \frac{n(black\ cards)}{n(\varepsilon)} = \frac{26}{52} = \frac{1}{2}$

   d) In 52 playing cards there are 13 spades all in all, so the probability of drawing a spade = $P(spade) = \frac{n(spades)}{n(\varepsilon)} = \frac{13}{52}$

3. a) when throwing a die, there are 6 possible outcomes being $\varepsilon = \{1,2,3,4,5,6\}$, and the cubic numbers

are $1^3 = 1, 2^3 = 8, 3^3 = 27,$ .... so the only cubic number that can come from a die is 1. So $P(cubic\ number) = \frac{n(cubic\ number)}{n(\varepsilon)} = \frac{1}{6}$

b) There are six possible outcomes in a die, and the square numbers are: $1^2 = 1, 2^2 = 4, 3^3 = 9$........ so there are only 2 ways in which a die can show a square number because the only square numbers that are present in die are and 1 and 4. $P(square\ no.) = \frac{n(square\ no.)}{n(\varepsilon)} = \frac{2}{6} = \frac{1}{3}$.

c) Multiples of 2 are: 2,4,6,8, so the multiples of two in die are, 2, 4, and6, therefore $P(multiple\ of\ 2) = \frac{n(multiples\ of\ 2)}{n(\varepsilon)} = \frac{3}{6} = \frac{1}{2}$.

4. a. $P(yellow) = \frac{n(yellow)}{n(\varepsilon)} = \frac{3}{10}$

b. $P(not\ brown) = \frac{n(non\ brown)}{n(\varepsilon)} = \frac{10}{10} = 1$ because there are 10 sweets all in all in the bag which are not brown.

c. There are 4 pink, 3 red and 3 yellow, so the probability that it is not a pink sweet is $P(not\ pink) = \frac{n(no\ pink)}{n(\varepsilon)} = \frac{6}{10} = \frac{3}{5}$

5. There were $\frac{4}{5} \times 60 = 48$ female candidates.

Solutions to Activity 3

No. 1

The combinations of possible outcomes are: (1,1)(1,2)(1,3)(1,4)(1,5)(1,6)(2,1)(2,2)(2,3)(2,4)(2,5)(2,6)(3,1)(3,2)(3,3)(3,4)(3,5)(3,6)(4,1)(4,2)(4,3)(4,4)(4,5)(4,6)(5,1)(5,2)(5,3)(5,4)(5,5)(5,6)(6,1)(6,2)(6,3)(6,4)(6,5)(6,6).So there are 36 possible combinations all in all and there is only one way in which the two dice can both show a 2. Therefore

$$P(2,2) = \frac{n(S)}{n(\varepsilon)} = \frac{1}{36}$$

No.2

The combinations of possible outcomes in this case are (H, H)(H,T)(T,H)(T,T).

There are 4 possible outcomes and there is only one (T,T), so

$$P(T,T) = \frac{n(S)}{n(\varepsilon)} = \frac{1}{4}.$$

No.3

The possible outcomes are:

(H,1)(H,2)(H,3)(H,4)(H,5)(H,6)(T,1)(T,2)(T,3)(T,4)(T,5)(T,6).

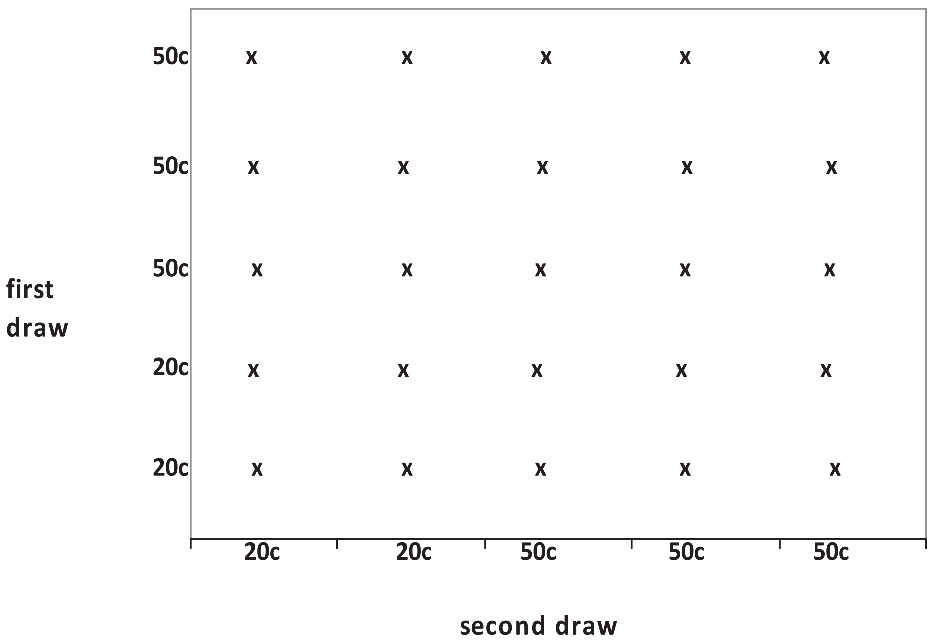
There two desirable outcomes (H,4) and (H,6). Therefore

$$P(H, even\ number) = \frac{n(S)}{n(\varepsilon)} = \frac{2}{12} = \frac{1}{6}$$

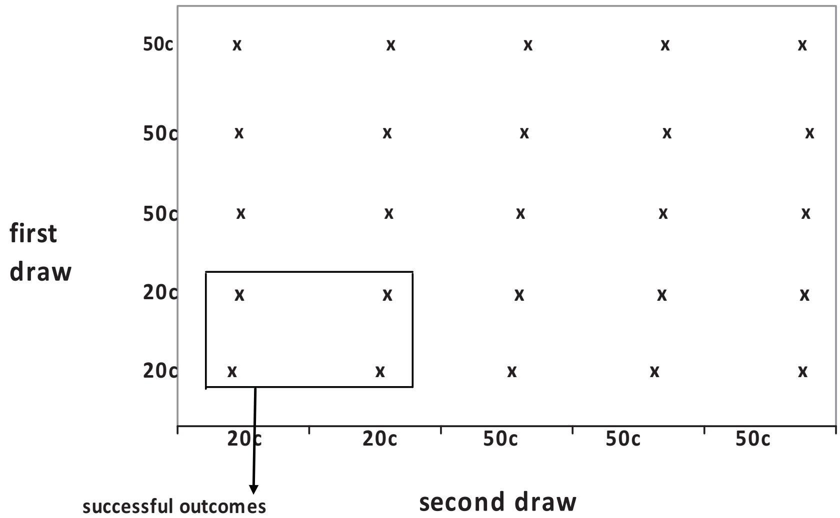
Solutions to activity 4:

No.1

a.



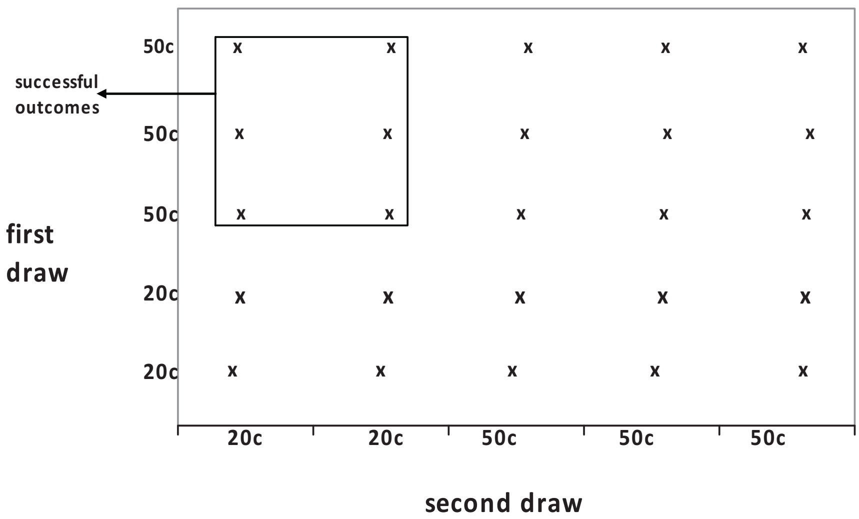
i.



The box of successful outcomes shows the ways in which an event of drawing two heads can occur.

We can see that there are four ways in which Mpho can draw two 20c coins from her bag. So the probability that she draws two 20c coin = $P(20c,20c) = \frac{n(S)}{n(\varepsilon)} = \frac{4}{25}$

ii.



There are 6 successful outcomes in this case, so $P(20c,50c) = \frac{n(S)}{n(\varepsilon)} = \frac{6}{25}$

iii. $P(30c,30c) = \frac{n(S)}{n(\varepsilon)} = \frac{0}{25}$, there are no 30c coins in her bag.

No. 2

a) There are 12 possible outcomes when looking at the possibility space.

b) The possible outcomes are:
(1,T),(1,H),(2,T),(2,H),(3,T),(3,H),(4,T),(4,H),(5,T),(5,H),(6,T),(6,H)

c) $P(2\ and\ tail) = \frac{n(S)}{n(\varepsilon)} = \frac{1}{12}$

d) $P(Head\ and\ even\ number) = \frac{n(S)}{n(\varepsilon)} = \frac{3}{12} = \frac{1}{4}$

Solutions to activity 5

a) When tossing a die, the set of possible outcomes is as follows: $\varepsilon = \{1,2,3,4,5,6\}$, so there are 6 possible outcomes.

b) There is only one 4 in a die, therefore $P(4) = \frac{n(S)}{n(\varepsilon)} = \frac{1}{6}$.

c) There is again only one 3 in a die, therefore $P(3) = \frac{n(S)}{n(\varepsilon)} = \frac{1}{6}$

d) With this question, we want the probability of either 3 or 4, that is whether the die rests as a 3 or a 4 it is fine. So there are two targeted outcomes here, 3 and 4 and the probability of an event occurring $= \frac{number\ of\ target\ outcomes}{total\ number\ of\ possible\ outcomes}$.

So $P(3\ or\ 4) = \frac{n(S)}{n(\varepsilon)} = \frac{2}{6} = \frac{1}{3}$

Solutions to activity 6

1. a) $P(blue) = \frac{no.\ of\ blue\ balls}{total\ no.\ of\ balls} = \frac{n(S)}{n(\varepsilon)}$ $\therefore P(blue) = \frac{n(S)}{n(\varepsilon)} = \frac{11}{20}$

b)

$$P(white\ or\ blue) = P(white) + P(blue) = \frac{n(white)}{n(\varepsilon)} + \frac{n(blue)}{n(\varepsilon)} = \frac{6}{20} + \frac{11}{20} = \frac{17}{20}$$

c) There are 20 balls all in all, 11 are blue and 9 are not blue

$$\therefore P(not\ blue) = \frac{n(S)}{n(\varepsilon)} = \frac{9}{20}$$

d) The targeted balls here are balls which are not red and also balls which are not white, that is blue balls. So

$$\therefore P(neither\ red\ nor\ white) = \frac{n(blue)}{n(\varepsilon)} = \frac{11}{20}$$

e) None of the balls is black or pink, so all of the 20 ball are the targeted outcomes;

$\therefore P(neither\ red\ nor\ white) = \frac{n(S)}{n(\varepsilon)} = \frac{20}{20} = 1$ remember at the beginning of the unit we said the maximum probability on the probability scale is 1.

2. a) $P(heart\ or\ black\ queen) = P(heart) + P(Black\ queen) = \frac{13}{56} + \frac{2}{56} = \frac{15}{54}$

b) $P(ace\ or\ king) = P(ace) + P(king) = \frac{4}{56} + \frac{4}{56} = \frac{8}{56} = \frac{1}{7}$

c)

$$P(heart\ or\ black\ queen) = P(heart) + P(Black\ queen) = \frac{13}{56} + \frac{2}{56} = \frac{15}{54}$$

d) There are 2 red jacks and 26 black cards in 52 playing cards, so we want none of those, so the targeted outcomes are the remaining when we subtract the 2 red jacks and the 26 black cards which is 24 cards.

$$P(neither\ red\ jack\ nor\ black\ card) = \frac{24}{56} = \frac{3}{7}$$

3. $P(neither\ red\ jack\ nor\ black\ card) = \frac{24}{56} = \frac{3}{7}$

a) In 1, 2, 3,4,5,6, the factors of 6 together with the factors of 5 are underlined. So the favourable outcomes are 5, we count 1 only ones and not twice though it is a factor of both 5 and 6 because there is only one '1' in a die. So

$$P(factors\ of\ 5\ or\ 6) = \frac{5}{6}$$

b) In 1, 2, 3,4,5,6, the square numbers are: 1and 4 and the multiples of 3 are: 3 and 6 , so those that are neither square numbers nor multiples of 3 are : 2and 5.

$$\therefore P(neither\ square\ number\ nor\ multiple\ of\ 3) = \frac{2}{6} = \frac{1}{3}$$

How was the activity, was it fun? I hope it was but if it wasn't, don't forget, you probably missed a concept somewhere, so just go back and see where.

Solutions to activity 7

1.

a. The rest possible outcomes are (3,H),(3,T),(4,H),(4,T),(5,H),(5,T)

b. Of the possible outcomes, there is only one (2,T) so

$$P(2,T) = \frac{no\ of\ (2,T)}{\varepsilon} = \frac{1}{12}$$

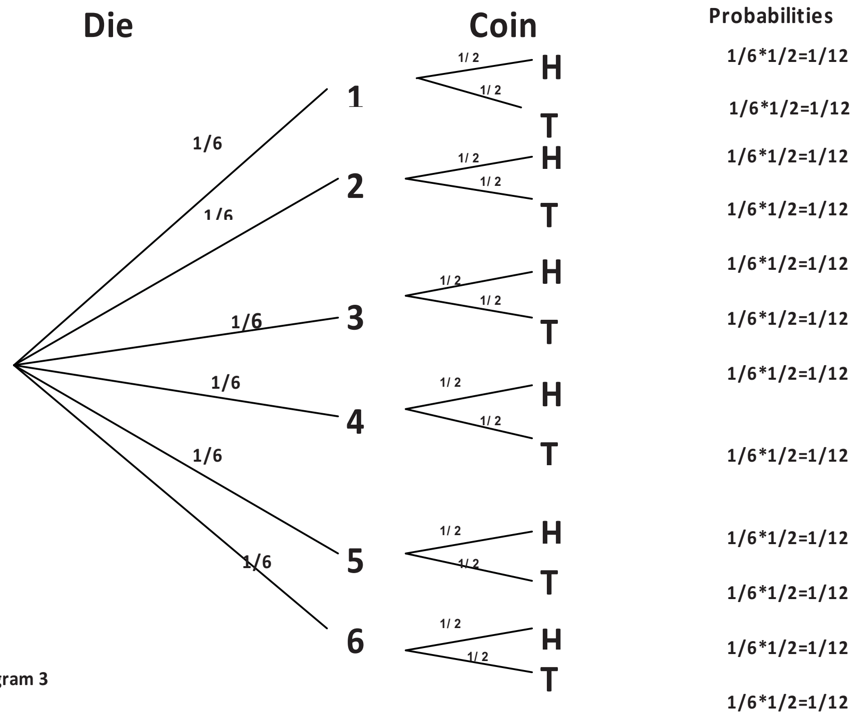
c. $P((2,T) or (3,H)) = P(2,T) + P(3,H) = \frac{1}{12} + \frac{1}{12} = \frac{2}{12} = \frac{1}{6}$

d. There are three combinations of heads and even numbers,

(2,H),(4,H),(6,H), so $P(even\ no., H) = \frac{3}{12} = \frac{1}{4}$

Solutions to activity 8:

1. i)



Tree diagram 3

I found that $P(2,T) = P(2) \times (PT) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$. This gives me the same answer to the one I found in activity 4b) where I found that $P(2,T) = \frac{1}{12}$.

2. i)

$P(\textit{even number and Head})$

$= P(2 \textit{ and } H) \textit{ or } P(4 \textit{ and } H) \textit{ or } P(6 \textit{ and } H)$

$= \frac{1}{6} \times \frac{1}{2} \textit{ or } \frac{1}{6} \times \frac{1}{2} \textit{ or } \frac{1}{6} \times \frac{1}{2}$

$= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$

2, 4 and 6 are even numbers that is why we have 3 even numbers.

When comparing the answers, I notice that
$P(\textit{Even number and Head}) = P(\textit{Even number}) \times P(\textit{Head})$

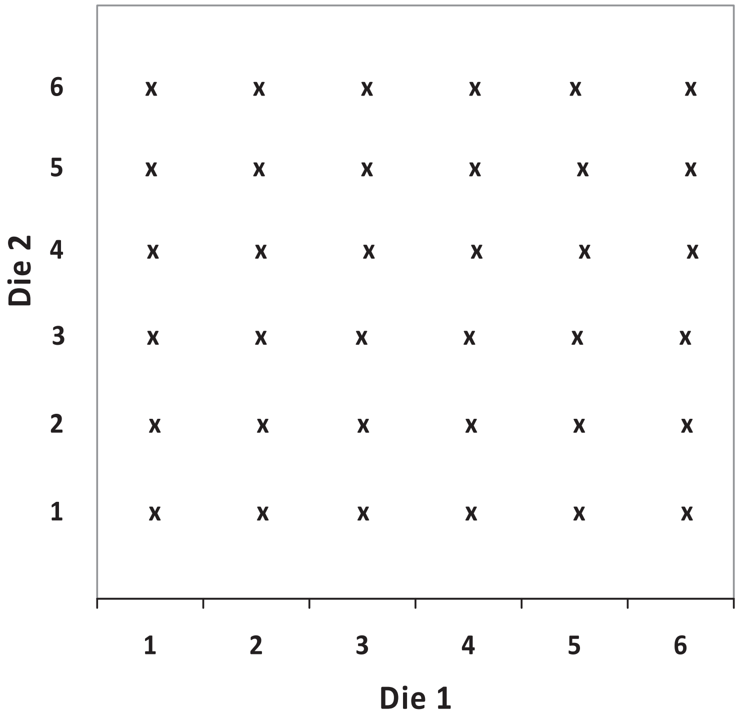
ii) $P(3,T) = P(3) \times P(T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

Solutions to activity 9

No.1

a)

The Possibility space



b) From the possibility space there are 36 possible outcomes of which there is only one way in which we can get a 2 and a 2. So $P(2{,}2)=\frac{n(S)}{n(\varepsilon)}=\frac{1}{36}$

c) To answer c) let us first list all the possible outcomes:

(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3,),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6).

i) Here we look for the combinations of which the sum is equal to ten and from our list above we have 3 combinations which are: (4,6),(5,5) and (6,6).

$$\therefore P(sum\ equals\ ten)=\frac{n(S)}{n(\varepsilon)}=\frac{3}{36}=\frac{1}{12}$$

ii) There are 26 combinations which heir sum is greater than 5.

$$P(sum\ is\ greater\ than\ 5)=\frac{n(S)}{n(\varepsilon)}=\frac{26}{36}=\frac{13}{18}$$

iii) $P(sum\ less\ than\ 3)=\frac{n(S)}{n(\varepsilon)}=\frac{1}{36}$

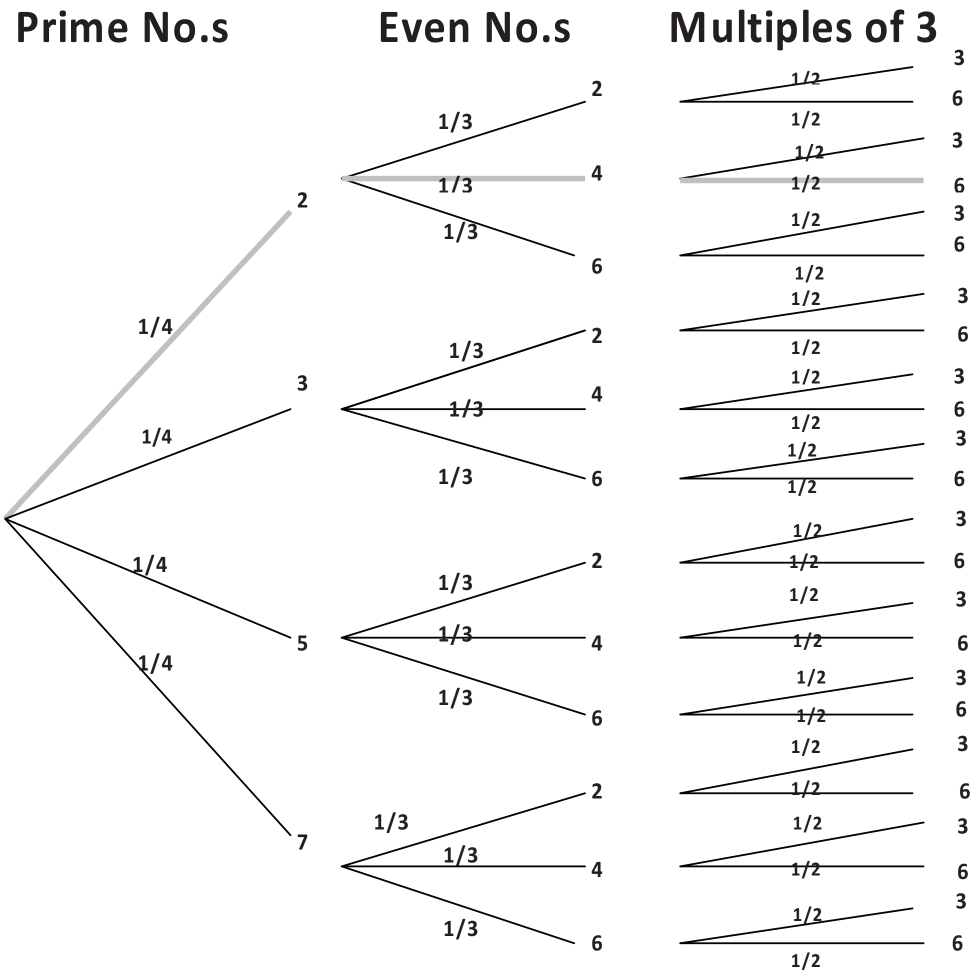
iv) Those that make multiples of 4 are highlighted
(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3,),(2,4),(2,5),(2,6),(3,1),(3,

2),(3,3),(3,4),
(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,
6),(6,1),(6,2), (6,3),(6,4), (6,5),(6,6).

$$P(sum\ is\ a\ multiple\ of\ 4) = \frac{n(S)}{n(\varepsilon)} = \frac{9}{36} = \frac{1}{4}$$

No 2.

a.



Tree diagram 7

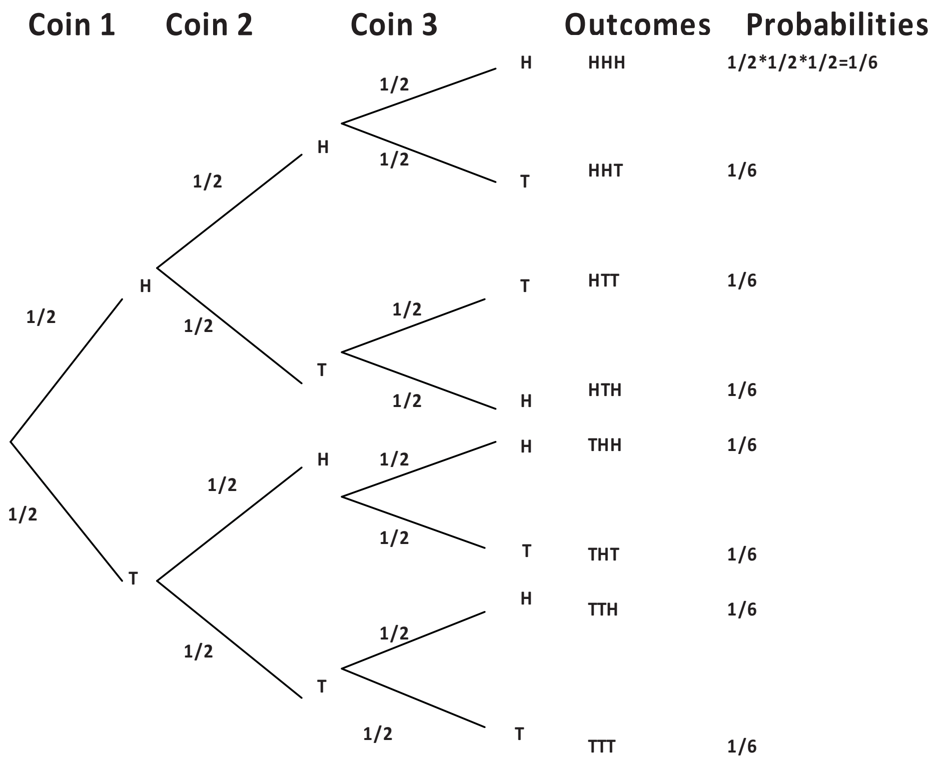
b. In a the branches showing the combination (2,4,6) are highlighted in grey. So

$P(\textit{prime no.}(2)\ \textit{and even number}(4)\ \textit{and multiple of}\ 3(6))$

$= P(2) \times P(4) \times P(6) = \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$

No 3.

a)



Tree diagram 8

b) There are 4 combinations with at least two heads each with probability 1/6.

So

$P(\textit{at least two heads}) = P(HHH) or P(HHT) or P(HTH) or P(THH)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

c)

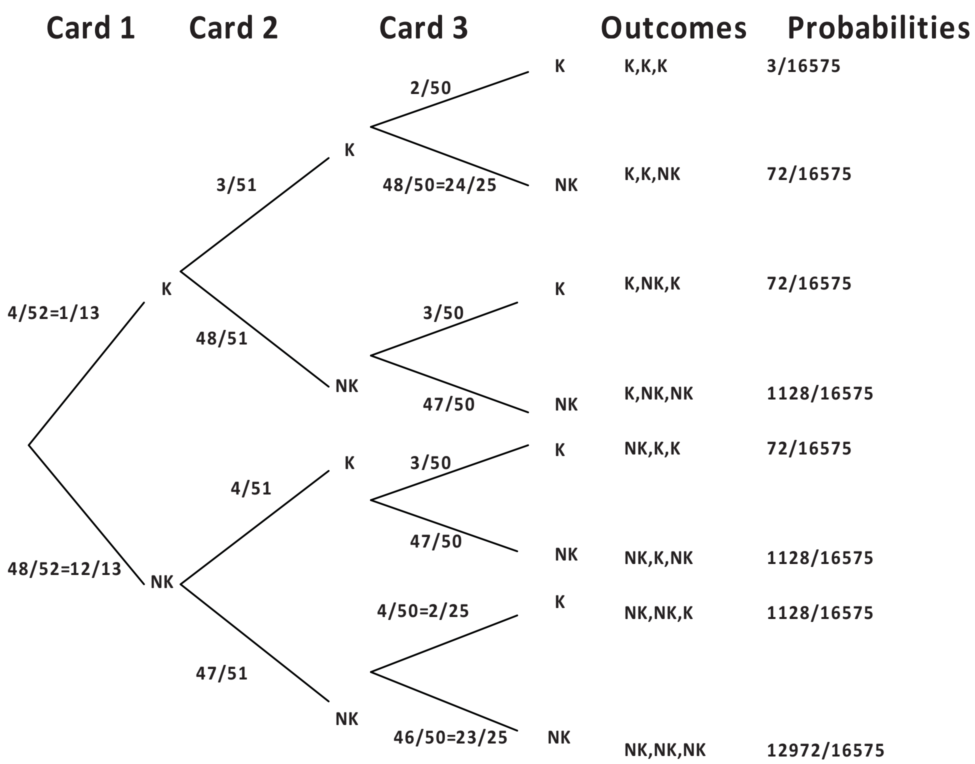
$P(exactly\ 2\ heads)\ or\ P(exactly\ 2\ tails) = P(HHT) or P(HTH) or P(HTT) or P(TTH)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6}$

Solutions to Activity 11

No. 1

a) To answer this question let us first draw the tree diagram.

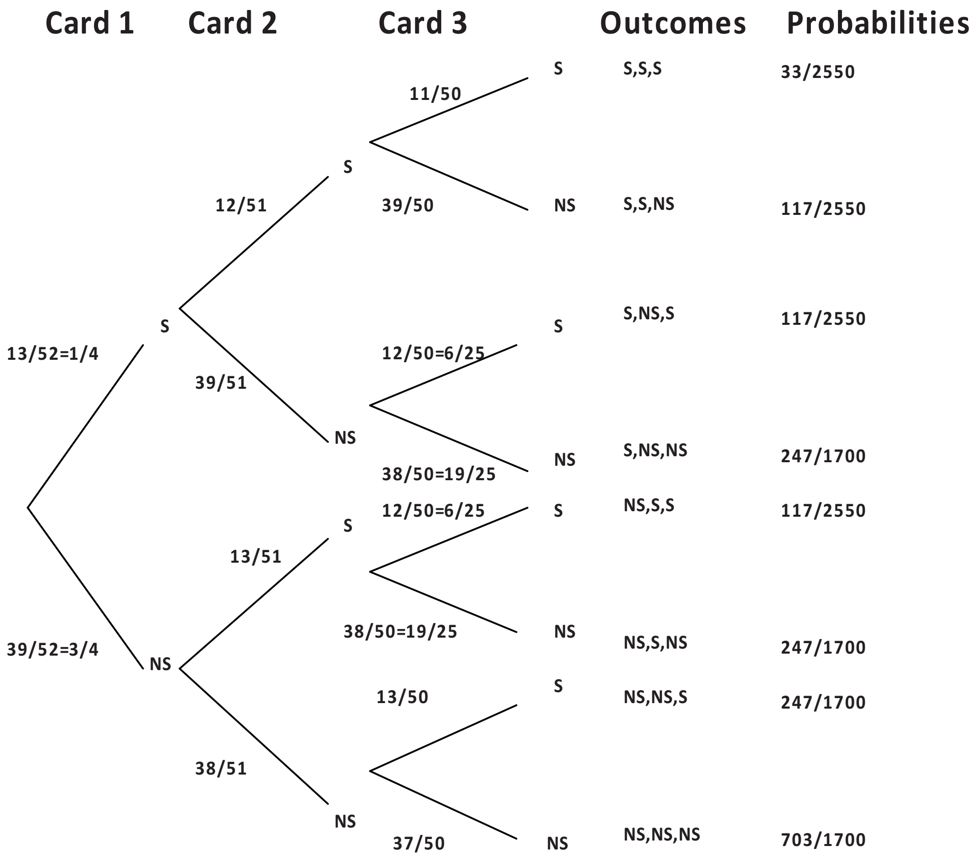


Tree diagram 5

let K represent a King and NK, not a King

Now from the tree diagram, the probability that all the three cards are kings is $\frac{3}{16575}$.

b) To find the probability that at least two are spade we will first draw the tree diagram.



Tree diagram 6

Let S represent a spade and NS, not a spade

Now the probability that at least two cards are spades from the tree diagram is given by:

$$P(at\ least\ two\ are\ spades) = P((S,S,S)or(S,S,NS)or(S,NS,S)or(NS,S,S))$$

$$= \frac{33}{2550} + \frac{117}{2550} + \frac{117}{2550} + \frac{117}{2550} = \frac{384}{2550} = \frac{192}{1275}$$

Unit Summary

Summary

In this unit you learned that:

1.

i) Probability is the numerical measure of the likelihood of an event to occur or not to occur.

ii) Probability scale ranges from 0 to 1and that the probability of an event is given either as a decimal or fraction in its simplest form.

2.

$$Theoretical\ probability = \frac{number\ of\ targeted\ outcomes}{total\ number\ of\ possible\ outcomes}$$

or

$$Theoretical\ probability = \frac{number\ of\ ways\ in\ which\ an\ event\ can\ occur}{total\ number\ of\ possible\ outcomes}$$

$$i.e.\ P(s) = \frac{n(S)}{n(\varepsilon)}$$

3.

i) With combined events two or more different events occurring are combined to give one probability.

ii) Possibility spaces and free diagrams can be used to display probabilities of combined events

iii) Under combined events we have

a) Mutually exclusive events: Whereby if A and B are mutually exclusive events $P(A\ or\ B) = P(A) + P(B)$

b) Independent events: Where if A and B are independent events $P(A\ and\ B) = P(A)\ x\ P(B)$

Conditional Probabilities: of which the probability of the subsequent event depends directly on the probability of the first event.

You have completed the material for this unit on probability. You should now spend some time reviewing the content. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

You are advised to spend 40 minutes on this assignment. It carries 50 marks in all and the marks for each question are shown.

Show all the necessary workings.

The use of calculators is permitted.

1. Two coins are tossed together.

   a) What is the probability that they will have the same score?
   (2 marks)

2. Two dice are tossed; draw the possibility space and find the probability that they will have different scores.
(4 marks)

3. A 5c stamp, a 10c stamp and a 20c stamp are put in a bag.

i) Assuming that each is equally likely to be drawn out, what is the probability that the 5c stamp is drawn?
(4 marks)

ii) The stamp is put back and a second one is drawn.

a) Draw a tree diagram to help you to answer the questions.
(4 marks)

b) Write down the outcomes and calculate the probabilities for each possible outcome. (4 marks)

c) What are the probabilities that the two stamps:

1. Are both 5c? (2 marks)

2. Total at least 20c? (3 marks)

3. Are neither 5c? (3 marks)

4. Are of the same value? (3 marks)

4. In a church there are two choirs, in choir A there are 20 females and 30 males, in choir B there are 15 females and 10 males. If a member is chosen at random from each choir, find the probability that:

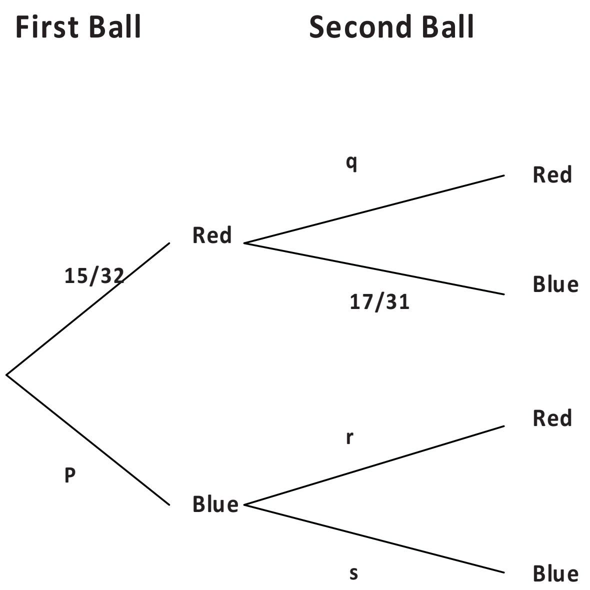
a) They are both females (4 marks)

b) It is a male and a female (5 marks)

5. A bag contains 15 red balls and 17 blue balls.

i. At random one ball is picked from the bag. Draw a tree diagram. What is the probability that it is a blue ball? (4 marks)

ii. Now the ball is put back, the balls are mixed thoroughly and at random two are withdrawn without replacement and the following tree diagram shows the probabilities.



Tree diagram 9

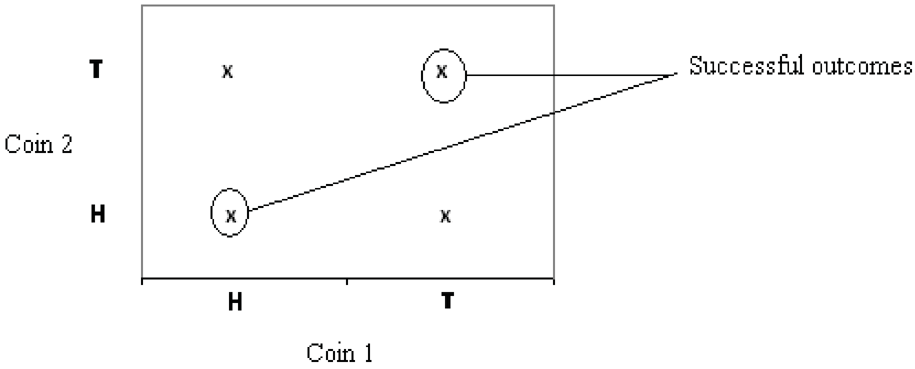
a) What are the values of p, q, r and s? (4 marks)

b) What is the probability of getting two red balls? (4 marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Solutions to the Assignment

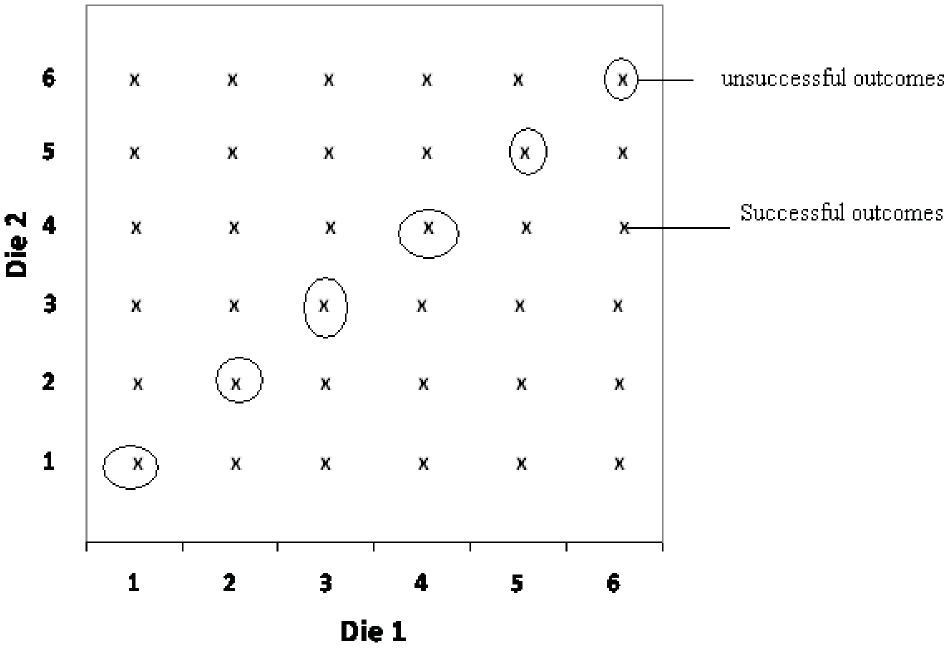
No.1



There are four possible outcomes and 2 successful outcomes, so

$$P(same\ score) = \frac{n(S)}{n(\varepsilon)} = \frac{2}{4} = \frac{1}{2}$$

No.2



The outcomes that are not circled are the successful ones. So

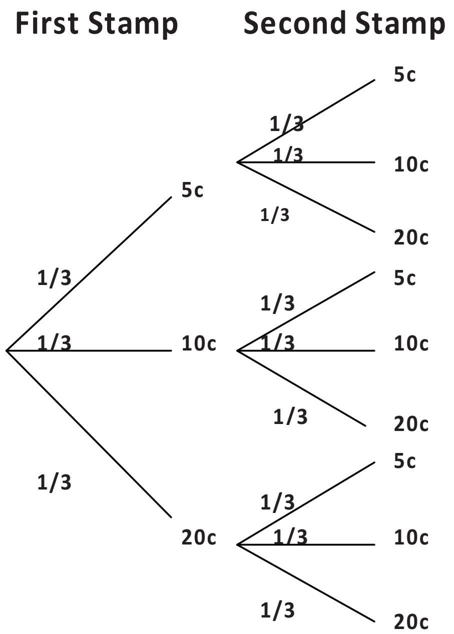
$$P(different\ scores) = \frac{n(S)}{n(\varepsilon)} = \frac{30}{36} = \frac{5}{6}$$

No.3

i) if they are equally likely to be drawn, then each has a probability of $\frac{1}{3}$ because they are 3.

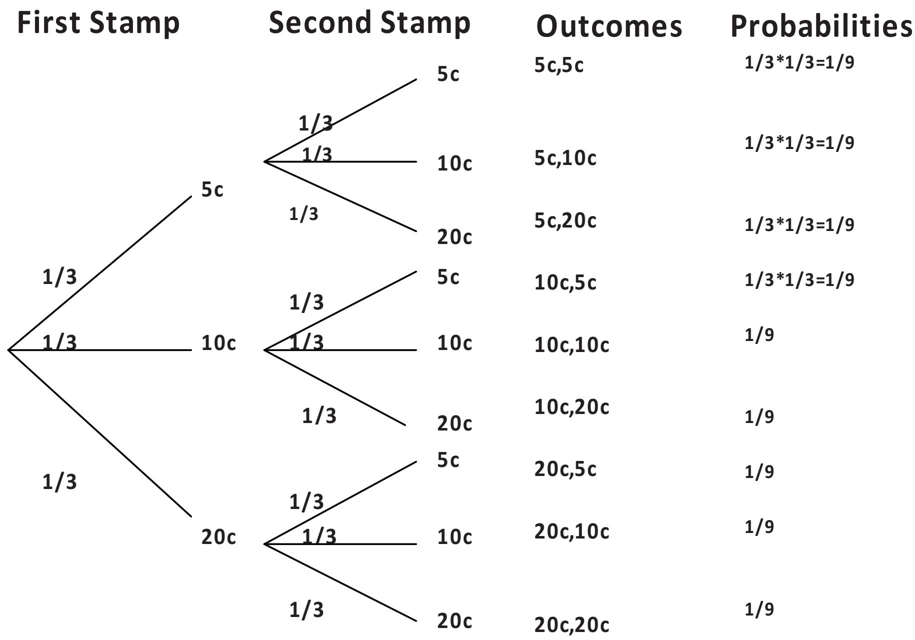
So $P(5c\ stamp) = \frac{1}{3}$.

ii) a)



Tree diagram 10

b)



Tree diagram 11

c) 1) $P(5c\ \&\ 5c)=\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$

2) $P(sum = at\ least\ 20c)=\frac{6}{9}=\frac{1}{3}$. There are nine possible outcomes and we look at combinations of which their sum is equal to 20 or greater 20 as our successful outcomes.

3) The successful outcomes are: (10c, 10c), (10c, 20c), (20c, 10c),

(20c, 20c), $P(both\ stamps\ are\ neither\ 5c)=\frac{4}{9}$

4) The successful outcomes are: (5c, 5c), (10c, 10c), (20c, 20c).

$P(stamps\ are\ the\ same\ value)=\frac{3}{9}=\frac{1}{3}$.

No. 4

a)

$$P(female\ and\ female) = P(f_A and\ f_B) = P(f_A) \times P(f_B) = \frac{20}{50} \times \frac{15}{25} = \frac{6}{25}$$

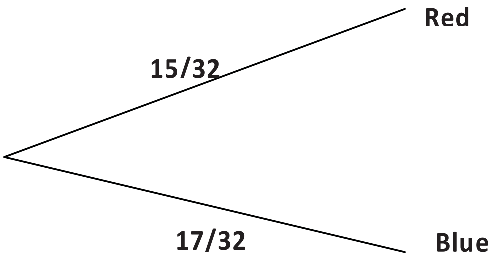
b)

$$P(male\ and\ female) = P(m_A and\ f_B)\ or\ P(f_A and\ m_B)$$

$$= P(m_A) \times P(f_B) + P(f_A) \times P(m_B) = \frac{30}{50} \times \frac{15}{25} + \frac{20}{50} \times \frac{10}{25} = \frac{13}{25}.$$

No.5

i) the tree diagram is as drawn below.



Tree diagram 12

$$P(blue) = \frac{17}{32}.$$

ii) a)

$$p = \frac{17}{32}$$

$$q = \frac{14}{31}\ the\ first\ ball\ that\ was\ taken\ was\ red\ so\ the\ red\ balls\ are\ now\ 14.$$

$$r = \frac{15}{31} the\ red\ balls\ remain\ 15\ because\ in\ this\ case\ a\ blue\ ball\ was\ taken$$

$$s = \frac{16}{31} a\ blue\ ball\ was\ taken\ so\ the\ blue\ balls\ are\ now\ 16$$

b) $P(r\ and\ r) = P(r) \times P(r) = \frac{15}{32} \times \frac{14}{31}$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

You are advised to spend at most 50 minutes on this assessment.

The assessment carries 60 marks and the marks for each question are shown.

The use of calculators is permitted.

1. In a village sports team, a member is picked at random, the probability that it is a female is $\frac{2}{7}$. If there are 10 females all in all, how many members make this team? (2 marks)

2. Tina has two fair, normal 6-sided dice. One is red and the other is blue. She throws both of them once.

You may find it helpful to draw a possibility space to answer the following questions.

Find as a simple fraction in its simplest form, the probability that:

i) The red die shows a 2 and the blue die does not show a 2.
(3 marks)

ii) The sum of the two numbers shown is equal to 5.
(4 marks)

iii) One die shows a 3 and the other an even number.
(3 marks)

3. A teacher has a box containing five mathematics books and seven science books.

Three of the maths books and four of the science books are defective, the others are good.

If pupil takes a book out of the box at random, what is the probability that it is:

a. A non-defective science book? (3 marks)

b. A maths book? (2 marks)

4. A pack contains a large number of flower seeds which look identical, but produce flowers with one of three colours, white, yellow and red.

One half of the seeds produce white flowers and one third produce yellow flowers. The remainder of the seeds produce red flowers.

a) Explain why the probability that a particular seed will produce a red flower is $\frac{1}{6}$.

(2 marks)

b) Find the probability that one particular seed will produce a seed which is not yellow.

(4 marks)

c) Two seeds are planted.

i) Draw a tree diagram and show the possible outcomes and their probabilities.

(4 marks)

ii) Find the probability that

1. Both will produce a yellow flower

(3 marks)

2. Both will produce a blue flower
(1 mark)

3. One will produce a yellow flower and the other a white flower,
(3 marks)

4. Neither will produce a red flower.
(4 marks)

5. A letter posted in Lesotho will be delivered in Botswana within 5 days with a probability of 0.7.Lineo posts 3 letters to Botswana from Lesotho, what is the probability that:

a) They all arrive in Botswana within five days (4 marks)

b) None arrives within five days (3 marks)

c) At least one arrives within five days (3 marks)

(Hint: draw a tree diagram to help you represent the probabilities)

6. There were 12 girls and 3 boys in a group of children. One child was chosen at random from the group. Another child was chosen from the remaining children. Expressing each answer as fraction in its simplest form, calculate the probability that:

i) The first child chosen was a girl (3 marks)

ii) The first child was a girl and the second a boy, (4 marks)

iii) A child of each sex was chosen. (5 marks)

Send your answers and work to your tutor for marking.

Solutions to the Assessment

No.1

$$P(f) = \frac{2}{7}$$

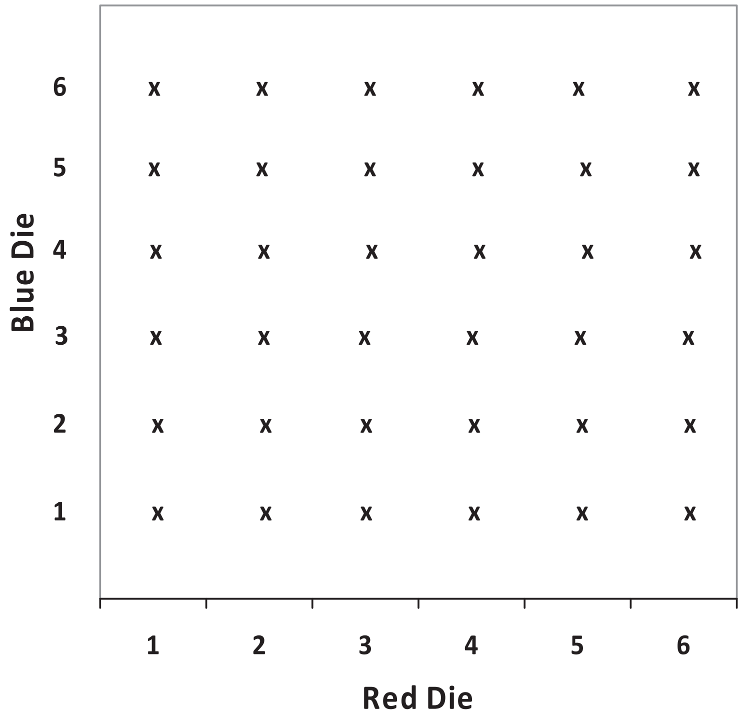
$so\ \frac{2}{7} \times x = 10$ Where x is the total number of members.

$\therefore 2x = 70$

$\Rightarrow x = 35$

There are 35 members of the team.

No.2



i) From the possibility space above, the successful outcomes are: (2,1)(2,3)(2,4)(2,5)(2,6).

So $P(red = 2 \text{ and } blue \neq 2) = \frac{n(S)}{n(\varepsilon)} = \frac{5}{36}$.

ii)The successful outcomes are (1,4)(2,3)(3,2)(4,1).

Therefore $P(Sum = 5) = \frac{n(S)}{n(\varepsilon)} = \frac{4}{36} = \frac{1}{9}$.

iii) The Successful outcomes are: (3,2)(3,4)(3,6)(2,3)(4,3)(6,3).

So $P(3 \text{ and an even number}) = \frac{n(S)}{n(\varepsilon)} = \frac{6}{36} = \frac{1}{6}$.

No.3

We have 12 books all in all, 5 are maths books of which 3 are defective and 2 are not, and 7 science books of 4 are defective and 3 are good.

a)

$$P(non - defective\ science\ book) = \frac{n(non\ defective\ science\ books)}{n(all\ books)} = \frac{3}{12} = \frac{1}{4}$$

b)

$$P(math\ book) = P(defective\ maths\ book\ or\ non-defective\ maths\ book)$$

$$P(defective\ maths\ book) + P(non-defective\ maths\ book) = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}.$$

No.4

$$P(w) = \frac{1}{2}$$

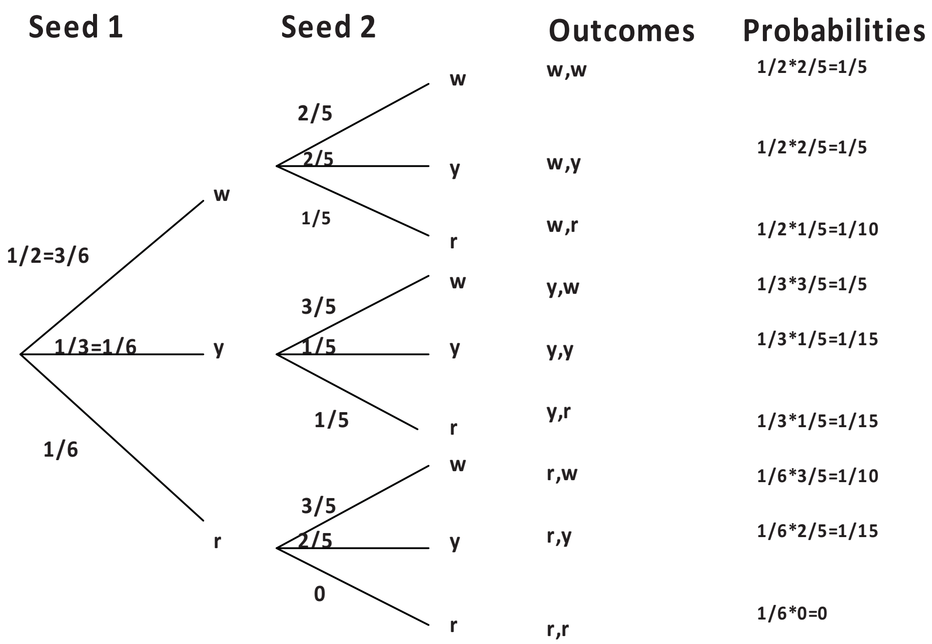
$$P(y) = \frac{1}{3}$$

$$P(r) = ?$$

a) It is because the probabilities must add up to one. So $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$.

b) $P(not\ yellow) = P(w\ or\ r) = P(w) + P(r) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

c) i)



Tree diagram 13

ii) 1) $P(y\ and\ y) = P(y, y) = \frac{1}{15}$

2) $P(blue) = 0$ There are no seeds that can produce blue flowers.

3)

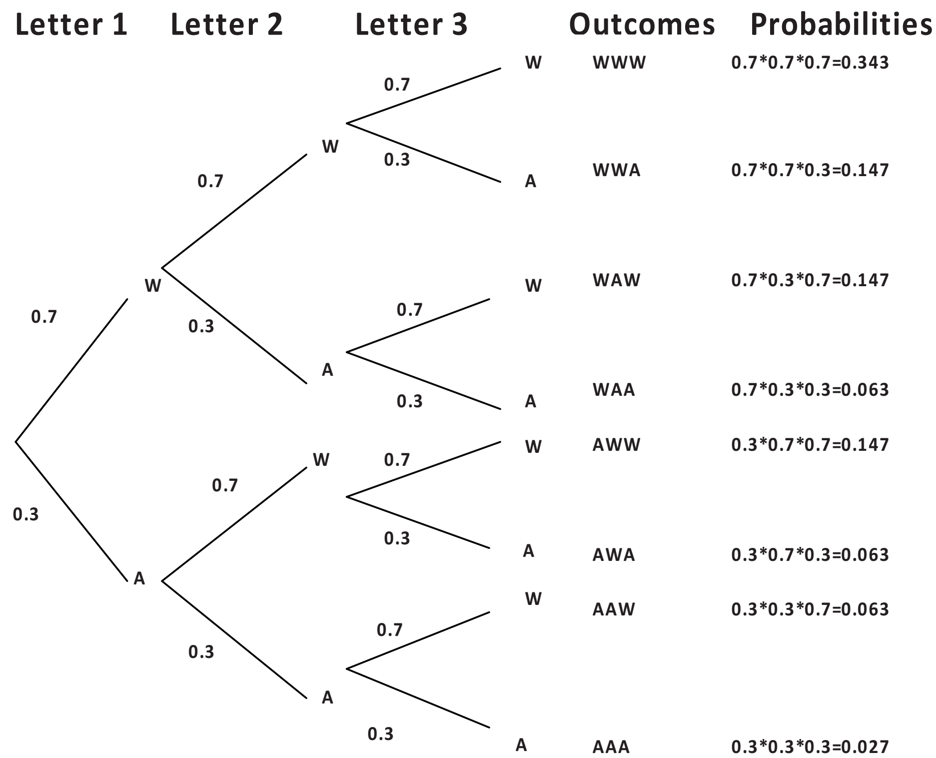
$$P(y\ and\ w) = P(w_1\ and\ y_2)\ or\ P(y_1\ and\ w_2)$$

$$= P(w_1) \times P(y_2) + P(y_1) \times P(w_2) = \frac{1}{2} \times \frac{2}{5} + \frac{1}{3} \times \frac{3}{5} = \frac{2}{5}$$

4)

$$P(no\ red) = P(w, w)\ or\ P(w, y)\ or\ P(y, w)\ or\ P(y, y) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{15} = \frac{10}{15} = \frac{2}{3}$$

No.5 First let us draw the tree diagram so that we can answer the questions with its help.



Tree diagram 14

W= Within 5 days
A= After 5 days

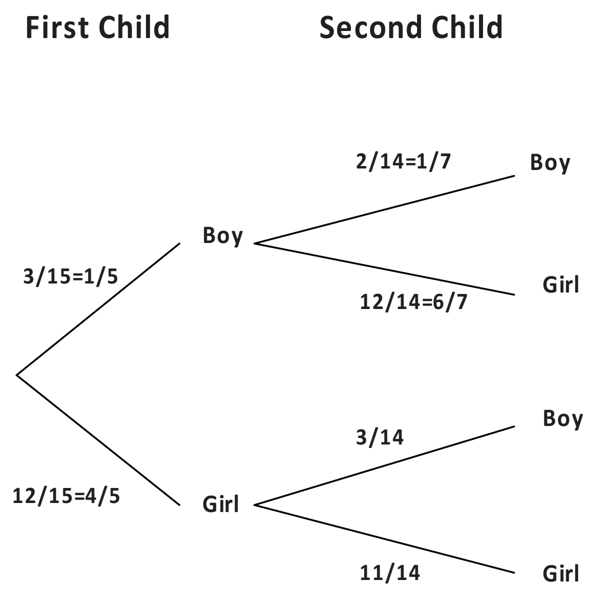
a) $P(all\ arrive\ within\ 5\ days) = P(w, w, w) = 0.343$

b) $P(non\ arrives\ within\ 5\ days) = P(A, A, A) = 0.027$

c)

$P(at\ least\ one\ arrives\ within\ 5\ days)$
$= P((W,W,W)\ or\ (W,W,A)\ or\ (W,A,W)\ or\ (W,A,A)\ or\ (A,W,W)\ or\ (A,W,A)\ or\ (A,A,W)$
$= 0.343 + 0.147 + 0.147 + 0.063 + 0.147 + 0.063 + 0.063 = 0.973$

No 6. Let us first draw a tree diagram to help us answer the questions.



Tree diagram 15

i) From the tree diagram the probability that the first child was a girl is $\frac{4}{5}$.

ii) And when we look at the tree diagram again, the probability of getting a girl first is: $P(g_1, b_2) = \frac{4}{5} \times \frac{3}{14} = \frac{6}{35}$, where

: g_1 *represents : the first child as a girl*
and b_2 *represents the second child as a boy*

iii)

$$P(g \text{ and } b) = P((g_1 \text{ and } b_2) \text{ or } (b_1 \text{ and } g_2)) = \frac{4}{5} \times \frac{3}{14} + \frac{1}{5} \times \frac{6}{7} = \frac{6}{35} + \frac{6}{35} = \frac{12}{35}$$

.

If you have scored at least 80% in this assessment, you are free to move to the next topic. But if you have scored anything less than the 80%, you need to revisit this unit to get a better understanding.

Unit Contents

Unit 24

Unit 24

Functions and Relations

Introduction

In everyday life we come across many situations that involve relationships between different aspects. For example, we may realize that when one travels at a higher speed they take a shorter time than when they travel through the same distance at a lower speed. This means that there is a relationship between speed and time. In this section we are going to learn about relations of numbers and how we may express them algebraically (as equations or mappings) and represent them as ordered pairs. This unit consists of 43 pages and is about 2% of the whole course. As reference, you will need to devote 15 hours to work on this unit, 10 hours for formal study and 5 hours for self-study and completing assessments/assignments.

Take a moment to read the following learning outcomes. You should focus on those skills while studying this unit.

This Unit is Comprised of Six Lessons:

Lesson 1 The Domain and Range of a Relation
Lesson 2 Functions and Non-functions
Lesson 3 Evaluating Functions
Lesson 4 Inverse of a Function
Lesson 5 Algebra of Functions
Lesson 6 Composite Functions

Upon completion of this unit you will be able to:

Outcomes

- *identify* the domain and range of a relation.
- *express* relations as ordered pairs.
- *represent* relations in the form of equations and using set notation.
- *distinguish* between functional and non-functional relations.
- *perform* operations on functions.
- *determine* composite functions.
- *find and recognise* inverse functions.

Terminology

Mapping: Relation between the input and the output.

Input: The first value in an ordered pair of a relation.

Output: The second value in an ordered pair of a relation.

Function: A relation in which each element of the domain has only one image.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 The Domain and Range of a Relation

At the end of this subunit you should be able to:

-Identify the domain and the range of any given function.

- Describe the domain and range of a relation using set notation.

-Determine the relationships that are represented by different mappings and express the relations in equation form.

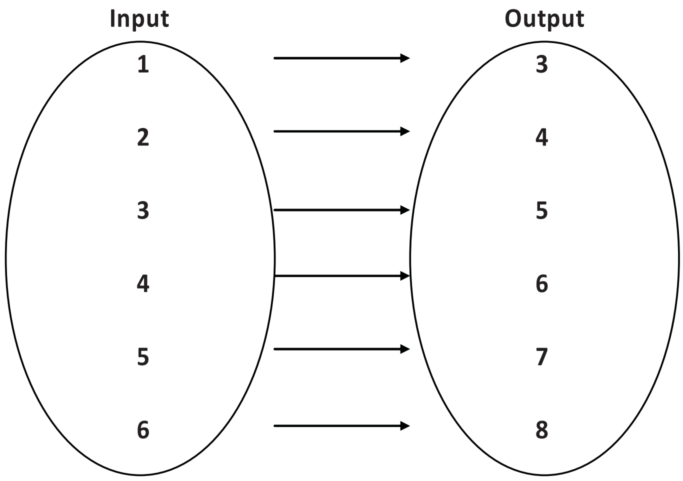
This subunit is about 3 pages in length.

Activity 1.1 will introduce you to the domain and the range.

Activity 1.1

Activity 1.2

Consider the following mapping.



a) List the members of the input set.

b) List the members of the output set.

c) Write the relation in the form (input, output), the first two are given for you as (1,3),(2,4). Continue with the rest.

d) Determine the relationship between each member of the input set and the corresponding member of the output set.

Compare your answers with those at the end of the subunit. I hope you got them all correct! Be sure that you understand each answer before continuing in this unit.

Note that there is a common rule that can be applied to all members of the input set to get the corresponding values in the output set. The rule expresses the relationship represented by the mapping. Members of the input set form the **domain** of the relation while members of the output set form the **range** of the relation.

A **relation** is a mapping that assigns each element of a **domain** to one or more elements of the **range**.

The members of the domain are usually assigned the letter "*x*", while those of the range are assigned the letter "*y*".

Thus the relation may be written as a set of ordered pairs (*x, y*) where $x \in domain$ and $y \in range$ (corresponding to x).

From the relation shown in Activity 1.1 we realise that when the input is 1 the output is 3, when the input is 2, the output 4 and so on and on, so the relation may be written algebraically as "x $\rightarrow$ x+2" which may be read *x is mapped onto* x+2 because 1 maps onto 1+2 which is 3, and 2 maps onto 2+2 which is 4. The equation $y = x+2$ may also be used to describe the relation.

Domain and Range

In this activity we are going to determine the domain and range of given relations and express them using set notation.

The domain is the largest subset of real numbers for which the relation makes sense.

The range is the set of all elements that correspond to each element of the domain under a given relation.

Consider the relation $y = \frac{2x+1}{x-1}$. Let's see how we use set notation to describe the domain and the range of this relation.

Solution

In this case, since division by zero is not defined (does not make sense), the domain (values of x) may be any of the numbers within set $\Re$ except the number 1. Substituting 1 for x in the denominator will lead to division *by zero*, which is undefined. We therefore denote the domain as

$$\{x : x \in \Re ;\ x \neq 1\}.$$

This means "a set of values of x such that x is an element of the set of real numbers, but $x \neq 1$" because 1-1 is zero.

Since y may take any value within the set of real numbers, the range may be denoted as:

$$\{y : y \in \Re\}.$$

Activity 1.2

Activity 1.2

Now use set notation to describe the domain and the range for the following relations:

a) $x \to 2x+1$

Domain = ____

Range = ____

b) $y = \dfrac{x^2+2x-1}{x}$

Domain = ____

Range = ____

Compare your answers with those given at the end of the subunit.

Key Points to Remember

The key points to remember in this subunit on the domain and range of are relation are:

-Members of the input set form the **domain** of the relation while members of the output set form the **range** of the relation.

- A **relation** is a mapping that assigns each element of a **domain** to one or more elements of the **range**.

- The relation may be written as a set of ordered pairs (*x, y*) where $x \in domain$ and $y \in range$ (corresponding to x).

In this sub-unit we have learned about relations, in the next subunit, our discussion is still on relations where we will learn different types of relations and functions.

Solutions to Subunit Activities

<u>Solutions to Activity 1.1</u>

a) Members of the input set are: 1,2,3,4,5,6

b) Members of the output set are: 3,4,5,6,7,8

c) Relations between output and input: (1,3),(2,4),(3,5),(4,6),(5,7),(6,8)

d) Let x represent the input and y represent the output, then the relationship between each member of the input set and the corresponding member of the output set is:

$y = x + 2$

Solutions to activity 1.2

a) Domain= $\{x : x \in \Re\}$ because the function is defined for any value of x in real numbers.

Range = $\{y: y \in \Re\}$.

b) Domain= $\{x : x \in \Re; x \neq 0\}$, because division by zero is undefined, so the function will be undefined if x=0.

Range = $\{y: y \in \Re\}$.

Lesson 2 Functions and Non-functions

At the end of this subunit you should be able to:

-Identify the different types of relations.

-Differentiate functions from non-functions.

This subunit is approximately 3 pages long.

There are different types of relations. We have **one-to-many**, **many-to-one** and **one-to-one** relations.

A **one-to-many** relation is a relation in which one element of the domain is mapped onto more than one member of the range.

Example 1 Consider the following relation.

(1, 1); (1, -1); (4, 2); (4, -2); (9, 3); (9, -3)

You should notice that: 1is mapped onto more than one values (1 and -1).

4 is mapped onto more than one values (2 and-2).

9 also has more than 2 output values (3 and -3).

A relation such as this one is called a *one-to-many* relation.

Note that:

- A **one-to-many** relation is a relation in which one element of the domain is mapped onto more than element of the range.
- A **many-to-one** relation is a relation in which more than one element of the domain is related to one element of the range.
- A **one-to-one** relation is a relation in which one element of the domain is mapped onto exactly one element of the range.

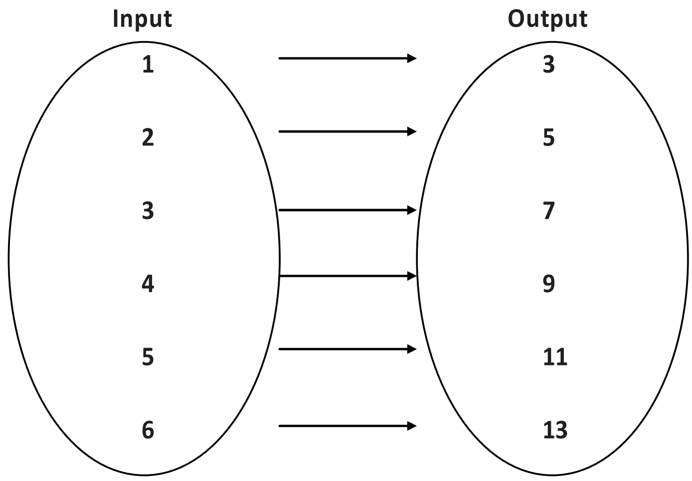
Activity

ACTIVITY 1.3

For this activity state whether the given relations is many-to-one, one-to-many or one-to-one based on what you have just learned.

a) (-1, 1), (1, 1), (-2, 4), (2, 4), (-3, 9), (3, 9), (-4, 16), (4, 16)

b)



c) $y = x^2 + 3$ (Hint: determine the set of ordered pairs for this relation for values of x: $-4 \leq x \leq 4$).

d) $y = \sqrt{x}$

Note it!

Compare your answers with those given at the end of the subunit. If you have got them all correct, you can move on, if not, consider reviewing this subunit.

Function:

A relation in which each element of the domain is mapped onto exactly one element of the range is a **function.**

We can also define a function as a set of ordered pairs in which no two or more pairs have the same first element. This definition can be applied when a relation is represented by a graph. If a vertical line drawn on the graph of the relation passes through no more than one point then the relation is a function. This is called a **vertical line test**.

Each of the graphs shown figures 1a, 1b and 1c represent a function.

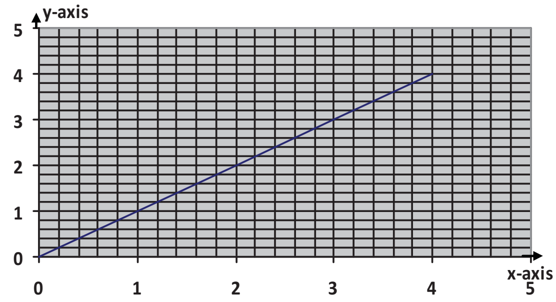


Figure 1

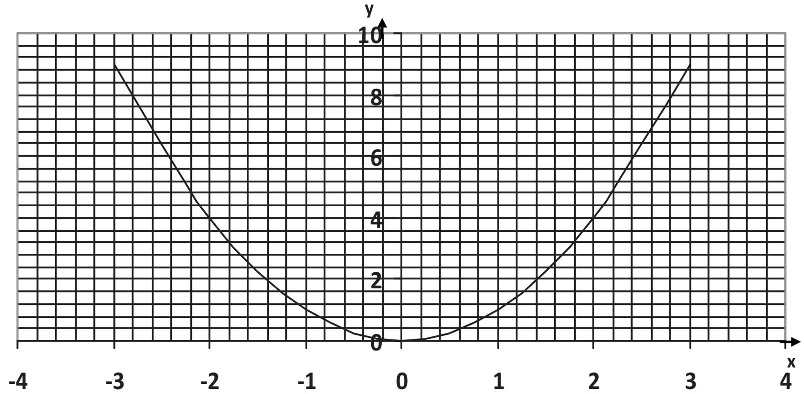


Figure 2

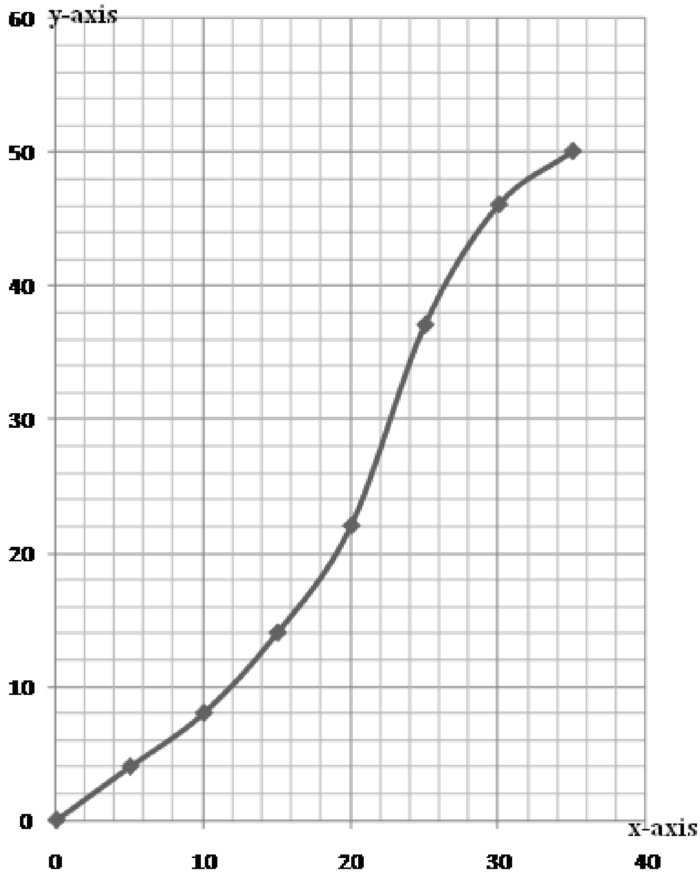


Figure 3

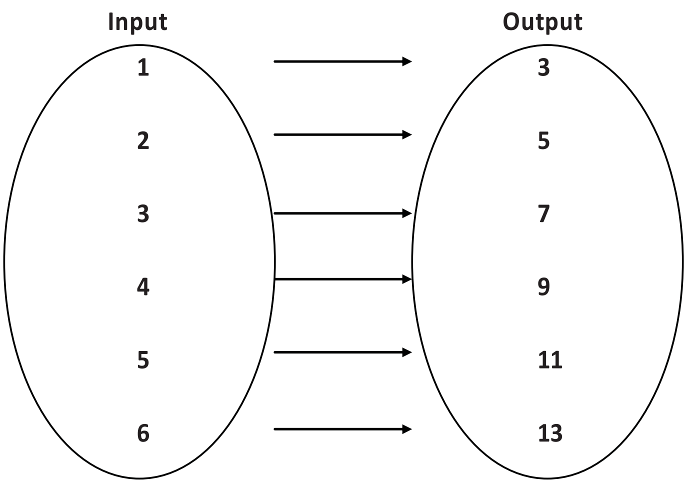
ACTIVITY 1.4

Now answer the following question:

1. Say which of the relations in Activity 1. 3 listed below are functions and which ones are non-functions.

a) a) (-1, 1), (1, 1), (-2, 4), (2, 4), (-3, 9), (3, 9), (-4, 16), (4, 16)

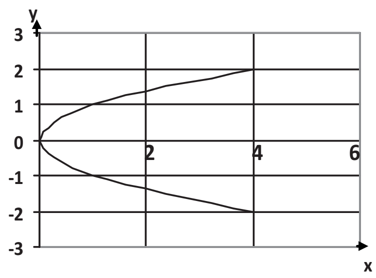
b)



c) $Y = x^2 + 3$

d) $y = \sqrt{x}$

2. Say whether or not the following graph is a function.



3. For each of the following state whether the information defines a function or not.

a) { (1, 5), (2, 4), (3, 3), (4, 2)}

____

____

______

b) { (1, 0), (2, 0), (3, 0), (4, 0)}

____

____

______

c) { (1, 5), (1, 4), (2, 2), (2, 3)}

____

____

______

When you have finished answering the questions, compare your answers with those given at the end of the subtopic. Be sure you understand how functions are defined before continuing.

Key Points to Remember:

The key points to remember in this subunit on functions and non-functions are:

- A **one-to-many** relation is a relation in which one element of the domain is mapped onto more than element of the range.
- A **many-to-one** relation is a relation in which more than one element of the domain is related to one element of the range.
- A **one-to-one** relation is a relation in which one element of the domain is mapped onto exactly one element of the range
- A relation in which each element of the domain is mapped onto exactly one element of the range is a **function.**

  Now that we know what a function is, in the next subtopic we will be evaluating functions.

<u>Solution to activity 1.3</u>

a) Many-to-one, 1 is mapped onto 1 and -1.

b) One-to-one, because each element of the domain is mapped onto exactly one element of the range.

c) Many-to-one, for instance if x=-4, $y = 4^2 + = 19$, and if x=4, $y = -4^2 + 3 = 19$,

d) One-to-many, for instance, if x=4, $y = \sqrt{4} = 2\ or\ -2$

Solutions to Activity 1.4

1. a) function b) function c) function d)non-function
2. Non-function-by the vertical line test, the vertical line passes through two points.
3. a) function b) function c) non-function

In activity 1.4 1a) note that even though two elements from the domain have the same image, each one of them is mapped onto exactly one image.

Lesson 3 Evaluating Functions

By the end of this subtopic you should be able to:

-Evaluate the value of any given function.

This subtopic is approximately one page long.

If one element of the domain of a function is given, we may determine the corresponding element of the range. Calculating the value of the element of the range for a given element of the domain is called evaluating the function.

For example:

Given f(x) = 2x + 5, calculate f (2).

Solution

In order to determine f (2), we substitute 2 for x in the expression for the function.

We write f(2) = 2(2) + 5

= 4 + 5

= 9.

Thus f (2) = 9

Now try these!

ACTIVITY 1.5

1. Given the function f(x) = x -7, evaluate:

   a) f(3)

   ____

   b) f(0)

   ____

   c) f(-4)

   ____

2. If $g(x) = \frac{3}{4}x + 3$, evaluate :

   a) g(1)

   ____

   b) g(4)

   ____

   c) g(-4)

   ____

If you have finished answering the questions, compare your answers with those given below. As needed, review any question that you got wrong.

Solutions to activity 1.5

1. a) $f(3) = 3 - 7 = -4$ b) $f(0) = 0 - 7 = -7$

   c) $f(-4) = -4 - 7 = -11$

2. a) $g(1) = \frac{3}{4}(1) + 3 = 3.75$ b) $g(4) = \frac{3}{4}(4) + 3 = 6$

c) $g(1) = \frac{3}{4}(-4) + 3 = 0$

You should remember that in order to get solutions in each of the cases above, you should substitute each of the given values for x in the expression for the function.

Lesson 4 Inverse of a Function

Introduction

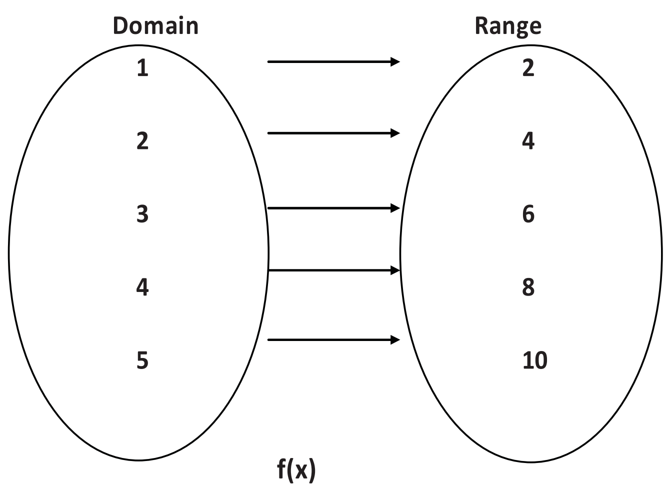
By the end of this subunit, you should be able to
- Determine and evaluate the inverse of any given function.

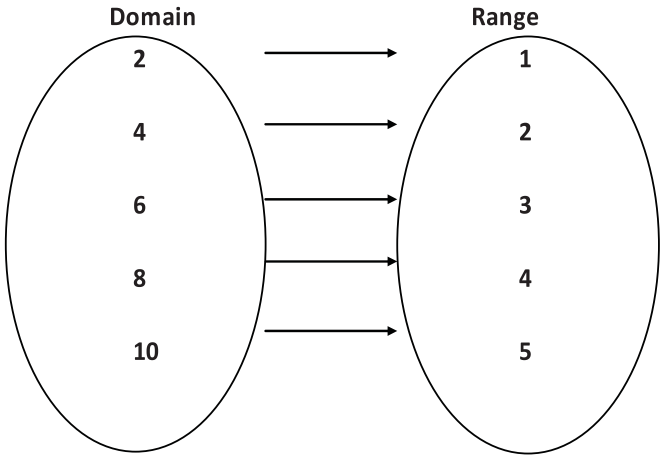
This subunit is about 3 pages in length.

Inverse of Functions

The notation $f^{-1}(x)$ is used to denote the inverse of the function $f(x)$. $f^{-1}(x)$ is read as f inverse of x.

The inverse of a function can be shown as a mapping.





$f^{-1}(x)$

Note that f (x) = {(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)} while $f^{-1}(x)$ = {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5)}.The inverse of any given function maps the range of the function onto its domain.

Now let us examine the relation that maps the domain of f(x) onto its range. Each element of the domain is mapped onto twice its value in the range. We can therefore write the function f as

$$f(x) = 2x,$$

Remember that f(x) is the definition for the output set.

Now considering the $f^{-1}(x)$, each element of the domain is mapped onto half its value, thus we write the inverse function $f^{-1}(x)$ as

$$f^{-1}(x) = \frac{1}{2}x$$

In order to determine the inverse of any given function you should first identify the operations that are involved in the original function, the order in which they operate on the variable and think of the corresponding inverse operations that can be performed to reverse the processes.

For example, for the above function, f: x → 2x, meaning that every element of the domain is multiplied by 2 to get the corresponding image in the range. This means therefore that in order to get from the image to the original entries we should divide (reverse of multiplying) values in the range by 2. Since this time our range is now the domain, we write the inverse function as: $f^{-1}: x \rightarrow \frac{1}{2}x$.

Example

Given that $f(x) = 2x + 5$, determine $f^{-1}(x)$.

Solution

There are two operations involved here; they are

a) multiplication by 2, followed

b) by addition of 5.

This tells you that to reverse these processes you should begin with the operation that was done last (addition of 5) and end with the operation that was done first (multiplication by 2). Thus the inverse function is

$$F^{-1}(x) = \frac{x-5}{2}.$$

Reflection

ACTIVITY 1.6

Now try these!

For each of the following functions determine the inverse.

a. $g(x) = x + 6$

b. $f(x) = 3x - 4$

c. $h(t) = t^2 + 3$

I hope you were able to answer all the questions. Compare your answers with those below. If you got all the questions correct continue to the next subtopic, if not revisit the subtopic.

Solutions to activity 1.6

a) . $g^{-1}(x) = x - 6$

b) $f^{-1}(x) = \frac{x+4}{3}$. We start by performing the reverse of subtracting 4, and then we undo multiplying by 3 which is dividing by 3. The operations are done in reverse order

c) $h^{-1}(t) = \sqrt{t-3}$. Two operations are involved here, first squaring of t, followed by addition of 3. So to get the inverse function you should subtract 3 first and then take the square root of the result.

I hope you are now able to get inverses of functions using this analysis of operations involved in any given functions.

Lesson 5 Algebra of Functions

Introduction

At the end of this subtopic you should be able to:

-Add, subtract, multiply and divide functions.

This subunit is approximately 4 pages long.

Combined Functions

We use the concept of combination of and / or composite functions in most of our daily life experiences.

Consider the following situation:

Limpho is buying a pair of jeans for M119.99. The jeans are on sale at a 20% discount, but she also has to pay the sales tax at 14%. How much will Limpho pay for the jeans?

There are two functions are involved in Limpho's problem – one involving a discount and the other involving sales tax. Remember that it is not necessary to use *f* for all functions. If we have more than one function in the same situation, we use different letters to name them. This now enables us to have fun with functions.

Note

To solve problems such as one above, you may need to add, subtract, multiply or divide functions**.** The functions could be f(x), g(x), h(x) or anything just to differentiate functions.

The following are the operations used in functions

Sum : $(f + g)(x) = f(x) + g(x)$

Difference: $(f - g)(x) = f(x) - g(x)$

Product : $(f \bullet g)(x) = f(x) \bullet g(x)$

Quotient : $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, if $g(x) \neq 0$

Now that we know the operations with combined functions, let us put them into practice in the following examples.

EXAMPLE 1

Given that f(x) = {(1, 2), (2, 4), (3, 6),(4, 8)}

g(x) = {(1, 3), (2, 5), (3, 7), (4, 9)}

then (f + g) (x) = f(x) + g(x) = { (1, 5), (2, 9), (3, 13), (4, 17)}and

(f • g) (x) = f(x) • g(x) = { (1, 6), (2, 20), (3, 42), (4, 72}

The domain of f + g and f • g is the intersection of the domains of f and g, i.e. f+g and f, and g have the same domain. The domain (D) of f + g, denoted as D (f + g) = D (f•g) = D (f) ∩ D (g). So in example 1 above, the first values in each bracket (which are the domain) have not changed because they are still the same values of x both in f(x) and in (f+g)(x).

i.e. D(f) ∩ D (g) = {1, 2, 3, 4}

Also $\frac{f}{g}(x) = \{(1, \frac{2}{3}), (2, \frac{4}{5}), (3, \frac{6}{7}), (4, \frac{8}{9})\}$

EXAMPLE 2

If f(x) = 2x + 5 and h(x) = x^2, find

i) (f + h)(x)

ii) (f • h) (x)

iii) $\frac{f}{h}(x)$

Solution

i) (f + h) (x) = f(x) + h(x)

$= (2x + 5) + (x^2)$

$= x^2 + 2x + 5$

ii) (f • h) (x) = f(x) • g(x)

$= (2x + 5)\, x^2$

$= 2x^3 + 5x^2$

iii) $\frac{f}{h}(x) = \frac{2x+5}{x^2}$

Now work on activity 1.7 and see how much you understand.

ACTIVITY 1.7

1. Given that f(x) = { (1, 3), (2, 4), (3, 5), (5. 0) and that

   h(x) = { (0, 1), (1, 2), (2, 3), (5, 1)},

   Determine: a) (f + h)(x)

   b) (f •h) (x)

   c) $\frac{f}{h}(x)$

   d) (f – h)(x)

2. If f(x) = $x^2 - 1$ and g(x) = x + 1, determine:

   a) (f + g)(x)

b) $(f \bullet g)(x)$

c) $\frac{f}{g}(x)$

d) $(f - g)(x)$

After completing the questions, compare your answers to the correct answers at the end of this subunit. Take the time needed to understand each answer before continuing.

Key Points to Remember

The key points to remember in this subunit on combined functions are:

- $(f+g)(x) = f(x) + g(x)$

- $(f-g)(x) = f(x) - g(x)$

- $(f \bullet g)(x) = f(x) \bullet g(x)$

- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, if $g(x) \neq 0$

Solutions to activity 1.7

1. a) {(1, 5), (2, 7), (5, 1)}
   b) {(1, 6), (2, 12), (5, 0)}

c) $\{(1, \frac{3}{2}), (2, \frac{4}{3}), (5, 0)$

d) { (1. 1), (2, 1), (5, -1)}

2. a) $x^2 -1+ x+1= x^2 +x$

b) $(x^2 – 1) (x + 1) = x^3 + x^2 – x – 1$

c) $\frac{x^2 - 1}{x+1} = \frac{(x+1)(x-1)}{(x+1)} = x - 1$

d) $(x^2 – 1) – (x + 1) = x^2 – x – 2$

Lesson 6 Composite Functions

Introduction

At the end of this subtopic you should be able to:

-Evaluate the compositions of functions.

This subunit is about 7 pages in length.

Evaluating the composition of functions

The composition of functions is an algebraic operation which is unique to functions. The composition of two functions f and g, denoted by f o g, is the function such that

$$f \circ g\ (x) = f(g(x))$$

its domain is $D(f \circ g) = \{x : x \in D(g) \text{ and } g(x) \in D(f)$.

Now let us carry the following activity together and learn how evaluate the composition of functions

Activity

ACTIVITY 1.8

Given that f(x) = 3x + 4 and that g(x) = x^2,

a) Find f(2).

Compare your answer with the one below.

f(2) = 3(2) + 4 = 10

b) What is the input of function f in a)?

Compare your answer with the following

The input of the function in a) is 2.

c) Explain how you calculated the output, which is f(2).

Compare your answer with the following.

By substituting 2 for x in $f(x) = 3x + 4$

d) In f(p), what is the input in the function f?

Compare your answer with the following.

p is the input

e) Write down an expression for f(p).

Compare your answer with the following:

$f(p) = 3p^2 + 4$

f) We read f(2) as "f of 2". This means that 2 is the input in the expression for the function f. Now, in f(g(x)) what is the input in the function f?

Compare your answer with the following

g(x) is the input in f

g) Hence determine the expression for f(g(x)) and simplify it.

Compare your answer with the following

$f(g(x)) = f(x^2) = 3x^2 + 4$

h) Determine the expression for g(f(x))

Compare your answer with the following

$$\begin{aligned} g(f(x)) &= g(3x + 4) \\ &= (3x + 4)^2 \\ &= 9x^2 + 24x + 16 \end{aligned}$$

Note that in composite functions, one function is the input of another function.

Now try the following activity under the composition of functions.

Activity

ACTIVITY 1.9

1. given that f(x) =(x+2) (2x-1), and g(x) = (2x-1), find an expression for g (f(x))

2. It is given that g(x) = $\frac{5x-4}{3}$ and that h(x) =5x+2, determine the expression for

g(h(x))

3. If f(x) =3x+4, and g(x) = x+2, evaluate

i) f (g (3)).

ii) g (f (4))

4. If f(x) = $\frac{5x-4}{3}$ and p(x) = $\frac{x+5}{5}$, find

i) The expression for f ∘ p (x)

ii) The value f (p (2))

When you are done answering all the questions, compare your answers with those given at the end of the subunit. Continue if you are satisfied with your ability to answer the questions. If not, review the above content and try the activity again.

ACTIVITY 1.10

1. Given that f(x) = 2x + 5 and g(x) = $\frac{x}{4}$, determine

a) i) (f + g)(x)

ii) (f • g) (x)

iii) (f – g) (x)

iv) (f o g) (x)

v) g(f(x))

Given also that h(x) = $\frac{x-5}{2}$, determine

b) i) f(h(x))

ii) h(f(x))

iii $f^{-1}(x)$

iv) $h^{-1}(x))$

v) What do realise about the results you got in b (i) and b(ii)?

vi) What do you notice about the functions f and h?

vii) Now complete the following statement such that it is true:

When two functions in x are __________________ of one another, the composite function is always equal to ____________

Compare your answers with those given at the end of the subunit. Continue on if you scored at least 80%. If not, review the above content and work through the activity again.

Key Points to Remember

The key points to remember in this subunit on composite functions are:

- The composition of two functions f and g, denoted by f o g, is the function such that

$$f \circ g\,(x) = f(g(x))$$

You have now completed the last subunit of this unit on functions and relations. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Solutions to activity 1.9

1.

$$g(f(x)) = 2((x+2)(2x+-1))-1$$
$$= 2(2x^2 - x + 4x - 2) - 1$$
$$= 4x^2 - 6x - 4 - 1$$
$$= 4x^2 - 6x - 5$$

2. $g(h(x)) = \dfrac{5(x+2)-4}{3} = \dfrac{25x+10-4}{3} = \dfrac{25x+6}{3}$

3. To find f(g(3)) we first need to find g(3)

i)

$$g(3) = 3 + 2 = 5$$
$$so$$
$$f(g(3)) = f(5) = 3\times 5 + 4 = 19$$

$$f(4) = 3\times 4 + 4 = 16$$

ii) so

$$g(f(4)) = g(16) = 16 + 2 = 18$$

4.i)

$f \circ p(x) = f(p(x))$

$$= \frac{5(\frac{x+5}{5}) - 4}{3} = \frac{x+5-4}{3} = \frac{x+1}{3}$$

ii)

$$f(p(x)) = \frac{x+1}{3}$$

$$\therefore f(p(2)) = \frac{2+1}{3} = 1$$

Solutions to activity 1.10

a) i) $2x + \frac{x}{4} + 5 = 2\frac{1}{4}x + 5$

ii) $\frac{x}{4}(2x + 5) = \frac{1}{2}x^2 + \frac{5}{4}x$ or $\frac{2x^2 + 5x}{4}$

iii) $(2x + 5) - (\frac{x}{4}) = 1\frac{3}{4} + 5$

iv) $f(\frac{x}{4}) = 2(\frac{x}{4}) + 5$

$= \frac{x}{2} + 5$

v) $g(2x + 5) = \frac{2x + 5}{4}$

b) i) x

ii) x

iii) $f^{-1}(x) = \frac{x-5}{2}$

iv) $h^{-1}(x) = 2x + 5$

v) Answers are the same = x

vi) They are inverses of one another.

vii) *When two functions in x are INVERSES of one another, the composite function is always equal to x.*

Unit Summary

Summary

In this unit you learned that:

- The domain of a relation is the set of input values for which the function is defined.
- The range is the set of all output values.
- A function is a relation in which each element of the domain has only one image.
- The following operation may be performed on functions:

  - **Sum** : $(f + g)(x) = f(x) + g(x)$
  - **Difference** : $(f - g)(x) = f(x) - g(x)$
  - **Product** : $(f \bullet g)(x) = f(x) \bullet g(x)$
  - **Quotient** : $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, if $g(x) \neq 0$

- The domain of $f + g$, $f - g$, $f \bullet g$ and $\left(\frac{f}{g}\right)$ is the intersection of the domains of f and g.
- The inverse of any function, f^{-1}, maps the range onto the domain
- When two functions in x are INVERSES of one another, the composite function is always equal to x.
- The composite function for any two functions f(x) and g(x) is denoted as

  $f(g(x)) = (f \circ g)(x)$

You have completed the material for this unit on functions and relations. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

This assignment consists of 50 marks.

Marks for each question are shown.

You are advised to spend no more than 1hour in answering these questions.

1. For each of the following relations state the domain and find the indicated value of the function:

   a) { (1, 2), (2, 0), (3, 3), (4, 1)}; find f(4) (2 marks)

   b) {(1, 0), (2, 1), (3, 2), (4, 3) }: find f(2) (2 marks)

2. Evaluate $f(x) = 3x^2 + 2x - 1$ at x = 1 and x = 2 (4 marks)

3. For each of the following decide whether the information defines a function. If it does, state the domain and the range of the function:

   a) { (1, 5), (1, 4), (2, 3), (2, 2) } (3 marks)

   b) {(0, 0), (1, 1), (2, 2), (3, 3) } (3 marks)

4. State the domain and the range for each of the following functions:

   a) { (x, y): y = 2x + 3} (2 marks)

   b) $\{(x, y) : y = \frac{12}{x+4}\}$ (3 marks)

5. Given that $f(x) = 5x - 4$ and $g(x) = 2x$ determine:

a) $(f + g)(x)$ (3 marks)

b) $(f \bullet g)(x)$ (3 marks)

c) $(f - g)(x)$ (3 marks)

d) $f^{-1}(x)$ (3 marks)

e) $(f \circ g)(x)$ (4 marks)

f) $g(f(x))$ (4 marks)

6. given that $f(x) = \frac{5x-4}{3}$, find $f(1\frac{1}{5})$ (3 marks)

7. Given that f(x)=6x+2 and g(x)= $\frac{x+1}{2}$

Evaluate

a) f(8) (1 mark)

b) f(g(1)) (2 marks)

c) g(f(3)) (3marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Solutions to the assignment:

1. a) Domain={1,2,3,4}

f (4) =1

b) Domain= {1, 2, 3, 4}

f (2) =1

2. $f(x) = 3x^2 + 2x - 1$ at x = 1

at x = 1

$f(1) = 3 \times 1^2 + 2 \times 1 - 1$

$= 3 + 2 - 1$

$= 4$

at x = 2

$f(2) = 3\times 2^2 + 2\times 2 - 1$
$= 3\times 4 + 4 - 1$
$= 15$

3. a) The information does not define any function.

b) The information defines the function $y = x$

Domain= $\{x : x \in \Re\}$

Range = $\{y: y\in \Re\}$.

4. a) Domain= $\{x : x \in \Re\}$

Range = $\{y: y\in \Re\}$.

b) Domain= $\{x : x \in \Re; x \neq -4\}$, because -4+4=0, and the function is undefined at x=-4

Range= $\{y : y\in \Re\}$

5. a) $(f + g)(x) = 5x - 4 + 2x = 7x - 4$

b) $(f \bullet g)(x) = (5x-4)(2x) = 10x^2 - 8x$

c) $(f - g)(x) = 5x - 4 - 2x = 3x - 4$

d) $f^{-1}(x) = \dfrac{x+4}{5}$

e) (f ∘ g) (x)= $f(g(x)) = 5(2x) - 4 = 10x - 4$

f) g (f(x))=2(5x-4)=10x-8

6.

$f(x) = \dfrac{5x-4}{3}$,

$so\ f(1\frac{1}{5}) = f(\frac{6}{5}) = \dfrac{5(\frac{6}{5}) - 4}{3} = \dfrac{2}{3}$

7. a) $f(8) = 6\times 8 + 2 = 48 + 2 = 50$

b)

$g(1) = \dfrac{1+1}{2} = 1$

$so\ f(g(1)) = f(1) = 6\times 1 + 2 = 8$

c)

$$f(3) = 6 \times 3 + 2 = 20$$

$$so\ g(f(3)) = g(20) = \frac{20+1}{2} = 10.5$$

Assessment

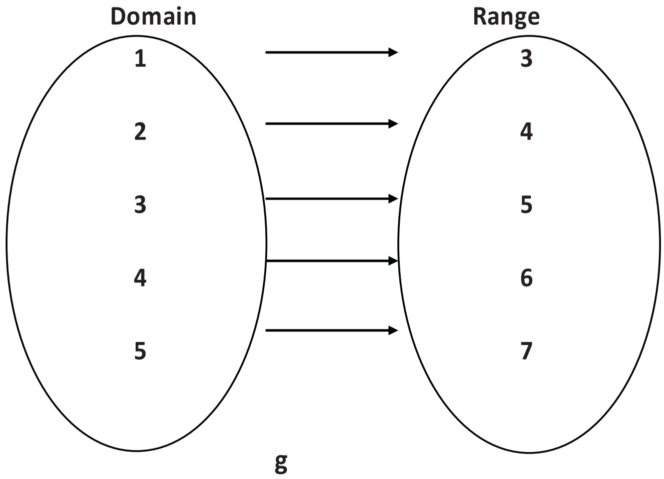
This assignment consists of 10 questions and 70 marks.

Marks for each question are shown.

You are advised to spend no more than 1hour 20 minutes in answering these questions.

Assessment

1. In the diagram below, state with reasons as to whether g is a function or not. (1 mark)



2. Consider the following functions and describe them using function notation again in each function describe the range and the domain.

   a) A function that adds 4 (2 marks)

   b) A function that multiplies by 3, adds 4 and divides the sum by 2. (3 marks)

c) A function ,s, that squares and adds 5 (3 marks)

3. State the domain of the following function and give ordered pairs with the values of x such that $1 \leq x \leq 3$:

$$g(x) = \frac{1}{3x^2 + 3}$$ (4 marks)

4. Given that $f(x) = 3x + 4$, $p(x) = x^2 - 2$ and $z(x) = \frac{x}{2}$

Find

a) $(f + p)(x)$ (3 marks)

b) $(z \bullet p)(x)$ (3 marks)

c) $f^{-1}(x)$ (3 marks)

d) p(z(x)) (3 marks)

e) z(f(x)) (3 marks)

f) p(z(f(x))) (4marks)

5. Using words describe the inverses of the following functions.

a) A function that divides x by 12. (2 marks)

b) A function that multiplies x by 10 (2 marks)

6. Given that $f(x) = 3x^2 - 4$

a) Find $f^{-1}(x)$ (3 marks)

b) Find $f(3)$ (3 marks)

c) Find $f^{-1}(5)$ (3 marks)

d) Find $f^{-1}(2)$ (3 marks)

7\.

f(x) =3x+4

Given that f (k) =k, find k. (3 marks)

8\. A function f is defined by $f(x) = \frac{x+5}{3}$

a) Given that f (1) =k, find the value of k. (4 marks)

b) Given that $f^{-1}(x) = cx + d$, find the values of c and d. (4 marks)

9\. given that f(x) =12-5x and g(x) =3x

a) Find f (4) (3 marks)

b) The value of x for which f(x) =17 (3 marks)

c) g(f(x)) (3 marks)

d) f(g(2)) (3marks)

10. Find the inverse of $f(x) = 3x + 2$ (3marks)

Solutions to the assessment

1) g is a function because each element of the domain is mapped onto exactly one element of the range.

2) a)

$$f(x) = x + 4$$
$$Domain = \{x : x \in R\}$$
$$let\ y = f(x),\ Range = \{y : y \in R\}$$

b)

$$f(x)=\frac{3x+4}{2}$$
$$Domain=\{x:x\in R\}$$
$$let\ \ y=f(x), so\ \ Range=\{y:y\in R\}$$

c)

$$s(x)=x^2+5$$
$$Domain=\{x:x\in R\}$$
$$Range=\{s:s\in R\}$$

3.

$$Domain=\{x:x\in R\}$$
$$Ordered\ \ pairs:(1,\frac{1}{6}),(2,\frac{1}{9}),(3,\frac{1}{12})$$

4.

a) $(f+p)(x)=3x+4+x^2-2=x^2+3x+2$

b) $(z\bullet p)(x)=\frac{x}{2}(x^2-2)=\frac{x^3-2x}{2}$

c) $f^{-1}(x)=\frac{x-4}{3}$

d) $p(z(x))=(\frac{x}{2})^2-2=\frac{x^2}{4}-2=\frac{x^2-8}{4}$

e) $z(f(x))=\frac{3x+4}{2}$

f)

$$first\ \ z(f(x))=\frac{3x+4}{2}$$
$$\therefore\ p(z(f(x)))=(\frac{3x+4}{2})^2-2=\frac{9x^2+24x+16-8}{4}=\frac{9x^2+24x+8}{4}$$

5)

a) A function that multiplies by 12.

b) A function that divides x by 10.

6)

a) $f^{-1}(x)=\sqrt{\frac{x+4}{3}}$

b) $f(3)=3(3)^2-4=27-4=23$

c) $f^{-1}(5)=\sqrt{\frac{5+4}{3}}=\sqrt{\frac{9}{3}}=\sqrt{3}$

d) $f^{-1}(2)=\sqrt{\frac{2+4}{3}}=\sqrt{2}$

7)

$f(x)=3x+4$

$a)\ f(k)=3k+4=k$

$\Rightarrow 3k+4=k$

$\Rightarrow 3k-k=-4$

$\Rightarrow 2k=-4$

$\Rightarrow k=-2$

8) a)

$f(1)=k=\frac{1+5}{3}$

$\Rightarrow k=\frac{6}{3}=2$

b)

$$f^{-1}(x) = 3x + 5$$
$$\Rightarrow c = 3 \text{ and } d = 5$$

9)

a) $f(4) = 12 - 20 = -8$

b)
$$f(x) = 17$$
$$\Rightarrow 12 - 5x = 17$$
$$\Rightarrow \frac{-5x}{-5} = \frac{17-12}{-5}$$
$$\Rightarrow x = \frac{5}{-5} = -1$$

c) $g(f(x)) = 3(12 - 5x) = 36 - 15x$

d)
$$g(2) = 3 \times 2 = 6$$
$$f(g(2)) = f(6) = 12 - 5 \times 6 = 12 - 30 = -18$$

10. $f^{-1}(x) = \dfrac{x-2}{3}$

Unit Contents

Unit 25

Unit 25

Loci

Introduction

Welcome to another interesting unit in this course. This unit consists of 54 pages. This is approximately 2% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 20 hours to work on this unit, 15 hours for formal study and 5 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Five Lessons:

Lesson 1 Locus of Points Equidistant from One Given Point
Lesson 2 Locus of Points Equidistant From A Given Straight Line
Lesson 3 Locus of Points Equidistant From Two Given Points
Lesson 4 Locus of Points Equidistant from Two Intersecting Lines
Lesson 5 Intersecting Loci

Locus (plural, loci) comes from Latin, meaning "position" or "place". A locus is **a set of points** that satisfy a **given condition**, or **rule**. In this unit, we are going to look at the locus of points in different situations.

When reading the following learning outcomes, think about them as a guide to what you should focus on while studying this unit.

Upon completion of this unit you will be able to:

Outcomes

- *determine* the locus of all points which are equidistant from one given point in two dimensions.
- *determine* the locus of all points which are equidistant from a given straight line in two dimensions.
- *determine* the locus of all points which are equidistant from two given points in two dimensions.
- *determine* the locus of all points which are equidistant from two intersecting lines in two dimensions.
- *determine* the locus of all points which satisfy more than one condition, when we have an intersection of two or more loci.

Terminology

Equidistant:	Having the same distance apart.
Locus:	(plural, loci) is a set of all possible points which satisfy some common conditions.
Circumference:	the boundary line of a figure, area or object.
Perpendicular:	At right angles.
Bisect:	Divide into two equal parts.
Parallel lines:	Lines that never meet, no matter how far they are extended.
Radius:	The distance of a straight line from the centre of a circle to its circumference.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Locus of Points Equidistant from One Given Point

By the end of this subunit, you should be able to:

- *Determine* the locus of all points which are equidistant from one given point in two dimensions.

This subunit is about 4 pages in length.

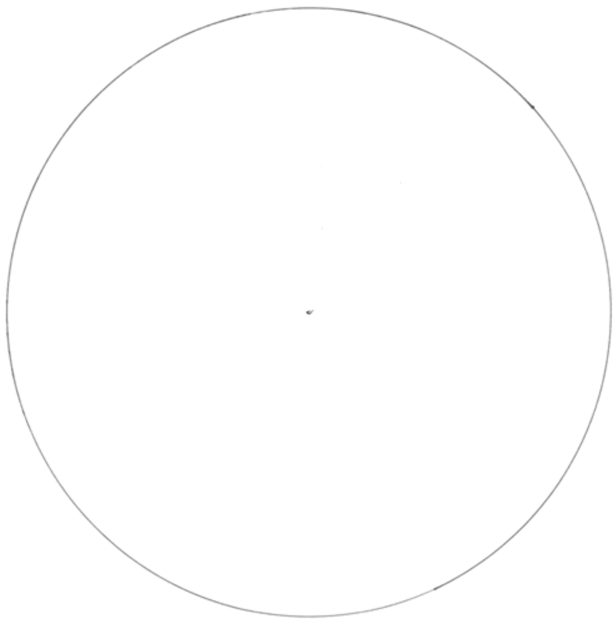
Think of a situation where a piece of string with a pencil on the end is tied to a fixed peg on a piece of paper. If we move the string, keeping it **completely stretched tight**, the pencil will trace out a set of points on the piece of paper in the shape of a circle.

We observe the following:

- The fixed peg is a fixed point, which we call O
- The string, which **is completely stretched tight**, allows us to say the distance from O, to any of this points, P, is constant, i.e OP is constant.
- When all these points are joined, we get a circle. The circumference of this circle is called the **locus.**

Take for example, a circle with centre, O, and radius 6 cm.

The circumference of this circle is the set of **all** points at a distance of 6 cm from the point O, i. e the circumference is the locus of all the points equidistant, meaning the same distance from the point O.



Therefore the locus of points at a given distance (d) from a given fixed point (O) is the circumference of a circle whose:

- centre O, is the given fixed point and
- radius is the given distance (d)

This can also be written as; the locus {P: OP = r } is the circumference of a circle centre O and radius r

Note it!

For a set of points to be a locus:

(a) The points must have a common property. This is often equality of distance from a point or line. We therefore speak of the points as being equidistant from the point.

(b) The set must contain all the points which have this common property.

From the example above, all the points are the same distance, 6 cm, from the centre O. This is the common property of having equality of distance from a point or line.

Also from the example above, the circumference of the circle contains all the points that are 6 cm from the centre O. This is the common property that the set contains all of the points.

Activity 1

1. Given the point A,

(a) draw the locus of all points that are 4cm from A

•A

(b) Describe the locus of all points that are 4cm from A

____

____

____

____

2. Given the point B, show and describe the locus of all points that are less than 4 cm from B

•B

Why do we exclude the circumference?

3. Draw all points that are more than 4 cm from a given point O.

O•

Describe the locus of these points.

Key Points to Remember

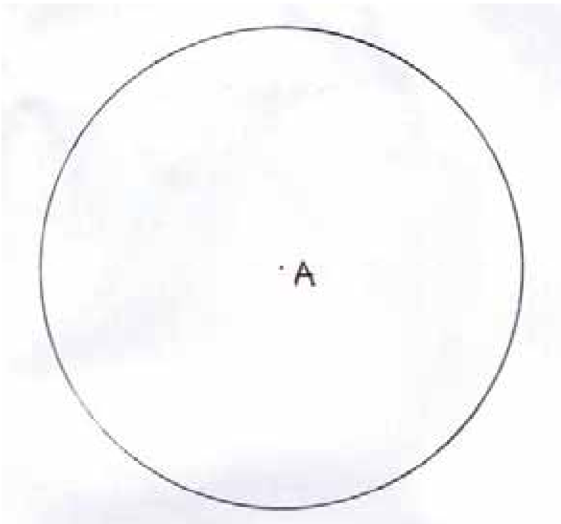
The key points to remember in this subunit on determining the locus of all points which are equidistant from one given point in two dimensions are:

- the locus of points at a given distance from a given point is the circumference of the circle.
- the locus of all points at less than a given distance from a point is the entire area inside the circumference.
- the locus of all points at more than a given distance from a point is the entire area outside the circumference.

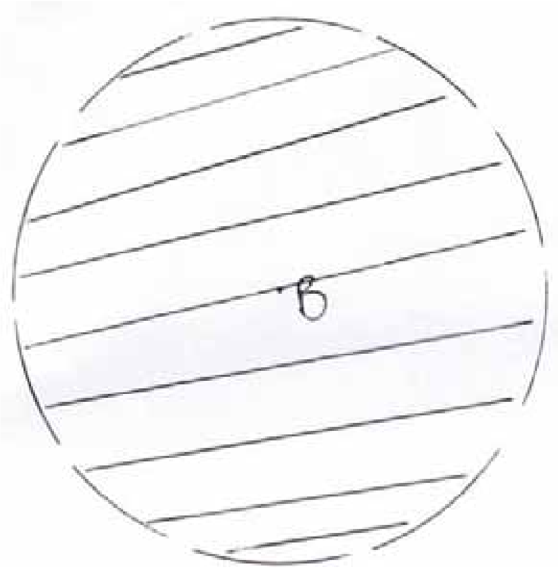
Answers

1. Your diagram is a circle that has a radius of 4cm.

   The locus of all points that are 4 cm from A is a circle with radius 4 cm.

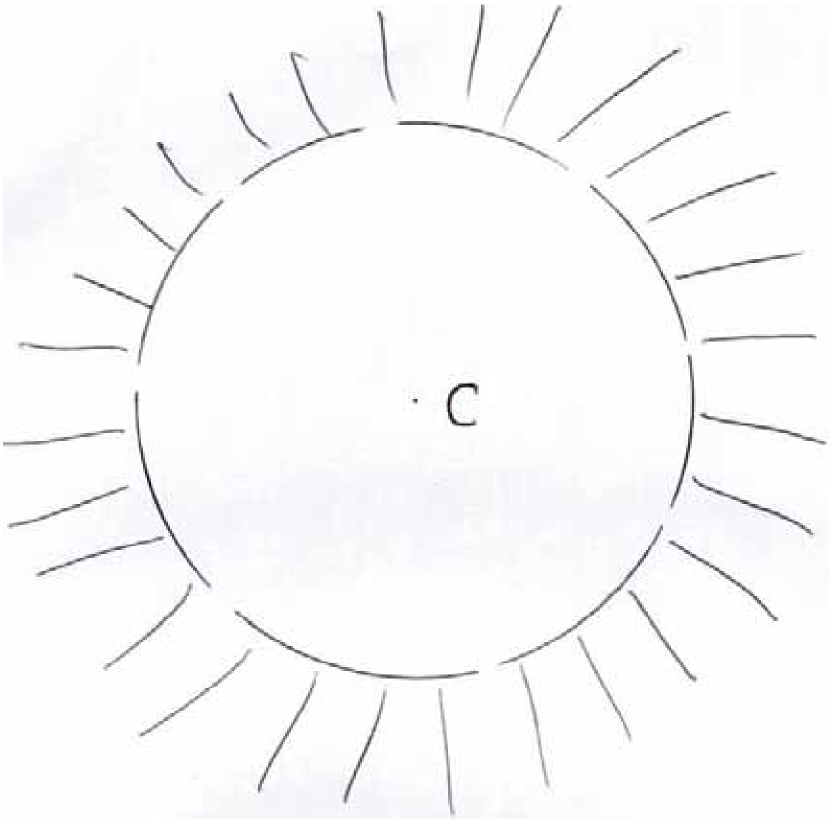


2. The locus is all points in the region that are less than 4 cm from the centre. This is the entire area of the circle excluding the circumference. This will be {P: OP < r }.



The circumference is shown as dashed lines because it is not included in the region. We exclude the circumference of the circle because the points on the circumference are exactly 4 cm from the centre.

3. The sky is the limit with this one! We will only be limited by the boundaries of our workbooks. This will be the whole area outside the circle, excluding the circumference. {P: OP > r }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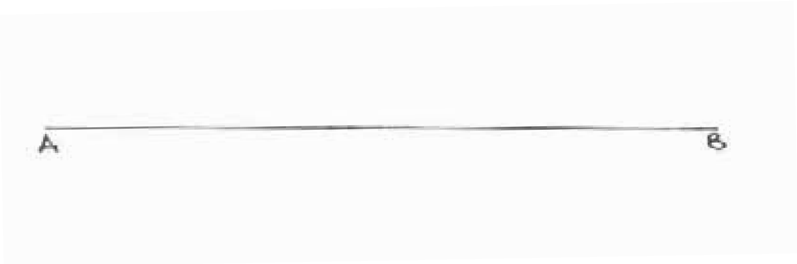
Lesson 2 Locus of Points Equidistant From A Given Straight Line

By the end of this subunit, you should be able to:

- *determine* the locus of all points which are equidistant from a given straight line in two dimensions.

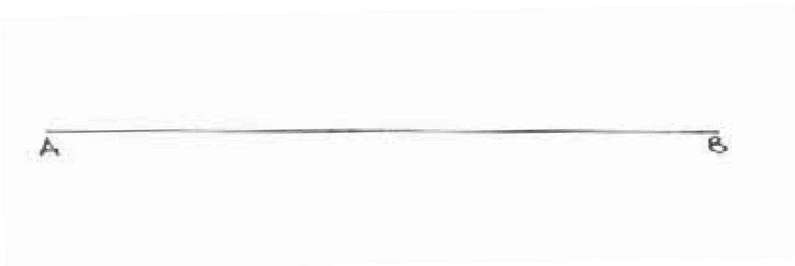
This subunit is about 4 pages in length.

Given the fixed line AB, 8 cm long, below, draw every point that is 2 cm from line AB. Yes, all of them!



Compare your answer with the answer below.

2. Line AB is given below, again. Draw all points that are 3 cm from it.



Compare your answer with the answer below.

In each case, when these points are joined, you should have found a pair of parallel lines to AB, one on either side of AB, and circles at the end of the lines; where the end of the line, A and B are the centres of the circles respectively.

This locus is said to have the “running – track” appearance.

The locus of all points equidistant from a given line is a pair of parallel lines to the line, one on either side of the line, and circles at the end of the lines; where the ends of the line, are the centres of the circles respectively.

Note it!

Most of the time, for practice purposes, when we want the locus of points which are equidistant from a given line; this given line is horizontal. Take note that any straight line can be used.

Lesson 3 Locus of Points Equidistant From Two Given Points

By the end of this subunit, you should be able to:

- *determine* the locus of all points which are equidistant from two given points in two dimensions.

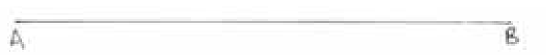
This subunit is about 4 pages in length

Note it!

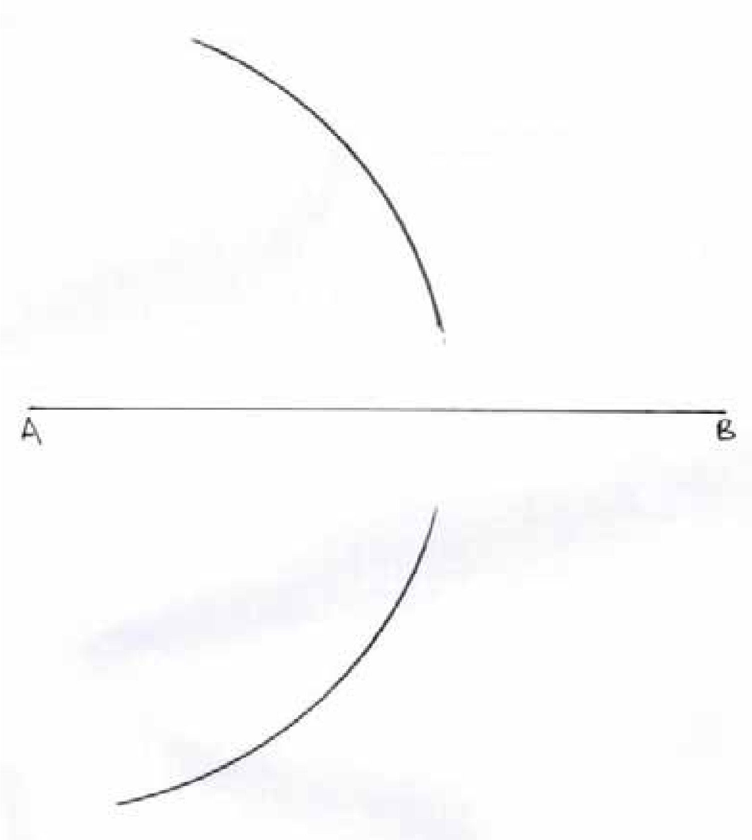
To illustrate how to go about determining this locus, we will show the steps in different diagrams. I will however expect you to to all the steps in one diagram.

Given the two points A and B, joined by a straight line, 10 cm long;

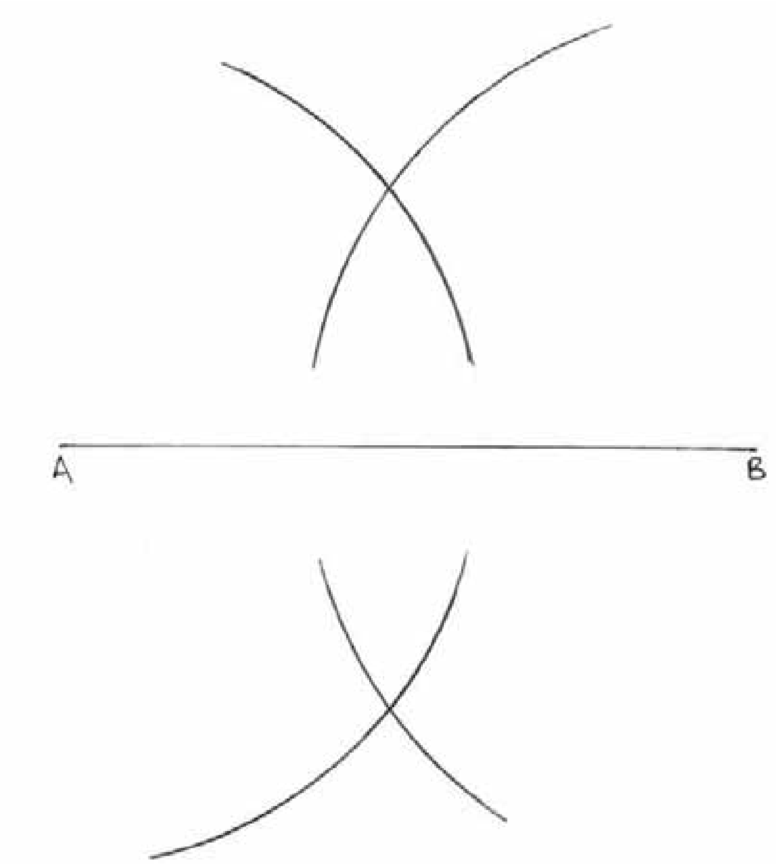
1. Open up your compass such that the radius is more than half of AB, i.e more than 5cm.



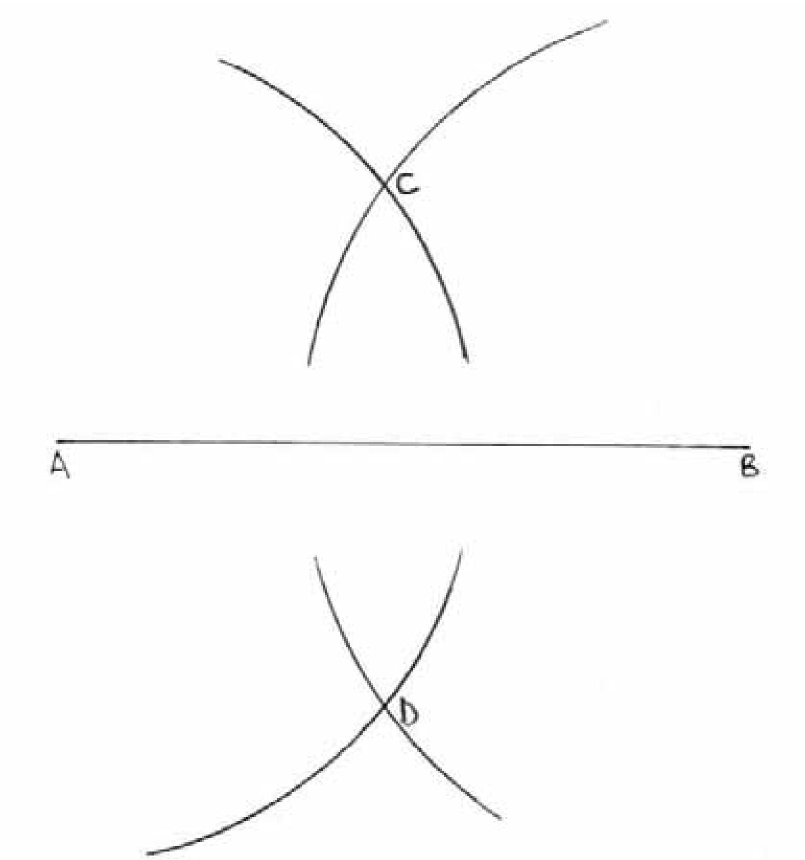
2. Put your compass at A, and mark an arc above and below line AB.



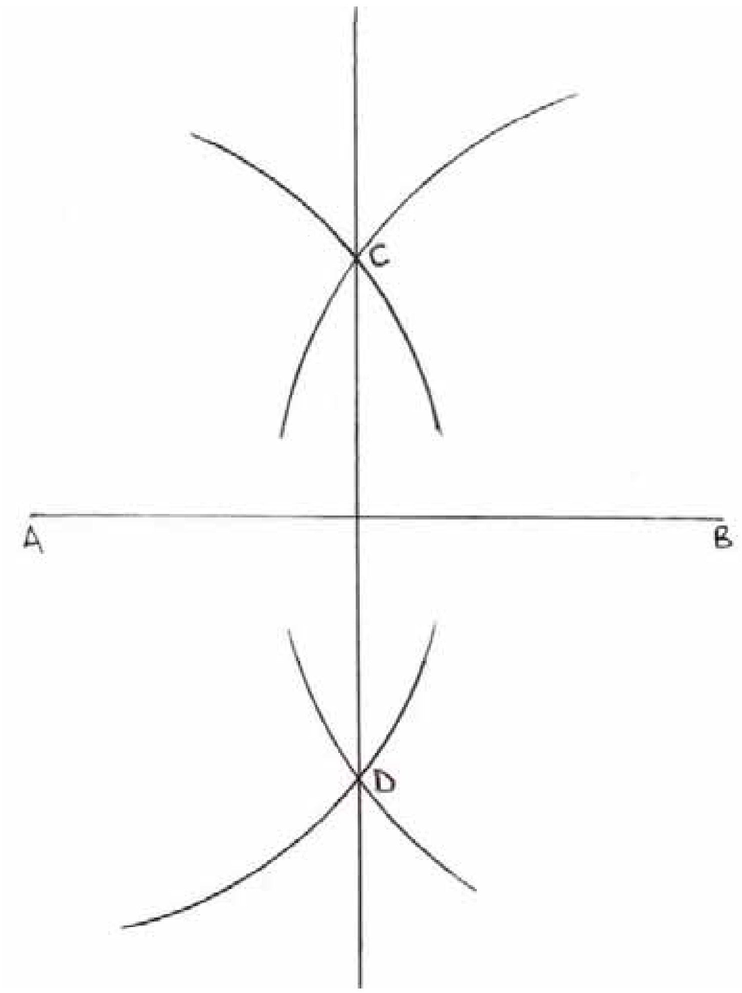
3. Do not adjust the radius. Then put the compass at B. Mark an arc above and below line AB, allowing the arcs to intersect.



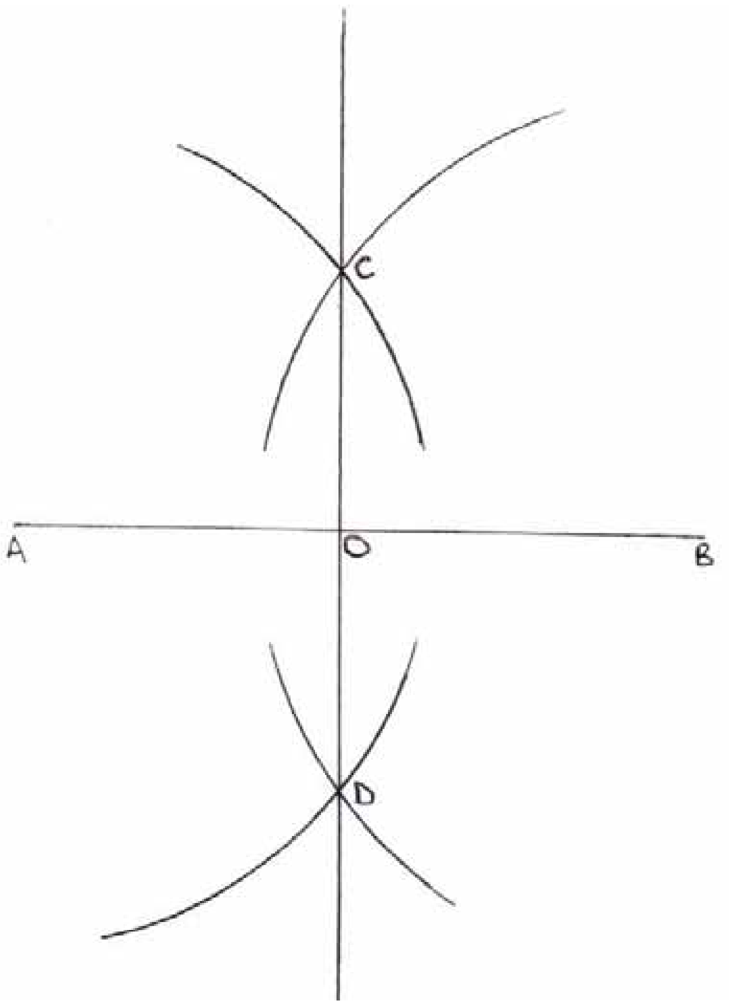
4. Mark the point of intersection of the arcs above the line C, and the point of intersection of the arcs below, D.



5. Join C and D with your ruler.



6. Mark the intersection of line AB and CD, O.



Compare your diagram with this one.

7. Measure AC and BC.

What do you notice about the distance AC and BC?

____

____

Compare your answer with the following:

AC = BC

8. Measure AD and BD.

What do you notice about the distance AD and BD?

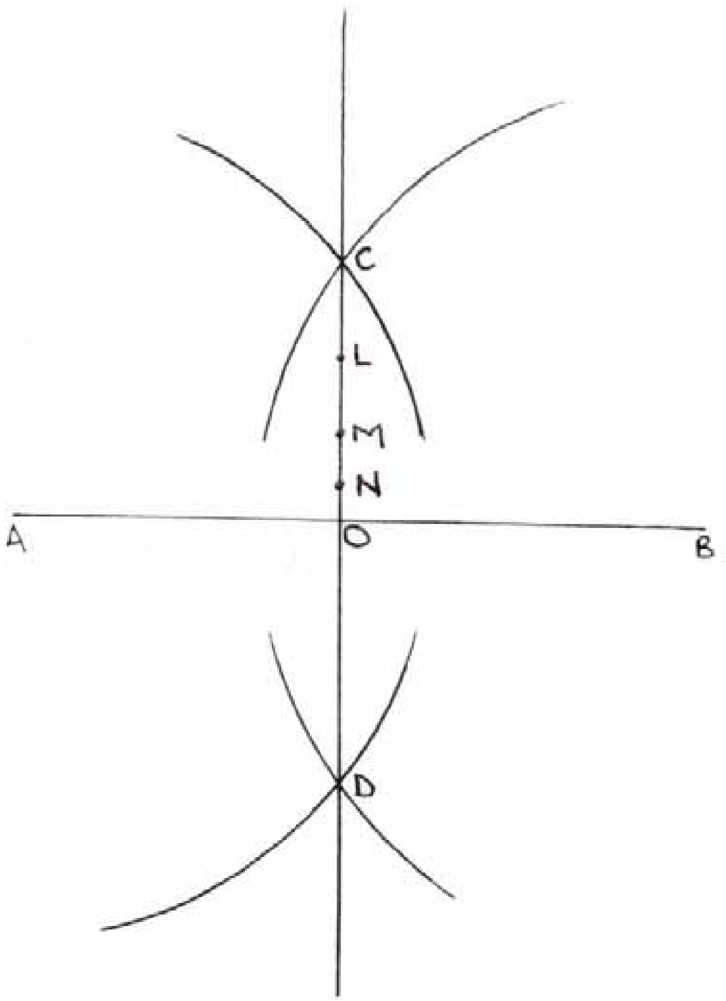
____

____

Compare your answer with the following:

AD = BD

9. Mark points L, M and N on line CO. Do not worry if your positions of L, M and N are not the same as the ones given in the diagram below. That will not in any way affect the result.



Measure : AL _________ BL ______________

AM __________BM _____________

AN __________BN _____________

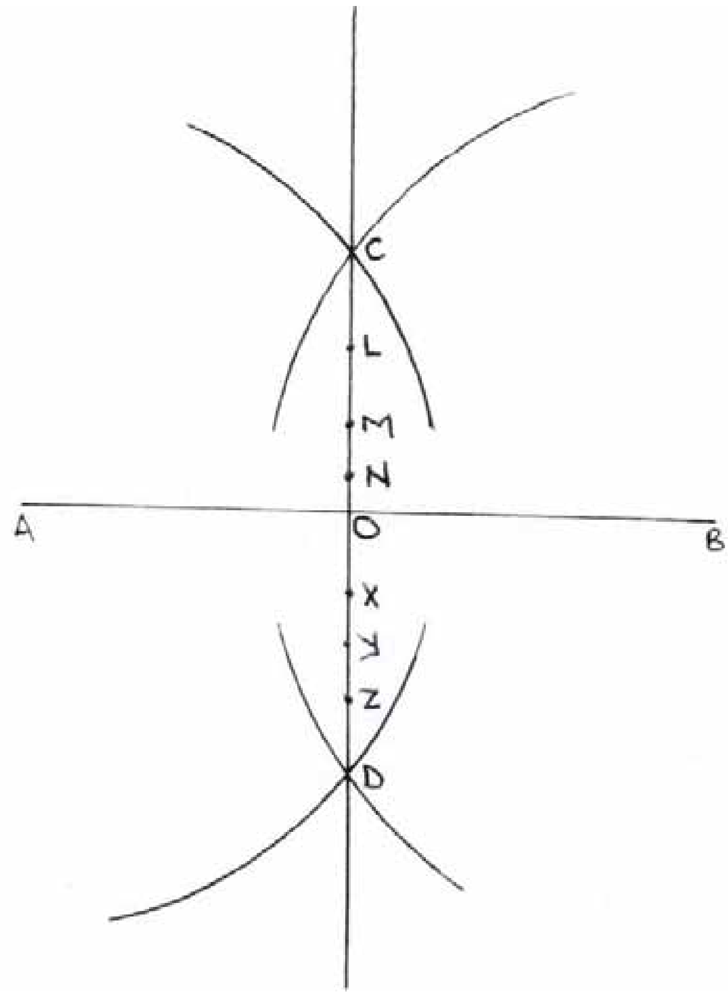
Compare your answers with the following:

AL = BL

AM = BM

AN = BN

10. Mark points X, Y and Z on line OD. Again do not worry if your positions X, Y and Z are not the same as the ones given in the diagram below. That will not in any way affect the result.



Measure: AX _________ BX ____________

AY __________BY ____________

AZ __________BZ _____________

Compare your answers with the following:

AX = BX

AY = BY

AZ = BZ

This is proof that all points on line CD, are equidistant from the points A and B.

Measure angle COA.

You should have found:

Angle COA = 090º

Therefore CD is perpendicular to AB.

Line CD is the perpendicular bisector of line AB, or its mediator.

Therefore, the perpendicular bisector of AB is the locus of all points equidistant from the points A and B.

Key Points to Remember

- The key points to remember in this subunit on determining the locus of all points which are equidistant from two given points in two dimensions are:

- the perpendicular bisector of AB is the locus of all points equidistant from the points A and B.

Lesson 4 Locus of Points Equidistant from Two Intersecting Lines

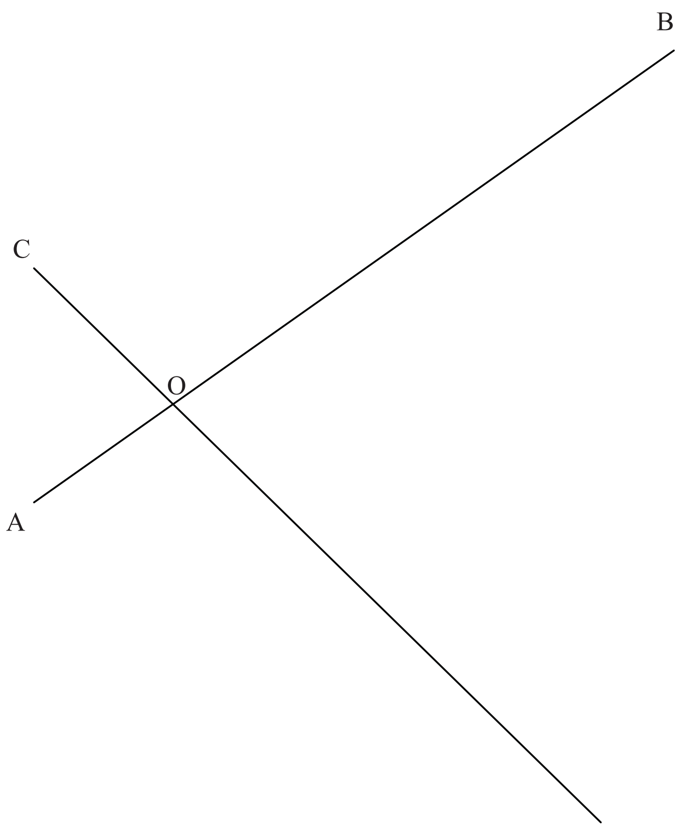
By the end of this subunit, you should be able to:

- *determine* the locus of all points which are equidistant from two intersecting lines in two dimensions

This subunit is about 4 pages in length

To illustrate how to go about determining this locus, we will show the steps in different diagrams. I will however expect you to all the steps in one diagram.

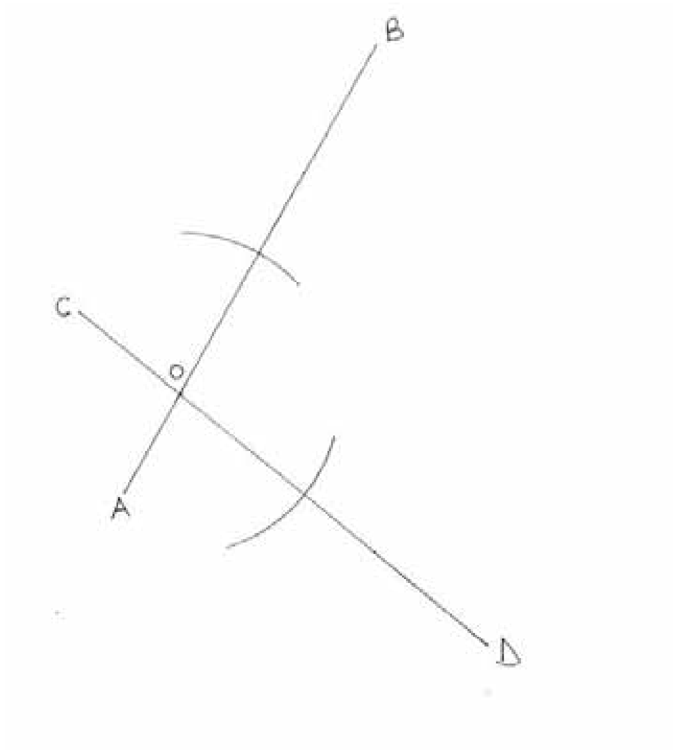
Given the two straight lines AB and CD, that meet at O.



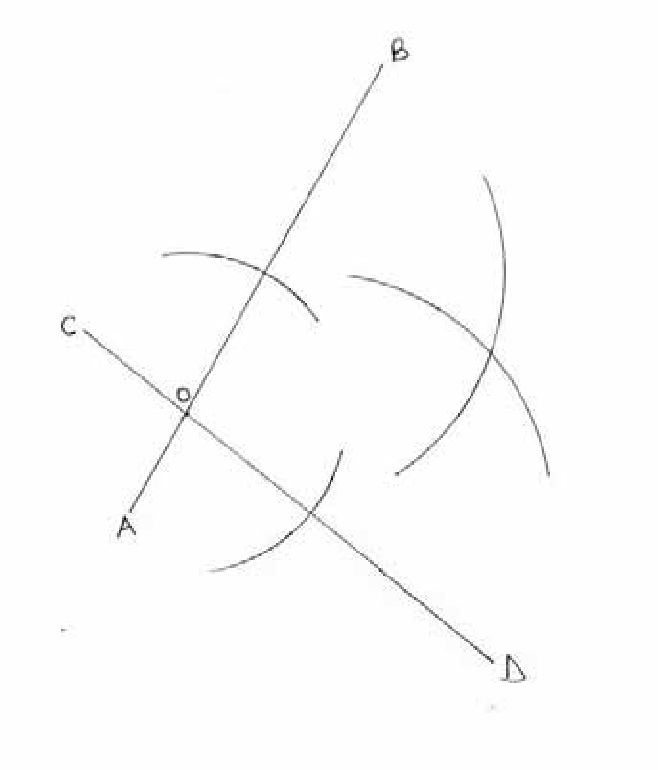
D

1. Open up your compass such that the radius is not more than half of OB and OD.

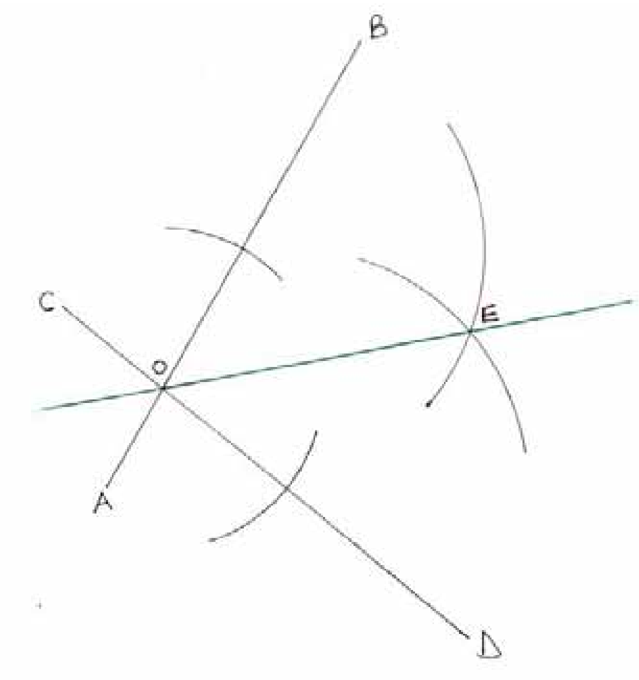
2. Put your compass at O, and mark arcs on OB and OD.



3. Put the compass at the arc on OB.
   - Make an arc inside BOC.
   - Without adjusting the radius of the compass, put the compass at the arc on OD.
   - Again make an arc inside BOC allowing the arcs to intersect.



4. Join the point of intersection of the arcs to O. Label this point E.



We now have a straight line going through EO.

5. Measure angle BOD.
6. Measure angle BOE and angle DOE.

What do you notice about these angles?

____

____

Angle BOD = angle BOE + angle DOE

Angle BOE = Angle DOE

This is confirmation that line EO bisects angle BOD, which is formed by line AB and line CD.

Therefore the locus of all points which are equidistant from two straight lines forming the arms of an angle is the bisector of the angle.

Key Points to Remember

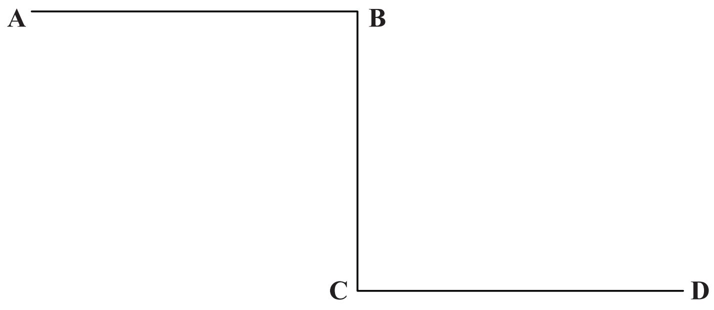
- The key points to remember in this subunit on determining the locus of all points which are equidistant from two intersecting lines in two dimensions are:

  - the locus of all points which are equidistant from two straight lines forming the arms of an angle is the bisector of the angle.

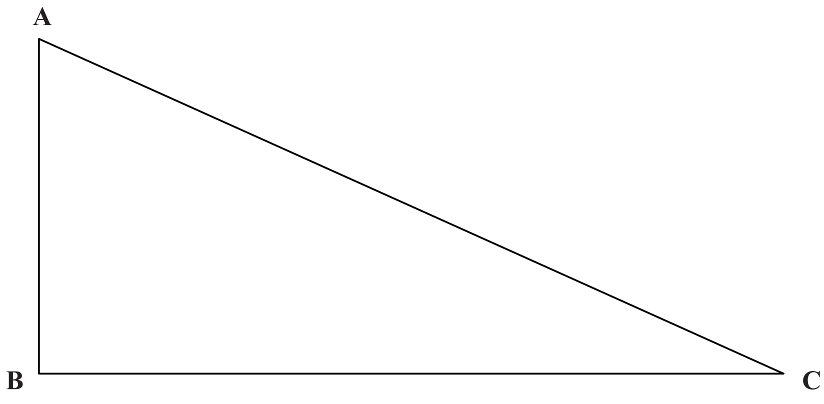
Activity 2

1. Draw the locus of points that are 4cm from the given line below.

2. Draw the locus of points that are
   - equidistant from A and B
   - equidistant from C and D

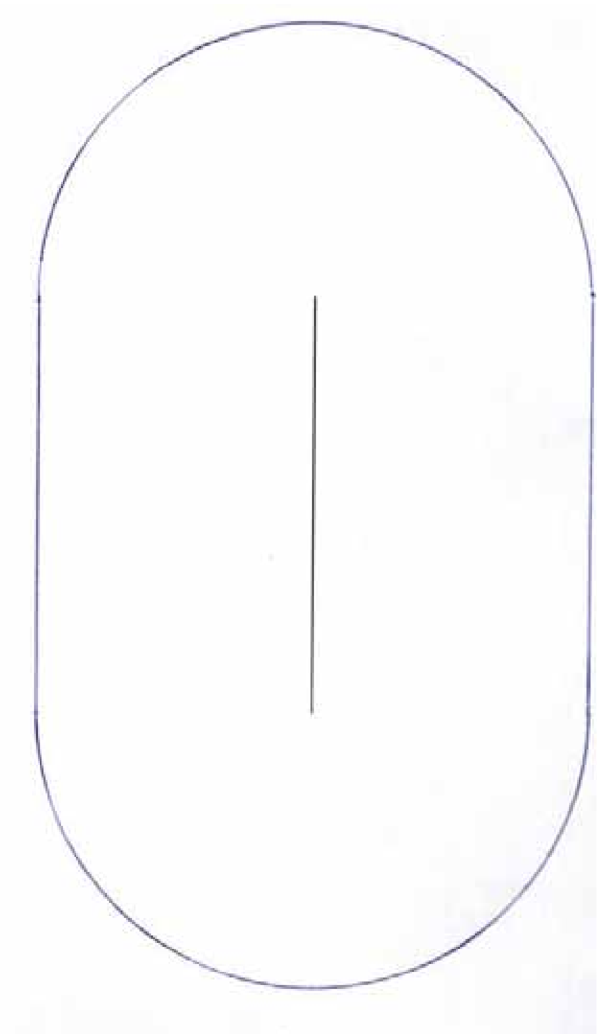


3. Draw the locus of points that are equidistant from line AB and line BC

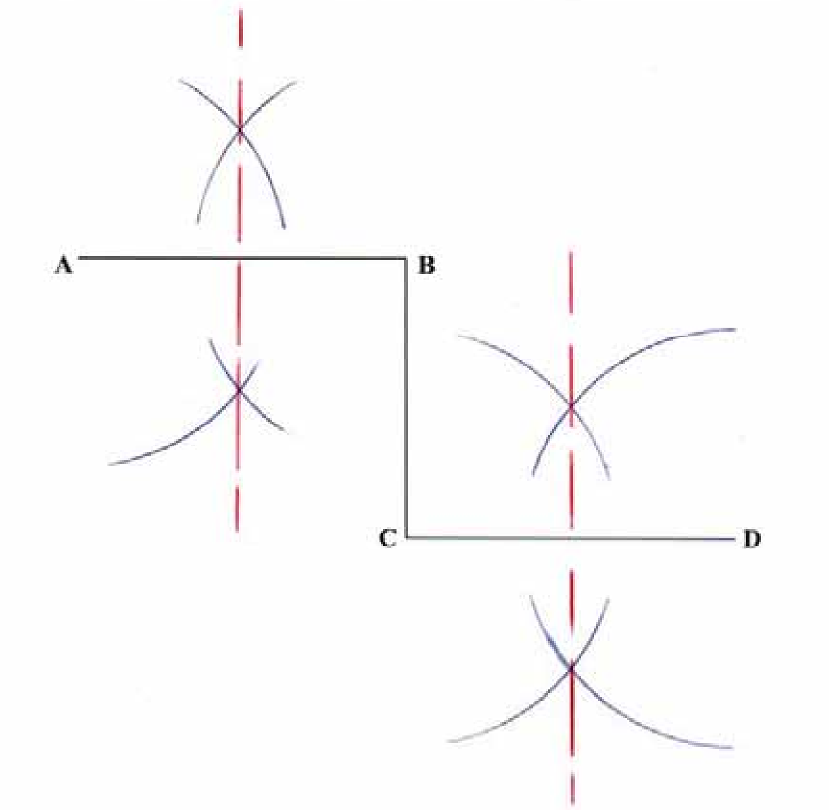


Answers

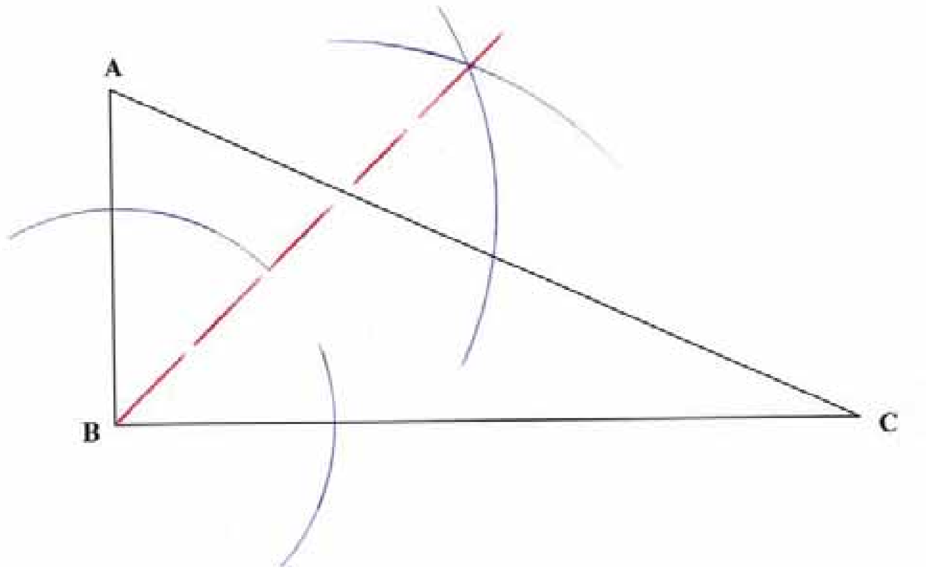
1.



2.



3.



Lesson 5 Intersecting Loci

By the end of this subunit, you should be able to

- *determine* the locus of all points which satisfy more than one condition, when we have an intersection of two or more loci.

This subunit is about 4 pages in length

So far we have worked with loci that were defining one set of points, at a time. When two sets of points are defined, the two loci are drawn, and they result in an intersection. **This intersection has the properties of all the loci in question.**

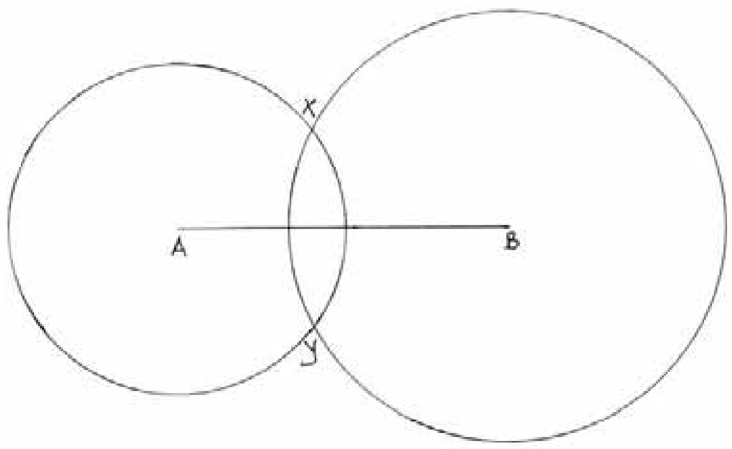
Example 1

Given a line AB, 6 cm,

- draw and describe the locus of points that are 3cm from A.
- draw and describe the locus of points that are 4cm from B.

This may seem like something new. It is not. We did work on the locus of points from one given point in the first subunit. You can go back and remind yourself of how it is done**.**

Compare your results with the ones given below.



The locus of points that are 4cm from A is the circle with radius 4cm

The locus of points that are 5cm from B is the circle with radius 5cm

The locus of points such that they are 4cm from A **and** 5cm from B are the intersections of the two circles marked X and Y.

This means the intersection has the properties of all the loci in question:

- The point marked X is 4 cm from A and 5 cm from B.
- The point marked Y is also 4 cm from B and 5 cm from B.

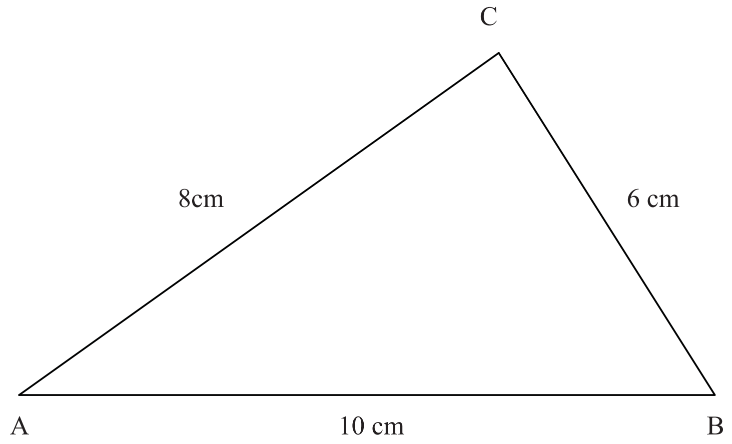
Example 2

Given triangle ABC, AB = 10cm, AC = 8cm and BC= 6cm:

show and describe the locus of points equidistant between A and B,

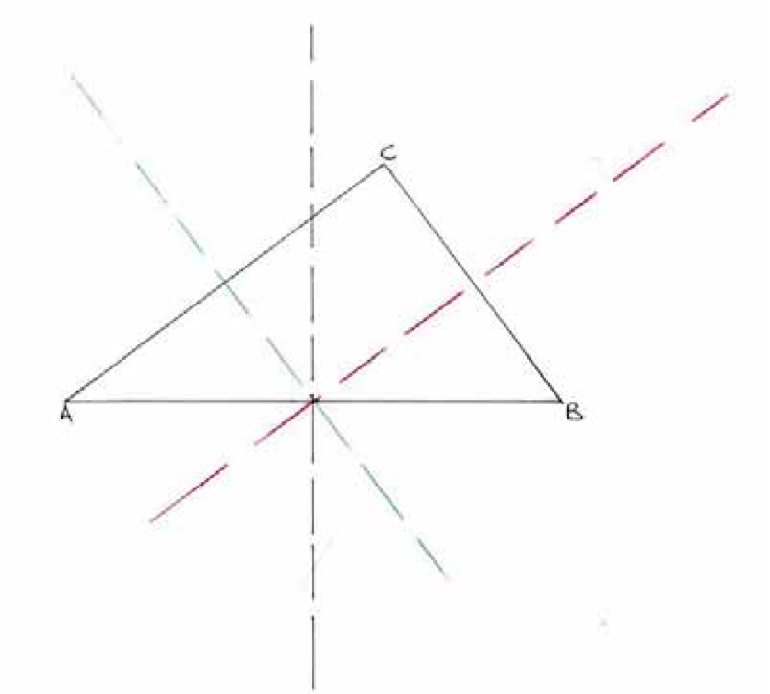
show and describe the locus of points equidistant between B and C, and

show and describe the locus of points equidistant between A and C.



This may again seem like something new. It is not! We did work on the locus of points equidistant between two points in the second subunit. We are just having the points joined together to give us a triangle.

You can go back and remind yourself of how it is done**.**



The loci in all three cases are perpendicular bisectors of the lines joining A and B, B and C, and A and C, which meet at one point, O. Point O has the properties of all the loci in question.

Therefore the locus of all points when we have an intersection of two or more loci is the point or all points which satisfy all the conditions.

These perpendicular bisectors meet at one point O. This point of intersection has the properties of all the loci in question.

Activity 3

1. Given a line AB, 6 cm

(a) Draw the locus of points which are:

(i) 4cm from point A

(ii) Equidistant between A and B

(b) Identify by shading, and describe the region that is less than 4 cm from point A and less than 3 cm from B, by shading it.

____

____

2. Given a fixed point O below,

•O

(a) Show the locus of points which are

(i) 3cm from point O

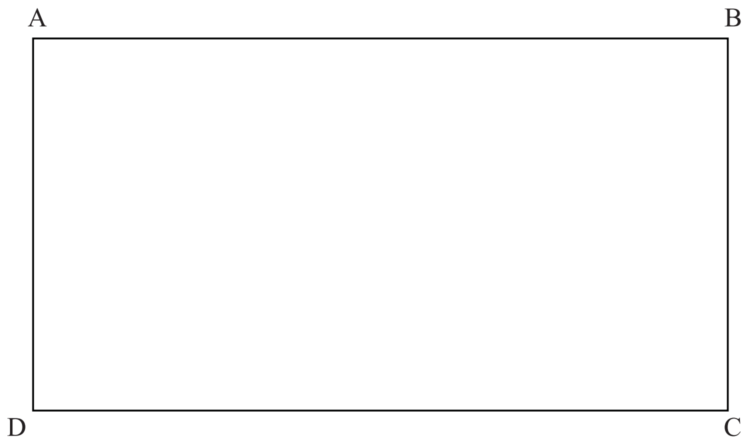
(ii) 5cm from point O

(b) Identify by shading, and describe the locus of points which are more than 3cm from O, but less than 5cm from O.

3.

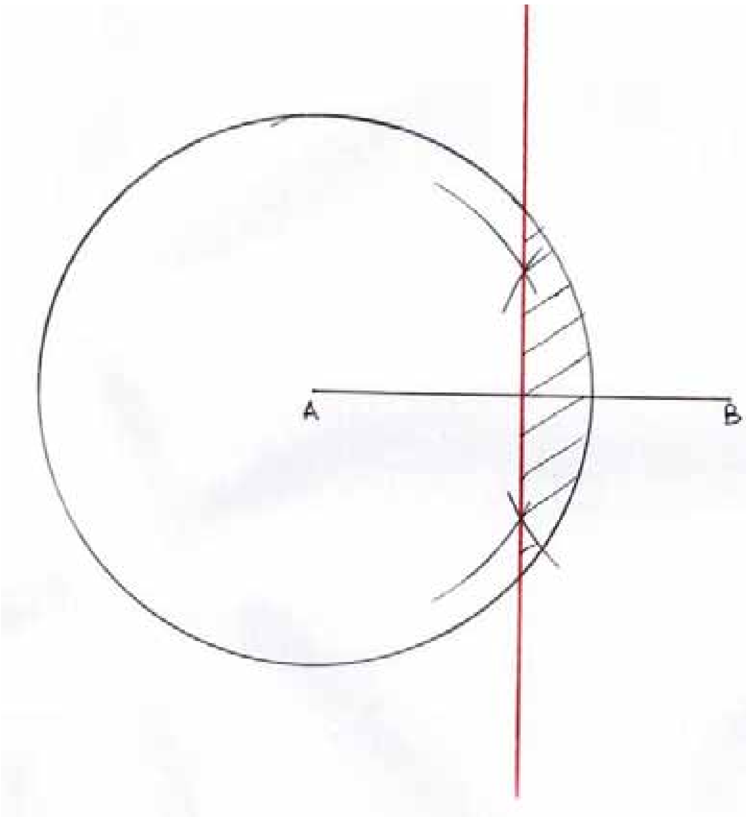
Construct the locus of a point which is equidistant from A and B, B and C.

Mark the point where these loci meet O.

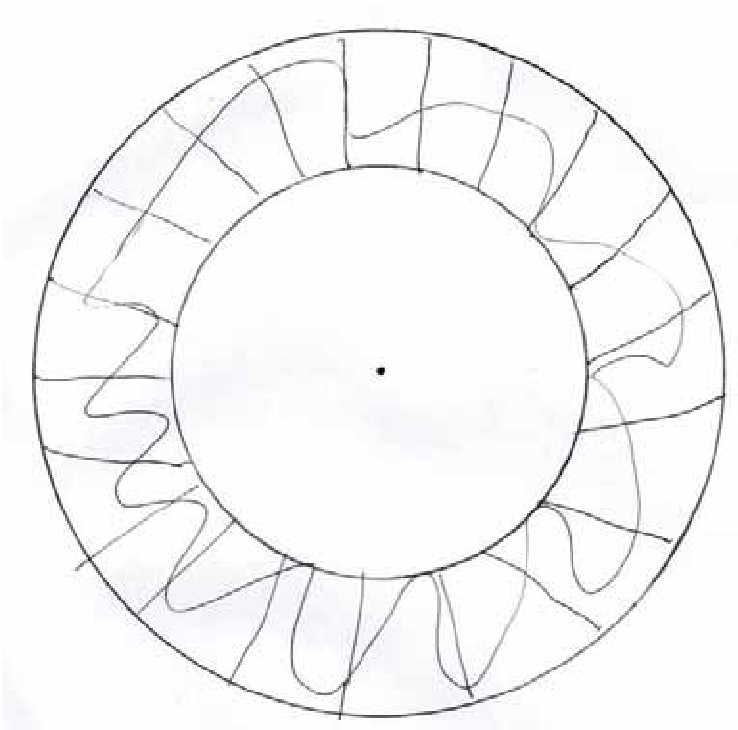


Answers

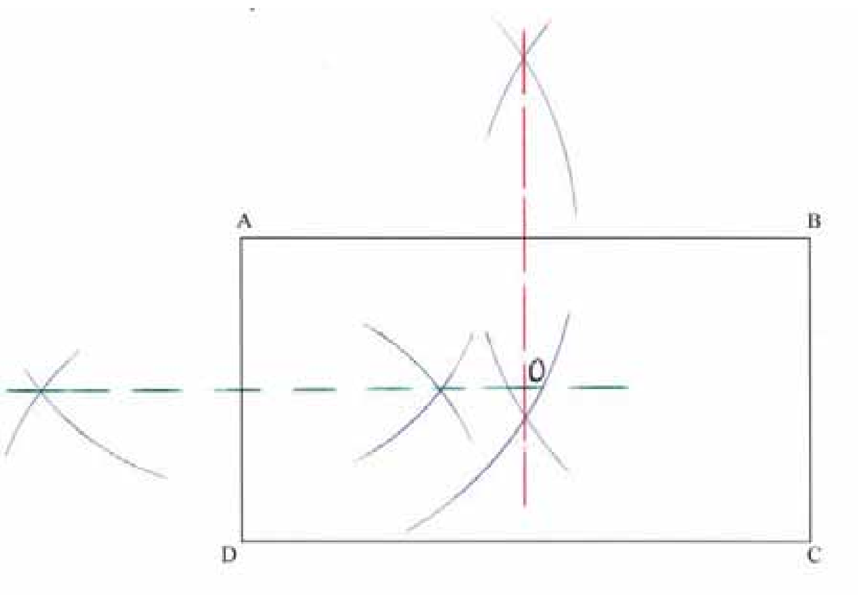
1. This is the region that is a little to the right of the perpendicular bisector of line AB, but within the circle.



2. This is the shaded region, the region between the two circles; it is more than 3cm from O, but less than 5cm from O.



3.



Unit Summary

Summary

In this unit you learned the loci of:

1. points from a given point, which is a circle
2. points from a straight line, are parallel lines plus circles
3. points equidistant from two given points, it is the perpendicular bisector of the line joining the two points
4. points equidistant from two straight intersecting lines; they are two straight lines that bisect the angles between the given lines.
5. two or more loci, which is an intersection that has the properties of all the loci in question.

You have completed the material for this unit on loci. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit. It covers symmetry.

Assignment

Assignment

Instructions

1. Answer All Questions.
2. Show all the necessary working.

Total marks = 20 **Time:** 20 minutes

1. Draw a line LM of length 9 cm. Construct the locus of a point which is equidistant from L and M. [4]

2. Draw two lines AB and AC of length 8 cm, where BAC is 80˚. Construct the locus of a point which is equidistant from the lines AB and AC. [6]

3. Two radio stations are 60 kilometres apart.

   Broadcasts from A can be heard up to 40kilometres away, while broadcasts from B can be heard up to 30kilometres.

   Shade the area where both broadcasts can be heard. Use a scale of 1cm = 1km. [4]

4. The point A is 5 cm from line, *l*.

   Mark the locus of points which are 3cm from A. Call this L.

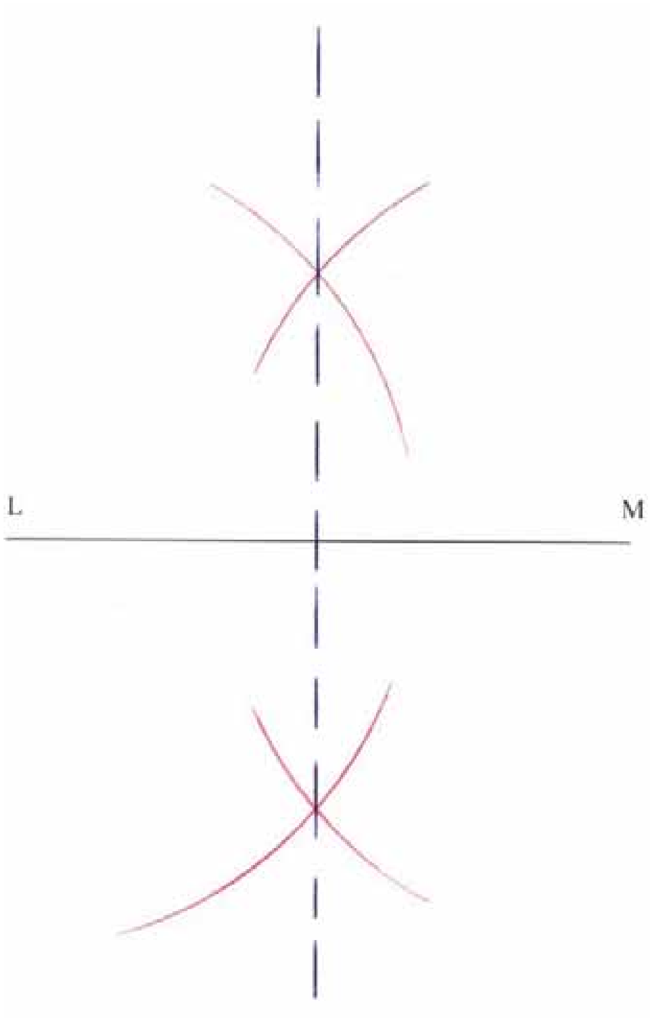
   Mark the locus of points which are 3 cm from the line. Call this set M.

   Mark with crosses (X) the points of intersection of L and M. [6]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

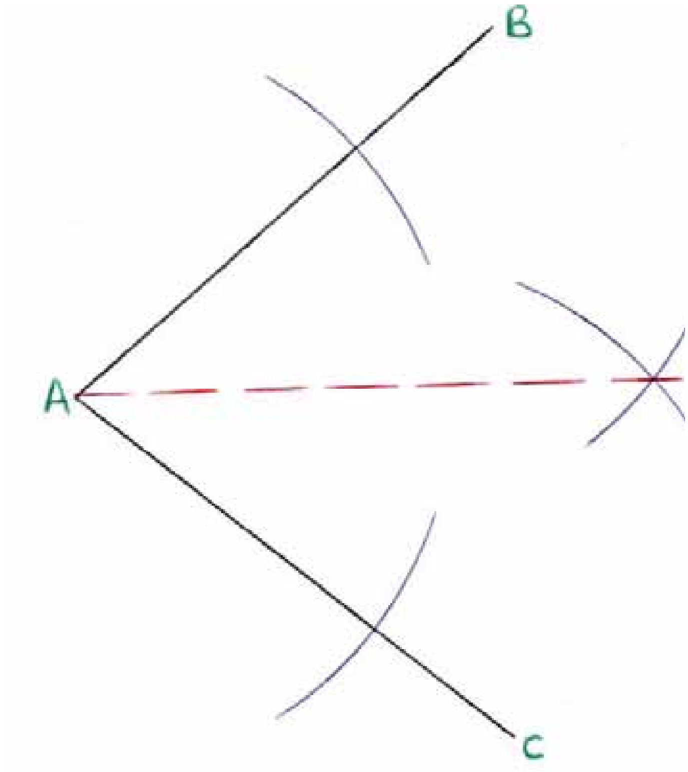
Answers

1.



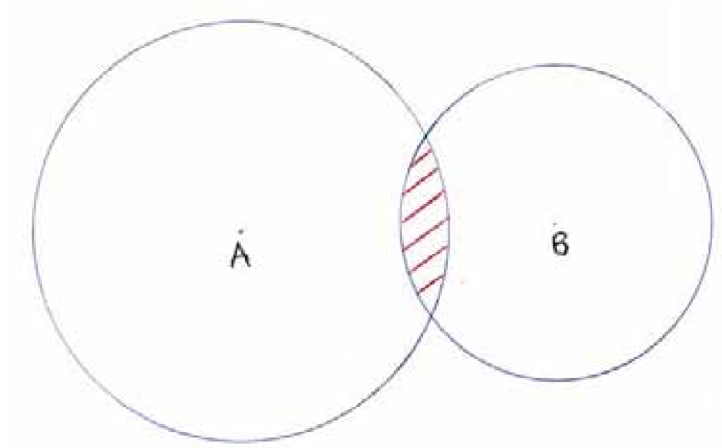
This is a perpendicular bisector of LM.

2.



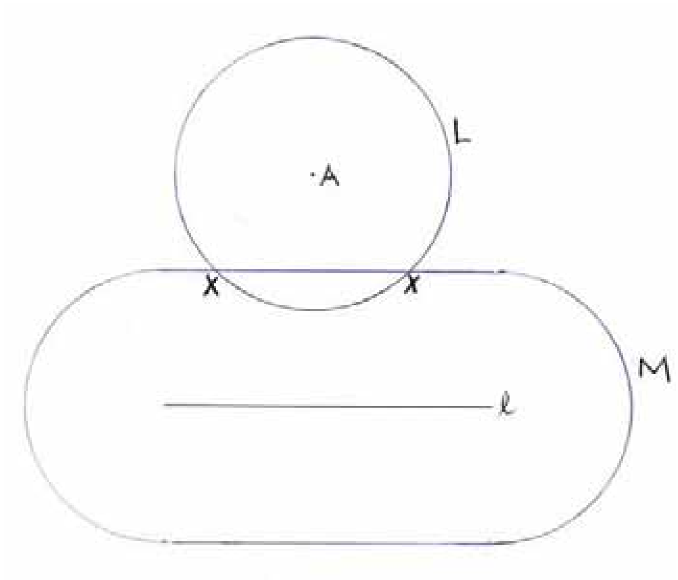
The locus is the straight line that bisects 80˚

3.



The shaded area is where both broadcasts can be heard.

4.



Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Instructions

1. Answer All Questions.
2. Show all the necessary working.

Total marks = 25 **Time:** 30 minutes

1. Draw a triangle ABC with AB = 8cm, BC = 9 cm and CA = 7cm.

   Construct the locus of a point that is equidistant from A and B.

   Construct the locus of a point which is equidistant from AB and AC.

   Mark with an X, the point which is the intersection of the two loci, and measure and record the length CX. [8]

2. Three donkeys are chained in a rectangular field that measures 80

metres by 60 metres shown by rectangle ABCD. One donkey is chained at corner D on a 20 metre rope; another is chained at the midpoint, E, of AD on a 15 metre rope, while the third donkey is chained using a 35 metre rope fastened to a post at the centre of the field, F.

Clearly show the areas where the donkeys are able to move.

Shade and label any area where more than one donkey can graze.

(1cm = 1m) [10]

3. There are developments that have to be made to Mr and Mrs Molapo's rectangular site.

   Whatever they do, they are planning not to temper with the well in their site. The well is within 5 metres of A and more than 3 metres from the line AD.

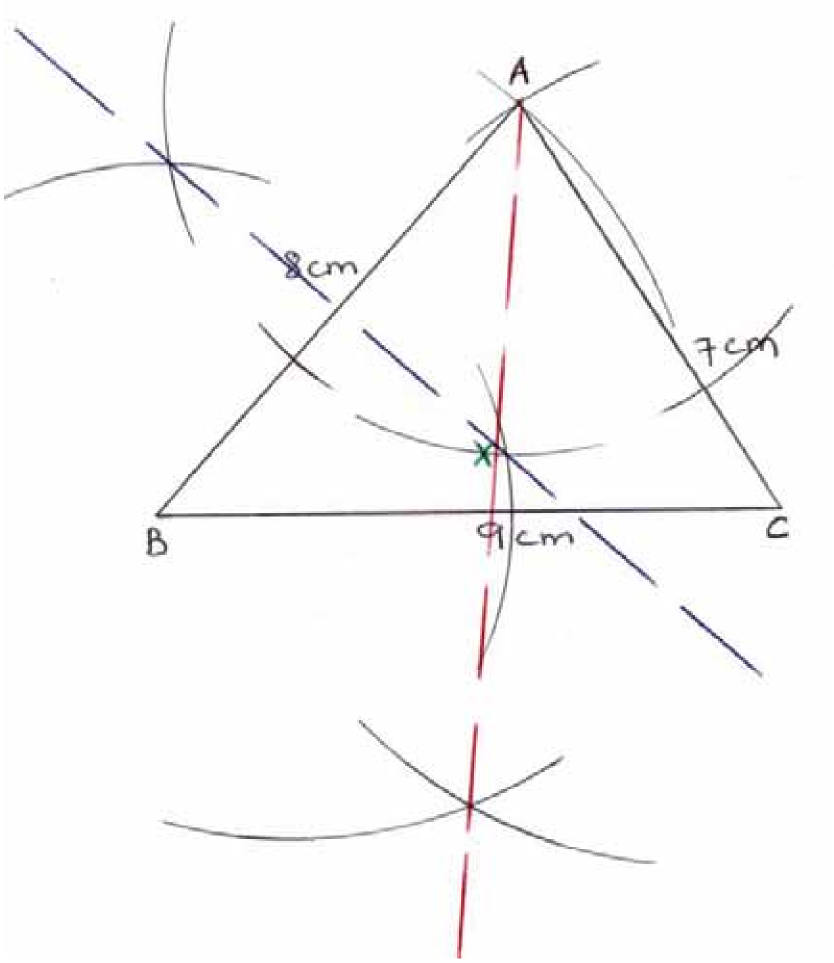
   Draw a plan of the garden and shade the points where the well could be.

   (1cm = 1m) [4]

4. Triangle XYZ has area 45 cm². Its base, XY, is 15 cm. Describe the locus of Z, if Z moves so that the area of the triangle XYZ is always 45 cm². [3]

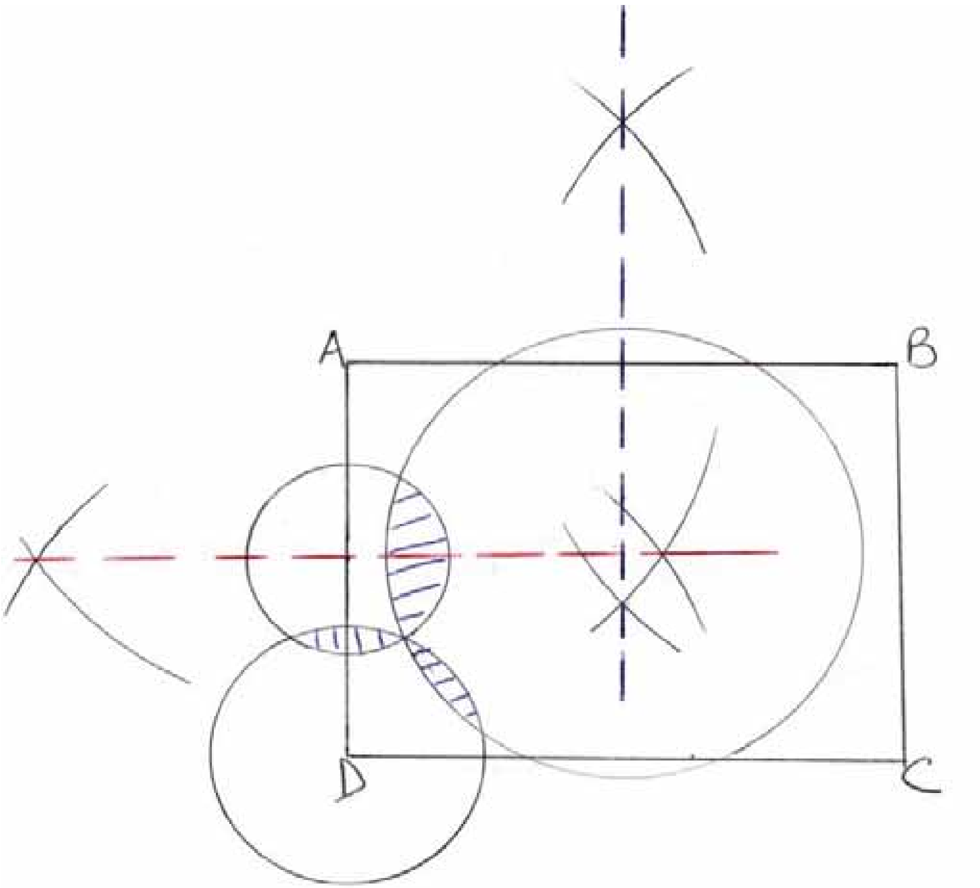
Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

1.

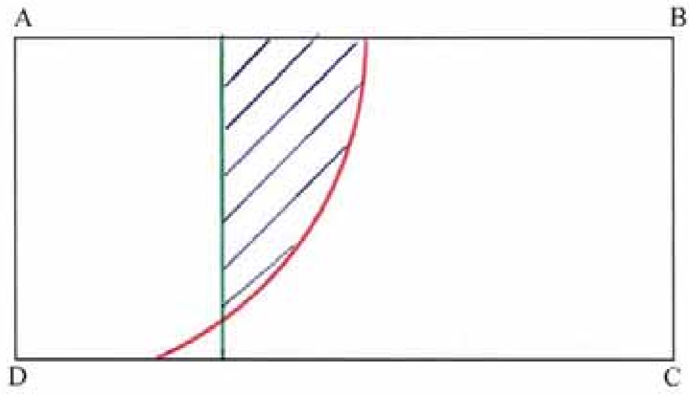


CX = 4.2cm

2.



3.



4.

Area of triangle XYZ = ½ × base × height

$45cm^2$ = ½ × 15cm × height

6cm = height

The locus of Z is the a line parallel to XY and 6 cm from XY.

Unit Contents

Unit 26

Unit 26

Symmetry

Introduction

Welcome to another interesting unit in this course. This unit consists of 61 pages. This is approximately 2% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 20 hours to work on this unit, 15 hours for formal study and 5 hours for self-study and completing assessments/assignments.

Shapes can be rotated or reflected to fit onto themselves. This is called **symmetry**. In this unit we are going to look at symmetry of two and three dimensional shapes.

It is necessary to make some solids as you will need them in the last subunit. Keep them safely until you need them.

These are the solids you should make:

- cube
- cuboid
- square – based pyramid
- cone
- cylinder
- equilateral triangular pyramid

When reading the following learning outcomes, think about them as a guide to what you should focus on while studying this unit.

This Unit is Comprised of Three Lessons:

Lesson 1 Line Symmetry
Lesson 2 Rotational Symmetry
Lesson 3 Symmetry Properties of Prisms and Pyramids

Upon completion of this unit you will be able to:

Outcomes

- *recognise* and *describe* line symmetry.
- *recognise* and *describe* rotational symmetry.
- *recognise* and *use* symmetry properties of plane shapes.
- *recognise* and *describe* symmetry properties of prisms (including cylinder) and pyramids (including cone).

Terminology

Symmetry:	The more ways a shape can be reflected and turned to fit onto itself.
Line symmetry:	Line such that when a mirror is placed on it one half of the shape reflects to the other half.
Rotational symmetry:	Rotation of a shape about a point through an angle of less than 360° to coincide with itself.
Plane shape:	A flat shape.
Prism:	A solid shape with uniform cross section and parallel ends whose shape and size are equal
Pyramid:	A polyhedron formed by joining the edges of a polygon to a point to form sloping triangular faces
Polyhedron:	A polyhedron (a solid shape whose faces are all polygons) formed by joining the edges of a polygon to a point to form sloping triangular faces

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Line Symmetry

Introduction

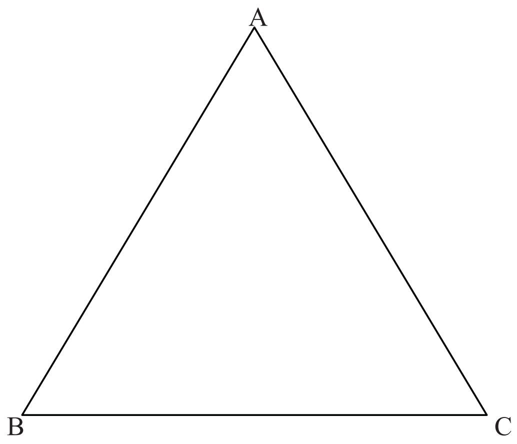
By the end of this subunit, you should be able to:

- recognise and describe line symmetry.

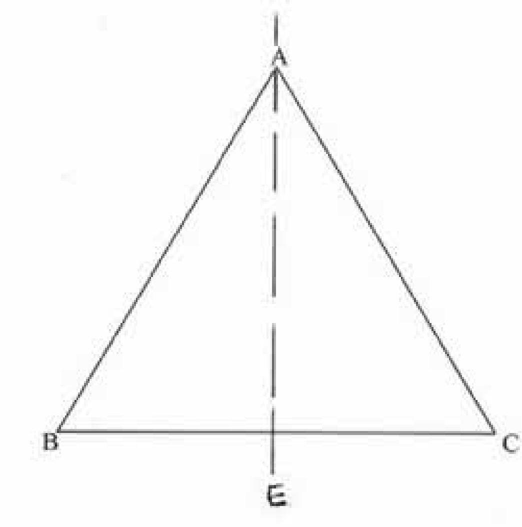
This subunit is about 4 pages in length.

Trace the isosceles triangle ABC given below. Fold it such that B is on C. Mark it with a dashed line.

Mark the other end of the fold, E.



This fold, which is labelled AE, is a **line of symmetry**.



Fold the isosceles triangle ABC such that C is on B.

Is this fold the same or different from the first one?

____

The second fold is exactly on the first fold.

Triangle ABC is now divided into two parts, triangle ABE and triangle ACE. Triangle ABE fits exactly over triangle ACE.

Fold the isosceles triangle ABC again, such that A is on B.
What do you notice?

____

____

____

Triangle ABC is still divided into two parts which do not fit exactly over each other. This fold is **not** a line of symmetry.

Fold the isosceles triangle ABC again, such that A is on C.
What do you notice?

Triangle ABC is still divided into two parts which do not fit exactly over each other. This fold is **not** a line of symmetry.

Does this isosceles triangle have other lines of symmetry?

The isosceles triangle does not have other lines of symmetry.

Isosceles triangle ABC is said to have **1 line of symmetry**, or **1 axis of symmetry.**

It has a vertical axis of symmetry.

When a plane shape is drawn on paper, the paper can be folded so that the folding line divides the shape into two congruent halves. If one half fits exactly over the other, the shape is then said to have **line symmetry**, or **bilateral symmetry** about an axis.

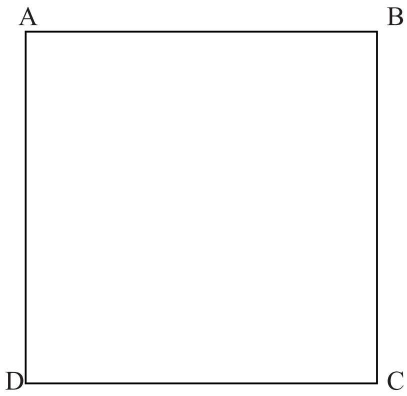
Reflection

Figures that have the same size and shape are called **congruent** figures.

So when we talk congruent halves, it means the halves have the same size and shape.

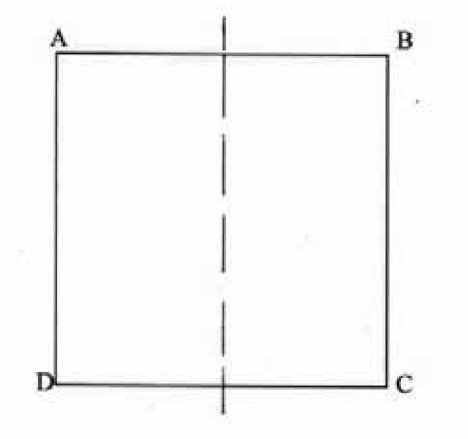
Example 1

How many lines of symmetry does square ABCD have?



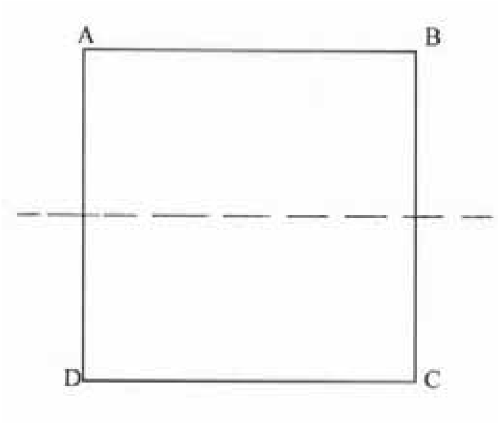
Solution

When it is folded such that A is on B and D is on C, the square is symmetric with the dashed line.

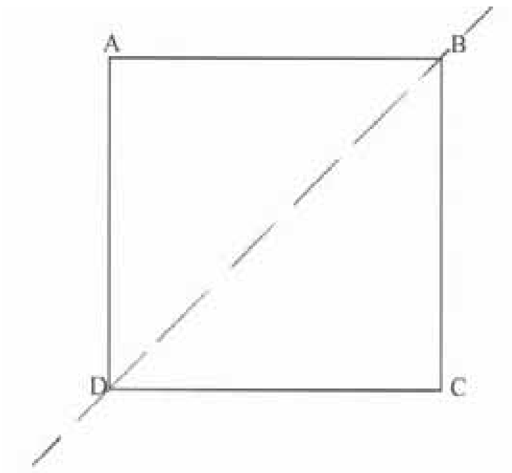


We will still get the same fold when B is on A and C is on D, the square is symmetric with the same dashed line.

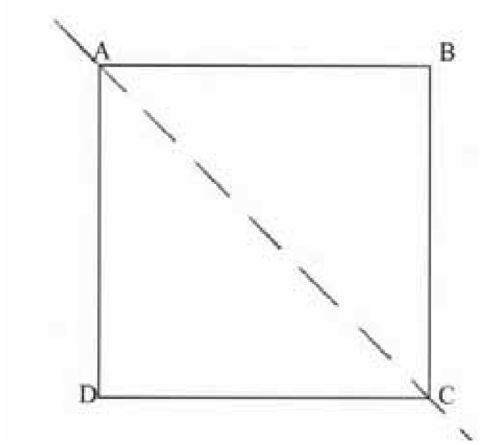
If the square is folded such that A is on D and B is on C, the square is symmetric with the dashed line.



Fold the square again such that A is on C, the square is symmetric with the dashed line.



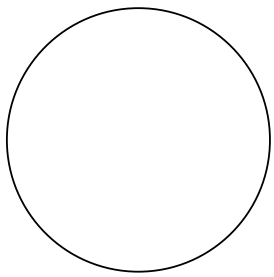
It can also be folded such that B is on D, the square is symmetric with the dashed line.



Square ABCD can still be divided into two parts which do not fit exactly over each other. There are many of these folds that we will get! Those folds are **not** lines of symmetry.

Therefore a square has 4 axes (singular: axis) of symmetry.

Example 2
How many lines of symmetry does this circle have?



Solution

A circle is symmetrical about every diameter, we therefore say it has an infinite number of axes of symmetry.

Example 3

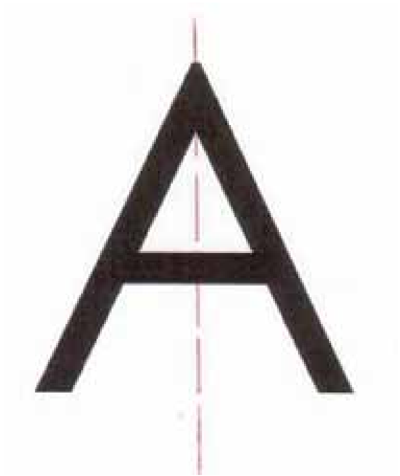
Mark the axes of symmetry in each of the letters of the alphabet given below.

A

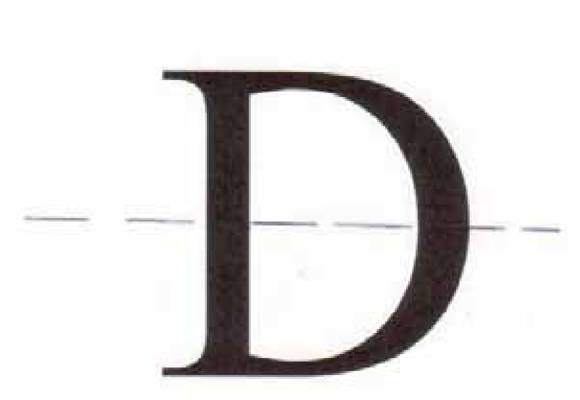
D

Solution

Letter A can only be folded along the dashed line to give two congruent parts. Therefore it has 1 axis of symmetry. It has a vertical axis of symmetry.



Letter D has 1 axis of symmetry. It can only be divided into two parts which fit exactly over each other. It has a horizontal axis of symmetry.

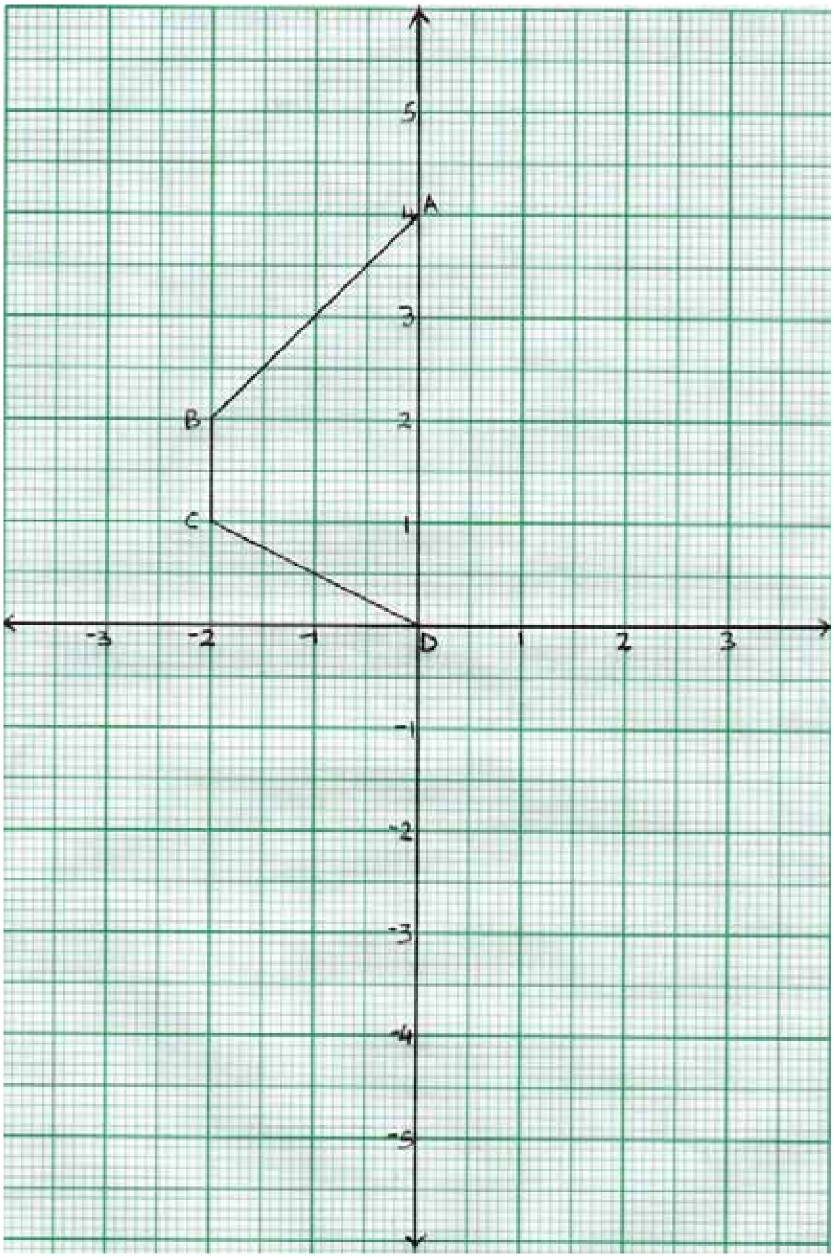


Example 4

Complete the figure so that it is symmetrical about the

(i) y axis

(ii) x axis



Solution

When the axis of symmetry is the y axis, we get the part that is labelled A' B' C' D'.

This axis of symmetry is the line of reflection. ABCD has been reflected along the y - axis.

When the axis of symmetry is the x axis, we get the parts labelled A"B' 'C' 'D" and the one labelled A'" B'" C'" D'". The x – axis is the line of reflection.

These axes of symmetry are the lines of reflection.

Given that we are talking reflection, all that we know about reflection still holds.

Under a reflection:

- The line of reflection is the perpendicular bisector of the line joining a point and its image.

  For example, taking point B and B', the line joining these points is the perpendicular bisector of the line of reflection.

- The shape and its image are the same size.

  For example, A' B' C' D' is the image of ABCD. They are the same size.

- The object and the image are the same distance from the mirror line.
  Taking for example points C' is the image of C. Each is two units from the mirror line.

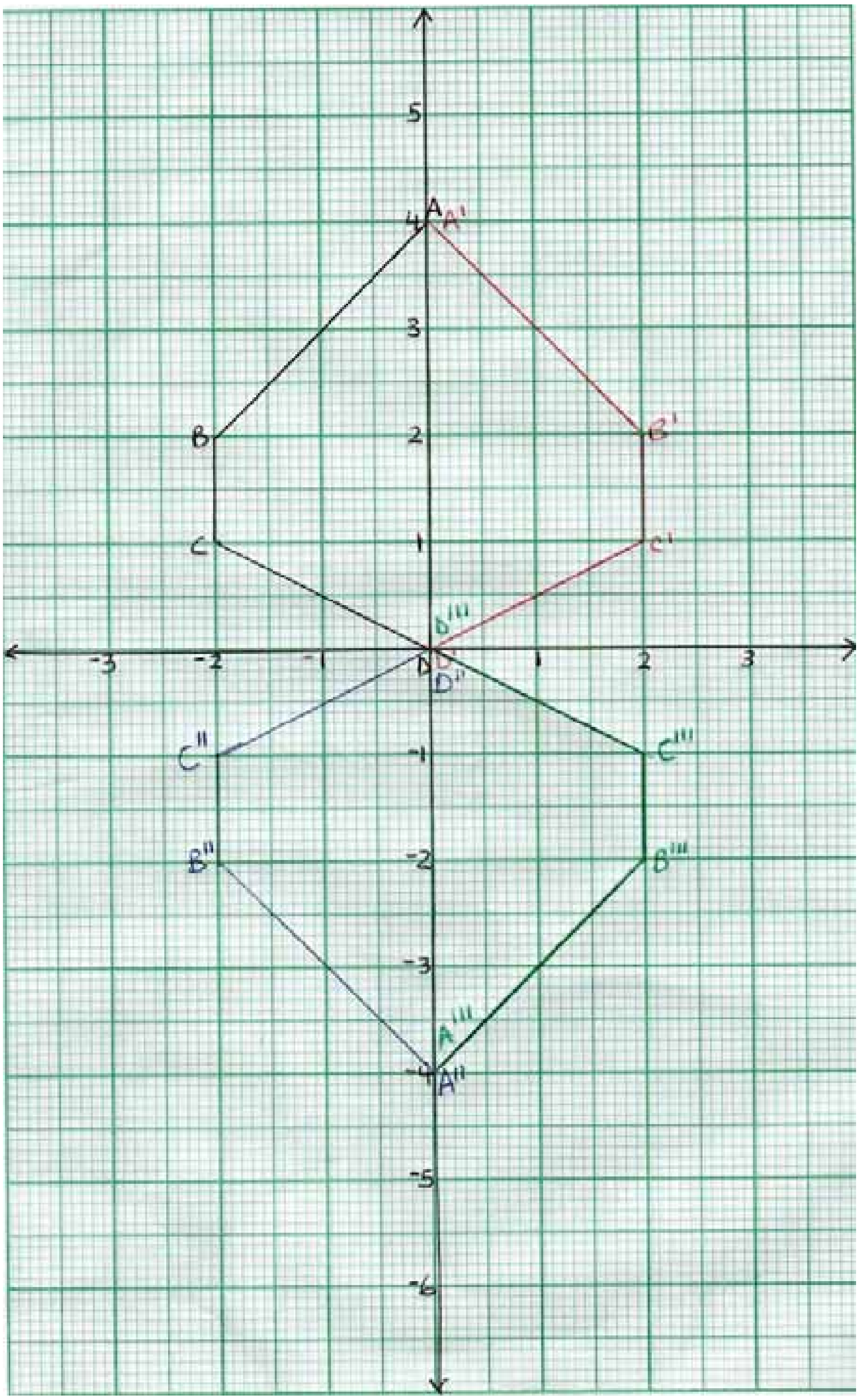
The shape is then said to have **reflective symmetry**.

Check what will happen if you start off by reflecting ABCD along the x – axis.

____

____

The result should be the same.


5
4
3
2
1
-1
-2
-3
-4
-5
-6
-3
-2
-1
1
2
3
A
A'
B
B'
C
C'
D
D'''
D''
C''
C'''
B''
B'''
A'''
A''

Activity 1

1. **How many axes of symmetry has**

(i) A square [1]

___

(ii) A rectangle [1]

___

(iii) A parallelogram [1]

___

(iv) A rhombus [1]

___

(v) An isosceles triangle [1]

___

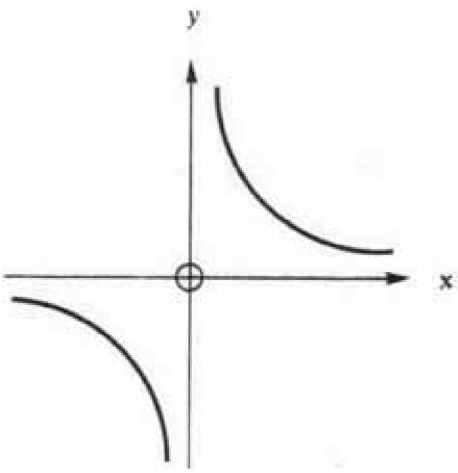
(vi) An equilateral triangle [1]

___

(vii) A circle [1]

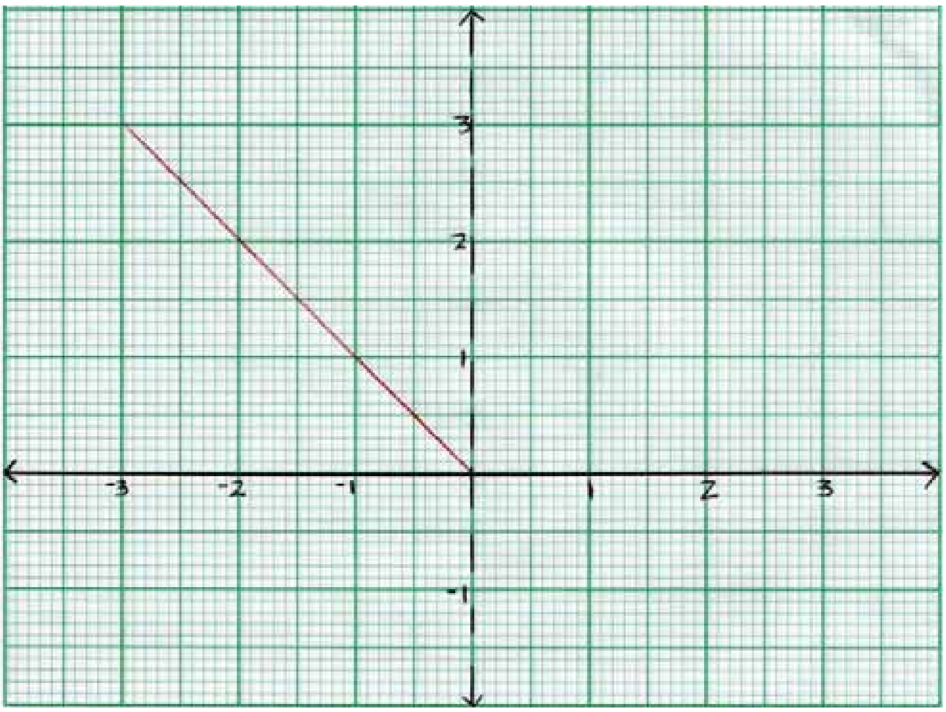
___

2. **Mark the axes of symmetry in these diagrams.**



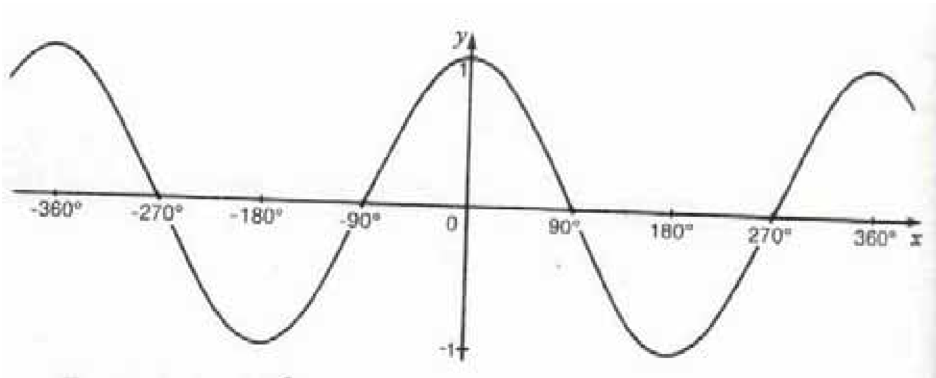
[5]

3. Complete the figure so that it is symmetrical about the dashed line.



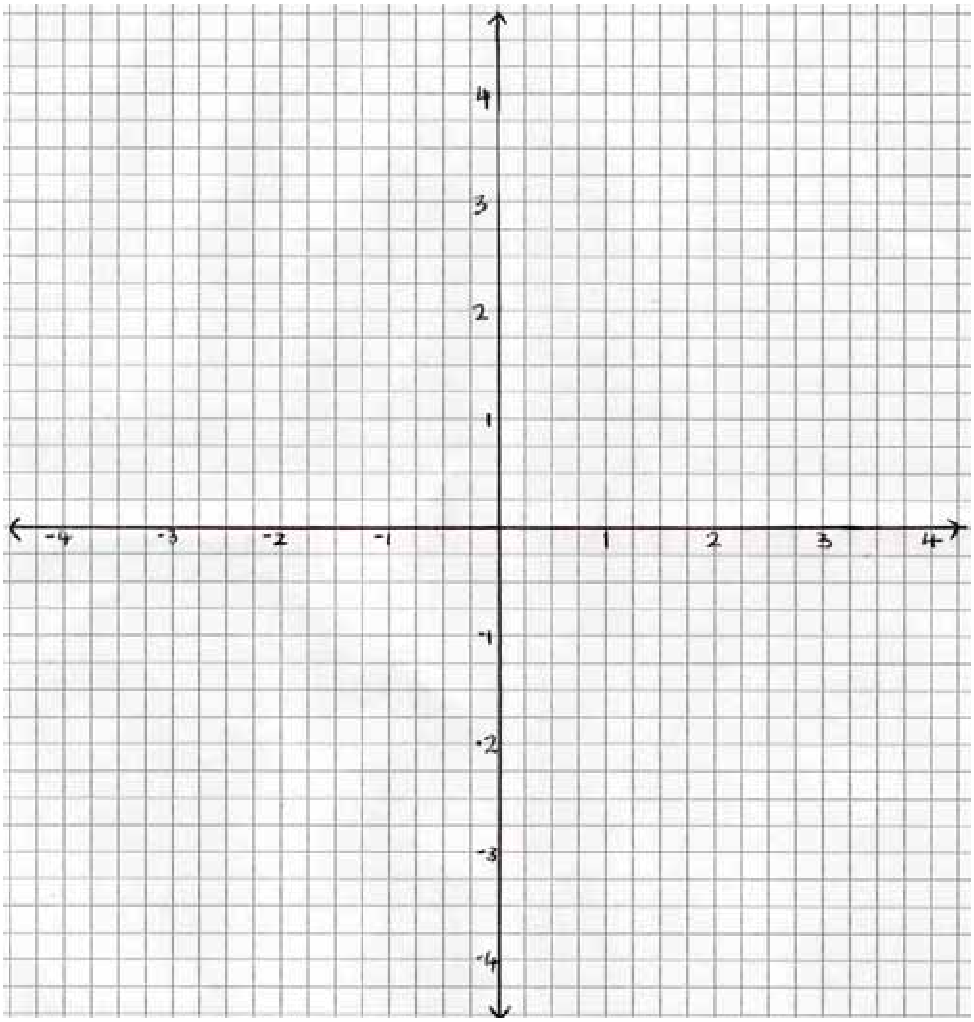
[3]

4. This curve is symmetrical about a line. Show it.



[1]

5. Plot the points A(0,0) B (-1,0), C(-1,1) D(-3,1) E(-3,2) and F(0,2).
This figure is part of a figure which is symmetrical about the y- axis and the x – axis. Complete the figure. [4]



Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on line symmetry are:

- If one half fits exactly over the other, the shape is then said to have line symmetry, bilateral symmetry or reflective symmetry about an axis. Each part has a corresponding part the same distance from the axis of symmetry.
- Figures with line symmetry are symmetric about a line.
- When joining corresponding points of a symmetric figure through the line of symmetry, the lines cross the line of symmetry.

In the next sub unit, we are going to look at changing the subject of the formula using factorisation.

Answers:

1.

(i) A square, 4 axes of symmetry

(ii) A rectangle, 2 axes of symmetry

(iii) A parallelogram, no axes of symmetry

(iv) A rhombus, 2 axes of symmetry

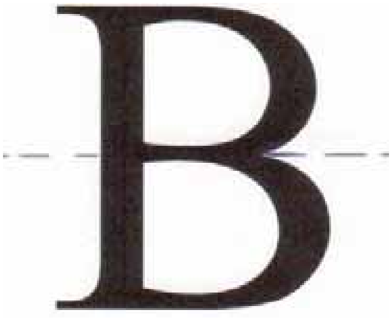
(v) An isosceles triangle, 1 axis of symmetry

(vi) An equilateral triangle, 3 axes of symmetry

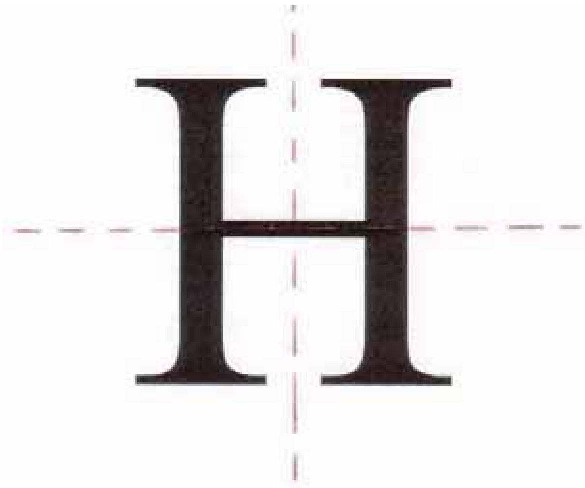
(vii) A circle, infinite number

2.

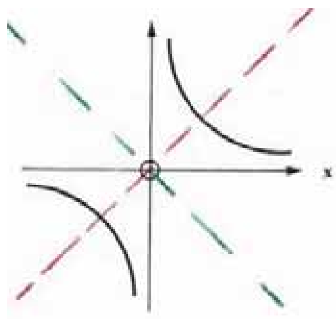
Letter B has 1 line of symmetry



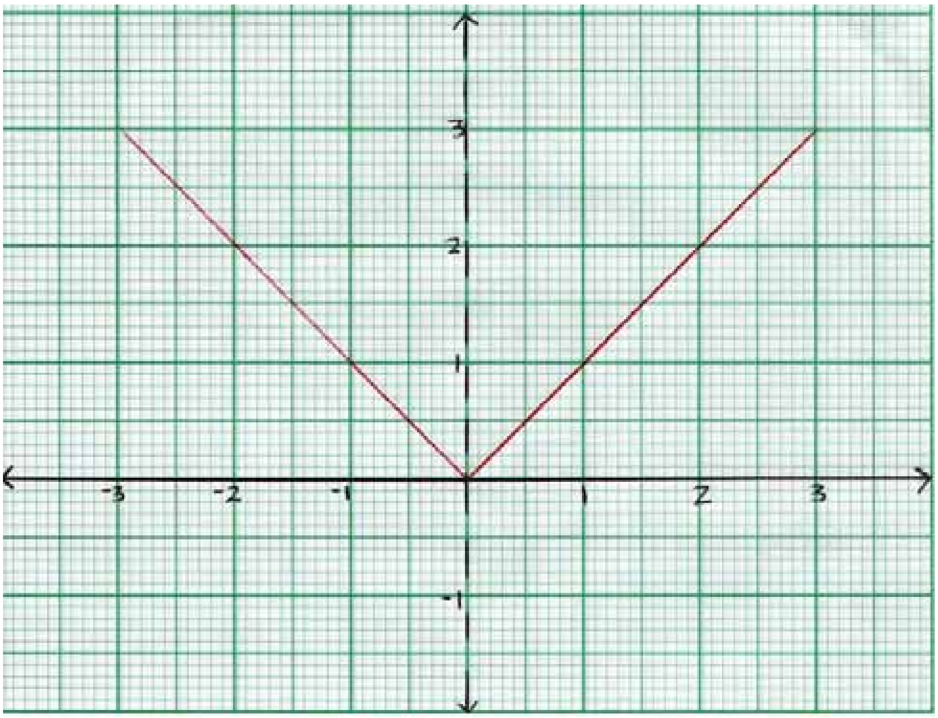
Letter H has 2 lines of symmetry



The curves have two lines of symmetry

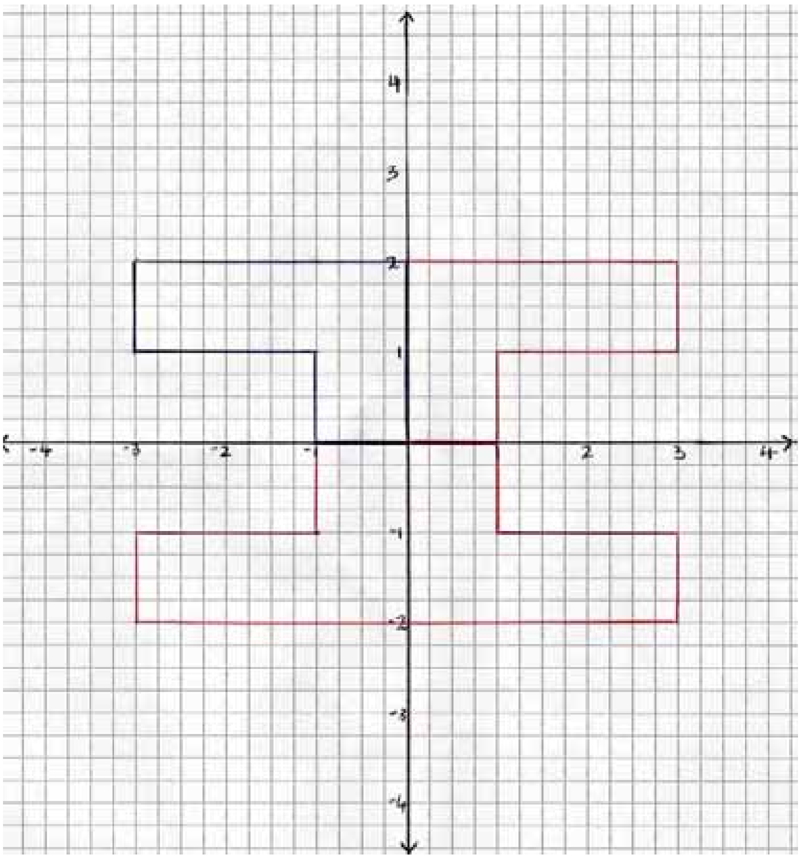


3.



4. This is the curve of y = cos x. It is symmetrical about the y – axis.

5.


4
3
2
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-1
-2
-3
-4
-4
-3
-2
-1
2
3
4

Lesson 2 Rotational Symmetry

By the end of this subunit, you should be able to:

- recognise and describe rotational symmetry.

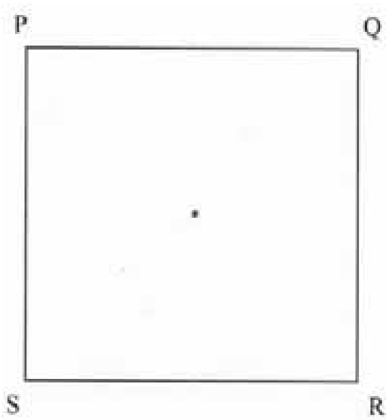
This subunit is about 5 pages in length.

For this sub unit, you will need tracing paper, please make sure that you have it before you start your work.

Some plane figures are not symmetrical about a straight line in their plane. They are such that, if rotated about a point, in a clockwise or in an anticlockwise direction, they fit onto the original pattern.

Every plane figure can be rotated through 360° about any point in its plane onto itself. So every plane figure is said to have rotational symmetry of order 1. The number of such rotations is its order of **rotational symmetry**.

Given square PQRS. Rotate it about the point inside it, in the clockwise direction. What will happen?



It will help to recall all you know about a square.

____

____

____

____

A square:

- is a quadrilateral, that is it has four sides.

- all of these sides are of equal length.

- is symmetrical about each of its diagonals and about the lines joining the mid – points of the opposite sides.

- has four angles, each is a right angle.

Trace square PQRS and rotate it.

When square PQRS is rotated about the point inside it, in the clockwise direction,
P will move to where Q is,
Q will move to where R is,
R will move to where S is and
S to where P is.

This is possible because of the facts given above.

Rotate the square until P, Q, R and S return to their original positions.

How many times did you rotate the square such that the square lands on itself?

____

The square was rotated 4 times.

Since we are talking rotation, we can't help it but talk about an angle of rotation.

What is the angle of rotation in this case?

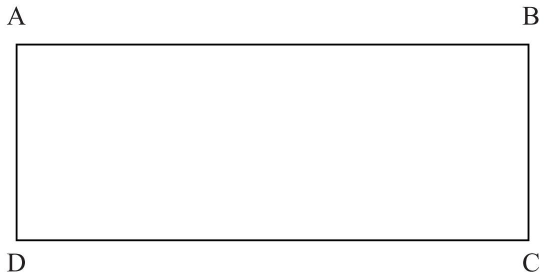
____

The angle of rotation is 90°.

Therefore a square has rotational symmetry of order 4.

Example 1

What is the order of rotational symmetry of rectangle ABCD ?



It will also help if you can trace the rectangle.

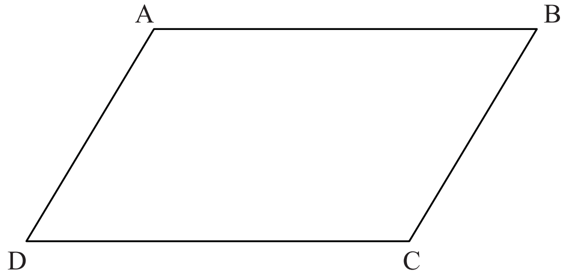
Remember a rectangle:

- has four sides, the opposite sides are parallel and of equal length.
- has four angles, all are right angles.
- is symmetrical about the lines joining the midpoints of the opposite sides.

A rectangle has rotational symmetry of order 2. Two rotations of 180° are required.

Example 2

What is the order of rotational symmetry of parallelogram ABCD?



Again it will help if you can trace the parallelogram.

A parallelogram has rotational symmetry of order 2. Two rotations of 180° are required.

Example 3

What is the order of rotational symmetry of a circle?

A circle has an infinite order. Any rotation will leave the circle as it was.

Note it!

The order of rotational symmetry can be different from the number of axes of reflective symmetry. Any figure has order of rotational symmetry 1, but it may have no axes of symmetry.

For example an isosceles triangle has 1 axis of symmetry and has order 1 of rotational symmetry.

A parallelogram has no axes of symmetry and has order 2 for rotational symmetry.

Discussion

Many works of art, designs and objects around us display symmetry.

Look around you. What are the things that display symmetry?

The people you are discussing with display symmetry!

If you were to draw a midline from the top of one's forehead, down the middle of the nose, over the lips and to the bottom of the chin; the eyes, ears, nostrils, and teeth all mirror each other on either side of the line! That is beauty at its best!

Look at the pictures below.

This is one of the many of Basotho blankets. It displays symmetry.

This leaf also displays symmetry. Well in biology, with some things the symmetry is "approximate."

This butterfly also displays symmetry.

Well in biology, with some things the symmetry is “approximate.”

Symmetry is simply a fundamental organizing principle.

Activity 2

1. **Draw the following shapes. Identify the centre of rotation and find the order of rotational symmetry.**

(i) square

(ii) equilateral triangle

(iii) regular pentagon

(iv) rhombus

(v) regular octagon

[20]

Compare your answers with those at the end of this subunit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content and work through the activity again.

Key Point to Remember

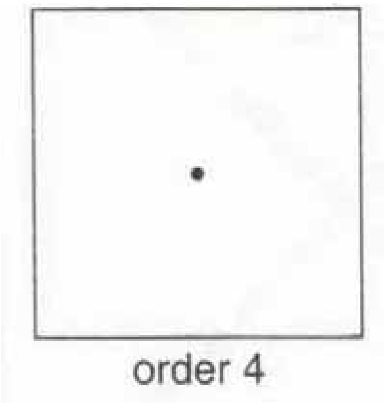
The key point to remember in this subunit on rotational symmetry are:

- when a plane figure is rotated about a point, in a clockwise or in an anticlockwise direction, they can fit onto the original pattern. The number of such rotations is its order of **rotational symmetry**.

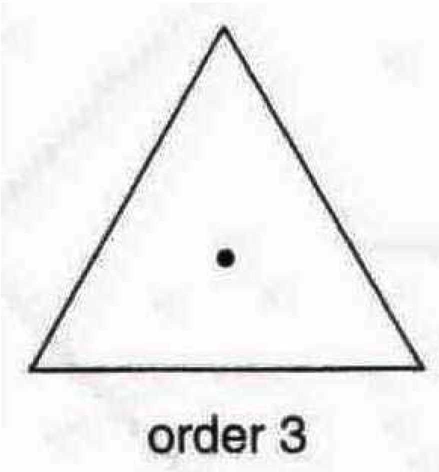
Answers:

1.

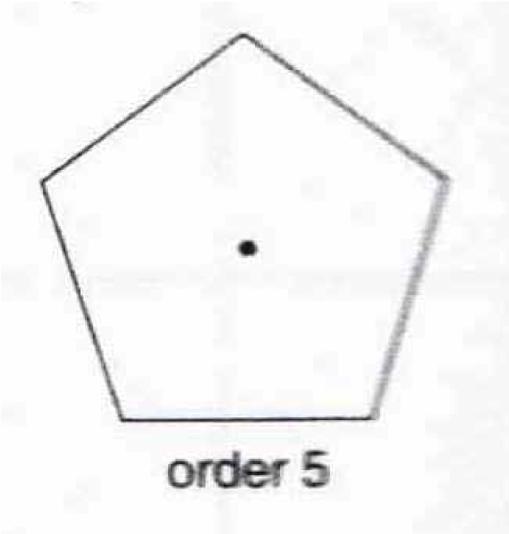
(i)



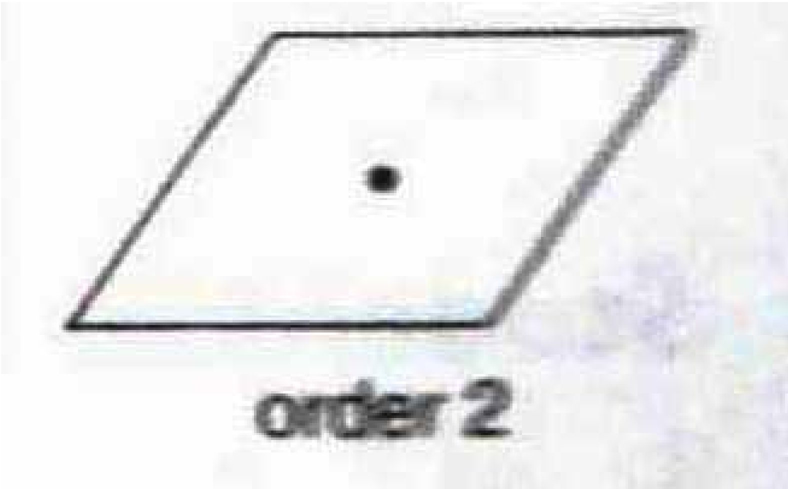
(ii)



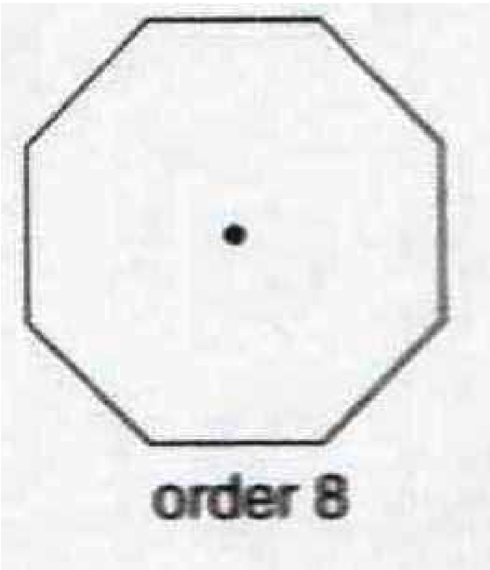
(iii)



(iv)



(v)



Lesson 3 Symmetry Properties of Prisms and Pyramids

Introduction

By the end of this subunit, you should be able to

- *recognise* and *describe* symmetry properties of prisms (including cylinder) and pyramids (including cone)

This subunit is about 4 pages in length.

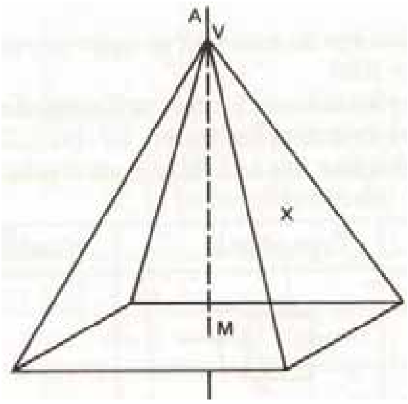
Rotational Symmetry

Symmetry is not only limited to two dimensional shapes; even three dimensional shapes can also exhibit symmetry.

Three dimensional shapes also have rotational symmetry. In three dimensions, a solid shape rotates about an **axis of rotation**.

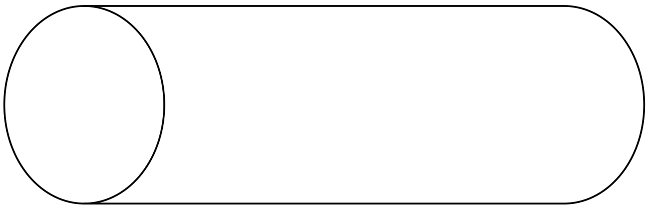
A three dimensional shape has rotational symmetry if when rotated about a central axis, it looks the same at certain intervals.

Take a square based pyramid. Mark the centre of the square base M.
Push a stiff grass stem through V and M.
Hold the stem at A and rotate the pyramid until it is back to its original position.
Mark one face to help you.



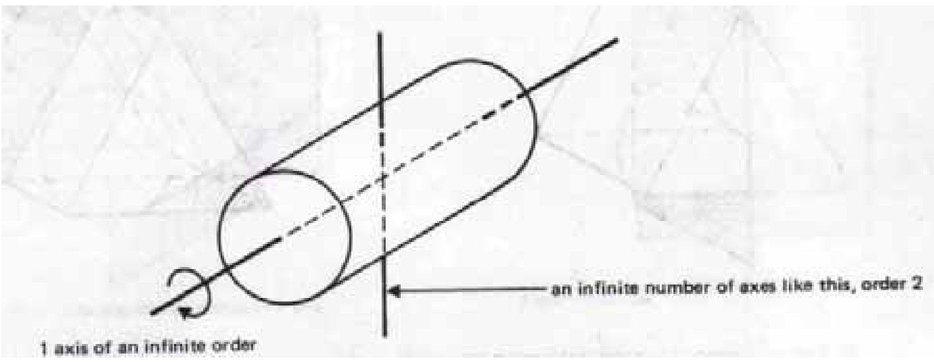
How many positions are there in which the pyramid fits onto the original pattern?

This is a cylinder.

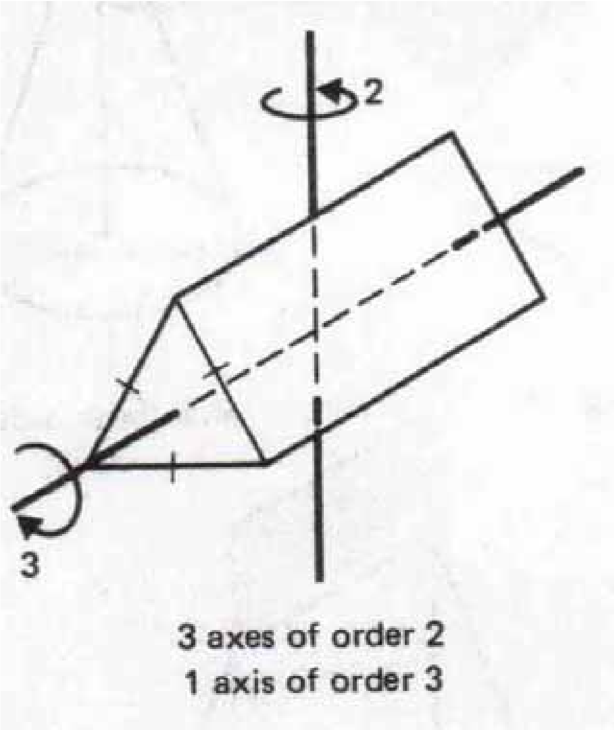


There is 1 axis of infinite order. This axis is a line through the centres of the circular faces.

There is an infinite number of axes like the one shown, they have order 2.



Here is another 3 dimensional shape that has been rotated as shown.



There is 1 axis of order 3. This axis is a line through the centres of the triangular faces.

There are three axes of order two. Each of these axes goes through the midpoint of the longitudinal edge.

It will help if you do these rotations yourself. Mark one face with X to help you.

Planes of Symmetry

Imagine a mirror passing through a three dimensional solid such that it divides it into two congruent halves; one half being the mirror image of the other half.

Lets use the cube below to help us. Imagine a mirror passing through the dotted line. This is a **mirror plane**.

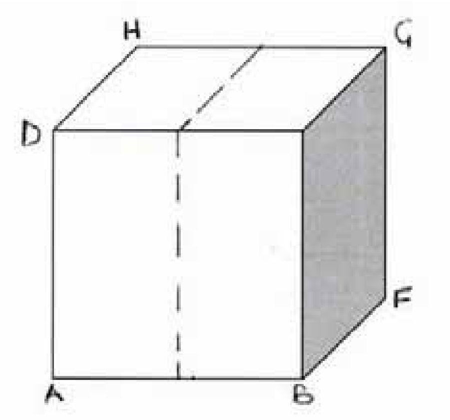
A is an image of B, and vice versa.

D is an image of C, and vice versa.

H is an image of G, and vice versa.

E is an image of F, and vice versa.

This mirror plane has divided the cube into two congruent solid shapes. This mirror plane is also called a **plane of symmetry**.



How many other planes of symmetry does the cube have?

____

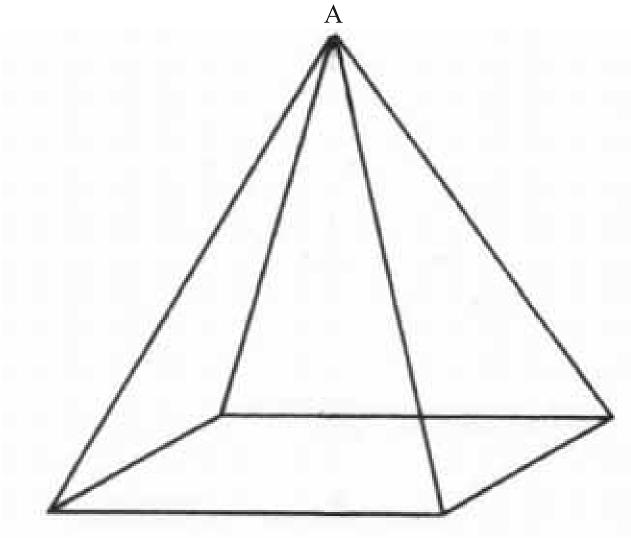
____

____

A cube has 9 planes of symmetry.
3 lie parallel to the sides and go through the centre
6 go through opposite edges and diagonals

Example 1

Given a square – based pyramid, how many planes of symmetry does it have?



It has some that pass through the vertex and the midpoints of opposite edges of the base.
It has some that pass through the vertex and opposite corners of the base. In total they are 16.

Example 2

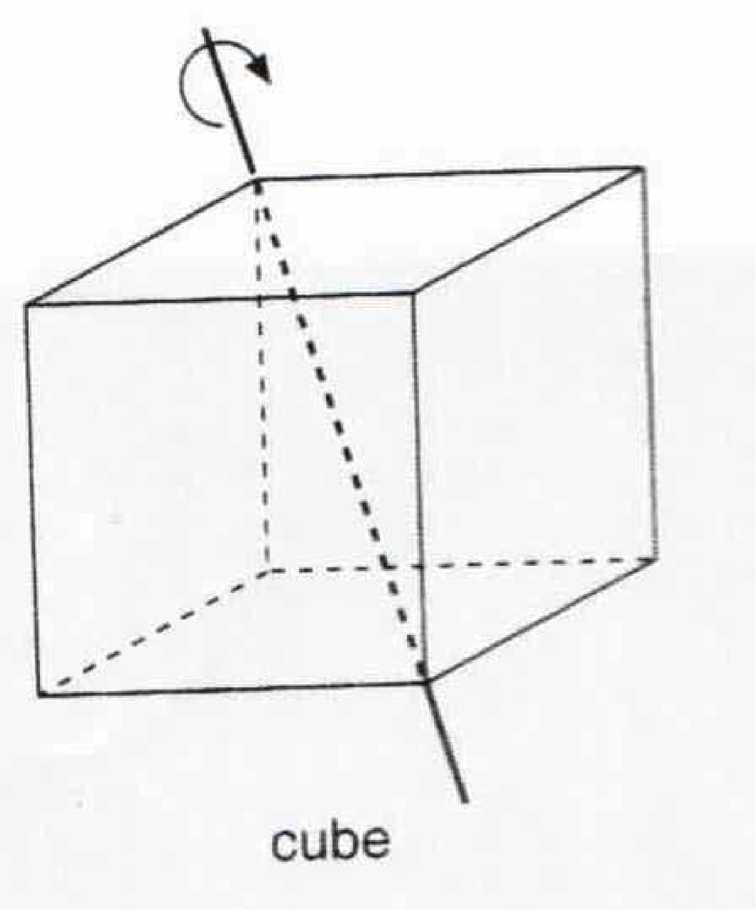
How many planes of symmetry does a cylinder have?

A cylinder can have an uncountable number of planes of symmetry. This is because a circle can have as many lines as symmetry as we care to draw.

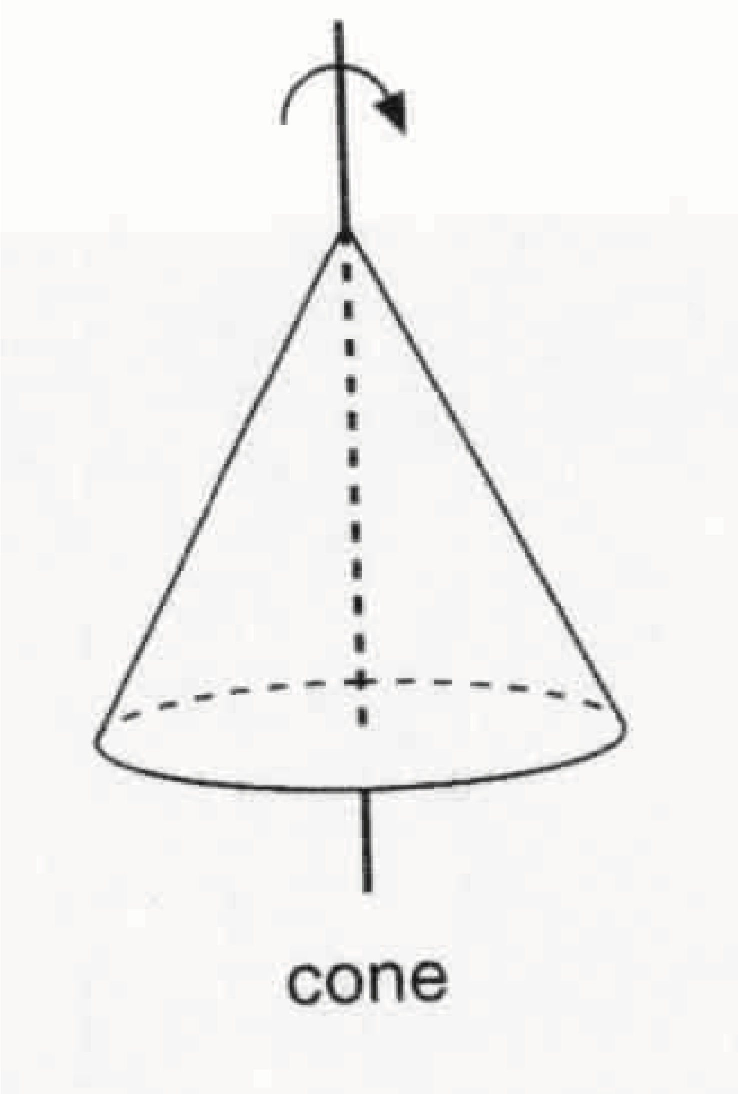
Activity 4

1. Determine the order of rotational symmetry about the axis shown

(i)



(ii)



[2]

2. Complete the table below.

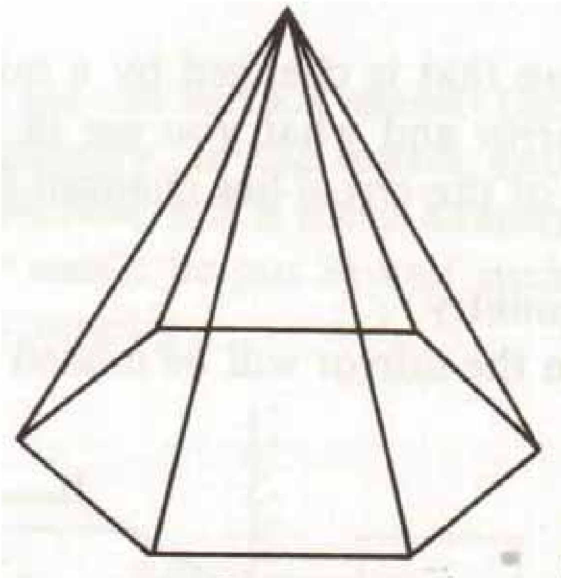
shape	**Number of planes of symmetry**
cuboid	
pyramid	
sphere	

3. How many planes of symmetry does:

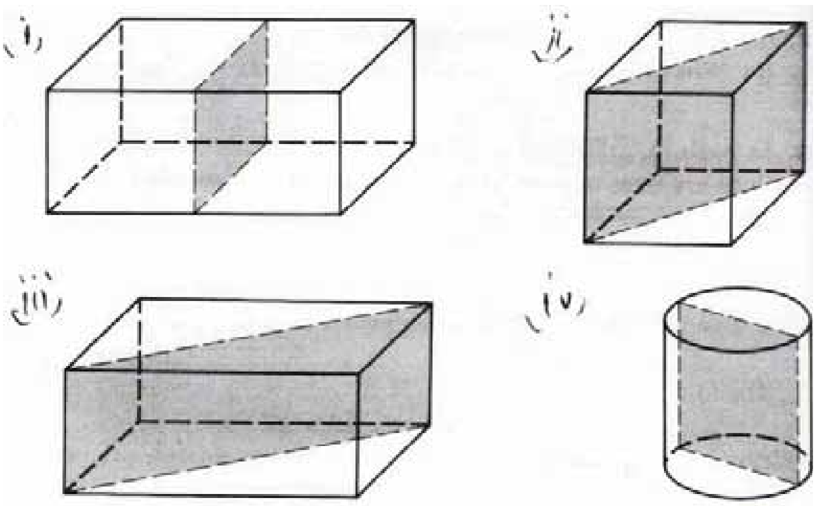
   (a) an unsharpened pencil, with circular ends have?

---

4. How many planes of symmetry of each type does this object have?



5. In each question, write whether or not the shaded part is a plane of symmetry for the solid shown



Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review the above content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on symmetry properties of prisms (including cylinder) and pyramids (including cone) are:

- a three dimensional shape has rotational symmetry if when rotated about a central axis, it looks the same at certain intervals.

- a three dimensional solid has a plane of symmetry if the plane divides the solid into two halves, one half being the mirror image of the other half.

You have now completed work on this unit on symmetry properties of prisms (including cylinder) and pyramids (including cone). Do a quick review of the entire content of this unit and then continue on to the unit summary.

Answers

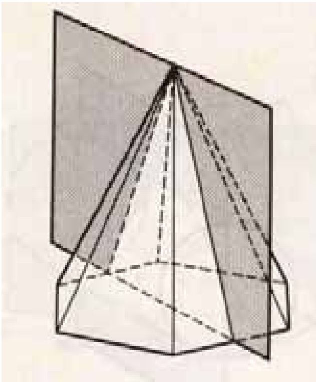
1. (i) cube - 4
   (ii) cone - infinite

2. Complete the table below.

shape	**Number of planes of symmetry**
cuboid	3
pyramid	4
sphere	infinite

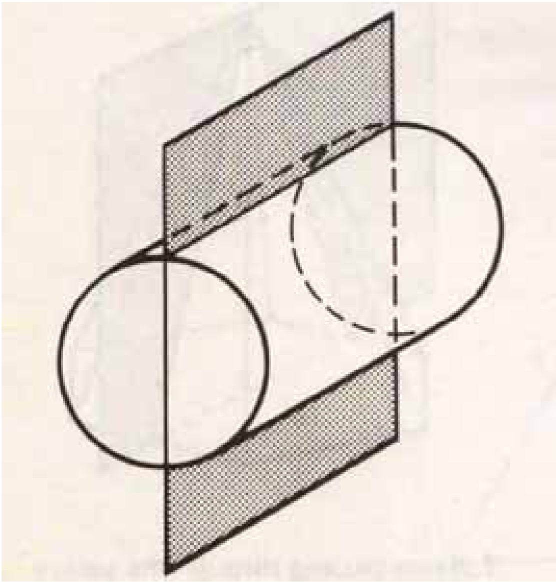
3. (a) an unsharpened pencil, with circular ends has an infinite number of planes like this

4.

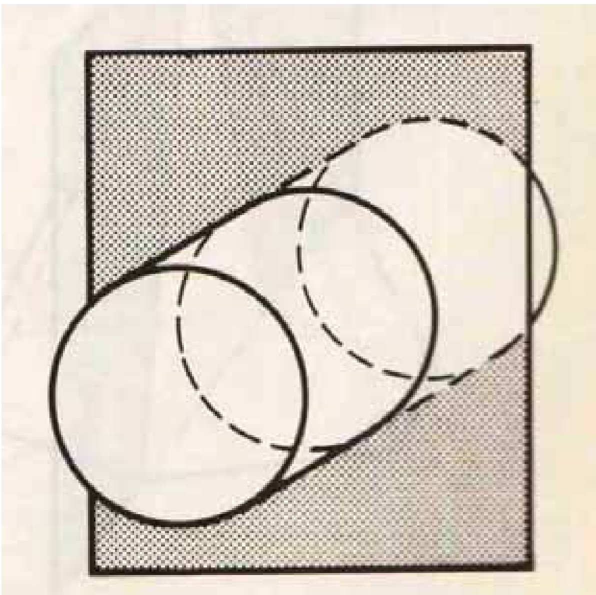


3 planes passing through the vertex and the midpoints of opposite edges of the regular hexagonal base

5. (i) yes
(ii)Yes
(iii) yes
(iv) yes



And 1 plane like this



6.

Unit Summary

Summary

In this unit you learned that:

- lines of symmetry divide two dimensional (plane) figures into identical parts.
- rotational symmetry turns or rotates a shape through some angle, and still look the same as it did originally.
- any figure has order of rotational symmetry 1.
- order of rotational symmetry can be different from the number of axes of reflective symmetry.
- a three dimensional shape has rotational symmetry if when rotated about a central axis, it looks the same at certain intervals.
- a three dimensional solid has a plane of symmetry if the plane divides the solid into two halves, one half being the mirror image of the other half.

You have completed the material for this unit on algebraic manipulation. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

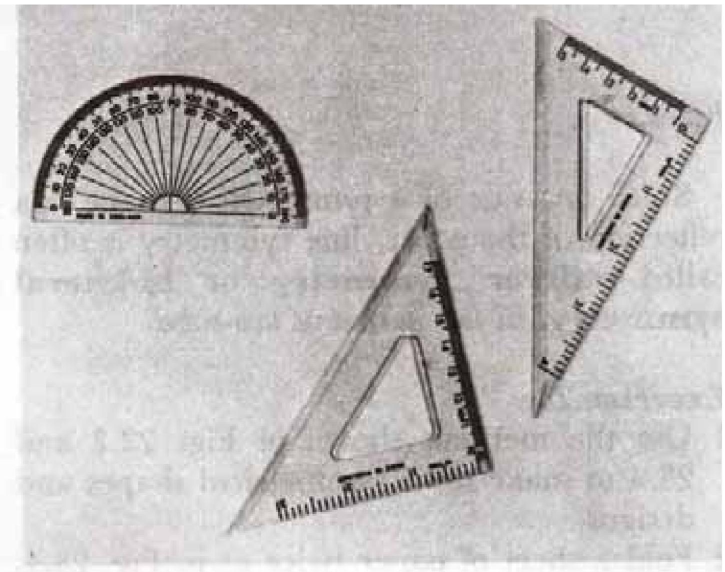
1. **Answer All Questions.**
2. **Show all the necessary working.**

Total marks = 30 **Time**: 40 mins

1. Write down those letters of the word SYMMETRY which have a vertical axis of symmetry [7]

2. A triangle has one angle 68° and the other 56°. Is the triangle symmetrical? [3]

3. Which of the mathematical instruments have bilateral symmetry? Ignore any writing on them.



[3]

4. Complete this shape so that the bold lines become lines of symmetry?

For the completed drawing:

(i) identify the centre of rotation

(ii) state the order of rotational symmetry

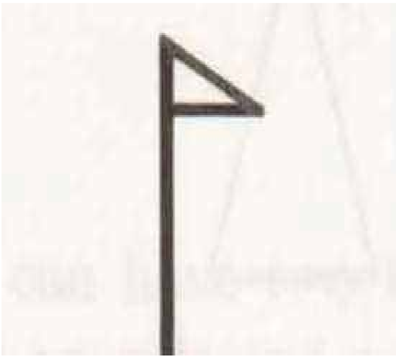
[5]

5. Complete this table

Shape	Axes of symmetry	Order of rotational symmetry
square		
equilateral triangle		
rectangle		
regular pentagon		
regular decagon		
circle		

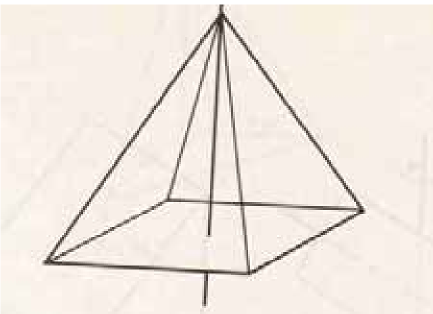
[12]

6. The flag is part of a figure which has order of rotational symmetry 3 about the point O. Complete the figure.



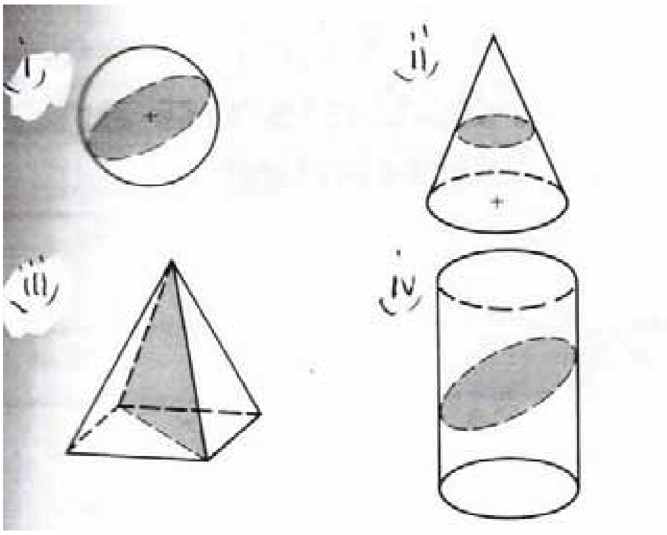
How many axes of symmetry does the completed figure have? [4]

7. Determine the order of rotational symmetry about the axis shown.



[2]

8. In each question, write whether or not the shaded part is a plane of symmetry for the solid shown.



Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

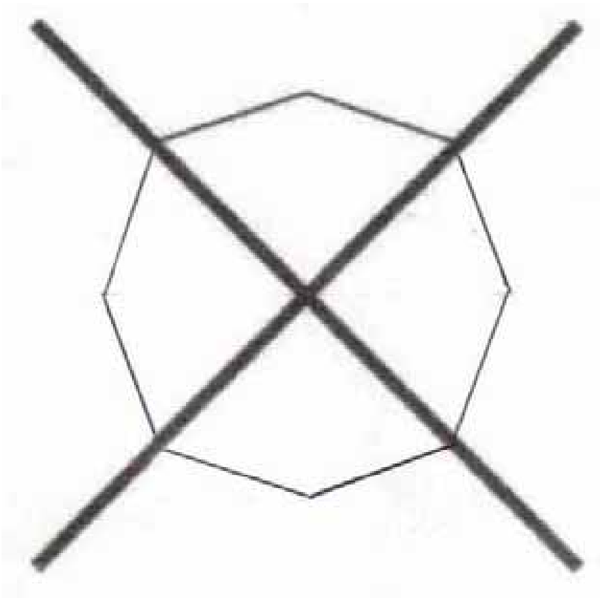
1. Y,M and T have vertical axes of symmetry
2. 68° + 56° = 124°

   180° - 124°° = 56°

   Two angles are equal, this is an isosceles triangle, it is symmetrical

3. 45° set square (isosceles triangle) and compass (half circle)

4.



(i) the centre of rotation is the point where the bold lines, the axes of symmetry meet

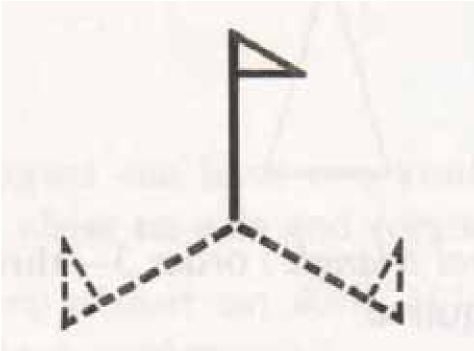
(ii) Rotational symmetry of order 4

5.

Shape	Axes of symmetry	Order of rotational symmetry
square	4	Order 4
equilateral triangle	3	Order 3
rhombus	2	Order 4
regular pentagon	5	5
regular decagon	10	10
circle	Infinite number	Infinite order

6.

Order 3 means 3 rotations , each is 120°



The completed figure has no axes of symmetry

7.

The order of rotational symmetry of the square – based pyramid about the given axis is order 4

8.
(i) yes
(ii) no
(iii) yes
(iv) yes

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

1. **Answer All Questions.**
2. **Show all the necessary working.**

Total marks = 38 **Time**: 45 mins

1. An equilateral triangle has 3 lines of symmetry. Describe fully another kind of symmetry this triangle has. [2]

____

____

2. Draw the lines of symmetry of the letters of the alphabet given below

M

E

T

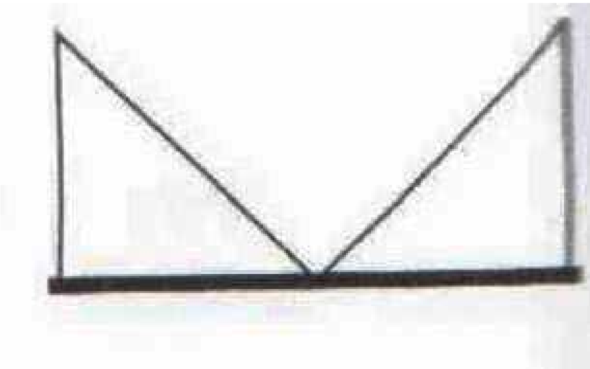
[6]

3. Complete this table:

Name of polygon	Axes of symmetry	Order of rotational symmetry
square		
isosceles triangle		
scalene triangle		
kite		
rhombus		
regular octagon		

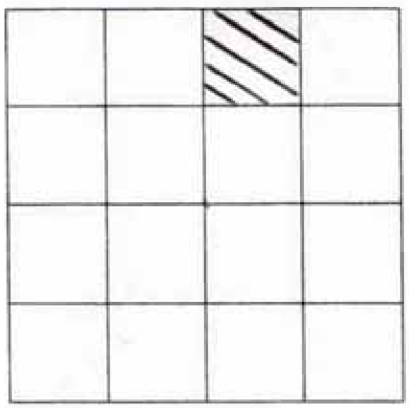
[12]

4. Complete the shape so that the bold line becomes line of symmetry.



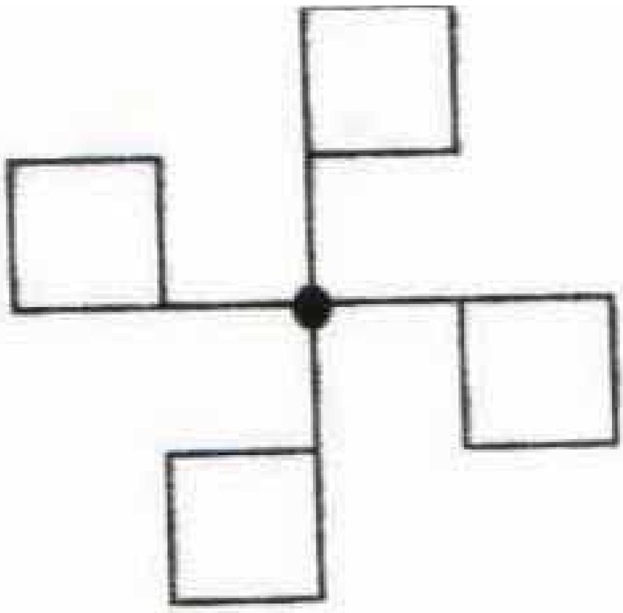
For the completed drawing, state the order of rotational symmetry. [4]

4. Shade three more squares so that the completed square grid has rotational symmetry of order 4.



[4]

5. What is the order of rotational symmetry of the diagram below?



[4]

6. How many planes of symmetry does each of the given figures below have?

   (i) A pyramid with a regular hexagon for its base

   (ii) A cuboid

   (iii) A cone

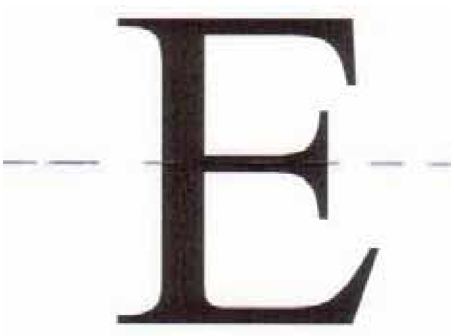
[6]

Answers

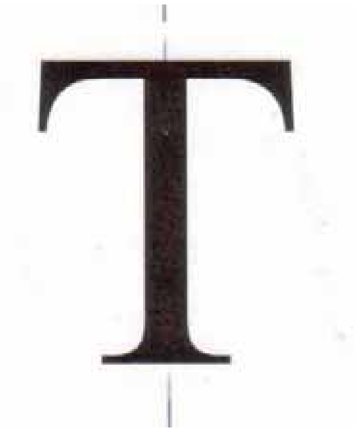
1. It has order of rotational symmetry 3

2. Letter M has 1 line of symmetry

Letter E has 1 line of symmetry



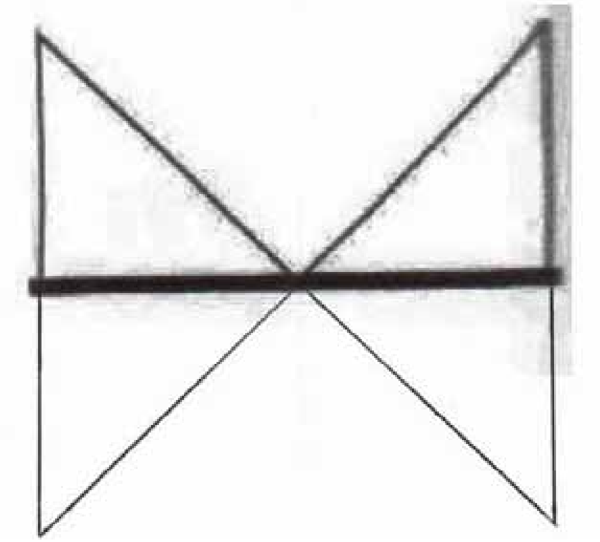
Letter T has 1 line of symmetry



3.

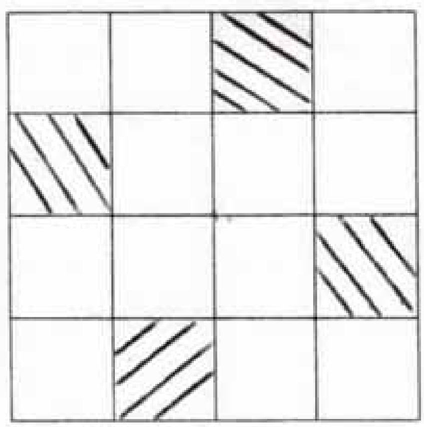
Name of polygon	Axes of symmetry	Order of rotational symmetry
square	4	4
isosceles triangle	1	1
scalene triangle	none	1
kite	1	1
rhombus	2	2
regular octagon	8	8

4.



Rotational symmetry of order 2

5.



Rotational symmetry of order 4

6.

(i) A pyramid with a regular hexagon for its base - 6

(ii) A cuboid - 3

(iii) A cone - Infinite

Unit Contents

Unit 27

Unit 27

Geometrical Terms

Introduction

This unit consists of 68 pages. This unit is approximately 3% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 20 hours to work on this unit, 15 hours for formal study and 5 hours for self-study and completing assessments/assignments.

This is one of the many units on geometry. Geometry is all about shapes, plane and solid shapes, and their properties. Previously we learned about the symmetry of three-dimensional shapes. Here we will go into more detail and describe these terms.

This Unit is Comprised of Four Lessons:

Lesson 1 Geometrical Terms
Lesson 2 Calculating Angles and/or Length of line segments including Angle Properties of Special Triangles and Qudrilaterals Using Properties of Angles
Lesson 3 Polygons
Lesson 4 Similar Polygons

When reading the following learning outcomes, think about them as a guide to what you should focus on while studying this unit.

Outcomes

- *interpret* geometrical terms
- *calculate* specified angles and/or length of line segments including angle properties of special triangles and quadrilaterals using properties of angles
- *calculate* specified angles and/or sides using properties of polygons
- *recognise* that the areas of similar figures are in proportion to the square of the corresponding sides with corresponding results for similar figures and extensions to volumes and surface areas of similar solids using the relationships between areas of similar triangles

Terminology

Congruent: Having identical shapes so that all parts correspond.

Plane figure: A two dimensional shape.

Solid figure: A three dimensional shape.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Geometrical Terms

By the end of this subunit, you should be able to:

- Interpret geometrical terms.

This subunit is about 4 pages in length.

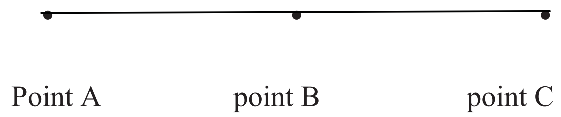
It is not the first time that you will be meeting some of these terms. It is very important that we look at them again at length, for we are going to use them a lot in this unit.

1. A **point** is the simplest figure studied in geometry:
   - It has position.
   - It is located at a definite place.
   - It is usually represented by a dot.
   - It is labelled with a capital letter.

Example

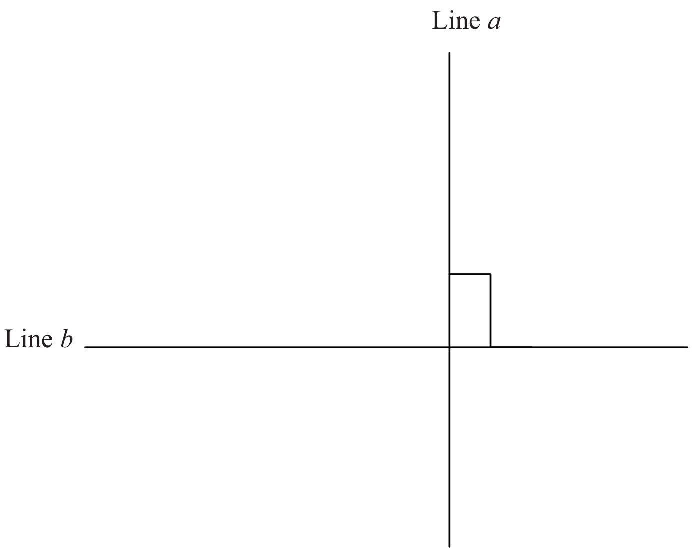
We have three points below, point A, B and C

2. Points joined together give a **line**.



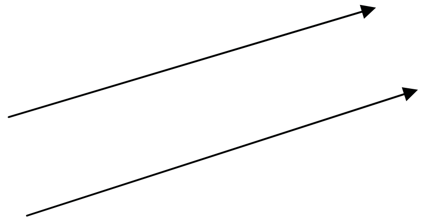
- It will always be understood to be a straight line, unless it is specifically mentined otherwise.
- It is infinite.
- It has a certain direction.
- We name a line after two of the points which lie on it, especially if they are end points. The line given above is line AC.
- Two lines are different when they occupy different positions on the plane.
- Lines can meet or intersect, that is they have one point in commom. Lines that meet at a right angle are **perpendicular lines.**

Example



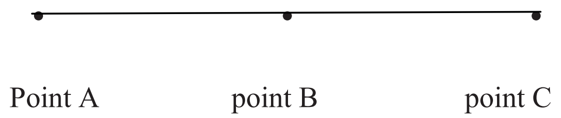
Lines may not intersect. Lines that do not meet are **parallel lines.**

Example

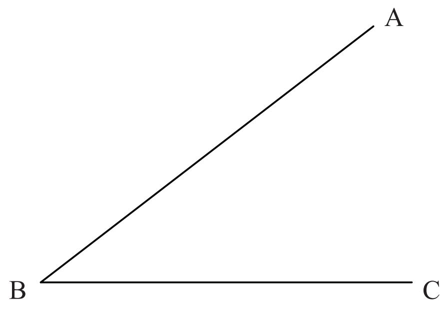


The arrows are used to show that the lines are parallel. There can be two or more parallel lines.

- A line may have segments. They are pieces of lines of given lengths and with two definite ends. Line AC has line segments AB and AC



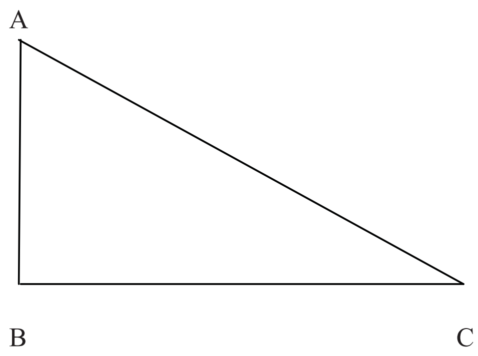
3. A **plane** is a perfectly flat surface.
4. An **angle** is the open – ended shape formed by two line segments joined at a common point, called the vertex. The angle below is formed by line segment AB and BC. They have a common point B.



This is angle ABC or angle CBA, written as ∠ ABC or angle CBA. This can also be written as ∠ B, naming only the vertex.

We will look at the different angles in detail in one of the next sub units.

5. Point at which two line segments meet is a **vertex**.
6. A triangle is a shape formed from three straight lines.



- It is named using the consecutive vertices in order. This is therefore triangle ABC, which can be written as ΔABC. It may also be named BAC, CAB , and so on.
- Every triangle has three angles and three sides.
- There are different types of triangles. We will look at them in detail in one of the next sub units.
- The sum of the three interior angles = 180 °.
- A degree (°) is a common unit used for giving the size of an angle.

7. A quadrilateral is a shape formed from four straight lines
   i. Every quadrilateral has four angles and four sides.
   ii. There are different types of quadrilaterals. We will look at them in detail in one of the next sub units.
   iii. The sum of the four interior angles = 360 °.

8. A polygon is a shape formed from three or more straight lines. It is a closed figure formed by connecting segments (sides of the polygon) at their endpoints (vertices of the polygon), all in a plane.
   i. Many of the polygons have special names which correspond to the number of lines forming their boundary.

ii. Every polygon has the same number of angles as it has sides.

iii. Polygons can be equilateral (all sides equal) equiangular (all angles equal), neither, or both.

iv. Polygons can have all interior angles equal and all sides equal. These are **regular** polygons.

v. Polygons can have all interior angles not equal and all sides not equal. These are **irregular** polygons.

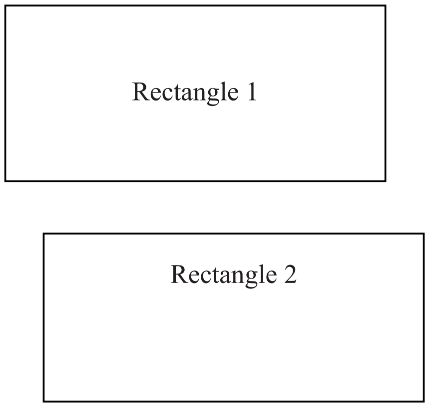
vi. Interior angles of a polygon = (n – 2) x 180° where n is the number of sides.

The table below gives some of the polygons.

Number of sides	Number of angles	Names	Sum of interior angles
3	3	triangle	180°
4	4	quadrilateral	360°
5	5	pentagon	540°
6	6	hexagon	720°
7	7	heptagon	900°
8	8	octagon	1080°
9	9	nonagon	1260°
10	10	decagon	1440°

9. **Congruent** polygons are identical in every aspect, except the positions they occupy. This says one shape can be placed on the other so that all their corresponding parts coincide.
   They have the same number of sides, and all corresponding sides are the same length and the interior angles are the same size.

Example



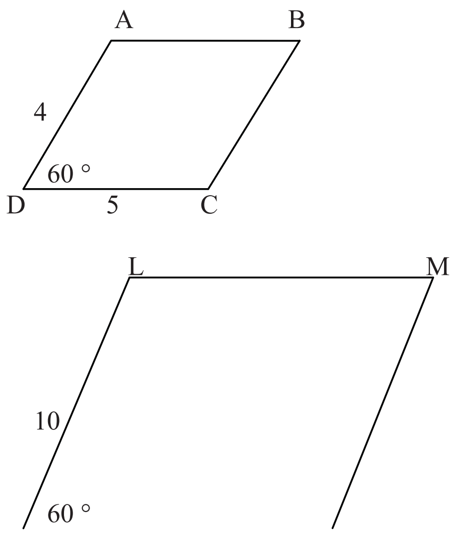
Rectangle 1 and rectangle 2 are congruent.

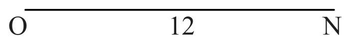
- They have the same number of sides.
- All corresponding sides are the same length .
- The interior angles are the same size.

Using tracing paper, check that when rectangle 1 is placed on rectangle 2, all their corresponding parts coincide.

10. **Similar** polygons have the same shape, but different sizes. The corresponding angles must be congruent, and the corresponding sides are proportional. Enlargements produce similar figures.

Example





Parallelogram ABCD and parallelogram LMNO

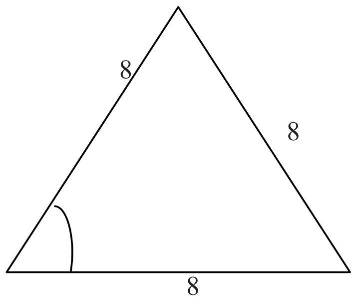
- have the same shape, but different sizes.
- have corresponding angles D and O that are the same size = 60 °
- the ratios of the lengths of the corresponding sides are.

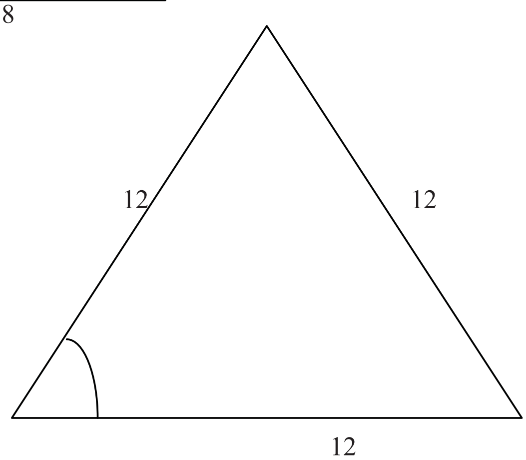
AD_ : LO
4 : 10
2: 5

CD_ : NO
5 : 12

They are not constant. Therefore Parallelogram ABCD and Parallelogram LMNO are not similar.

The first triangle has been enlarged to give the second triangle.





The triangles above:

- have the same shape, but different sizes.
- have corresponding angles that are the same size.
- the ratios of the lengths of the corresponding sides are constant.

They are:

8 : 12 or $\frac{12}{8} = 1.5$
2: 3

8 : 12 or $\frac{12}{8} = 1.5$
2: 3

8 : 12 or $\frac{12}{8} = 1.5$
2: 3

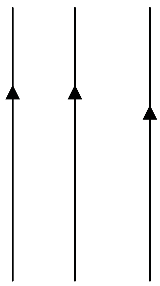
1.5 is the scale factor of enlargement

Therefore the triangles are similar.

Activity 1

1. Which of the sets of lines given below are perpendicular, which are parallel

(a)



(b)

90°

____

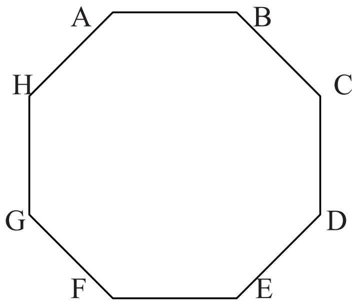
2. Which of the following are the right ways of naming the polygon below

octagon ABCDEFGH

octagon GHABCDEF

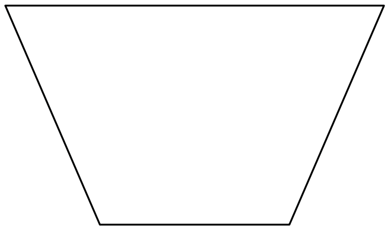
octagon CDBAHGFE

octagon EFGHDCBA

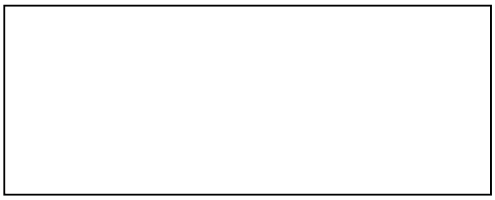


3. Which of the following polygons are regular polygons?

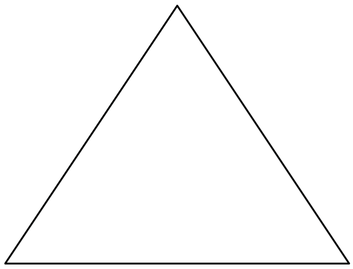
(a)



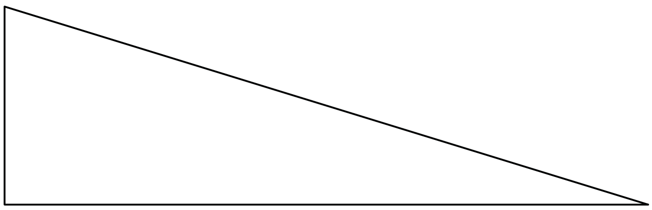
(b)



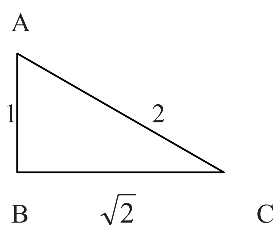
(c)

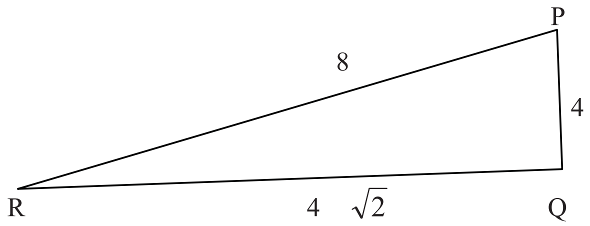


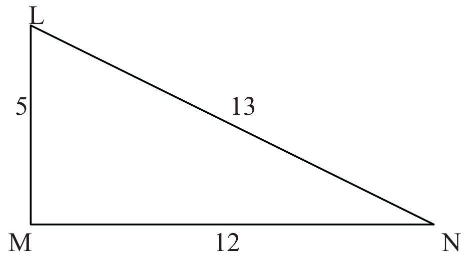
(d)



4. (a) Which of the triangles are similar to triangle ABC?







Compare your answers with those at the end of the subunit. Review the related content for any question that you missed.

Answers to activity 1

1. (a) they are parallel

   (b) they are perpendicular

2. octagon ABCDEFGH

   octagon GHABCDEF

3. (c) is the only polygon regular, all sides are equal

4. Triangle ABC and triangle PQR

The ratio of corresponding sides

$$\frac{AB}{PQ} = \frac{1}{4} = \frac{1}{4}$$

$$\frac{BC}{QR} = \frac{\sqrt{2}}{4\sqrt{2}} = \frac{1}{4}$$

$$\frac{AC}{PR} = \frac{2}{8} = \frac{1}{4}$$

They are constant. Therefore Triangle ABC and triangle PQR are similar.

Triangle ABC and triangle LMN.

The ratio of corresponding sides.

$$\frac{AB}{LM} = \frac{1}{5} =$$

$$\frac{BC}{MN} = \frac{\sqrt{2}}{12}$$

$$\frac{AC}{LN} = \frac{2}{13}$$

They are not constant. Therefore Triangle ABC and triangle LMN are not similar.

Key Points to Remember

The key points to remember in this subunit on geometrical terms are:

- Points joined together form a line
- Parallel lines never meet.
- Lines can meet. Those that meet at a right angle are perpendicular lines
- A plane is a flat surface
- When lines meet at a vertex, an angle is formed

- There are regular polygons and irregular polygons
- Congruent polygons have the same number of sides, and all corresponding sides are the same length and the interior angles are the same size
- Similar polygons have the same shape, but different sizes.

Lesson 2 Calculating Angles and/or Length of line segments including Angle Properties of Special Triangles and Qudrilaterals Using Properties of Angles

By the end of this subunit, you should be able to

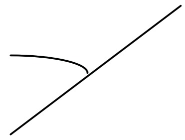
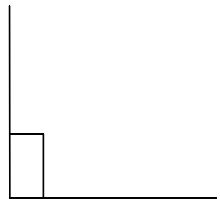
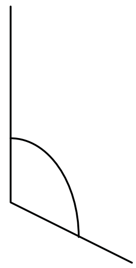
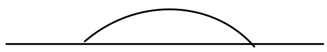
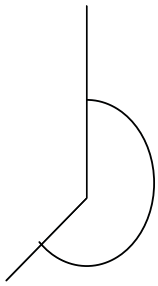
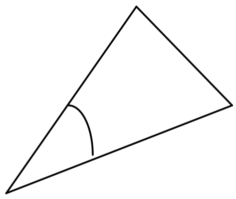
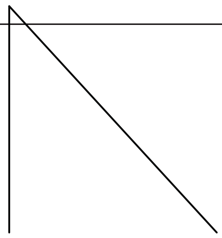
- calculate specified angles and/or length of line segments including angle properties of special triangles and quadrilaterals using properties of angles

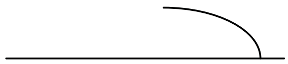
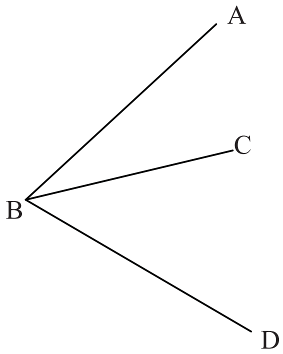
This subunit is about 4 pages in length.

An **angle** is the open – ended shape formed by two line segments joined at a common point, called the vertex.

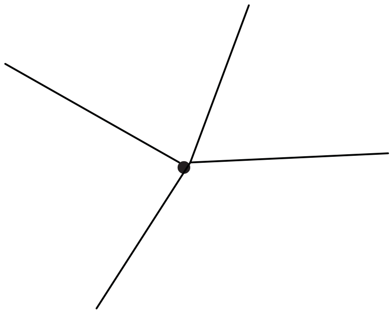
The table below gives different angles.

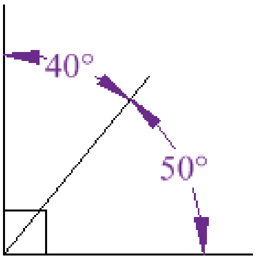
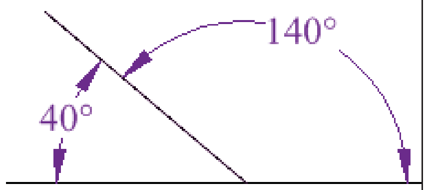
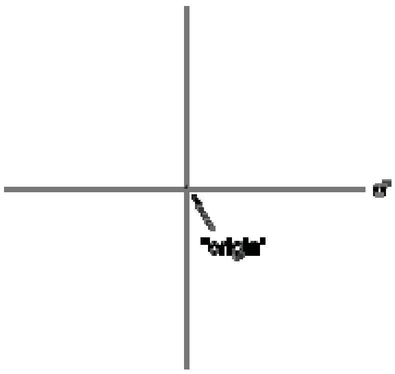
Name of Angle	Definition	Example
acute	An angle whose	

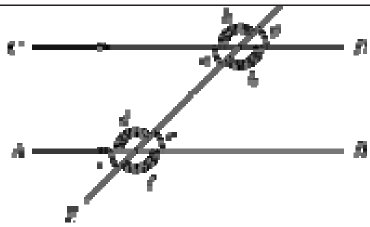
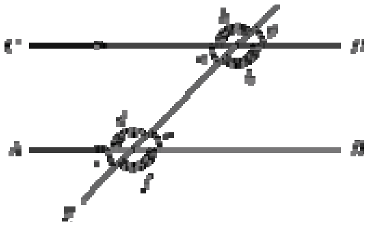
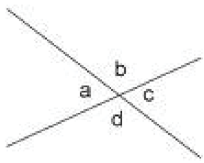
	size is less than 90°	
right angle	An angle whose size is 90°	
obtuse	An angle whose size is more than 90° but less than 180°	
straight angle	An angle whose size is 180°	
reflex	An angle whose size is more than 180° but less than 360°	
Interior angle	An angle inside a shape	
Exterior angle	An angle formed	

	externally between adjacent sides, when one of the sides is extended	
Adjacent angles	Adjacent angles have a common side and a common vertex. Angle ABC and angle CBD are adjacent angles	

Properties of Angles

Property	Example
Angles at a point add up to 360°	

When two angles add up to 90°, they are called complementary angles.	
When two angles add up to 180°, they are called supplementary angles.	
Perpendicular lines meet at a right angle	
When two parallel lines are intersected by a third line, the **corresponding**	

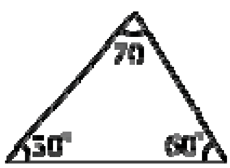
angles are equal. Corresponding angles occupy similar positions at the two intersections.	 - parallel lines are lines CD and lines AB. They are intersected by line E. Angle ***a*** and angle ***e*** are corresponding angles, they are equal. Angle ***b*** and angle ***f*** are corresponding angles, they are equal, etc
When two parallel lines are intersected by a third line, the **alternate** angles are equal.	 - parallel lines are lines CD and lines AB. They are intersected by line E. Angle ***a*** and angle ***c*** are alternate angles, they are equal. Angle ***b*** and angle ***d*** are alternate angles, they are equal, etc
When two straight lines intersect, vertically opposite angles are equal	 - angle ***a*** and angle ***c*** are vertically opposite angles, they are equal - angle ***b*** and angle ***d*** are vertically opposite angles, they are equal

Interior angles of a polygon = (n – 2) x 180° where n is the number of sides	A triangle has 3 sides. Interior angles of a triangle = (3 – 2) x 180° = (1) x 180° = 180° A square has 3 sides. Interior angles of a square = (4 – 2) x 180° = (2) x 180° = 360°

Properties of Triangles

1. The sum of the interior angles in a triangle is always180°

   Example

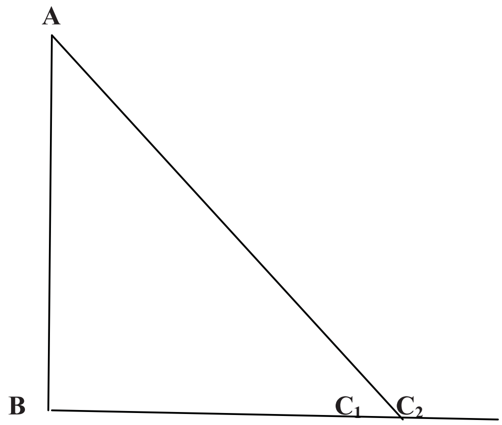


50° + 60°+ 70°= 180°

2. Sum of an interior angle and its adjacent exterior angle add up to 180°

Example

In the triangle given below,



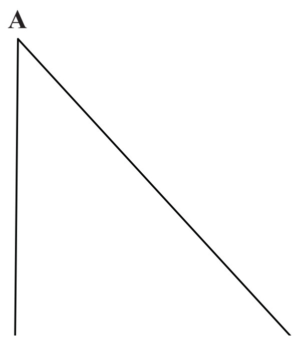
Angle $\mathbf{C_1}$ is an interior angle, angle $\mathbf{C_2}$ is an exterior angle, their sum is 180°.

This is further justified by the fact that angle $\mathbf{C_1}$ and angle $\mathbf{C_2}$ are angles in a straight line. Angles in a straight line add up to 180°.

3. An exterior angle of a triangle is equal to the sum of the two opposite interior angles.

Example

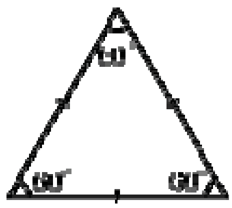
In the triangle given below,



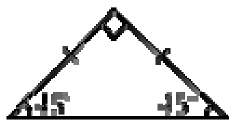
Angle A + Angle B = Angle C_2

4. An equilateral triangle has three equal angles, each is 60° and three equal sides; an isosceles triangle has two equal angles and two equal sides; a scalene triangle has all three angles not the same size and all three sides not the same size.

Examples



equilateral triangle

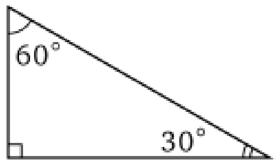


isosceles triangle

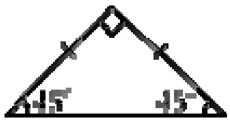
scalene triangle

5. A right angled triangle can be scalene or isosceles.

Examples



A right angled triangle that is scalene.



A right angled triangle that is isosceles.

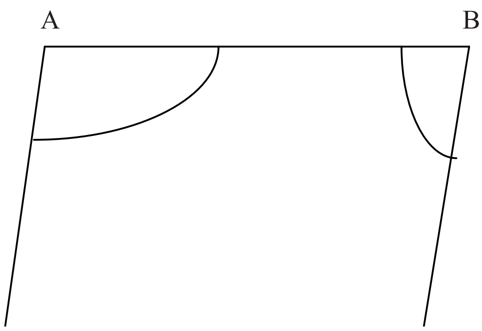
6. In a right angled triangle, the right angle is opposite the longest side, the hypotenuse.

Properties of Quadrilaterals

Like triangles, quadrilaterals come in all shapes and sizes!

1. The sum of the interior angles in a quadrilateral is always 360°

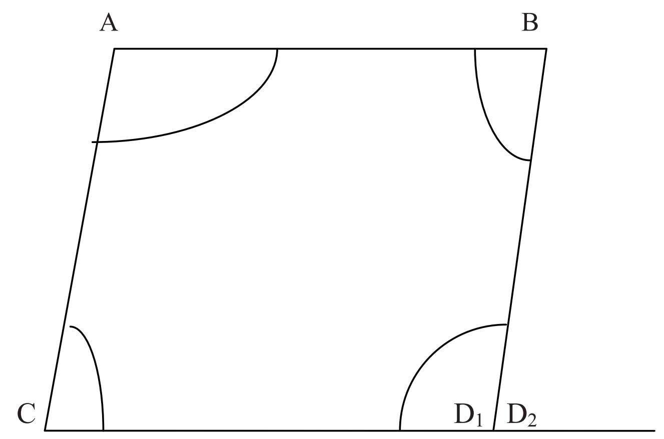
Example



Angle A + Angle B + Angle C + Angle D = 360°

2. At each vertex, the sum of an interior angle and its adjacent exterior angle add up to 180°.

Example



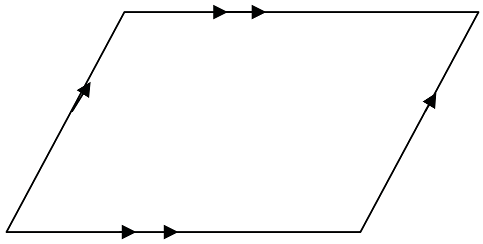
Angle $\mathbf{D_1}$ is an interior angle, angle $\mathbf{D_2}$ is an exterior angle, their sum is 180°.

This is further justified by the fact that angle $\mathbf{D_1}$ and angle $\mathbf{D_2}$ are angles in a straight line. Angles in a straight line add up to 180°.

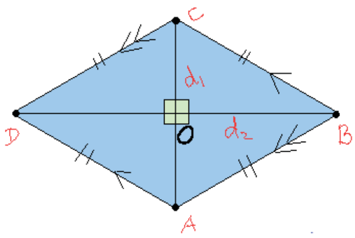
3. Quadrilaterals are classified into parallelograms if the opposite sides of the quadrilateral are parallel and equal.

- **Parallelogram**
  - A parallelogram is a quadrilateral having two pairs of parallel sides.
  - Opposite angles of a parallelogram are equal in size.

- A square, a rhombus, and a rectangle are all examples of parallelograms.



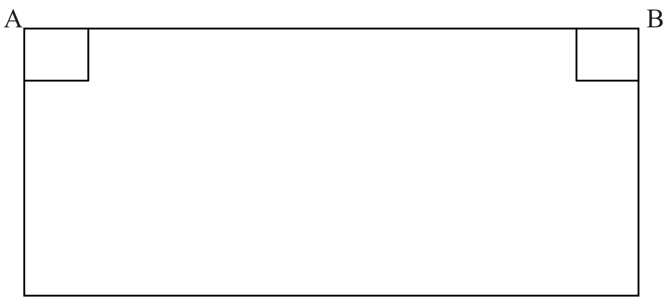
- **Rhombus**
  - A rhombus is a quadrilateral of which all fours sides are the same length.



- **Rectangle**
  - a parallelogram of which all four angles are 90° .

  Example

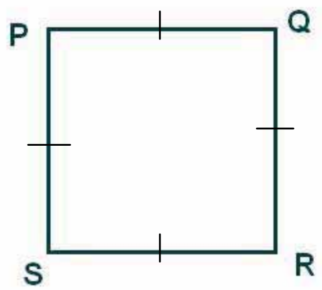
  This is rectangle ABCD



- **Square**
  - a parallelogram with all four sides the same length, and all four angles are 90°.
  - A square is a rectangle, a rhombus, and a parallelogram.

Example

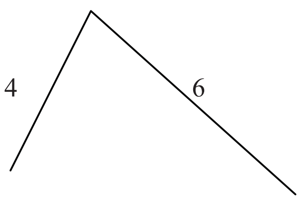
This is square PQRS



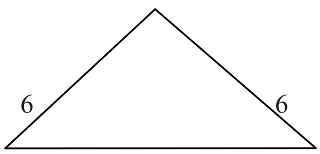
Example 1

Classify each triangle by its sides

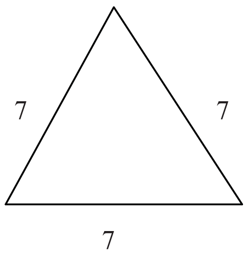
(i)



(ii)



(iii)



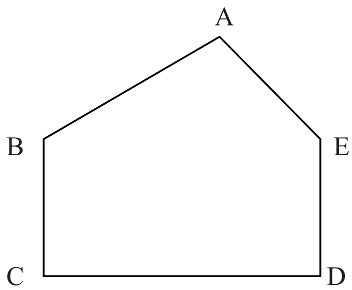
(i) This is a scalene triangle as it has all three sides not the same size.

(ii) This is an isosceles triangle as it has two sides equal.

(iii) This is an equilateral triangle as it has all three sides the same size.

Example 2

Which of the following are acceptable names for this polygon?



(a)ABCDE

(b) CBAED

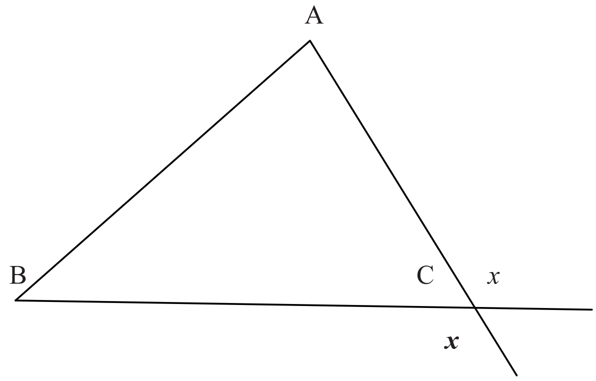
(c) DECBA

(a) and (b) are acceptable names

Example 3

Angle C = 60° and angle B = 50°. Calculate

a. Angle A
b. Angle x



(a) Sum of angles in a triangle = 180°

Angle A + Angle B + Angle C = 180°

Angle A + 50° + 60° = 180°

Angle A + 110° = 180°

Angle A = 70°

(b) Angle C and angle x (either of the angles) are angles in a straight line, i.e they add up to 180°

Angle C + Angle x = 180°

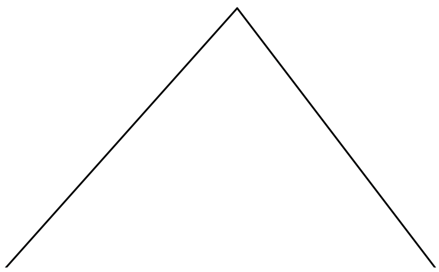
60° + Angle x = 180°

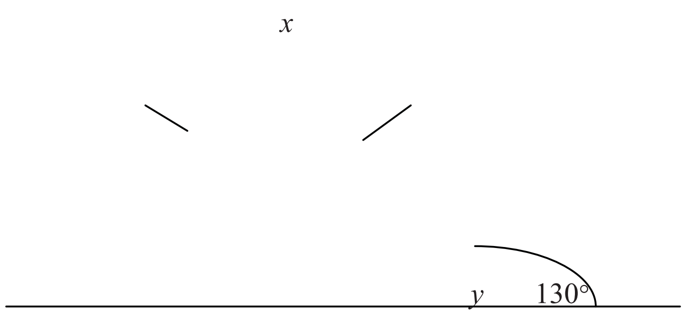
Angle x = 120°

NB: Either of the angles marked x are equal. They are vertically opposite.

Example 4

Find the value of *x* and *y*





130° and angle y are angles in a straight line, i.e they add up to 180°.

$$130° + \text{angle } y = 180°$$

$$\text{angle } y = 50°$$

This is an isosceles triangle.

Therefore angle y = the third interior angle not labelled

angle y = the third interior angle not labelled = 50°

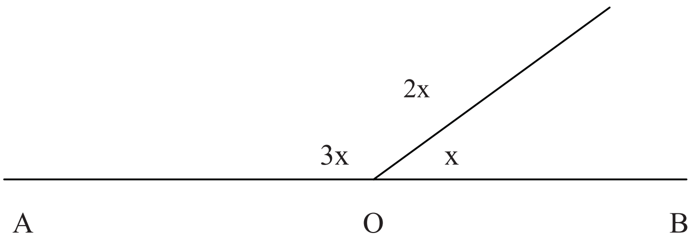
angle y + the third interior angle not labelled + angle x = 180°

50° + 50° + angle x = 180°

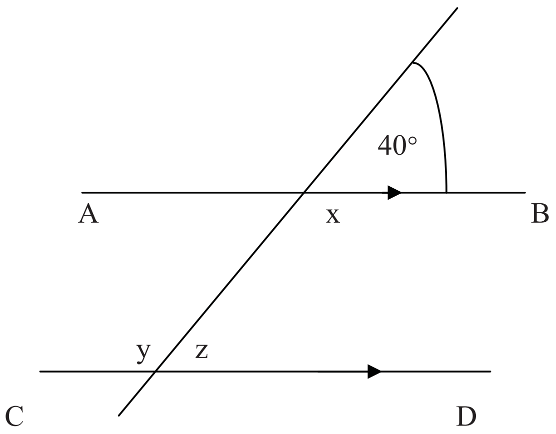
angle x = 80°

1. In the figure below, AOB is a straight line. Find x.

Activity 2

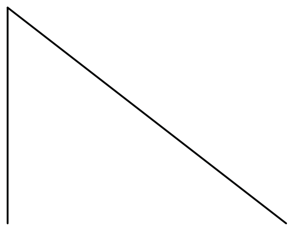


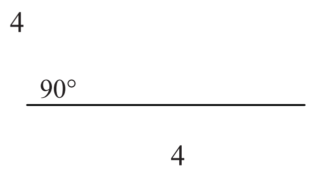
2. Given that AB is parallel to CD, find x, y and z.



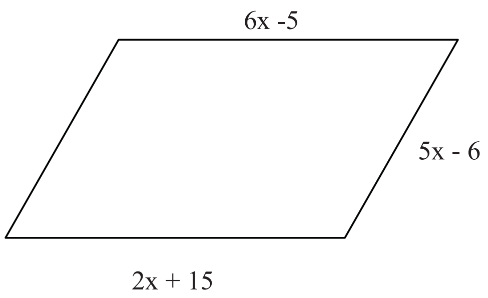
3. Classify each of the following by sides and/or angles.

(a)





4. Find the length of the side of the parallelogram marked 5x – 6.



Compare your answers with those at the end of the subunit. Review the related content for any question that you missed.

Answers to activity 2

1. These are angles in a straight line

$3x + 2x + x = 180°$

$6x = 180°$

$x = 30°$

2. - z corresponds with 40°,therefore z = 40°

   z + y = 180° (angles in a straight line)

- 40° + y = 180° (angles in a straight line)

  y = 140°

- 40° + x = 180° (angles in a straight line)

  x = 140°

  or

x alternates with y = 140°,therefore x =140°

3.

(a) A right – angled triangle that is isosceles

(b) A kite. It has two pairs of congruent adjacent sides.

4. Opposite sides of a parallelogram are equal

$6x - 5 = 2x + 15$

Solving for x

$x = 5$

Key Points to Remember

The key points to remember in this subunit on angle properties of special triangles and quadrilaterals using properties of angles are:

- there are different angles with different sizes
- triangles come in different shapes and sizes
- quadrilaterals also come in different shapes and sizes

Lesson 3 Polygons

By the end of this subunit, you should be able to

Calculate specified angles and/or sides using properties of polygons

This subunit is about 4 pages in length.

Reflection

A polygon is any plane shape bounded by straight line segments.

Below are some of the familiar polygons.

Number of sides	Number of angles	Names
3	3	triangle
4	4	quadrilateral
5	5	pentagon
6	6	hexagon
7	7	heptagon
8	8	octagon
9	9	nonagon
10	10	decagon

In this subunit, we are going to look at **regular** polygons.

A regular polygon is a polygon that has **all angles equal** and **all sides equal.**
An **irregular polygon** is therefore everything that a regular polygon is not! It has sides of different lengths and angles of different sizes.

An equilateral triangle is therefore a regular polygon, so is the square.

Facts about regular polygons

We have already established that interior angles of a polygon, regular or irregular, of n sides = $(n - 2) \times 180°$ where n is the number of sides.

1. In a regular polygon, each angle = $\frac{(n - 2) \times 180°}{n}$

Example

The sum of angles of a hexagon, regular or not, = $(n - 2) \times 180°$

$$= (6 - 2) \times 180°$$

$$= 4 \times 180°$$

$$= 720°$$

In a regular hexagon, all the angles are the same size. Each angle is therefore = $\frac{(n - 2) \times 180°}{n}$

$$= \frac{(6 - 2) \times 180°}{6}$$

$$= \frac{4 \times 180°}{6}$$

$$= \frac{720°}{6}$$

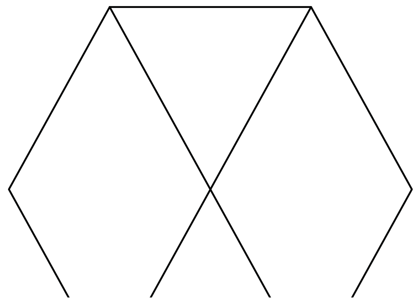
$$= 120°$$

Each of the regular polygons.

2. Each of the n angles at the centre of a regular polygon is $\frac{360°}{n}$

Example

All the vertices have been joined to the centre of the hexagon with straight lines.

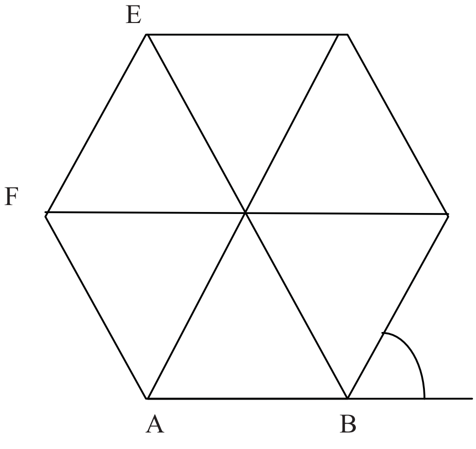


- The size of the angle at the centre is 360°
- This is made up of 6 equal angles.
- Each of these angles = $\frac{360°}{6}$ = 60°

3. The exterior angle of a regular polygon is of the same size as each of the angles at its centre.

Example

All the vertices have been joined to the centre of the hexagon with straight lines.



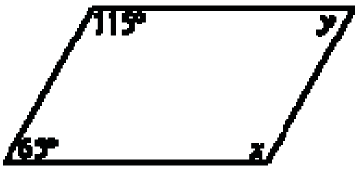
- The size of the angle at the centre is 360°
- This is made up of 6 equal angles.
- Each of these angles = $\frac{360°}{6}$ = 60°

Therefore the exterior angle = 60°

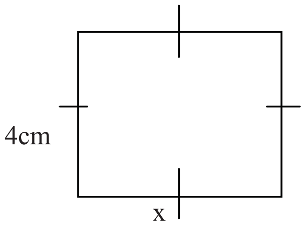
Activity 3

1. Find the missing terms in the following. Give reasons for your answers.

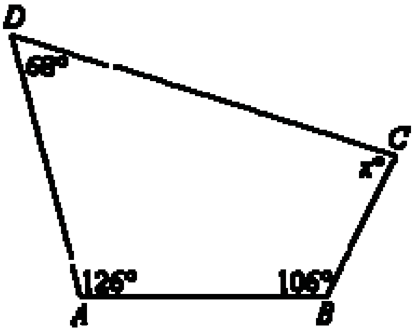
(a)



(b)



(c)



2. Complete this table

Regular polygon	No of sides	Size of each interior angle	Size of each angle at the centre	Size of each exterior angle
triangle	3	120°		
pentagon	5		72°	
octagon		135°		45°
decagon				36°
20 sided	20	162°	18°	

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers to activity 3

1. (a) x = 115° (opposite angles of a parallelogram are equal in size)
y =65°(opposite angles of a parallelogram are equal in size)

(b) x = 4cm (all sides of a square are equal)

(c) 65° + 126° + 106 + x° = 360°

297° + x° = 360°

x° = 63°

2.

Regular polygon	No of sides	Size of each interior angle	Size of each angle at the centre	Size of each exterior angle
triangle	3	120°	**120°**	**120°**
pentagon	5	**108°**	72°	**72°**
octagon	**8**	135°	**45°**	45°
decagon	**10**	**144°**	**36°**	36°
20 sided	20	162°	18°	**18°**

Key Points to Remember

The key points to remember in this subunit on are:

1. In a regular polygon, each angle = $\frac{(n - 2) \times 180°}{n}$

2. Each of the *n* angles at the centre of a regular polygon is $\frac{360°}{n}$

3. The exterior angle of a regular polygon is of the same size as each of

the angles at its centre.

Lesson 4 Similar Polygons

By the end of this subunit, you should be able to

- recognise that the areas of similar figures are in proportion to the square of the corresponding sides with corresponding results for similar figures and extensions to volumes and surface areas of similar solids using the relationships between areas of similar triangles.

This subunit is about 4 pages in length.

You have already done work on area, surface area and volume of different figures. In this unit, we are going to look at area, surface area and volume in relation to similar figures.

As was the case in the previous sub units, we are not expected to prove any of the facts given to us in this subunit.

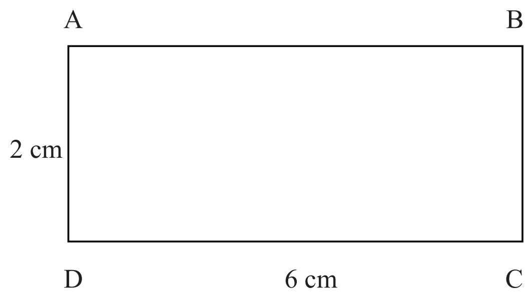
Area of Similar Figures

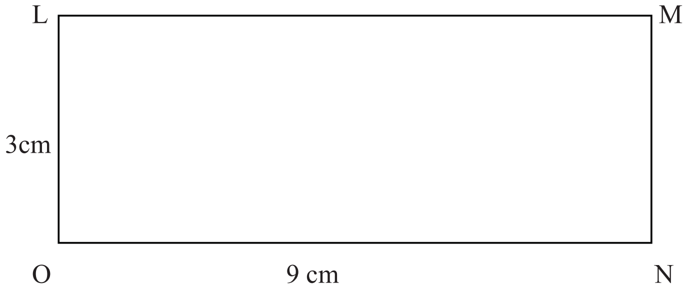
Area is the amount of flat surface a two-dimensional figure takes up. Area is always measured in square units.

Surface Area is the amount of flat surface the faces of a three-dimensional figure take up. Surface area is always measured in square units.

If the ratio of the corresponding sides of two similar figures is a:b, the ratio of their areas is $a^2:b^2$.

Example 1





Solution:

Rectangle ABCD and rectangle LMNO are similar because they

- have the same shape, but different sizes.
- have corresponding angles that are the same size. Each of the angles = 90°
- the ratios of the lengths of the corresponding sides are constant.

Starting with what we already know:

The area of rectangle ABCD = 2 x 6

= 12 cm^2

The area of rectangle LMNO = 3 x 9

= 27 cm^2

Confirming the answers with the fact that if the ratio of the corresponding sides of two similar figures is a:b, the ratio of their areas is $a^2:b^2$.

The corresponding sides are AB : LM

6 : 9

2 : 3

1 : 1.5

The first rectangle has been enlarged to give the second rectangle using scale factor is 1.5
The ratio of their areas is

$2^2:3^2$

4: 9

The area of rectangle ABCD = 12 cm^2

The area of rectangle LMNO = 27cm^2

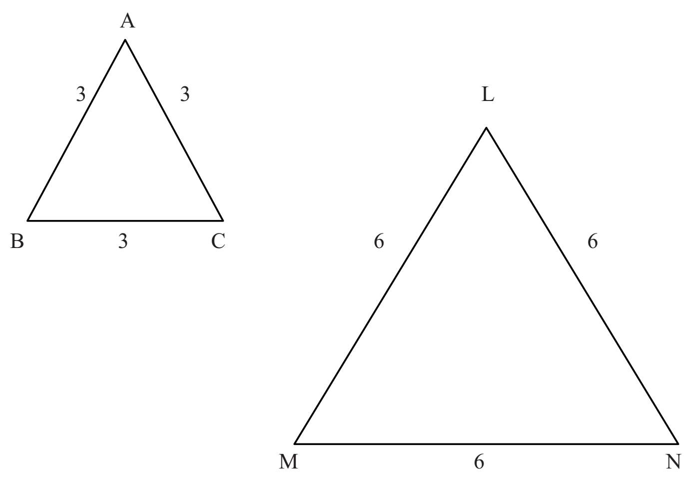
The ratio of their areas is

12 cm^2 : 27 cm^2

12 : 27

4 : 9

Example 2



Solution

- The corresponding sides are AB : LM

3 : 6

1 : 2

The ratio of their areas is $1^2:2^2$

1 : 4

- The corresponding sides are BC : MN

3 : 6

1 : 2

The ratio of their areas is $1^2:2^2$

1 : 4

- The corresponding sides are AC : LN

3 : 6

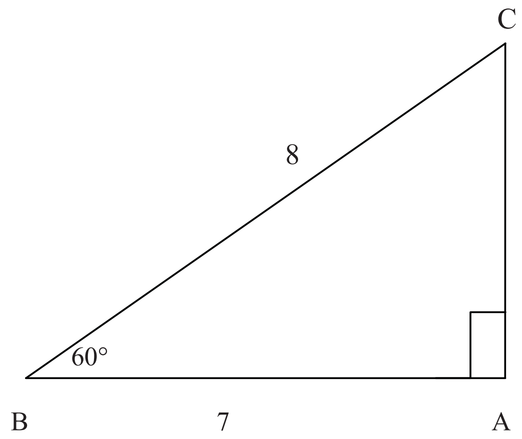
1 : 2

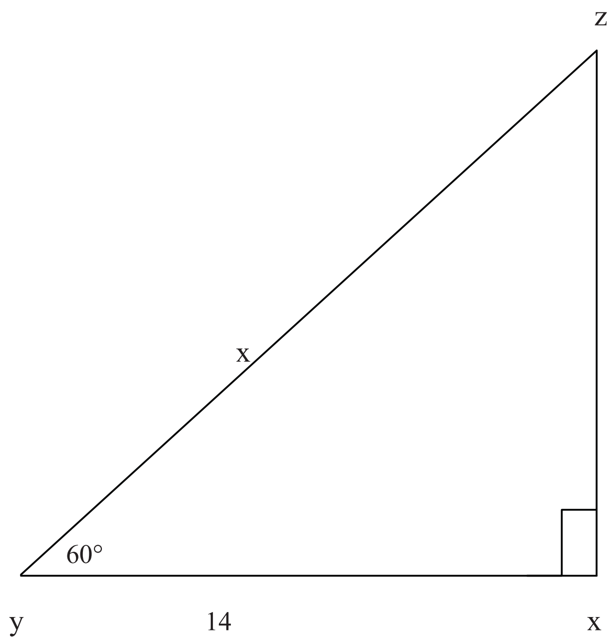
The ratio of their areas is $1^2:2^2$

1 : 4

Example 3

Find the value of x in the following pair of triangles.





Solution:

Triangle ABC and triangle XYZ are similar as they are equiangular.

Therefore 7 : 14

1: 2

The scale factor is 2

These can be written as

$$\frac{7}{14} = \frac{1}{2}$$

8: x should simplify to 1: 2

These can be written as

$$\frac{8}{x} = \frac{1}{2}$$

Solving for x

$8 \times 2 = 1\ (x)$

$x = 16$

Volume of Similar Figures

Volume is the amount of space that a three-dimensional figure takes up. Volume is always measured in cubic units.

If the ratio of the corresponding sides of two similar figures is a:b, the ratio of their volumes is a^3:b^3.

Example 1

Two glasses are similar. The smaller glass has volume of 250cm^3 and has a height of 10cm. The height of the larger glass is 12cm.

What is the volume of the larger glass?

Solution

The ratio of their heights is 10 cm : 12 cm

10: 12

5 : 6

The ratio of their areas is $a^3 : b^3$.

$5^3 : 6^3$

125 : 216

The volume of the smaller glass is 250 cm^3

Therefore

125 : 216

250 : x

Calculating x

250 x 216 = (125) x

$54\,000 = 125x$

$432 = x$

The volume of the larger glass is 432 cm^3

Example 2

A solid figure has a height of 5 cm and a volume of 120 cm^3. A similar solid has a volume of 3240 cm^3. What is its height?

Solution

Given that they are similar, the ratio of their volumes is

$120\ cm^3 : 3240\ cm^3$

$120 : 3240$

$1 : 27$

$1 : (3)^3$

Therefore the scale factor is 3

The height of the second solid = 3 x 5

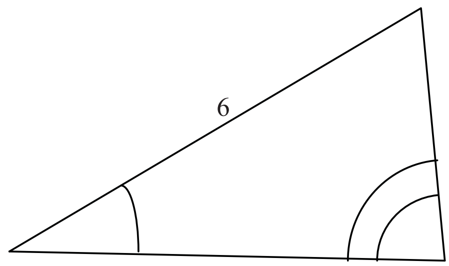
= 15 cm

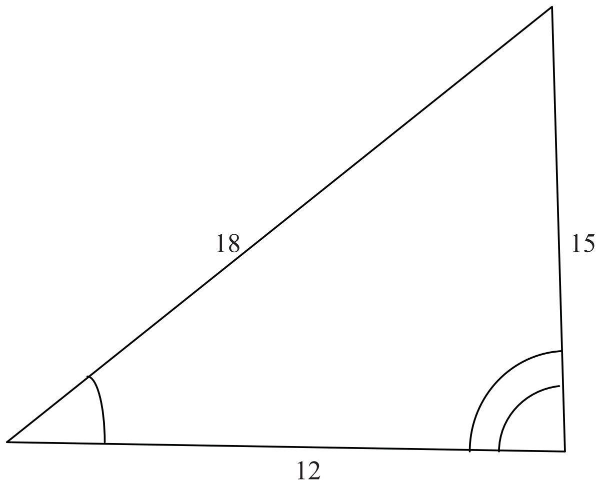
Activity 4

1. The sides of one triangle are 6cm, 7 cm and 9 cm respectively, and those of another triangle 14 cm, 18cm and 15 cm respectively.

Are these triangles similar or not? Give reasons.

2. Find the missing angles in the second triangle





3. A bottle 20cm high holds 1 litre of water.

What will a similar jug which is 40 cm high hold?

4. The volumes of two similar solids are $343cm^3$ and 1331 cm^3. If the larger solid has a surface area of 605 cm^2, find the surface area of the larger.

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers to activity 4

1. The ratios of corresponding sides are

First set

$\frac{6}{14} = \frac{3}{7}$

Second set

$\frac{7}{18}$

Third set

$\frac{9}{15}$

Given that the ratio of corresponding sides is not constant, the triangles are not similar.

2. The triangles are similar for corresponding sides are equal

The scale factor of enlargement is $\frac{18}{6} = 3$

The side that corresponds with the one with length 4 cm is

4 x 3 = 12 cm

The side that corresponds with the one with length 5 cm is

5 x 3 = 12 cm

3. The ratio of the lengths is 20 : 40

1 : 2

Ratio of volumes $1^3 : 2^3$

1 : 8

The other bottle will hold 1 : 8

1 000 : A

Area = 8 000 cm^3

4. The volumes of two similar solids are 343cm^3 and 1331 cm^3. If the larger solid has a surface area of 605 cm^2, find the surface area of the larger.

Finding the cube roots of the volumes, will give the proportional sides.

$\sqrt[3]{343} = 7$

$\sqrt[3]{1331} = 11$

The volumes of similar figures are in the ratio of $7^3 : 11^3$

The areas are in the ratio of $7^2 : 11^2$

The larger area is 605 cm^2,

Using the proportion.

$7^2 : 11^2 = A : 605$

$49 : 121 = A : 605$

Solving for A

A = 169(15) = 245 cm^2.

Unit Summary

Summary

In this unit you learned that:

- Points joined together form a line.
- Parallel lines never meet.
- Lines can meet. Those that meet at a right angle are perpendicular lines.
- A plane is a flat surface:

- When lines meet at a vertex, an angle is formed. An angle is the open – ended shape formed by two line segments joined at a common point, called the vertex. There are different angles with different sizes.

- A polygon is any plane shape bounded by straight line segments.
- A regular polygon is a polygon that has all angles equal and all sides equal. An irregular polygon has sides of different lengths and angles of different sizes.
- Congruent polygons are identical in every aspect, except the positions they occupy. This says one shape can be placed on the other so that all their corresponding parts coincide.
- Similar polygons have the same shape, but different sizes. The corresponding angles must be congruent, and the corresponding sides are proportional.
- Enlargements produce similar figures.
- interior angles of a polygon, regular or irregular, of n sides = $(n – 2) \times 180°$ where n is the number of sides.
- In a regular polygon, each angle = $\frac{(n-2) \times 180°}{n}$
- If the ratio of the corresponding sides of two similar figures is a:b, the ratio of their areas is $a^2:b^2$.
- If the ratio of the corresponding sides of two similar figures is a:b, the ratio of their volumes is $a^3:b^3$.

You have completed the material for this unit on solving linear inequalities. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit. It covers Indices.

Assignment

Assignment

1. **Answer All Questions.**
2. **Show all the necessary working.**

Total marks = 30 **Time:** 45 minutes

1. **Write down whether the following are true or false. If false, write down the correct statement.**

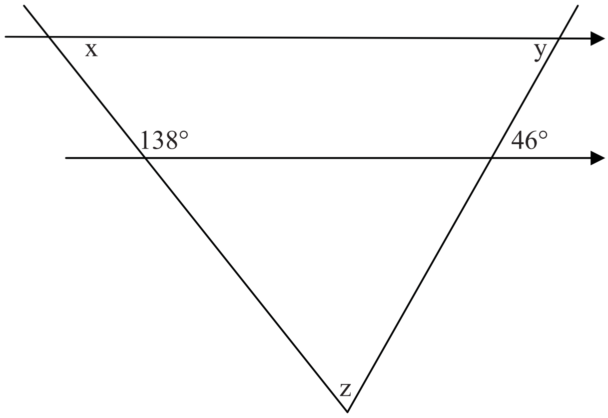
(a) Similar polygons have the same size and shape. [2]

(b) In congruent polygons, corresponding angles and corresponding sides are equal. [2]

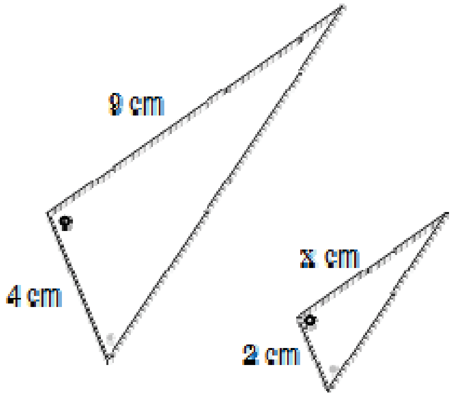
(c) sum of interior angles of a polygon = (n + 2) x 180°. [2]

(d) If the ratio of the corresponding sides of two similar figures is a:b, the ratio of their areas is $a^2:b^2$. [2]

2. Find the missing angles marked with letters. [6]

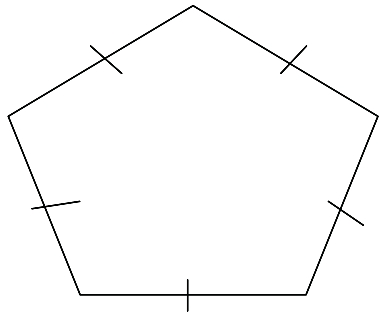


3. The two triangles given below are similar.
   (a) What is the scale factor? [2]
   (b) Find the value of x. [3]



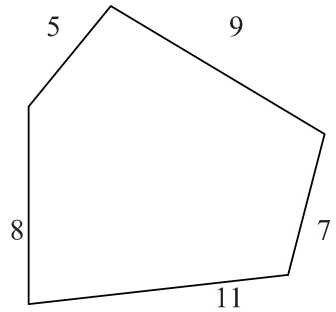
4. Classify each of the following by sides and/or angles.

(a)



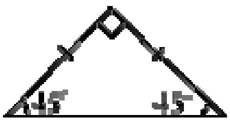
[2]

(b)



[2]

(c)



[2]

5. Two similar polygons have sides that are in the ratio 2: 3 The sum of their areas is 52 cm^2. What is the area of each of the polygons? [5]

Answers to assignment

1.

(a) False

Congruent polygons have the same size and shape

(b) True

(c) True

(d) True

5. - Angle y alternates with angle = 46°

Therefore angle y = 46°

- 180° - 138° = 42° (angles in a straight line)

  Angle x = 42° (angle x and angle 42° are alternate angles)

- angle x + angle y + angle z = 180° (angles in a straight line)

3.

(a) scale factor $= \frac{4}{2}$

$= 2$

(b)

$$\frac{4}{2} = \frac{9}{x}$$

Solving for x

x = 4.5

4. (a) This is a regular pentagon. All five sides are equal

(b) This is an irregular pentagon. Sides are different in size.

(c) An isosceles triangle that is right angled.

5.

The ratio of their areas is $2^2 : 3^2$

$4 : 9$

$4 + 9 = 11$

The smaller polygon: $\frac{4}{11} \times 52\ cm^2 = 16\ cm^2$

The larger polygon: $\frac{9}{11} \times 52\ cm^2 = 36\ cm^2$

The sum of their areas is $52\ cm^2$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

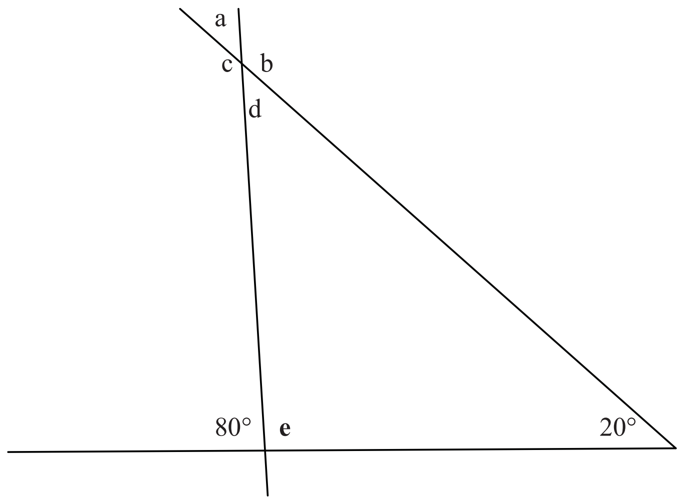
Assessment

Assessment

1. **Answer All Questions.**
2. **Show all the necessary working.**

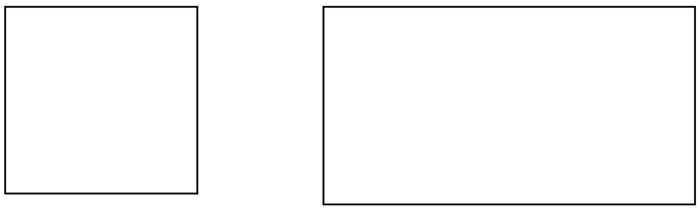
Total marks = 22 **Time:** 45 minutes

1. Find the missing angles marked with letters. [10]



2. A cuboid is enlarged by scale factor 4 to give an image of another cuboid. What is the ratio of their volumes? [3]

3. Are these two figures congruent? Explain your answer. [3]



4. Two containers shown are similar. Their diameters are 9 cm and 12 cm. The height of the larger container is 36 cm. Calculate the height of the smaller container. [6]

Unit Contents

Unit 28

Linear Programming

Introduction

Linear programming is a method of **solving problems** involving **two variables** that are **subject to certain conditions** using the **graphical representation of inequalities**.

Linear programming is employed in order to find which choice is **best**. It is used for determining a way to achieve the **bes**t outcome. This can be a maximum value, for example profit, or a minimum value, for example cost. For example, in business, making maximum profit is the best outcome. It can also be found in numerous areas including business and industry, agriculture and military.

This unit consists of 69 pages. It covers approximately 3% of the course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 30 hours to work on this unit, 20 hours for formal study and 10 hours for self-study and completing assessments/assignments.

Spend a few moments reading the following learning outcomes. They are a guide to what you should focus on while studying this unit.

This Unit is Comprised of One Lesson:

Lesson 1 Solving Real Life Problems

Upon completion of this unit you will be able to:

Outcomes

- *Solve* real – life problems which involve maximising (making as big as possible) or minimising (making as small as possible) a quantity.

Terminology

Variable: Letter that represents a value in an algebraic expression.

Inequality: A mathematical sentence in which the value of the expression on the left hand side is not equal to that on the right hand side. Symbols used with inequalities are $<, >, \leq,$ and $\geq$.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Solving Real Life Problems

By the end of this subunit, you should be able to:

- *solve* real – life problems which involve maximising (making as big as possible) or minimising (making as small as possible) a quantity.

This subunit is about 35 pages in length.

We will start off with the terminology used in linear programming.

1. **Constraints** are a set of inequalities with variables subject to certain stated conditions.

2. **Feasible solution** or **region** is the set of all points which satisfy all the constraints.

3. **Objective function** is a linear function which is to be optimised, that is maximised or minimised.

4. **Optimal point** is the one that gives the **bes**t outcome.

These are the steps to follow in solving linear programming problems:

1. Identify the quantities to be determined. We usually give them the symbols x and y.
2. Formulate the inequalities.
3. Formulate the objective function.
4. Graph the inequalities.
5. Identify all the points in the feasible region. These points satisfy **all** the conditions. The points that give the best results are at the vertices of the feasible region.
6. Substitute the coordinates of these vertices in order to see which one produces the best results.

Reflection

- When we graph inequalities, we shade the **unwanted** region. This is done in order to allow the required region to be clear, in order to facilitate the identification of the points that satisfy all the conditions. We do not want to obscure these points.

- Solid (unbroken lines) are used when the inequalities have the symbols $\leq$ or $\geq$.

- Broken lines are used when the inequalities have the symbols $<$ or $>$.

Please note that in other parts of the world you maybe asked to shade the wanted region.

Linear programming is best illustrated with examples. We will begin with examples that:

- start at a point where we graph the inequalities,

- which will be followed by identification of the points at the vertices of the feasible regions,

- finally we will do the substitution in order to get the minimum and the maximum values.

Example 1

(a) Graph the inequalities

$x \geq 0$; $y \geq 0$; $x + 3y \leq 6$;

(b) Find all the vertices

(c) Find the maximum and minimum values of $p = x + y$

Solution

- We change the inequalities to equations;

Inequality	Equation
$x \geq 0$	$x = 0$
$y \geq 0$	$y = 0$
$x + 3y \leq 6$	$x + 3y = 6$ $y = -\frac{1}{3}x + 2$

- We pick the x –values that we want to use.

  When we were working with inequalities, we could choose x values that were both negative and positive. In linear programming, we will only work with positive x- values

  We have picked x = 0, 1, 2, 3, 4, 5, 6

- We use the equations to calculate the coordinates.

With the equation $x = 0$ and $y = 0$, there is no need to do any calculations. The equation $x = 0$ is the y – axis, and the equation $y = 0$ is the x – axis.

We will then only do calculations for the equation $y = -\frac{1}{3}x + 2$

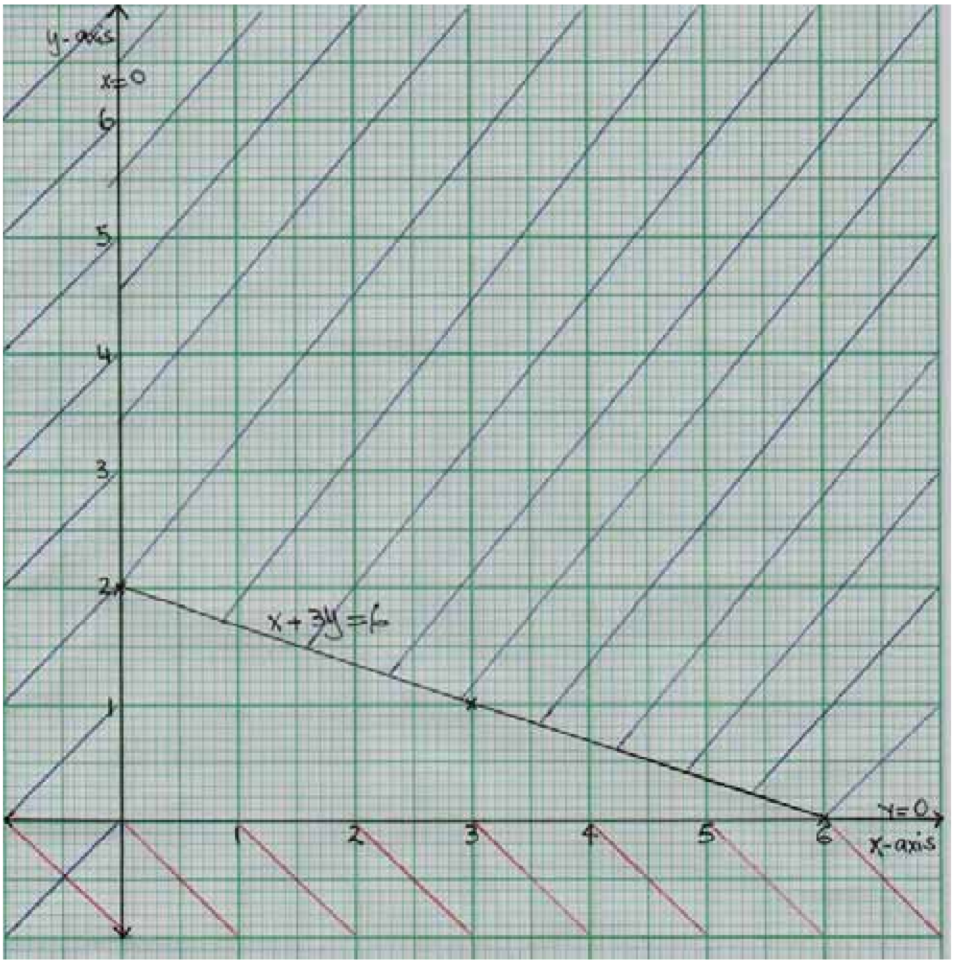
The corresponding y – values are in the table below:

x	**y**
0	2
1	$1\frac{2}{3}$
2	$1\frac{1}{3}$
3	1
4	$\frac{2}{3}$
5	$\frac{1}{3}$
6	0

- All the inequalities have the symbols $\leq$ or $\geq$. We are therefore going to have solid lines.

- By shading the unwanted region, show the region that represents the inequalities $x \geq 0$; $y \geq 0$; $x + 3y \leq 6$;

- Always label the lines with their equations.



The vertices are (0,0), (0,2) and (6,0)

Using them to get the maximum and minimum values of p = x + y , the objective function

p= x + y	p= x + y	p= x + y
p= 0 + 0	p= 0 + 2	p= 6 + 0
p = 0	p = 2	p= 6
(the minimum value)		(the maximum value)

.

Example 2

(a) Graph the inequalities

$y \geq 0$; $y \leq x$; $3x + 4y \geq 12$; $3x + 4y \leq 18$;

(b) Find all the vertices.

(c) Find the maximum and minimum values of $q = 6x + 10y$.

Solution

- We change the inequalities to equations;

Inequality	Equation
$y \geq 0$	$y = 0$
$y \leq x$	$y = x$
$3x + 4y \geq 12$	$3x + 4y = 12$ $y = -\frac{3}{4}x + 3$
$3x + 4y \leq 18$	$3x + 4y = 18$ $y = -\frac{3}{4}x + 4\frac{1}{2}$

- We pick the x –values that we want to use.

  We have picked x = 0, 1, 2, 3, 4, 5, 6

- We use the equations to calculate the coordinates.

  With the equation $y = 0$, there is no need to do any calculations. The equation $y = 0$ is the x – axis.

  We will only do calculations for the equations $y = x$, $y = -\frac{3}{4}x + 3$ and $y = -\frac{3}{4}x + 4\frac{1}{2}$

  The corresponding y – values are in the tables below:

$y = x$

x	y
0	0

1	1
2	2
3	3
4	4
5	5
6	6

$y = -\frac{3}{4}x + 3$

Given the practice that we have had of calculating the coordinates, we can now only use the intercepts. We will therefore only use x = 0 and y = 0

x	**y**
0	3
4	0

$$y = -\frac{3}{4}x + 4\frac{1}{2}$$

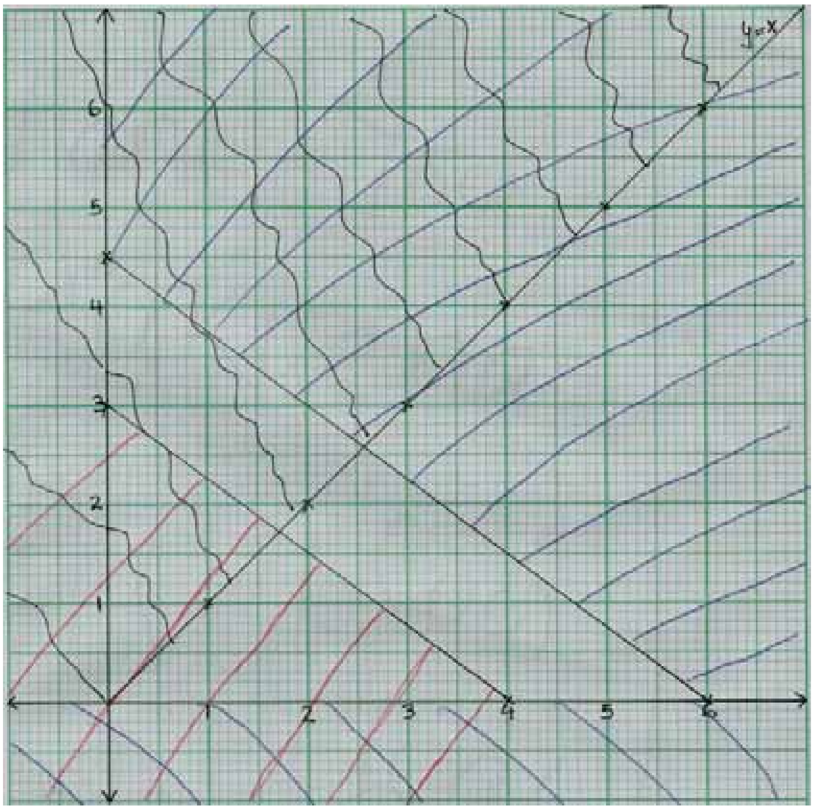
For this equation, we will only use x = 0 and y = 0

x	**y**
0	4½
6	0

- All the inequalities have the symbols ≤ or ≥. We are therefore going to have solid lines.

By shading the unwanted region, show the region that represents the inequalities $y \geq 0$; $y \leq x$; $3x + 4y \geq 12$; $3x + 4y \leq 18$;

- Always label the lines with their equations.



The vertices are (1.7, 1.7), (2.6, 2.6), (4,0) and (6,0)

Using them to get the maximum and minimum values of q= 6x + 10y, the objective function

q= 6x + 10y
q= 6(1.7) + 10(1.7)
q= 10.2 + 17
q= 27.2

q= 6x + 10y
q= 6(2.6) + 10(2.6)
q= 15.6 + 26
q= 41.6
(the maximum value)

q= 6x + 10y
q= 6(4) + 10(0)
q= 24 + 0
q= 24
(the minimum value)

q= 6x + 10y
q= 6(6) + 10(0)
q= 36 + 0
q= 36

The above examples have reminded you of the plotting of inequalities.

The next examples are going to be with real situations.

Example 3

A manager wishes to hire two machines, A and B.

He considers the following facts:

	Machine A	Machine B
Floor space required	$2m^2$	$3m^2$
Number of men required to operate	4	3

He has a maximum of 24 m^2 of floor space and a maximum of 36 men available.

In addition, he is not allowed to hire more machines of type B than of type A.

The profit from using machine A is M28 and that from using machine B is M56. How many machines of each type should he get in order to get maximum profit?

Solution

1. We identify the quantities to be determined. Remember we usually give them the symbols *x* and *y*.

   Machines of type A are *x* and machines of type B are *y*. That says our variables are *x* and *y*, and they cannot be negative.

2. We use all the given facts to formulate inequalities.

(a) Machines of type A are *x* and machines of type B are *y*. That says our variables are *x* and *y*, and they cannot be negative.

So $x \geq 0$, and $y \geq 0$

(b)

The space that machine A will use is $2m^2 \times x$.

The space that machine B will use is $3m^2 \times y$.

There is a maximum space of $24\ m^2$. This cannot be exceeded.

So $(2m^2 \times x) + (3m^2 \times y) \leq 24\ m^2$

$= 2x + 3y \leq 24$

(c)

Machine A needs 4 men, so x machines requires 4x men

Machine B needs 3 men, so y machines requires 3y men

There are 36 men required. This cannot be exceeded.

So $4x + 3y \leq 36$

(d)

Machine B type should not be more than machine A type.

So $y \leq x$

We have 5 inequalities, the constraints;

$x \geq 0$

$y \geq 0$

$2x + 3y \leq 24$

$4x + 3y \leq 36$

$y \leq x$

3. We then formulate the objective function

   The profit from using machine A is M28 and that from using machine B is M56.

   The maximum profit is (M28 × x) + (M56 × y)

   P = (M28 × x) + (M56 × y)

   $P = 28x + 56y$

4. Graph the inequalities.

   We change the inequalities to equations:

Inequality	Equation
$x \geq 0$	$x = 0$
$y \geq 0$	$y = 0$
$y \leq x$	$y = x$
$2x + 3y \leq 24$	$2x + 3y = 24$ $y = -\frac{2}{3}x + 8$
$4x + 3y \leq 36$	$4x + 3y = 36$ $y = -\frac{4}{3}x + 12$

- We pick the x –values that we want to use

  We can just use the intercepts to calculate the coordinates. We will therefore only use x = 0 and y = 0

- We use the equations to calculate the coordinates

  $x = 0$ is the y axis

  $y = 0$ is the x axis

  some of the coordinates for $y = x$ are (0,0), (1,1), (2,2), (3,3), (4,4),

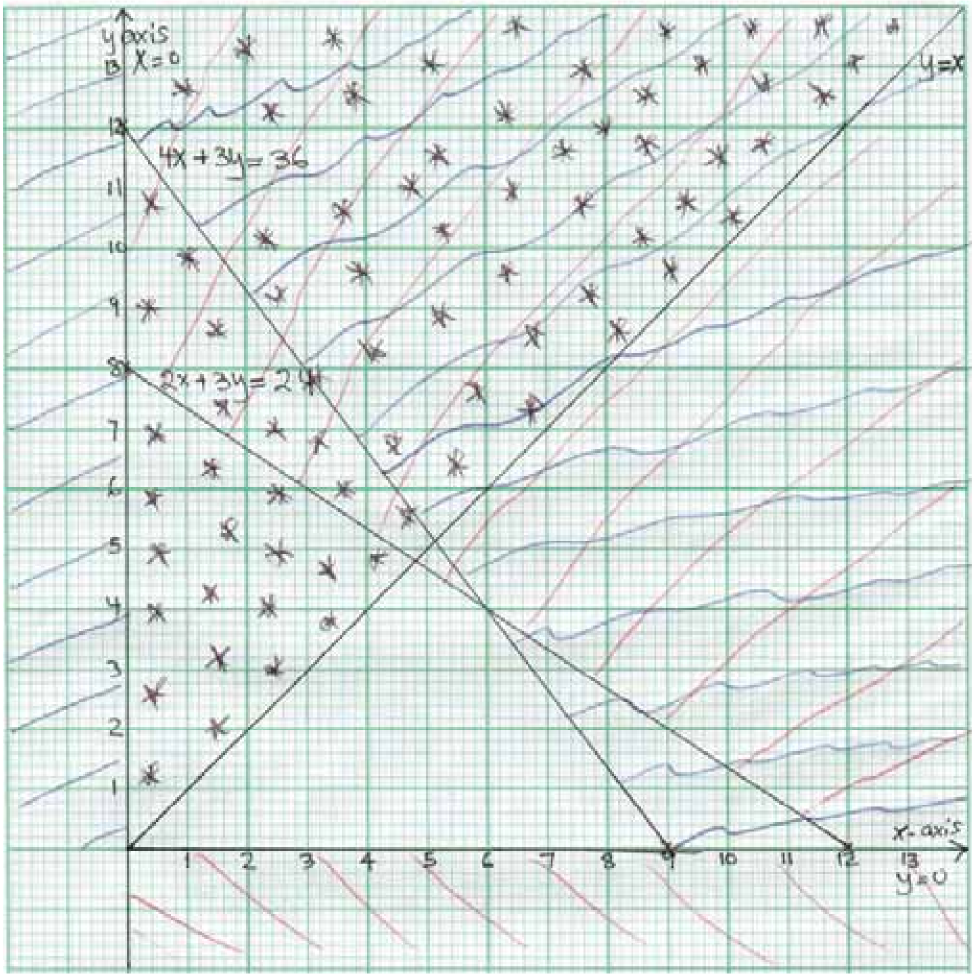
$2x + 3y = 24$

x	y
0	8
12	0

$4x + 3y = 36$

x	y
0	12
9	0

All the boundary lines are drawn below. Always label them with their equations. The unwanted regions are shaded.


y axis
X = 0
4X + 3y = 36
2x + 3y = 24
y=x
x-axis
y=0
1
2
3
4
5
6
7
8
9
10
11
12
13

5. Identify all the points in the feasible region. These points satisfy **all** the conditions. The points that give the best results are at the vertices of the feasible region. They are:

   (0,0); (1,1); (2,2); (3,3); (4,4); (6,4); (9,0); (8,0); (7,0); (6,0); (5,0); (4,0); (3,0); (2,0); (1,0);

Note it!

There are other points at the vertices of the feasible region which we have not listed above. They are not whole numbers. We have only listed whole numbers for we cannot have, for example, half a machine, quarter of a machine.

Substitute the coordinates of these vertices in order to see which one produces the best results in

P = (M28 × x) + (M56 × y)

Using (0,0)

P = (M28 × 0) + (M56 × 0)

P = 0 + 0

P = 0

Using (1,1)

P = (M28 × 1) + (M56 × 1)

P = 28 + 56

P = 84

Using (2,2)

P = (M28 × 2) + (M56 × 2)

P = 56 + 112

P = 168

Using (3,3)

P = (M28 × 3) + (M56 × 3)

P = 84 + 168

P = 252

Using (4,4)

Using (6,4)

$P = (M28 \times 4) + (M56 \times 4)$
$P = 112 + 224$
$P = 336$

$P = (M28 \times 6) + (M56 \times 4)$
$P = 168 + 224$
$P = 392$

Using (9,0)

$P = (M28 \times 9) + (M56 \times 0)$
$P = 252 + 0$
$P = 252$

Using (8,0)

$P = (M28 \times 8) + (M56 \times 0)$
$P = 224 + 0$
$P = 224$

Using (7,0)

$P = (M28 \times 7) + (M56 \times 0)$
$P = 196 + 0$
$P = 196$

Using (6,0)

$P = (M28 \times 6) + (M56 \times 0)$
$P = 168 + 0$
$P = 168$

Using (5,0)

$P = (M28 \times 5) + (M56 \times 0)$
$P = 140 + 0$
$P = 140$

Using (4,0)

$P = (M28 \times 4) + (M56 \times 0)$
$P = 112 + 0$
$P = 112$

Using (3,0)

$P = (M28 \times 3) + (M56 \times 0)$
$P = 84 + 0$
$P = 84$

Using (2,0)

$P = (M28 \times 2) + (M56 \times 0)$
$P = 56 + 0$
$P = 56$

Using (1,0)

P = (M28 × 1) + (M56 × 0)

P = 28 + 0

P = 28

The highest number calculated is 392. Attaching the relevant units it is M392; which was calculated using (6,4), the optimal point.

Therefore the manager must get 6 machines of type A and 4 machines of type B in order to get **maximum** profit.

Example 4

A decision has to be made on the number of ladies and children's shoes to be produced by the shoe manufacturer. Here are the facts to be considered:

- each of the 6 skilled workers can make, per hour, 2 pairs of ladies' shoes or 3 pairs of children's shoes.
- the machines they have can manufacture, per hour, 7 pairs of ladies' shoes and 12 pairs of children's shoes
- the total cost of employing these 6 workers is M70 per hour, whether any shoes are made or not; and this has to be met from a profit of M7 on a pair of children's shoes and M10 on a pair of ladies shoes.

Solution

1. We identify the quantities to be determined. Remember we usually give them the symbols x and y.

Children's shoes are x pairs that can be manufactured per hour and ladies shoes are y pairs that can be manufactured per hour. Our variables are x and y, and they cannot be negative.

2. We use all the given facts to formulate inequalities

Children's shoes are x pairs that can be manufactured per hour and ladies shoes are y pairs that can be manufactured per hour. Our variables are x and y, and they cannot be negative.

So $x \geq 0$, and $y \geq 0$

- 1 worker can make 3 pairs of children's shoes per hour. Therefore the number of workers required to make x pairs of children's shoes per hour is $\frac{x}{3}$; and

- 1 worker can make 2 pairs of ladies' shoes per hour, so the number of workers required to make y pairs of ladies shoes per hour is $\frac{y}{2}$

There are only 6 workers. This cannot be exceeded.

So the total number of workers required is $\frac{x}{3} + \frac{y}{2} \leq 6$

$= 2x + 3y \leq 36$

(c) Only 12 pairs of children's shoes can be made in one hour;

so $x \leq 12$;

and only 7 pairs of ladies' shoes can be made in one hour,

so $y \leq 7$

(d) The profit on x pairs of children's shoes is M7 × x and that on y pairs of ladies shoes is M10 × y

For there to be any profit, (M7 × x) + (M10 × y) must be more than the M70, which is the labour cost

So (M7 × x) + (M10 × y) ≥ 70

So $7x + 10y \geq 70$

We have 6 inequalities, which we know are sometimes called constraints

$x \geq 0$

$y \geq 0$

$2x + 3y \leq 36$

$x \leq 12$

$y \leq 7$

$7x + 10y \geq 70$

3. We then formulate the objective function

   The profit on x pairs of children's shoes is M7 × *x* and that on *y* pairs of ladies shoes is M10 × *y*

   So Profit = [(M7 × *x*) + (M10 × *y*)] - M70, which is the labour cost

4. Graph the inequalities

We change the inequalities to equations.

Inequality	Equation
$x \geq 0$	$x = 0$
$y \geq 0$	$y = 0$
$2x + 3y \leq 36$	$2x + 3y = 36$
$x \leq 12$	$x = 12$
$y \leq 7$	$y = 7$
$7x + 10y \geq 70$	$7x + 10y = 70$

- We pick the x –values that we want to use.

  We can just use the intercepts to calculate the coordinates. We will therefore only use x = 0 and y = 0

- We use the equations to calculate the coordinates.

x = 0 is the y axis

y = 0 is the x axis

x = 12 is the vertical line x = 12

y = 7 is the horizontal line y = 7

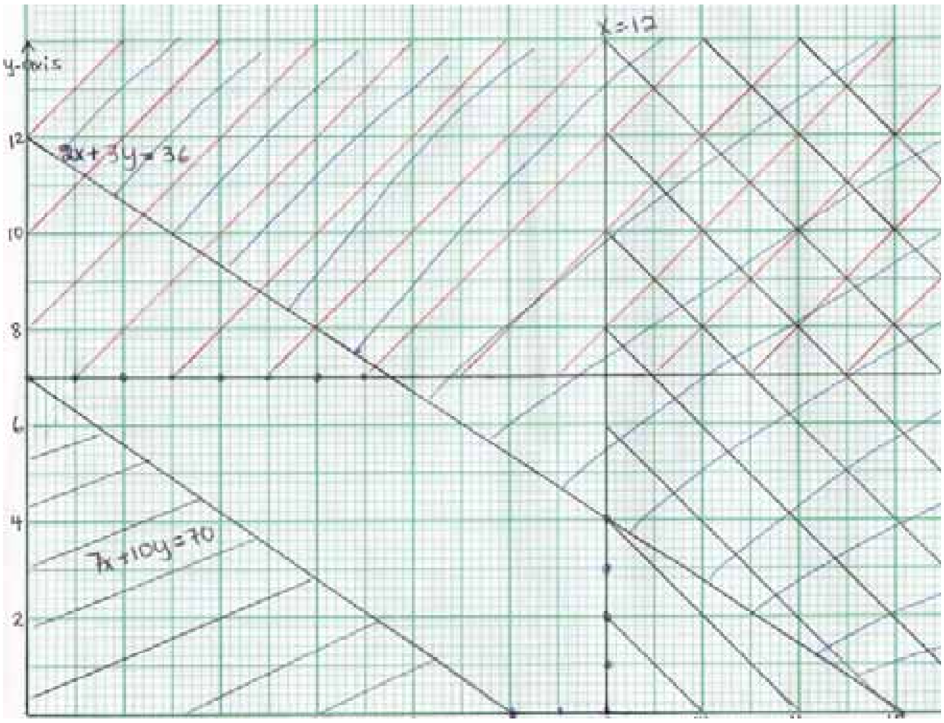
2x + 3y = 36

x	y
0	12
18	0

$7x + 10y = 70$

x	y
0	7
10	0

All the boundary lines are drawn below. Always label them with their equations. The unwanted regions are shaded.



5. Identify all the points in the feasible region. These points satisfy **all** the conditions. The points that give the best results are at the vertices of the feasible region. They are:

(0,7); (1,7); (2,7); (3,7); (4,7); (5,7); (6,7); (7,7); (9,6); (12,4); (12,2); (12,0); (10,0); (11,0); (2,0); (1,0);

Note it!

There are other points at the vertices of the feasible region which we have not listed above. They are not whole numbers. We have only listed whole numbers.

Substitute the coordinates of these vertices in order to see which one produces the best results in

Profit = [(M7 × *x)* + (M10 × *y)*] - M70

Using (0,7)

$P=[(M7 \times x) + (M10 \times y)] - M70$

$P=[(M7 \times 0) + (M10 \times 7)] - M70$

$P= [M0 + 70] - M70$

$P= M70 - M70$

$P= M0$

Using (1,7)

$P=[(M7 \times x) + (M10 \times y)] - M70$

$P=[(M7 \times 1) + (M10 \times 7)] - M70$

$P= [M7 + 70] - M70$

$P= M77 - M70$

$P= M7$

Using (2, 7)

$P=[(M7 \times x) + (M10 \times y)] - M70$

$P=[(M7 \times 2) + (M10 \times 7)] - M70$

$P= [M14 + 70] - M70$

$P= M84 - M70$

$P= M14$

Using (3, 7)

$P=[(M7 \times x) + (M10 \times y)] - M70$

$P=[(M7 \times 3) + (M10 \times 7)] - M70$

$P= [M21 + 70] - M70$

$P= M91 - M70$

P= M21

Using (4, 7)

P=[(M7 × x) + (M10 × y)] - M70

P=[(M7 × 4) + (M10 × 7)] - M70

P= [M28 + 70] - M70

P= M98 - M70

P= M28

Using (5, 7)

P=[(M7 × x) + (M10 × y)] - M70

P=[(M7 × 5) + (M10 × 7)] - M70

P= [M35 + 70] - M70

P= M105 - M70

P= M35

Using (6,7)

P=[(M7 × x) + (M10 × y)] - M70

P=[(M7 × 6) + (M10 × 7)] - M70

P= [M42 + 70] - M70

P= M112 - M70

P= M42

Using (7,7)

P=[(M7 × x) + (M10 × y)] - M70

P=[(M7 × 7) + (M10 × 7)] - M70

P= [M49 + 70] - M70

P= M119 - M70

P= M49

Using (9, 6)

P=[(M7 × *x*) + (M10 × *y*)] - M70

P=[(M7 × 9) + (M10 × 6)] - M70

P= [M63 + 60] - M70

P= M123 - M70

P= M53

Using (12,4)

P=[(M7 × *x*) + (M10 × *y*)] - M70

P=[(M7 × 12) + (M10 × 4)] - M70

P= [M84 + 40] - M70

P= M124 - M70

P= M54

Using (12, 2)

P=[(M7 × *x*) + (M10 × *y*)] - M70

P=[(M7 × 12) + (M10 × 2)] - M70

P= [M84 + 20] - M70

P= M104 - M70

P= M34

Using (12, 0)

P=[(M7 × *x*) + (M10 × *y*)] - M70

P=[(M7 × 12) + (M10 × 0)] - M70

$P = [M84 + 0] - M70$

$P = M84 - M70$

$P = M14$

Using (10,0)

$P = [(M7 \times x) + (M10 \times y)] - M70$

$P = [(M7 \times 10) + (M10 \times 0)] - M70$

$P = [M70 + 0] - M70$

$P = M70 - M70$

$P = M0$

Using (11, 0)

$P = [(M7 \times x) + (M10 \times y)] - M70$

$P = [(M7 \times 11) + (M10 \times 0)] - M70$

$P = [M77 + 0] - M70$

$P = M77 - M70$

$P = M7$

Using (2,0)

$P = [(M7 \times x) + (M10 \times y)] - M70$

$P = [(M7 \times 2) + (M10 \times 0)] - M70$

$P = [M14 + 0] - M70$

$P = M14 - M70$

$P = -M56$

Using (1,0)

$P=[(M7 \times x) + (M10 \times y)] - M70$

$P=[(M7 \times 1) + (M10 \times 0)] - M70$

$P= [M7 + 0] - M70$

$P= M7 - M70$

$P= -M63$

The biggest number calculated is 54. Attaching the relevant units it is M54; which was calculated using (12,4), the optimal point.

Therefore 12 pairs of children's shoes and 4 pairs of ladies' shoes must be made in order to get **maximum** profit.

Activity 1

1. Find the maximum value of k, given the following constraints.

   $5x + 4y \geq 40$ $\quad$ $x + 4y \leq 40$ $\quad$ $7x + 4y \leq 112$

   $$k = x + y$$

   $$k = 2x + y$$

2. Find the minimum value of p, given the following constraints.

$-x + 2y \leq 2$; $3x + 2y \leq 10$; $x + 6y \geq 6$; $3x + 2y \geq 6$

$$p = 4x + 5y$$

3. A man has two types of truck, A and B.

He has three of type A and five of type B.

Type A will carry 40 bags of mealie meal and type B will carry 30 bags.

He has to deliver 180 bags to a customer.

Each truck makes only one journey.

What choice does he have?

Key Points to Remember

The key points to remember in this subunit on linear programming are:

- linear programming problems are solved using the graphical representation of inequalities.

- when we graph inequalities, we shade the **unwanted** region.

- linear programming involves real – life problems which involve maximising or minimising a quantity

- constraints are a set of inequalities used in linear programming.

- the set of all points which satisfy all the constraints is the feasible solution or region.

- a linear function which is to be optimised, that is maximised or minimised is called the objective function.

- optimal point is the one that gives the best outcome.

Answers to Activity 1

1.

$5x + 4y = 40$

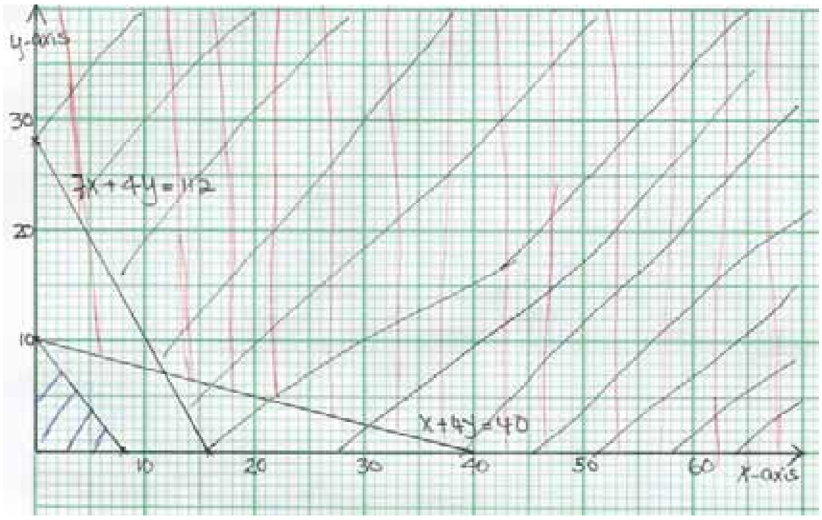
x	y
0	10
8	0

$x + 4y = 40$

x	y
0	10
40	0

$7x + 4y = 112$

x	y
0	28
16	0



$k = x + y$

$= 12 + 7$

$= 19$

$k = x + 2y$

$= 12 + (2)7$

$= 12 + 14$

$= 26$

2.

$-x + 2y = 2$

x	y
0	1
4	3

$3x + 2y = 10$

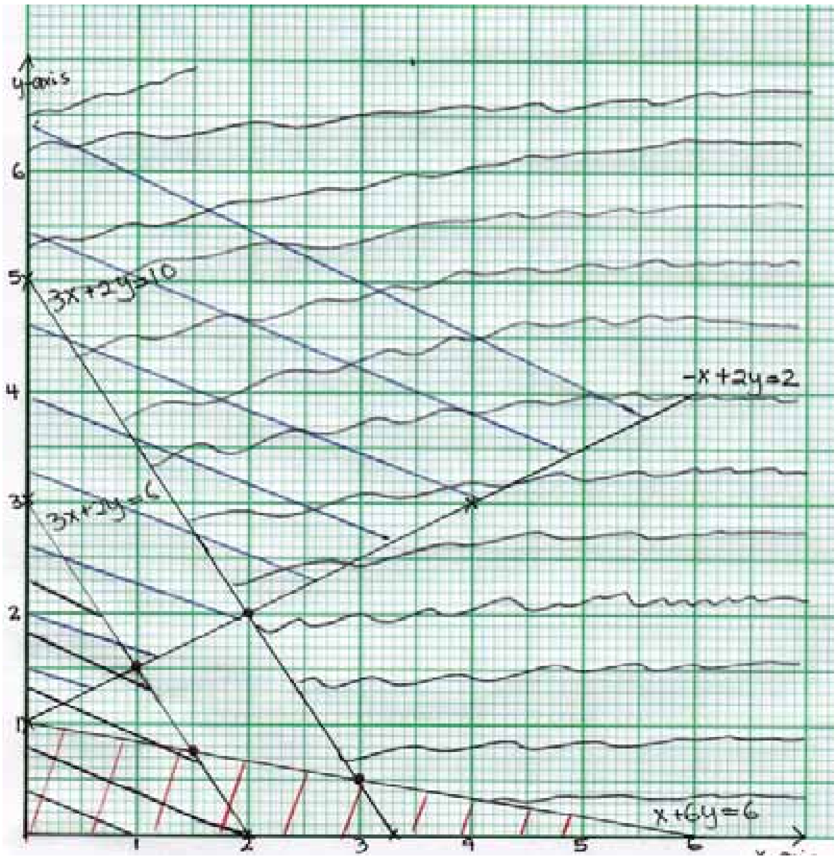
x	y
0	5
$3\frac{1}{3}$	0

$x + 6y = 6$

x	y
0	1
6	0

$3x + 2y = 6$

x	y
0	3
2	0



$p = 4(1.5) + 5(0.75)$

$= 6 + 3.75$

$= 9.75$

3. Let x be the number of type A trucks he uses.

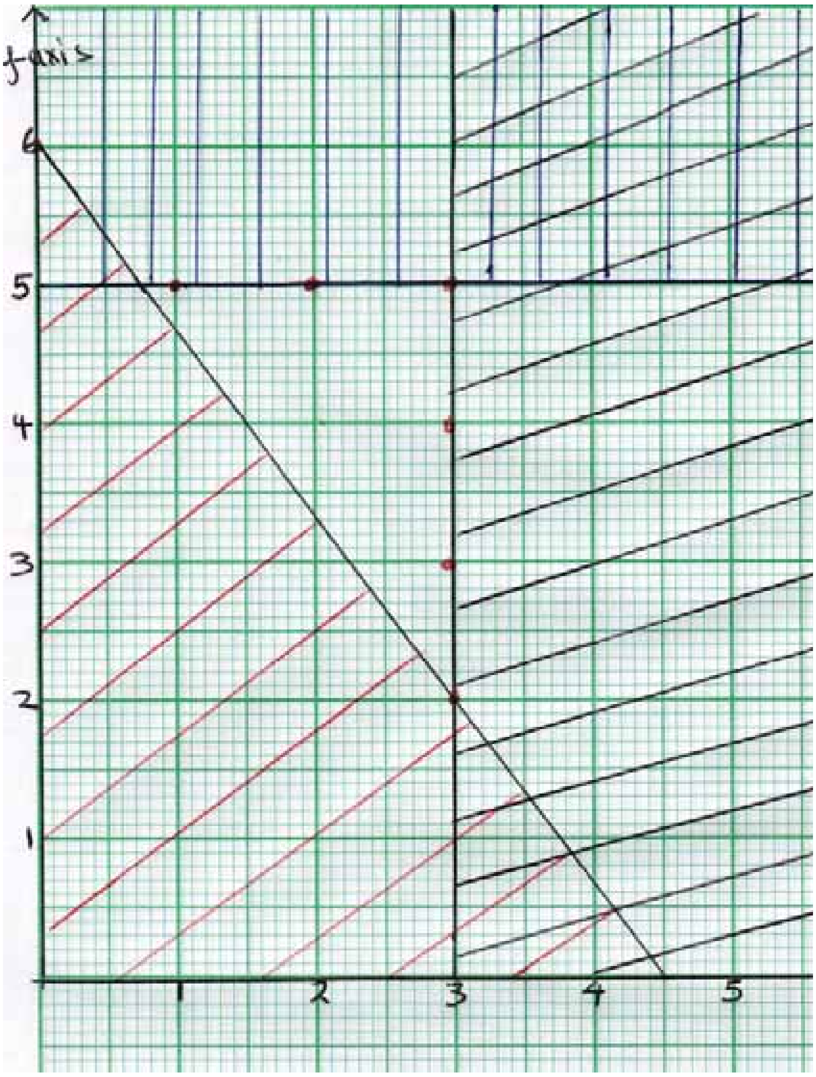
Let y be the number of type B trucks he uses.

The inequalities are

- $x \le 3$ for type A trucks

- $y \le 5$ for type B trucks

and $40x + 30y \ge 180$



Of the many possibilities that we have (3,2) is the best as it gives

$40x + 30y \geq 180$

$40(3) + 30(2) \geq 180$

$120 + 60 \geq 180$

$180 \geq 180$

So 3 type A trucks will be used and 2 type B trucks will be used.

Unit Summary

Summary

In this unit you learned that:

- linear programming problems are solved using the graphical representation of inequalities. When we graph inequalities, we shade the **unwanted** region.

- linear programming involves real life problems which involve maximising or minimising a quantity.

- constraints are a set of inequalities used in linear programming.

- the set of all points which satisfy all the constraints is the feasible solution or region

- a linear function which is to be optimised, that is maximised or minimised is called the objective function

- optimal point is the one that gives the best outcome.

- the steps to follow in solving linear programming problems are:

1. Identify the quantities to be determined. They are usually given the symbols x and y
2. Formulate the inequalities
3. Formulate the objective function, if it is not given
4. Graph the inequalities
5. Identify all the points in the feasible region.
6. Substitute the coordinates of these vertices in order to see which one produces the best results

You have completed the material for this unit on linear programming. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit. It covers polynomials.

Assignment

Assignment

Instructions

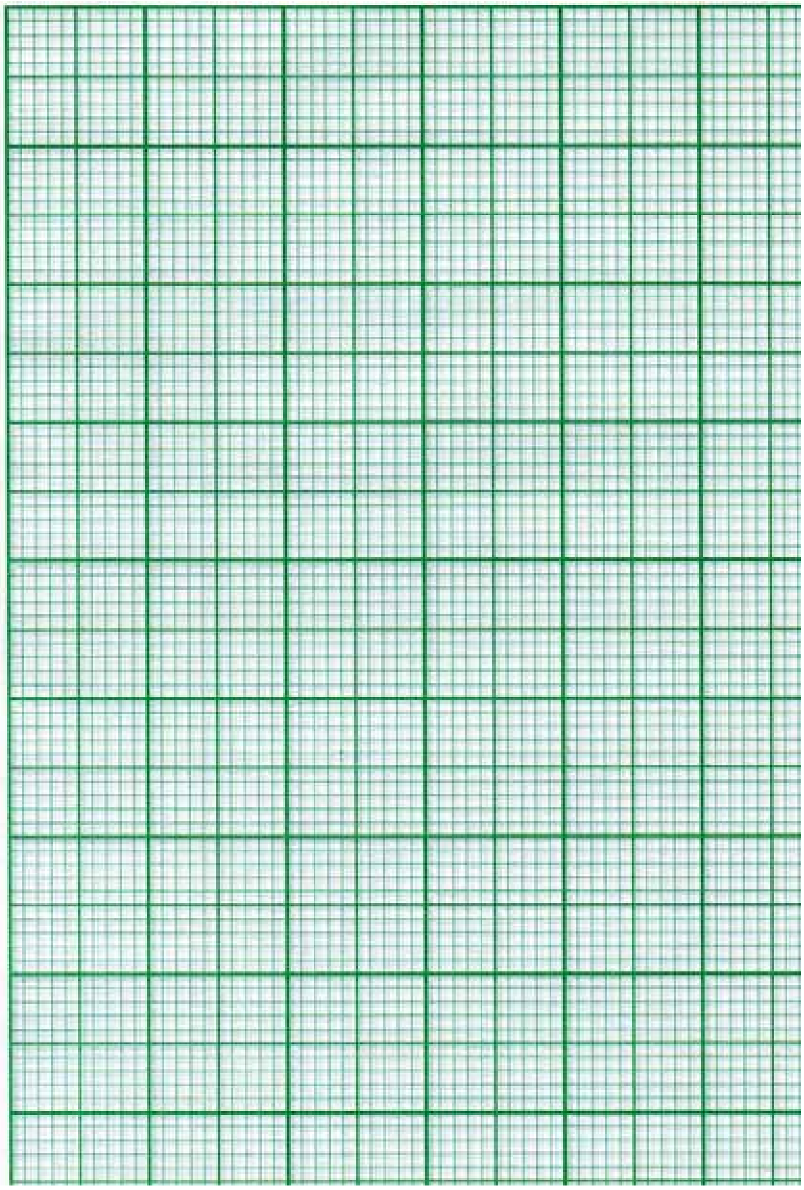
1. Answer All Questions.
2. Show all the necessary working.

Total marks = 60 **Time:** 1½ hours

1. A bar of chocolate costs M10 and a packet of chips costs M20.

   Khotso buys at least two bars of chocolate and one packet of chips.

   If Khotso has only M80 to spend, list the possible numbers of each that he can buy.



2. In an examination, there are two papers each with a total of 50 marks. To pass the examination, a candidate must score at least 20 marks on each paper and at least 50 marks on the two papers combined.

   A candidate scores 20 marks in paper 1 and y marks in paper 2. Find the number of possible values of y which would allow the candidate to pass.

   Find the number of ways in which it is possible for a candidate to score at least 20 marks in each paper and yet to fail the examination.

3. A car park, area $1400m^2$, is laid out for cars and for vans. $10m^2$ of space is allowed for a car, $15m^2$ for a van. It is estimated that the number of vans will never be less than half the number of cars, nor more than twice the number of cars. Find how many spaces must be marked for cars, in order to park as many vehicles as possible.

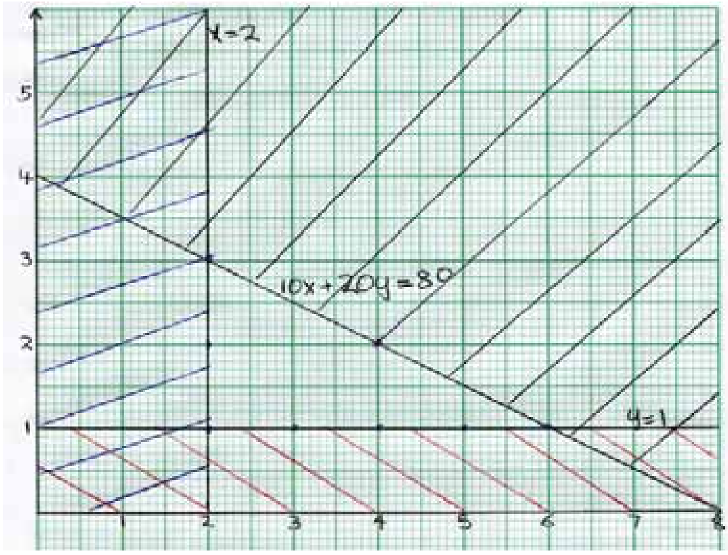
Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers to Assignment

1. -*x* is the number of the bars of chocolate; so x ≥ 2

   - *y* is the number of the packet of chips; so y ≥ 1

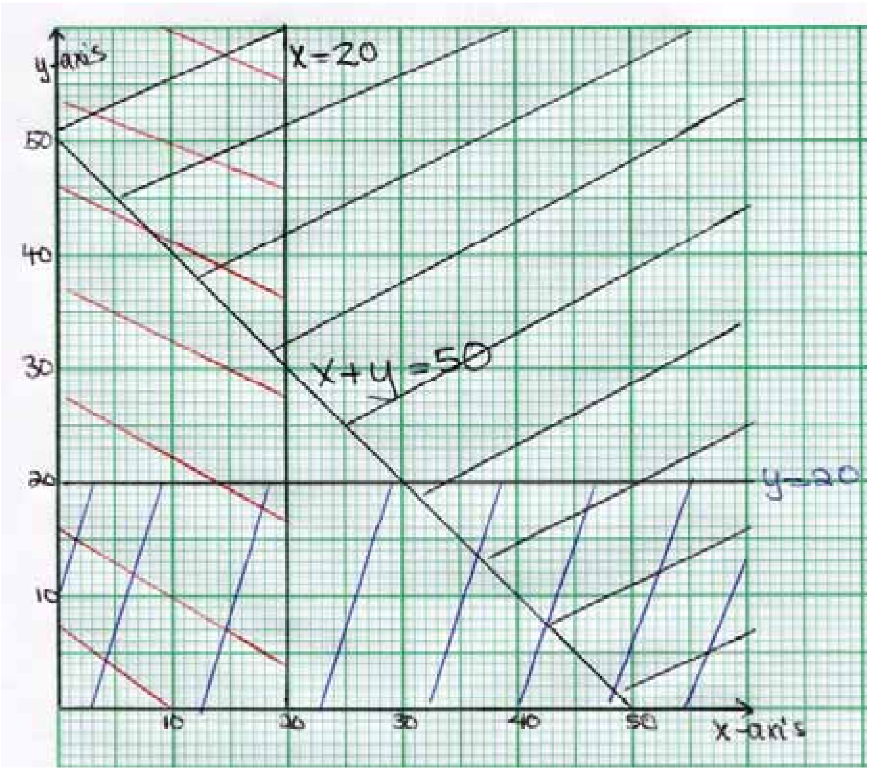
   10x + 20y ≤ 80



These are the possibilities

Bars of chocolate	Packet of chips
2	1
2	2
2	3
3	1
3	2
4	1
4	2
5	1
6	1

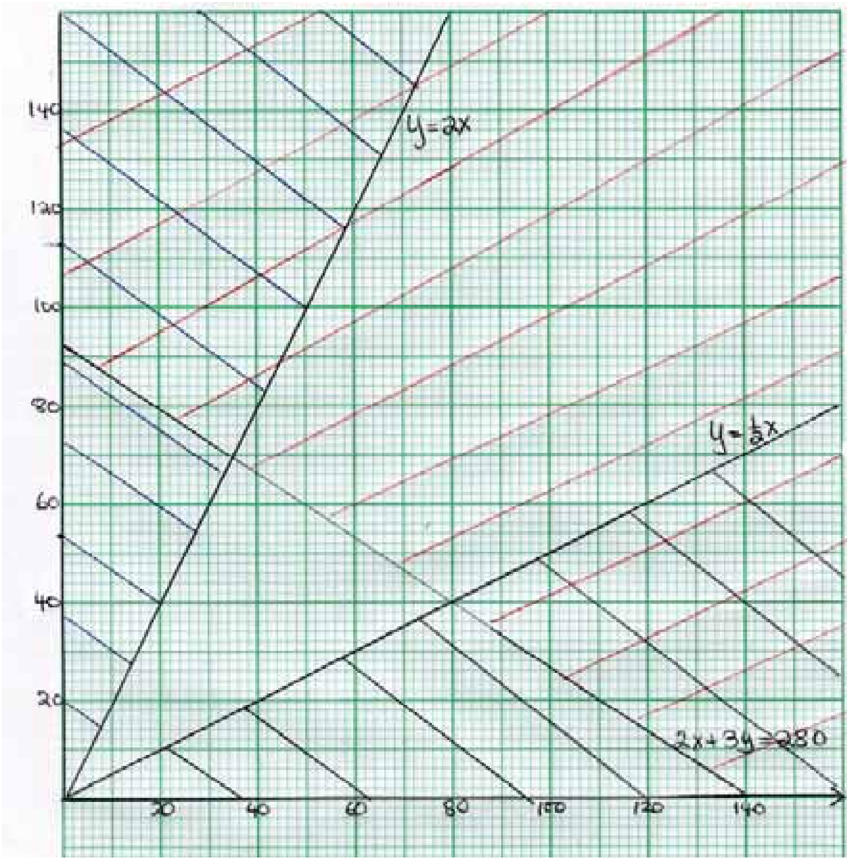
2. - *x* is the least marks that can be scored in paper 1; so $x \geq 20$
   - *y* is the least marks that can be scored in paper 2; so $y \geq 20$
   - 50 is the least marks that can be scored on the two papers combined: so $x + y \geq 50$



If $x = 20$, then the possible y values which would allow the candidate to pass are between 30 and 50, both numbers included; which we can write as $30 \leq y \leq 50$ there are in total, 21 possible ways.

All the possible ways are the whole number values inside the triangle, excluding the line x + y = 50. They are 55 in total.
For example, x + y = 50
20 + 20 = 40

3.- x is the number of cars that can be parked
- Y is the number of vans that can be parked
- space that can be used: $10x + 15y \leq 1400$
$2x + 3y \leq 280$
- number of vans is not less than half the number of cars: $y \geq \frac{1}{2}x$
number of vans is not greater than twice the number of cars:
$y \leq 2x$



The greatest number of vehicles that can be parked is 120; 80 cars and 40 vans.

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Instructions

1. Answer All Questions.
2. Show all the necessary work.

Total marks = 40 **Time:** 1 hour

1. Find the region defined by the inequalities

   $2x + y \leq 8$

   $3x + 4y \leq 24$

   $x > 0$

   $y > 0$

2. A gardener wishes to plant not more than $23m^2$ of ground with two different bushes. He allows $1m^2$ of ground for one of type A and $2m^2$ for one of type B. Type A bush costs M120 and type B costs M40. He does not wish to spend more than M1160. What is the greatest number of bushes he can plant?

3. A manufacturer makes two kinds of appliances, model A and model B.

   Model A earns a profit of M1000 and model B earns M1200 profit.

   Each month the manufacturer can produce up to 600 units of model A and up to 500 of units model B.

   If there are only enough man-hours available to produce no more than a total of 900 appliances per month, how many of each kind should be produced to obtain the maximum profit?

Unit Contents

Unit 29

Unit 29

Polynomials

Introduction

The content of this unit is closely related to the work on Algebraic Expressions. Polynomials are algebraic expressions that have many terms ("poly" means "many"). In this unit, we are going to work with polynomials with the four basic operations.

This unit consists of 49 pages. This is approximately 2% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 20 hours to work on this unit, 15 hours for formal study and 5 hours for self-study and completing assessments/assignments.

Spend a few moments reading the following learning outcomes. They are a guide to what you should focus on while studying this unit.

This Unit is Comprised of Five Lessons:

Lesson 1 Operations of Addition and Subtraction of Polynomials
Lesson 2 Operations of Multiplication of Polynomials
Lesson 3 Dividing a Polynomial by a Binomial
Lesson 4 Remainder Theorem
Lesson 5 Factor Theorem

Outcomes

Upon completion of this unit you will be able to:

- *carry* out operations of addition, subtraction and multiplication of polynomials.
- *divide* a polynomial by a binomial expression, and identify the quotient and the remainder.
- *apply* the remainder theorem and the factor theorem.

Terminology

Term: Constant term and a variable term.

Expression: Is formed when terms are combined by either addition or subtraction. It is any well-formed combination of mathematical symbols ,variables and constants. An algebraic expression is only a phrase, not a whole sentence, so it cannot contain an equality sign (=).

Factor: Factors of numbers are numbers that divide into another exactly.

Factorisation: Writing a number or an expression as a product of its factors

Monomial: An algebraic expression with one term.

Binomial: An algebraic expression with two terms.

Trinomial: An algebraic expression with three terms.

Polynomial: An algebraic expression that has many terms. ("poly" means "many")of the form $c_nx^n + c_{n-1}x^{n-1} +$ ....$c_1x + c_0$ with

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Operations of Addition and Subtraction of Polynomials

By the end of this subunit, you should be able to:

- add polynomials
- subtract polynomials

This subunit is about 3 pages in length.

Polynomials are expressions that can be written in the form

$$c_nx^n + c_{n-1}x^{n-1} + \text{........} + c_2x^2 + c_1x + c_0$$

where n is a non-negative integer called the **degree**.

x is the **variable**.

$c_n x^n$, $c_{n-1}x^{n-1}$, c_2x^2, c_1x, c_0 are called **terms.**

A polynomial has a definite number of terms.

c_n, c_{n-1}, c_2, c_1, c_0 are non zero real numbers called **coefficients**.

$c_n x^n$, $c_{n-1}x^{n-1}$, c_2x^2, c_1x, c_0 are called **terms.**

Although the terms of a polynomial can be written in any order, we usually write them in descending powers of x. This is called **standard form.**

The term containing the highest power of x is called the **leading term**.

The coefficient of the leading term is called the **leading coefficient**.

The **degree** of the polynomial is the power of x contained in the leading term.

Example 1

$5x + x^3 + - 1$

- written in standard form is $x^3 + 5x - 1$
- the leading term is x^3
- this is a third - degree polynomial
- it has one variable, x

Polynomials of the first few degrees have special names given below:

Degree	Name	Example
0	constant	7
1	linear	$11x + 3$
2	quadratic	$x^2 - 4xy + 6$
3	cubic	$2x^3 - 7x - 56$
4	quartic	$x^4 - 16$
5	quintic	$x^5 + 8x^4 - x^3 - 2x^2 + 5$

Adding and subtracting polynomials is simply the adding and subtracting of like terms.

Addition of polynomials is also called **collecting like terms**.

Note it!

The properties of real numbers also apply to polynomials since polynomials are expressions that represent real numbers. This says that all that you know about addition and subtraction with real numbers, will be true with polynomials.

Example 1

Add $2x – 5$ and $3x + 2y + 7$

$2x$ and $3x$ are like terms, -5 and 7 are like terms, $2y$ is on its own.

$2x + 3x -5 + 7 + 2y$

$= 5x + 2 + 2y$

$= 5x + 2y + 2$

Example 2

Add $3x^2y – 2x + 1$ and $4x^2y + 6x - 3$

$= 3x^2y + 4x^2y – 2x + 6x + 1 - 3$

$= 7x^2y + 6x - 2$

Example 3

Add $x^2 + ax + 12y - 4pq$ and $4x^2 + 5ax – 5$

$= x^2 + 4x^2 + ax + 5ax + 12y - 4pq – 5$

$= \quad 5x^2 + 6ax \qquad + 12y \ - 4pq – 5$

Example 4

Subtract 2s – rs from 8s + 3rs

$$(8s + 3rs) – (2s – rs)$$

We use the idea that subtracting a number is the same as adding its additive inverse. We therefore change the subtraction sign to addition, and change the signs of all of the terms being subtracted. Then follow the rules for adding signed numbers.

$$(8s + 3rs) + (-2s + - rs) = (8s + 3rs) + (-2s - rs)$$

Identify like terms and group them

$$= 8s -2s + 3rs - rs$$

Add like terms

$$= 6s + 2rs$$

Example 5

Subtract $2x^2 + 7$ from $5x^2 + 3x - 5$

$$(5x^2 + 3x - 5) – (2x^2 + 7)$$

Change the subtraction sign to addition, and change the signs of all of the terms being subtracted.

$$(5x^2 + 3x - 5) – (2x^2 + 7) = (5x^2 + 3x - 5) - (-2x^2 - 7)$$

Identify like terms and group them

$$= 5x^2 -2x^2 + 3x - 5 - 7$$

Add like terms

$$= 3x^2 + 3x - 12$$

Example 6

Add 2x – 5 and 3x + 2y + 7

(2x – 5) + (3x + 2y + 7)

Removing the brackets.

$2x - 5 + 3x + 2y + 7$

$= 2x + 3x + 2y + 7 - 5$

$= 5x + 2y - 12$

Example 7

Subtract $2s - rs$ from $8s + 3rs$

$(8s + 3rs) - (2s - rs)$

Removing the brackets:

$8s + 3rs - 2s + rs$

$= 8s - 2s + rs + 3rs$

$= 6s + 4rs = 4rs + 6s$

Activity 1

Perform the indicated operations

1. $(a + b) + (2a + 5b)$

2. $(a + b + c) + (a^2 - 5b)$

3. $(17a^4b - a^3 + 23a^2b + ab - 11) + (5a^3 + 5)$

4. $(5w + 4x + y) + (w - x + y) + (5x + 2y + z)$

5. $(a + b) - (3a + 7b)$

6. $(2x^2y + 3xy) - (x^2y - 5)$

7. $(6xyz + 14) - (xy + yz - 7)$

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1. $(a + b) + (2a + 5b) = a + b + 2a + 5b$
   $= a + 2a + b + 5b$
   $= 3a + 6b$

2. $(a + b + c) + (a^2 + 5b) = a + b + c + a^2 - 5b$
   $= a + b - 5b + c + a^2$
   $= a - 4b + c + a^2$

3. $(17a^4b - a^3 + 23a^2b + ab - 11) + (5a^3 + 5)$
   $= 17a^4b - a^3 + 23a^2b + ab - 11 + 5a^3 + 5$
   $= 17a^4b - a^3 + 5a^3 + 23a^2b + ab - 11 + 5$
   $= 17a^4b + 4a^3 + 23a^2b + ab - 6$

4. $(5w + 4x + y) + (w - x + y) + (5x + 2y + z)$
$= 5w + 4x + y + w - x + y + 5x + 2y + z$
$= 5w + w + 4x - x + 5x + y + y + 2y + z$
$= 6w + 8x + 4y + z$

5. $(a + b) - (3a + 7b) = a + b - 3a - 7b$
$= a - 3a + b - 7b$
$= -2a -6b$

6. $(2x^2y + 3xy) - (x^2y -5) = 2x^2y + 3xy - x^2y + 5$
$= 2x^2y - x^2y + 3xy + 5$
$= x^2y + 3xy + 5$

7. $(6xyz + 14) – (xy + yz – 7) = 6xyz + 14 – xy - yz + 7$
$= 6xyz + 14 + 7– xy - yz$
$= 6xyz + 21– xy - yz$
$= 6xyz– xy - yz + 21$

Key Points to Remember

The key points to remember in this subunit on addition and subtraction of polynomials are:

- The properties of real numbers also apply to polynomials since polynomials are expressions that represent real numbers.
- Adding and subtracting polynomials is simply the adding and subtracting of like terms.
- Addition of polynomials is also called **collecting like terms**.

Lesson 2 Operations of Multiplication of Polynomials

By the end of this subunit, you should be able to

- multiply polynomials

This subunit is about 4 pages in length.

All of the work for this subunit is the work that we have dealt with in the unit on algebraic manipulation. We will therefore only review that work.

This is what you should remember when you multiply polynomials:

- multiply numerical coefficients together, then;
- list all the variables that occur in the terms being multiplied and write them in alphabetical order; since it makes it easier to read when the problems become more involved;
- apply the laws of indices; add the exponents of like variables.

Example 1

1. $3a \times 5c = 15ac$
2. $4t^2(7t^7 - 8) = 28t^2 \times t^7 - 32t^2$

$= 28t^{2+7} - 32t^2$

$= 28t^9 - 32t^2$

There are some special products that we looked at. They are:

1. **Square of a sum :** $(a + b)^2$

$(a + b)^2 = (a + b)(a + b)$

$= a^2 + 2ab + b^2$

2. **Square of a difference :** $(a - b)^2$

$(a - b)^2 = (a - b)(a - b)$

$= a^2 - 2ab + b^2$

3. **Difference of two squares :** $(a + b)(a - b) = a^2 - b^2$

We have an additional special product called cube of a binomial:

$(a + b)^3 = (a + b)(a + b)(a + b)$
$= a^3 + 3a^2b + 3ab^2 + b^3$

$(a - b)^3 = (a - b)(a - b)(a - b)$
$= a^3 - 3a^2b + 3ab^2 - b^3$

Example 2 - Find each product:

1. $(3x + 7)^2 = (3x + 7)(3x + 7)$

$= 3x.3x + 3x.7 + 7.3x + 7.7$

$= 9x^2 + 21x + 21x + 49$

$= 9x^2 + 42x + 49$

2. $(3a – 5b)(3a + 5b) = 3a.3a + 3a.5b.3a – 5b.3a – 5b.5b$
$= 9a^2 + 15ab – 15ab + 25b^2$
$= 9a^2 + 25b^2$

3. $(7 – x)^3 = (7 - x)(7 - x)(7 - x)$
$= 7^3 + 3.7^2\ x + 3.7.x^2 - \ x^3$
$= 343 + 147x + 21x^2 - \ x^3$

Activity 2

Work out:

1. 3 (a + b)

2. 6y(y + z)

3. $(a^2)\ (b^2)$

4. (2x + y) (x + z)

5. 3(a + 4) (a + 2)

6. $-2x(3x^3 – x^2 – x -5)$

7. $(2x+ 5)^2$

8. $(2x - 5)^2$

9. $(3a - 10)(3a + 10)$

10. $(x + 3)(x + 3)(x + 3)$

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Key Points to Remember

The key points to remember in this subunit on multiplication of polynomials are:

- numerical coefficients are multiplied together.
- all the variables that occur in the terms being multiplied are to be written in alphabetical order.
- add the exponents of like variables.

Answers

1. $3(a + b) = 3a + 3b$

2. $6y(y + z) = 6y^2 + 6yz$

3. $(a^2)(b^2) = a^2b^2$

4. $(2x + y)(x + z) + (3x + y)(x + 2z)$
$= (2x^2 + 2xz + xy + yz) + (3x^2 + 6xz + xy + 2yz)$
$= 2x^2 + 3x^2 + 2xz + 6xz + xy + xy + yz + 2yz$
$= 5x^2 + 8xz + 2xy + 3yz$

5. $3(a + 4)(a + 2) = 3(a^2 + 2a + 4a + 8)$
$= 3(a^2 + 6a + 8)$
$= 3a^2 + 18a + 24$

6. $-2x(3x^3 – x^2 – x - 5) = -6x^4 + 2x^3 + 2x^2 + 10x$

7. $(2x + 5)^2 = (2x)^2 + 2(2x)5 + (5)^2$
$= 4x^2 + 20x + 25$

8. $(2x - 5)^2 = (2x)^2 - 2(2x)5 + (5)^2$
$= 4x^2 - 20x + 25$

9. $(3a – 10)(3a + 10) = (3a)^2 - (10)^2$
$= 9a^2 – 100$

10. $(x + 3)(x + 3)(x + 3) = x^3 + 3(x^2)3 + 3(x)3^2 + 3^3$
$= x^3 + 9x^2 + 27x + 27$

Lesson 3 Dividing a Polynomial by a Binomial

By the end of this subunit, you should be able to
- divide a polynomial by a binomial

This subunit is about 4 pages in length.

Division is finding out how many times one quantity is contained in another. It is also said to be the opposite of multiplication.

There are several ways of writing a division problem. Each statement below means “**a** divided by **b**”.

a ÷ b

$$\frac{a}{b}$$

a/b

a | b

Reflection

1. Divide 12 by 3

12 ÷ 3 = 4

12 is called the dividend, 3 is the divisor, 4 is the quotient.

3 is contained in 12, exactly 4 times.

We check if the answer is correct by multiplying 4 by 3.
4 x 3 = 12

4 and 3 are said to be factors of 12.

2. Divide 20 by 6

20 ÷ 6 = 3

6 is contained in 20, not exactly 3 times. It has a remainder of 2.

20 is called the dividend, 6 is the divisor, 3 is the quotient, 2 is the remainder.

We check if the answer is correct by multiplying 3 by 6.
3 x 6 = 18

18 + 2 = 20
6 is not a factor of 20 for it does not divide exactly 3 times into it.

Division of polynomials can be done very much like the division of real numbers.

Example 1

$x^2 - 9x - 10 \div x + 1$

This we can do with a number of methods. The first method is by factorising. The problem with the method is that it only works if the dividend can be factored, with one factor being the same as the divisor.

$$\frac{x^2 - 9x - 10}{x + 1}$$

We factorise $x^2 - 9x - 10$

$$\frac{x^2 - 9x - 10}{x + 1} = \frac{(x - 10)(x + 1)}{x + 1}$$

x + 1 in the numerator will cancel with x + 1 in the denominator

$$= \frac{(x - 10)}{1}$$

$$= x - 10$$

We can also get the result using the **“long division**” method.

This is used when the power of the divisor is less than or equal to the power of the dividend.

We always stop when the remainder is zero or a polynomial whose degree is less than the degree of the divisor.

You can verify the division using the following equation:

(quotient) x (divisor) + remainder = dividend

Notice that if the remainder is 0, then the divisor is a factor of the dividend.

Step 1

(a) Write the terms in the dividend and the divisor in order of descending powers. Place a zero for any missing terms.

$x^2 - 9x - 10 \div x + 1$

In this case, the terms in the dividend and the divisor are already written in descending order.

Step 2

Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.

$$\begin{array}{r|l} & x \\ \hline x + 1 & x^2 - 9x - 10 \end{array}$$

Step 3

- multiply the complete divisor by the quotient just obtained.
- write the terms of the product under the like terms of the dividend, and subtract this expression from the dividend.

$$\begin{array}{r|l} & x \\ \hline x + 1 & x^2 - 9x - 10 \\ & \underline{-x^2 + x} \\ & \quad -10x \end{array}$$

Step 4

Consider the remainder as a new dividend and divide it by the first term of the divisor and write the result as the second term of the quotient.

The new dividend is -10x

We divide it by x

$$\begin{array}{r|l} & x - 10 \\ \hline x + 1 & x^2 - 9x - 10 \\ & \underline{-x^2 + x} \\ & \quad -10x \end{array}$$

Step 5

Multiply the complete divisor by the quotient **just obtained**, which in this case is the second term of the quotient.

$$\begin{array}{r|l} & x - 10 \\ \hline x + 1 & x^2 - 9x - 10 \\ & \underline{-x^2 + x} \\ & \quad -10x \\ & \quad \underline{-10x - 10} \\ & \quad\ \ -\quad - \end{array}$$

(x + 1) and (x – 10) are factors of $x^2 - 9x - 10$

Let us multiply them out to check if they are correct.

(quotient) x (divisor) + remainder = dividend

$(x - 10)(x + 1) + 0 = (x^2 - 10x + x - 10) + 0$

$= x^2 - 9x - 10 + 0$

$= x^2 - 9x - 10$ (This was the starting expression.)

Example 2

$x^3 - x^2 - 9x - 6 \div x + 2$

$$
\begin{array}{r|l}
 & x^2 - 3x - 3 \\
\hline
x + 2 & x^3 - x^2 - 9x - 6 \\
 & \underline{x^3 + 2x^2} \\
 & \quad -3x^2 - 9x \\
 & \quad \underline{-3x^2 - 9x} \\
 & \qquad -3x - 6 \\
 & \qquad \underline{-3x - 6} \\
 & \qquad - \quad -
\end{array}
$$

$(x + 2)$ and $(x^2 - 3x - 3)$ are factors of $x^3 - x^2 - 9x - 6$

Let us multiply them out to check if they are correct.

$$
\begin{aligned}
(x + 2)(x^2 - 3x - 3) + 0 &= (x^3 - 3x^2 - 3x + 2x^2 - 6x - 6) + 0 \\
&= (x^3 - 3x^2 + 2x^2 - 3x - 6x - 6) + 0 \\
&= x^3 - x^2 - 9x - 6 + 0 \\
&= x^3 - x^2 - 9x - 6
\end{aligned}
$$

Example 3

$x^3 - 2x^2 + 3x - 4 \div x + 2$

$$
\begin{array}{r|l}
 & x^2 - 4x + 11 \\
\hline
 &
\end{array}
$$

$x + 2 \quad x^3 - 2x^2 + 3x - 4$

$\underline{\quad\quad x^3 + 2x^2}$

$-4x^2 + 3x$

$\underline{-4x^2 - 8x}$

$11x - 4$

$\underline{11x + 22}$

-26

$(x + 2)$ and $(x^2 - 4x + 11)$ are **not** factors of $x^3 - 2x^2 + 3x - 4$ for there is a remainder of -26

We can still, however, multiply out to check if they give us the original expression.

$$(x + 2)(x^2 - 4x + 11) + -26 = (x^3 - 4x^2 + 11x + 2x^2 - 8x - 22) + -26$$

$$= (x^3 - 4x^2 + 2x^2 + 11x - 8x - 22) + -26$$

$$= (x^3 - 2x^2 + 3x + 22) + -26$$

$$= x^3 - 2x^2 + 3x - 4$$

The divisor is a factor only if the remainder = 0

Activity 3

Work out each of the following using long division

1. $x^2 - 3x - 4 \div x + 1$

2. $2x^2 + x - 6 \div x + 2$

3. $x^3 + 5x^2 - 9x - 45 \div x - 3$

4. $3x^4 - 25x^2 - 18 \div x - 3$

5. $x^4 + 5x \div x + 5$

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Key Point to Remember

The key point to remember in this subunit on division of polynomials are that division of polynomials can be done very much like the division of real numbers:

(a) arrange the terms in descending powers
(b) place a zero for any missing terms
(c) divide by the long division method

Answers

1.

$$\begin{array}{r|l} & \quad x - 4 \\ \hline x + 1 & \quad x^2 - 3x - 4 \\ & \quad \underline{x^2 + x} \\ & \qquad -4x - 4 \\ & \qquad \underline{-4x - 4} \end{array}$$

$x^2 - 3x - 4 \div x + 1 = x - 4$

2. $2x^2 + x - 6 \div x + 2$

$$\begin{array}{r|l} & \quad 2x - 3 \\ \hline x + 2 & \quad 2x^2 + x - 6 \\ & \quad \underline{2x^2 + 4x} \\ & \qquad -3x - 6 \\ & \qquad \underline{-3x - 6} \end{array}$$

$2x^2 + x - 6 \div x + 2 = 2x - 3$

3. $x^3 + 5x^2 - 9x - 45 \div x - 3$

$$
\begin{array}{r|l}
 & x^2 + 8x + 15 \\
\hline
x - 3 & x^3 + 5x^2 - 9x - 45 \\
 & \underline{x^3 - 3x^2} \\
 & 8x^2 - 9x \\
 & \underline{8x^2 - 24x} \\
 & 15x - 45 \\
 & \underline{15x - 45}
\end{array}
$$

$x^3 + 5x^2 - 9x - 45 \div x - 3 = x^2 + 8x + 15$

4. $3x^4 - 25x^2 - 18 \div x - 3$

$$
\begin{array}{r|l}
 & 3x^3 + 9x^2 + 2x + 6 \\
\hline
x - 3 & 3x^4 + 0x^3 - 25x^2 + 0x - 18 \\
 & \underline{3x^4 - 9x^3} \\
 & 9x^3 - 25x^2 \\
 & \underline{9x^3 - 27x^2} \\
 & 2x^2 + 0x \\
 & \underline{2x^2 - 6x} \\
 & 6x - 18 \\
 & \underline{6x - 18} \\
 & - \quad -
\end{array}
$$

$3x^4 - 25x^2 - 18 \div x - 3 = 3x^3 + 9x^2 + 2x + 6$

5. $x^4 + 5x \div x + 5$

$$
\begin{array}{r|l}
 & x^3 - 5x^2 + 25x - 120 \\
\hline
x + 5 & x^4 + 0x^3 + 0x^2 + 5x + 0 \\
 & \underline{x^4 + 5x^3} \\
 & \quad -5x^3 + 0x^2 \\
 & \quad \underline{-5x^3 - 25x^2} \\
 & \qquad 25x^2 + 5x \\
 & \qquad \underline{25x^2 + 125x} \\
 & \qquad\qquad -120x + 0 \\
 & \qquad\qquad \underline{-120x - 600} \\
 & \qquad\qquad\qquad 600
\end{array}
$$

$x^4 + 5x \div x + 5 = x^3 - 5x^2 + 25x - 120$ remainder 600

Lesson 4 Remainder Theorem

By the end of this subunit, you should be able to

- apply the remainder theorem

This subunit is about 3 pages in length.

The higher the degree of an expression gets, the harder, and the more difficult the division of polynomials can get. We will still be expected to do it, though!

There is however a much easier way of finding the remainder without actually performing the division. It can also be used as a simple test in factorisation. In other words, if the remainder is 0 then using the factorisation method of division is faster. This is called the **Remainder Theorem**.

The Remainder Theorem says:

When you divide a polynomial f(x) by the linear expression x-c, until the remainder does not contain x, then the remainder, R = f(c).

We will test the remainder theorem with some of the examples we used with long division in the previous subunit.

Example 1

Determine the remainder when we have $x^3 - 2x^2 + 3x - 4 \div x + 2$

- The polynomial $f(x) = x^3 - 2x^2 + 3x - 4$
- The divisor is x + 2. We write it in the form x - c

$x + 2 = x - (-2)$

$f(-2) = x^3 - 2x^2 + 3x - 4$

$= (-2)^3 - 2(-2)^2 + 3(-2) - 4$

$= -8 - 8 - 6 - 4$

$= -26$

The remainder is -26. This says x + 2 is not a factor of $x^3 - 2x^2 + 3x - 4$

Example 2

Determine the remainder when we have $x^3 - x^2 - 9x - 6 \div x + 2$

- The polynomial $f(x) = x^3 - x^2 - 9x - 6$
- The divisor is x + 2. We write it in the form x - c

$x + 2 = x - (-2)$

$f(-2) = x^3 - x^2 - 9x - 6$

$= (-2)^3 - (-2)^2 - 9(-2) - 6$

$= -8 - 4 + 18 - 6$

$= 0$

This says that x + 2 is a factor of $x^3 - x^2 - 9x - 6$

Example 3

Determine the remainder when we have $x^3 - 6x + 1 \div x - 3$

- The polynomial $f(x) = x^3 - 6x + 1$
- The divisor is x - 3. We write it in the form x - c

$x - 3 = x - (+3)$

$= x - 3$

$f(3) = x^3 - 6x + 1$

$= x^3 + 0x^2 - 6x + 1$

$= (3)^3 + 0(3)^2 - 6(3) + 1$

$= 27 + 0 - 18 + 1$

$= 10$

This says x - 3 is not a factor of $x^3 - 6x + 1$

With the help of the remainder theorem, we can be able to determine the value of the unknowns in a polynomial.

Example 4

If $f(x) = x^3 + 3x^2 + kx + 1$ leaves a remainder of 3 when divided by x -1, determine the value of *k*.

- The polynomial $f(x) = x^3 + 3x^2 + kx + 1$
- The divisor is x -1. We write it in the form x – c. It is x – (1)

$f(1) = x^3 + 3x^2 + kx + 1$

$= (1)^3 + 3(1)^2 + k(1) + 1$

$= 1 + 3 + k + 1$

$= 5 + k$

$= 5 + k$

Remainder = f(c)

$3 = 5 + k$

$-2 = k$

Activity 4

Find the remainder when

1. $2x^3 - 4x^2 + x \div x - 3$

2. $x^3 + x^2 + x + 1 \div x + 3$

3. $x^4 - 6x^2 + 1 \div x - 3$

4. $3x^3 - 2x^2 - 7x - 2 \div 3x + 1$

5. $6x^3 - 5x^2 - 12x - 4 \div 2x + 1$

Compare your answers with those at the end of this subunit. Continue on if you had at least four out of five correct. If not, review the above content and work through the activity again.

Key Point to Remember

The key point to remember in this subunit on factorisation is:

- The Remainder Theorem is a shortcut to polynomial long division. It states that when you divide a polynomial f(x) by a linear expression x-c, until the remainder does not contain x, then the remainder, R = f(c).

Answers

1. $2x^3 - 4x^2 + x \div x - 3$

- The polynomial $f(x) = 2x^3 - 4x^2 + x$
- The divisor is x - 3. We write it in the form x - c

$x - 3 = x – (+3)$

$= x - 3$

$f(3) = 2x^3 - 4x^2 + x$

$= 2x^3 - 4x^2 + x + 0$

$= 2(3)^3 - 4(3)^2 + (3) + 0$

$= 2(27) – 36 + 3+ 1$

$= 54 – 36 + 3+ 1$

$= 22$

This says x - 3 is **not** a factor of $x^3 - 6x + 1$

2. $x^3 + x^2 + x + 1 \div x + 3$

- The polynomial $f(x) = x^3 + x^2 + x + 1$
- The divisor is x + 3. We write it in the form x - c

$x + 3 = x – (-3)$

$f(-3) = x^3 + x^2 + x + 1$

$= (-3)^3 + (3)^2 + (-3) + 1$

$= -27 + 9 - 3+ 1$

$= -20$

This says x + 3 is not a factor of $x^3 + x^2 + x + 1$

3. $x^4 - 6 x^2 + 1 \div x – 3$

- The polynomial $f(x) = x^4 - 6 x^2 + 1$
- The divisor is x – 3. We write it in the form x - c

$x – 3 = x – (3)$

$f(3) = x^4 - 6x^2 + 1$

$= x^4 + 0x^3 - 6x^2 + 0x + 1$

$= (3)^4 + 0(3)^3 - 6(3)^2 + 0(3) + 1$

$= 81 + 0 - 54 + 0 + 1$

$= 28$

This says $x - 3$ is not a factor of $x^4 - 6x^2 + 1$

4. $3x^3 - 2x^2 - 7x - 2 \div 3x + 1$
5. $6x^3 - 5x^2 - 12x - 4 \div 2x + 1$

- The polynomial $f(x) = 6x^3 - 5x^2 - 12x - 4$
- The divisor is $x - 3$. We write it in the form $x - c$

$x - 3 = x - (+3)$

$= x - 3$

$f(3) = 2x^3 - 4x^2 + x$

$= 2x^3 - 4x^2 + x + 0$

$= 2(3)^3 - 4(3)^2 + (3) + 0$

$= 2(27) - 36 + 3 + 1$

$= 54 - 36 + 3 + 1$

$= 22$

Lesson 5 Factor Theorem

By the end of this subunit, you should be able to
- apply the factor theorem

This subunit is about 4 pages in length.

The **Factor Theorem** is a special case of the remainder theorem.

The Factor Theorem states that

$x - a$ is a factor of the polynomial, f(x), if f(a) = 0

Just as with the Remainder Theorem, the point here is not to do the long division of a given polynomial by a given factor.

Example 1

Determine whether x + 4 is a factor of $x^3 - 16x$

The polynomial, f(x), = $x^3 - 16x$

x + 4 written in the form $x - a$ = x – (-4)

a = -4

$f(a) = x^3 - 16x$

$= (-4)^3 - 16(-4)$

$= (-4)^3 - 16(-4)$

$= -64 - (-64)$

$= -64 + 64$

$= 0$

Therefore x + 4 is a factor of $x^3 - 16x$

Example 2

Determine whether x - 1 is a factor of $x^2 + 5x + 6$

The polynomial, f(x), = $x^2 + 5x + 6$

x - 1 written in the form $x - a$ = x – (1)

a = 1

$f(a) = x^2 + 5x + 6$

$= (1)^2 + 5(1) + 6$

$= 1 + (5) + 6$

$= 1 + 5 + 6$

$= 12$

Therefore x - 1 is **not** a factor of $x^2 + 5x + 6$

There are times when we are asked to find factors of a given polynomial. We will work towards finding a factor with the help of factorization by trial and inspection.

Example 3

Find the factors of $x^3 - 6x^2 + 11x - 6$

Possible values of x are1, 6, 2, 3, -1, -6, -2, or -3

Putting x = 1,

$x^3 - 6x^2 + 11x - 6 = 1 - 6 + 11 - 6$
$= 0$

Therefore (x- 1) is a factor.

By long division method,

$$\begin{array}{r|l} & x^2 - 5x + 6 \\ x - 1 & x^3 - 6x^2 + 11x - 6 \\ & \underline{x^3 - x^2} \\ & \quad -5x^2 + 11x \\ & \quad \underline{-5x^2 + 5x} \\ & \qquad\quad 6x - 6 \\ & \qquad\quad \underline{6x - 6} \end{array}$$

Factors of $x^3 - 6x^2 + 11x - 6$ are x- 1 and $x^2 - 5x + 6$

We know that $x^2 - 5x + 6$ can be factorised to give (x - 2) (x - 3)

Therefore factors of $x^3 - 6x^2 + 11x - 6$ are x- 1, x – 2 and x – 3

With the help of the factor theorem, we can be able to determine the value of the unknowns in a polynomial.

Example 4

If x – 2 is a factor of $x^3 - 3x^2 - px + 3p$, prove that x – 3 is another factor of $x^3 - 3x^2 - px + 3p$

Given that x -2 is a factor $x^3 - 3x^2 - px + 3p$,

$f(2) = 0$

$2^3 - 3.2^2 - p.2 + 3.p = 0$

$8 - 12 - 2p + 3p = 0$

$-4 + p = 0$

$p = 4$

Therefore $f(x) = x^3 - 3x^2 - px + 3p$,

$= x^3 - 3x^2 - 4x + 3.4$

$= x^3 - 3x^2 - 4x + 12$

$f(x) = x^3 - 3x^2 - 4x + 12$

$f(3) = 3^3 - 3.3^2 - 4.3 + 12$

$= 27 - 27 - 12 + 12$

$= 0$

Therefore x – 3 is a factor of f(x)

Example 5

If x + 3 is a factor of $f(x) = x^3 + 2x^2 - 5x - p$, determine the value of p

$f(x) = x^3 + 2x^2 - 5x - p$,

$f(-3) = 0$

$(-3)2 + 2(-3)2 - 5(-3) - p = 0$

$-27 + 18 + 15 - p = 0$

$6 - p = 0$

$p = 6$

Activity 1

1. Which of the following are factors of $f(x) = x^3 - 5x^2 + 3x + 9$

   (a) $x - 1$ (b) $x + 3$ (c) $x - 3$

2.If $x - 3$ is a factor of $2x^3 + px^2 - 28x - 15$, determine the value of p

After completing the questions, compare your answers to the correct answers at the end of this subunit. Take the time needed to understand each answer before continuing.

Answers

1.

(a) The polynomial, f(*x*), $= x^3 - 5x^2 + 3x + 9$

x - 1 written in the form $x - a$ = x – (1)

a = 1

f(a) = $x^3 - 5x^2 + 3x + 9$

= $(1)^3 - 5(1)^2 + 3(1) + 9$

= 1 -5 + 3 + 9

= 8

Remainder ≠ 0, therefore x - 1 is **not** a factor of $x^3 - 5x^2 + 3x + 9$

(b) The polynomial, f(x), = $x^3 - 5x^2 + 3x + 9$

x + 3 written in the form $x - a$ = x – (-3)

a = -3

f(a) = $x^3 - 5x^2 + 3x + 9$

= $(-3)^3 - 5(-3)^2 + 3(-3) + 9$

= (-27) -5(9) +(-9) + 9

= -27 -45 - 9 + 9

= -72

Remainder ≠ 0, therefore x + 3 is **not** a factor of $x^3 - 5x^2 + 3x + 9$

(c) The polynomial, f(x), = $x^3 - 5x^2 + 3x + 9$

x - 3 written in the form $x - a$ = x – (3)

a = 3

f(a) = $x^3 - 5x^2 + 3x + 9$

= $(3)^3 - 5(3)^2 + 3(3) + 9$

= 27 -45 + 9 + 9

$= 0$

Remainder = 0, therefore x - 3 is a factor of $x^3 - 5x^2 + 3x + 9$

2.$f(x) = 2x^3 + px^2 - 28x - 15$

$f(3) = 2.(3)^3 + p(3^2) - 28(3) - 15$

$= 54 + p.9 - 84 - 15$

$= 9p - 45$

$9p = 45$

$p = 5$

Key Points to Remember

The key points to remember in this subunit on factorisation are:

- The Factor Theorem is a special case of the Remainder Theorem. It is a shortcut to polynomial long division.

The Factor Theorem states that

$x - a$ is a factor of the polynomial, $f(x)$, if $f(a) = 0$

You have now completed the last subunit of this unit on polynomials. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Unit Summary

Summary

In this unit you learned that

- Polynomials are expressions that can be written in the form

$c_n x^n + c_{n-1} x^{n-1} + \dots\dots\dots + c_2 x^2 + c_1 x + c_0$

Although the terms of a polynomial can be written in any order, we usually write them in descending powers of x. This is called **standard form.**

- Adding and subtracting polynomials is simply the adding and subtracting of like terms.
- In multiplication of polynomials are:

  - numerical coefficients are multiplied together

  - all the variables that occur in the terms being multiplied are to be written in alphabetical order

  -add the exponents of like variables.

- division of polynomials can be done very much like the division of real numbers.
- remainder Theorem states that when you divide a polynomial f(x) by a linear expression x-c, until the remainder does not contain x, then the remainder, R = f(c).
- factor theorem states that $x - a$ is a factor of the polynomial, f(x), if f(a) = 0

You have completed the material for this unit on polynomials. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Assignment

Instructions

1. Answer All Questions.
2. Show all the necessary working.

Total marks = 32 **Time:** 1 hour

1. Perform the indicated operations

   (a) $(2x^2 – 3x + 4) + (9 - 2x – x^2)$ [2]

   (b) $(3s + 2t) - (9s + 4t^2)$ [2]

   (c) $(s + t^2)(s^2 – 3t)$ [2]

2. Perform the indicated divisions. Check the division using

   (quotient) x (divisor) + remainder = dividend

   (a) $(x^2 + 2x - 24) \div (x + 6)$ [3]

(b) $(x^3 - 3x^2 + x - 55) \div (x + 3)$ [3]

3. Which of the following are factors of $f(x) = x^4 - 3x^3 + 5x - 2$

(a) $x + 2$ (b) $x - 2$ (c) $x + 4$

[12]

4. Use the Remainder Theorem to determine the remainder when

(a) $x^3 - 2x^2 + 3x - 4$ is divided by $x + 2$ [4]

(b) $2x^2 - 3x - 1$ is divided $x - 5$ [4]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1.

(a) $2x^2 - 3x + 4$

$+$

$-x^2 - 2x + 9$

$x^2 - 5x + 13$

(b) $3s + 2t + 0t^2$

$-$

$9s + 0 + 4t^2$

$-6s + 2t - 4t^2$

(c) $(s + t^2)(s^2 - 3t)) = s^3 - 3st + t^2s^2 - t^2.3t$

$= s^3 - 3st + t^2s^2 - 3t^3$

2.

(a)

$$
\begin{array}{r|l}
 & x - 4 \\
x + 6 & x^2 + 2x - 24 \\
 & \underline{x^2 + 6x} \\
 & \quad -4x - 24 \\
 & \quad \underline{-4x - 24} \\
 & \quad\quad - \quad -
\end{array}
$$

or

-6	1	2	-24
		-6	24
	1	-4	0

Check

(quotient) x (divisor) + remainder = dividend

$(x - 4) \times (x + 6) + 0 = (x^2 + 6x - 4x - 24) + 0$

$= (x^2 + 2x - 24) + 0$

$= x^2 + 2x - 24$

(b)

$$
\begin{array}{r|l}
 & x^2 - 6x + 19 \\
x + 3 & x^3 - 3x^2 + x - 55 \\
 & \underline{x^3 + 3x^2} \\
 & -6x^2 + x \\
 & \underline{-6x^2 - 18x} \\
 & 19x - 55 \\
 & \underline{19x + 57} \\
 & -112
\end{array}
$$

or

-3	1	-3	1	-55
		-3	18	-57
	1	-6	19	-112

Check

(quotient) x (divisor) + remainder = dividend

$(x^2 - 6x + 19) \times (x + 3) + 2 \quad = (x^3 + 3x^2 - 6x^2 - 18x + 19x + 57) + -112$

$= (x^3 - 3x^2 + x + 57) + -112$

$= x^3 - 3x^2 + x - 55$

3.

(a) The polynomial, f(x), $= x^4 - 3x^3 + 5x - 2$

x + 2 written in the form $x - a = x - (-2)$

$a = -2$

$f(a) = x^4 - 3x^3 + 5x - 2$

$= (-2)^4 - 3(-2)^3 + 5(-2) - 2$

$= 16 + 24 - 10 - 2$

$= 28$

Remainder $\neq 0$, therefore x + 2 is not a factor of $x^4 - 3x^3 + 5x - 2$

(b) The polynomial, f(*x*), $= x^4 - 3x^3 + 5x - 2$

x - 2 written in the form $x - a = x - (2)$

$a = 2$

$f(a) = x^4 - 3x^3 + 5x - 2$

$= (2)^4 - 3(2)^3 + 5(2) - 2$

$= 16 - 24 + 10 - 2$

$= 0$

Remainder = 0, therefore x - 2 is a factor of $x^4 - 3x^3 + 5x - 2$

(c) The polynomial, f(*x*), $= x^4 - 3x^3 + 5x - 2$

x + 4 written in the form $x - a = x - (-4)$

$a = -4$

$f(a) = x^4 - 3x^3 + 5x - 2$

$= (-4)^4 - 3(-4)^3 + 5(-4) - 2$

$= 256 + 192 - 20 - 2$

$= 426$

Remainder $\neq 0$, therefore $x + 4$ is not a factor of $x^4 - 3x^3 + 5x - 2$

4.

(a) $x^3 - 2x^2 + 3x - 4$ is divided by $x + 2$ [4]

The polynomial $f(x) = x^3 - 2x^2 + 3x - 4$

$f(-2) = (-2)^3 - 2(-2)^2 + 3(2) - 4$

$= -8 - 8 - 6 - 4$

$= -26$

The remainder = -26

(b) $2x^2 - 3x - 1$ is divided $x - 5$ [4]

The polynomial $f(x) = 2x^2 - 3x - 1$

$f(5) = 2(5)^2 - 3(5) - 1$

$= 50 - 15 - 1$

$= 34$

The remainder = 34

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Instructions

3. Answer All Questions.
4. Show all the necessary working.

Total marks = 31 **Time:** 1 hour

1. Perform the indicated operations

(a) $(7x^2 + 11x - 5) + 3x^2$ [2]

(b) $(x^2 \text{ -}3x - 5) - (3x^2 - x)$ [2]

(c) $(25xy)\,(8x^2y^3z)$ [2]

2. Work out each of the following using

(a) long division

(b) the remainder theorem, to find the remainder R.

(i) $2x^3 + x^2 - 3x + 1 \div x - 2$ [6]

(ii) $5x^3 + 6x^2 - 3x + 1 \div x - 3$ [6]

(iii) $2x^4 - 5x^2 + 7x + 9 \div x + 2$ [6]

3. Use the Factor Theorem to determine if the second expression is a factor of the first.

(a) $7x^3 - 6x^2 + 9x - 1 \, ; \, x + 1$ [3]

(b) $3x^4 + x^3 - 5x + 8$; $x - 2$ [3]

(c) $3x^3 - 8x^2 + 5x - 2$; $x - 2$ [3]

4. Find expressions that divide into $x^2 - 4x - 6$ and leave a remainder of -9 [4]

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1. (a) $10x^2 + 11x - 5$

(c) $-2x^2 - 2x - 5$

(d) $200x^3y^4z$

2.

polynomials	**Long division**	**Remainder Theorem**
$2x^3 + x^2 - 3x + 1$ $\div x - 2$	$2x^2 + 5x + 7$ remainder 15	f(2) = 15
$5x^3 + 6x^2 -3x + 1$ $\div x - 3$	$5x^2 + 21x + 60$ remainder 181	f(3) = 181
$2x^4 -5x^2 + 7x + 9$ $\div x + 2$	$2x^3 - 4x^2 +3x + 1$ remainder 7	f(-2) = 7

3.

(a) Remainder = -23, therefore x + 1 is **not** a factor of $7x^3 - 6x^2 +9x - 1$;

(b) Remainder = 54, therefore x-2 is **not** a factor of $3x^4 + x^3 -5x + 8$

(c) Remainder = 0, therefore x – 2 is a factor of $3x^3 -8x^2 + 5x - 2$

4.

The expressions are x – c

f(c) = $x^2 -4x - 6$

f(c) = -9

$c^2 -4c - 6 = -9$

$c^2 -4c + 3 = 0$

$(c - 3)(c - 1) = 0$

The expressions are (x –) and (x – 1)

Unit Contents

Unit 30

Unit 30

Logarithms

Introduction

You have already learned how to change a number into index form. In this unit you will learn how to change the index form of a number into a log form, evaluate terms written in log form, and use them to perform several calculations.

"Logs were invented as a means of calculating and, in fact, were used as such until only a few years ago when calculators were allowed to be used…….They were later found to have other uses in more advanced mathematics and also in Science" (A. D. Abbott 1994 p102). In Science logs are used on pH scales. The logarithmic scale is marked off in distances proportional to the logarithms of the values being represented. When data that covers the large range of values is presented on a logarithmic scale it is reduced to a more manageable range. Logarithms are also applied in measuring earthquake intensity on a logarithmic scale called the **Richter scale**. In **astronomy**, the **apparent magnitude** measures the brightness of **stars** logarithmically. **Musical intervals** are measured logarithmically as **semitones**. In advanced mathematics logarithms are helpful in calculations that involve a large range of values. In this unit you will learn about how to use logs in calculations such as solving equations.

This Unit is Comprised of Seven Lessons:

Lesson 1 Expressing Indices in Log Form
Lesson 2 Expanding Logarithms
Lesson 3 Evaluating logs
Lesson 4 Change the Base of a Log
Lesson 5 Solve Log Equations Involving the Same
Lesson 6 Solve Log Equations Using a Change of Base
Lesson 7 Sketching Log Graphs

Upon completion of this unit you will be able to:

Outcomes

- *express* indices in log form.
- *evaluate* terms written in log form
- *add and subtract* terms that are written in log form
- *evaluate* logs
- *change* the base of logs
- *solve* log equations involving terms that have the same bases
- *solve* log equations involving terms that have different bases
- *sketch* log graphs

Terminology

Log:	Exponent or index or power e.g. in x^3, 3 is an exponent or index or power.
Base:	The number raised to an index e.g. in x^3, x is a base.
Asymptote:	A straight line which a curve approaches arbitrarily closely, but never reaches, as they go to infinity.
Index form:	The index form a number is when it is written to a base. For example, the index form of 16 to base 2 is 2^4. The base needs to be indicated as 16 can be expressed in other bases, such as in $16 = 4^2$, here the base is 4.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Expressing Indices in Log Form

Reminder:
Let us start by reminding ourselves about how a number is changed to index form:
Change the following numbers into index form using the given base. The first two have been done for you as a review.

Activity 1

Number	base	Number expressed in Index form using the base	equation
4	2	2^2	$4 = 2^2$
9	3	3^2	$9 = 3^2$
25	5	__________	__________
64	4	__________	__________
49	7	__________	__________
27	3	__________	__________
100	10	__________	__________

Compare your answers with those at the end of the subunit. Be sure you understand how to express a number in index form using the base before you continue.

The equations generated in the activity above can be written in log form. Let us see how this is done.

Converting index form to log form examples

An equation $8 = 2^3$ when written as log form is $\log_2 8 = 3$. Remember that 2 is the base and 3 the exponent. Study the following examples to learn how we create the index form of a number and how the index form is converted into a log form.

Index form	Log form
$25 = 5^2$	$\log_5 25 = 2$
$100 = 10^2$	$\text{Log}_{10} 100 = 2$
$9 = 3^2$	$\log_3 9 = 2$
$16 = 2^4$	$\log_2 16 = 4$
$64 = 4^3$	$\log_4 64 = 3$
$49 = 7^2$	$\log_7 49 = 2$
$27 = 3^3$	$\log_3 27 = 3$

Activity 2

Activity 2

Write the following equations in log form

(a) $8 = 2^3$ ______________________________

(b) $49 = 7^2$ ______________________________

(c) $1 = 4^0$ ______________________________

(d) $a = b^m$ ______________________________

Compare your answers with those at the end of the subunit. Be sure you understand how to change equations to log form express indices in log form before you continue to the next section.

Converting Log Form to Index Form Example and Activity

Can you reverse the process and change a log into an index form? Using the example, $\log_9 81 = 2$ is another way of writing $81 = 9^2$, do the following activity.

Activity 3

Change the following into index form

(a) $\log_2 64 = 6$ ______________________________

(b) $\log_2 16 = 4$ ______________________________

(c) $\log_4 x = 1$ ______________________________

(d) $\log_{10} x = 3$ ______________________________

Compare your answers with those at the end of the subunit. Be sure you understand how to express the log form as indices.

Summary

Before you continue to the next section, note the following summary of the relationships learned in this subunit:

If $a = b^m$ then $\log_b a = m$

If $\log_b a = m$ then $a = b^m$

Answers to subunit activities

Answers to activity 1

Number	Base	Number expressed in Index form using the base	Equation
4	2	2^2	$4 = 2^2$
8	2	2^3	$9 = 3^2$
25	5	5^2	$25 = 5^2$
64	4	4^3	$64 = 4^3$
49	7	7^2	$49 = 7^2$
27	3	3^3	$27 = 3^3$
100	10	10^2	$100 = 10^2$

Answers to activity 2

(a) $\log_2 8 = 3$

(b) $\log_7 49 = 2$

(c) $\log_4 1 = 0$

(d) $\log_b a = m$

Answers to activity 3

(a) $64 = 2^6$ (b) $16 = 2^4$ (c) $x = 4^1$ (d) $x = 10^3$

Adding and Subtracting Logs

By the end of this subunit, you should be able to add and subtract logs.

Adding and subtracting logs having the same base resembles what happens, as you have done previously, when you multiply and divide indices having the same base. For example:

Multiplication: $2^2 \times 2^4 = 2^{2+4} = 2^6$

Division: $2^5 \div 2^3 = 2^{5-3} = 2^2$

In multiplication the indices/logs are added, while in division the indices/logs are subtracted.

Note that when the base is not written you should take the base to be 10. For example, log5 means $\log_{10} 5$.

Adding Logs

Let us show that $\log_a b + \log_a d = \log_a (b \times d)$ by substituting numbers into the formula**.**

For example, $\log_2 8 + \log_2 4 = \log_2(8 \times 4)$.

Remember that the statement $\log_2 8$ and $\log_2 4$ are each equal to an exponent as will be illustrated below. Let us see what exponent we get from each statement.

$\log_2 8 = 3$

$\log_2 4 = 2$

Adding the two equations we get $\log_2 8 + \log_2 4 = 5$. This shows that

$\log_2 8 + \log_2 4 = \log_2 8 \text{x} 4 = \log_2 32 = 5$

Subtracting Logs

Similarly, let us show that $\log_a b - \log_a d = \log_a \frac{b}{d}$ by substituting numbers in the formula.

For example, $\log_3 81 - \log_3 9 = \log_3 \frac{81}{9}$

$\log_3 81 = 4$

$\log_3 9 = 2$

The right hand side is as follows:

$4 - 2 = 2$

$\log_3 81 - \log_3 9 = \log_3 \frac{81}{9} = \log_3 9 = 2$

Study the following examples to learn how add and subtract logs:

Logs	Evaluat e each term and add	Original statement reduced	Reduc ed statement reduced further

	the values to get the log		
(a) $\log_3 9 + \log_3 3$	2 + 1= 3	$\log_3 9 \times 3 = 3$	$\log_3 27 = 3$
(b) $\log_5 25 + \log_5 5$	2 + 1 = 3	$\log_5 25 \times 5 = 3$	$\log_5 125 = 3$
(c) $\log_2 16 + \log_2 4$	4 + 2 = 6	$\log_2 16 \times 4 = 6$	$\log_2 64 = 6$
(d) $\log_4 16 + \log_4 64$	2 + 3 = 5	l $\log_4 16 \times 64 = 5$	$\log_4 1024 = 5$
(e) $\log_2 8 + \log_2 32$	3 + 5 = 8	$\log_2 8 \times 32 = 8$	$\log_2 256 = 8$
(f) $\log_3 27 + \log_3 9$	3 + 2 = 5	$\log_3 27 \times 9 = 5$	$\log_3 243 = 5$
Subtraction			
(g) $\log_3 9 - \log_3 3$	2 – 1 = 1	$\log_3 \frac{9}{3} = 3$	$\log_3 3 = 1$
(h) $\log_5 25 - \log_5 5$	2 – 1 = 1	$\log_5 \frac{25}{5} = 1$	$\log_5 5 = 1$
(i) $\log_2 16 - \log_2 4$	4 – 2 = 2	$\log_2 \frac{16}{4} = 2$	$\log_2 4 = 2$
(j) $\log_2 8 - \log_2 4$	3 – 2 = 1	$\log_2 \frac{8}{4} = 1$	$\log_2 2 = 1$
(k) $\log_3 27 - \log_3 9$	3 – 2 = 1	$\log_3 \frac{27}{9} = 1$	$\log_3 3 = 1$
(l) $\log_4 64 - \log_4 4$	3 – 1 = 2	$\log_4 \frac{64}{4} = 2$	$\log_4 16 = 2$

Summary

For a>0, a $\neq 1$, b, c $\in$ Real numbers

$\log_a b + \log_a c = \log_a b \times c$

$\log_a b - \log_a c = \log_a \frac{b}{c}$

Activity 4

Add and subtract the following logs by reducing them to a single log term:

Activity 1

1.

(a) $\log_2 3 + \log_2 4 = ?$

(b) $\log_5 5 + \log_5 2 = ?$

(c) $\log_3 6 + \log_3 4 = ?$

(d) $\log_4 2 + \log_4 3 = ?$

(e) log 4 + log 5 = ?

2.

(a) $\log_7 2 - \log_7 5 = ?$

(b) log 8 – log 3 =?

(c) $\log_5 8 - \log_5 2 = ?$

(d) $\log_6 36 - \log_6 6 = ?$

Compare your answers with those at the end of the subunit. If you scored 8 or 9 out of 9, you understand how to add and subtract logs and should continue to the next subunit. If you scored 7 or less, you should review this subtopic and try the activity again.

Answers to activity 4:

1.

(a) $\log_2 3 + \log_2 4 = \log_2 3 \times 4 = \log_2 12$

(b) $\log_5 5 + \log_5 2 = \log_5 5 \times 2 = \log_5 10$

(c) $\log_3 6 + \log_3 4 = \log_3 6 \times 4 = \log_3 24$

(d) $\log_4 2 + \log_4 3 = \log_4 2 \times 3 = \log_4 6$

(e) $\log 4 + \log 5 = \log 4 \times 5 = \log 20$

2.

(a) $\log_7 2 - \log_7 5 = \log_7 \frac{2}{5}$

(b) $\log 8 - \log 3 = \log\frac{8}{3}$

(c) $\log_5 8 - \log_5 2 = \log_5 \frac{8}{2} = \log_5 4$

(d) $\log_6 36 - \log_6 6 = \log_6 \frac{36}{6} = \log_6 6 = 1$

Lesson 2 Expanding Logarithms

By the end of this subunit, you should be able to expand logs.

Let us now reverse the process by "expanding the log".

Note the $\text{Log}_a 12$ can be expanded in several ways because 12 has more than one pair of factors:
$12 = 3 \times 4$ or $12 = 6 \times 2$ So $\log_a 12 = \log_a(3 \times 4)$ or $\log_a 6 \times 2$)

Example 1

Expand $\log_2 6$:

$\log_2 6 = \log_2 3 \times 2$

In this example 3 and 2 are the only factors of 6 so this will be the only way that we can expand the logarithm.

Example 2

Expand $\log_3 12$:

$\log_3 3 \times 4$ or $\log_3 6 \times 2$

Since 12 has factors 3×4 and 6×2 as its factors, we can expand it as either.

Activity 5

Expand the following logs:

(a) $\log_6 18$

(b) $\log_6 \frac{4}{5}$

(c) $\log_2 10$

(d) $\log_5 \frac{2}{3}$

(e) $\log_{10} \frac{3}{4}$

(f) $\log_4 \frac{x}{y}$

Compare your answers with those at the end of the subunit. Continue to the next section if you had at least 5 correct. Otherwise, review the content and try the activity again.

Key Points to Remember

The key points to remember in this subunit on working with logarithms are:

- $\log_b a = m$ means that $a = b^m$
- $\log_2 8 = \log_2 4 \times 2$
- $\log_a b - \log_a c = \log_a \frac{b}{c}$
- $\log_a b + \log_a c = \log_a b \times c$

These rules that we investigated will be used in evaluating logs in the next section.

Answers to activity 5:

(a) $\log_6 18 = \log_6 3 + \log_6 6$ OR $\log_6 9 + \log_6 2$

There are two possible answers. You need to give every possible answer for these.

(b) $\log_6 \frac{4}{5} = \log_6 4 - \log_6 5$

(c) $\log_2 10 = \log_2 5 + \log_2 2$

(d) $\log_5 \frac{2}{3} = \log_5 2 - \log_5 3$

(e) $\log_{10} \frac{3}{4} = \log_{10} 3 - \log_{10} 4$

(f) $\log_4 \frac{x}{y} = \log_4 x - \log_4 y$

Lesson 3 Evaluating logs

By the end of this subunit, you should be able to evaluate logs.

To evaluate logs we use the following facts about logs:

$\log_a a = 1$

$\log_b a^y = y \log_b a$

Let us show that $\log_a a = 1$ and that $\log_b a^y = y \log_b a$ are true by substituting numbers for the variables in the formula.

Let us first show that: $\log_a a = 1$

When 2 is substituted for a in the equation, the equation becomes:

$\log_2 2 = 1$

$2^1 = 2$

When 3 is substituted for a in the equation, the equation becomes:

$\log_3 3 = 1$

$3^1 = 3$

This shows that $\log_2 2 = \log_3 3 = \log_a a = 1$

Let us now show that: $\log_b a^y = y \log_b a$ by substituting numbers for variables in the equation and evaluating each side of the equation as shown below:

$\log_b a^y = y \log_b a$

Substituting 4 for “a”, 2 for “b”, and 3 for “y” in the left hand side of the equation we get:

$\log_2 4^3$

To show that

$\log_2 4^3 = 3\log_2 4$

Let us evaluate the left hand side as follows:

$\log_2 4^3$

$= \log_2 64$

$= \log_2 2^6$

$= 6$

Evaluating the left hand side we get:

$= 3\log_2 2^2$

$= 3 \times 2\log_2 2$

$= 6 \times 1$

$= 6$

Therefore $\log_2 4^3 = 3\log_2 4 = 6$

To state that generally we say $\log_b a^y = y\log_b a$

Let us now use these facts to evaluate logs

Example1:

Evaluate $\log_4 16$

Step 1 - Express 16 in index form in base 4, which is the same base as the original log:

$\log_4 16 = \log_4 4^2$

Step 2: Use the rule $\log_b a^y = y\log_b a$ that was discussed above:

$\log_4 4^2 = 2\log_4 4$

Step 3 – Reduce

Since $\log_4 4 = 1$

$2\log_4 4 = 2 \times 1 = 2$

Example 2: Evaluate $\log_3 81$

Step 1: Express 81 in index form in base 3, which is the same base as the original log:

$\log_3 81 = \log_3 3^4$

Step 2: Use the rule $\log_b a^y = y\log_b a$:

$\log_3 3^4 = 4\log_3 3$

Step 3 – Reduce

Since $\log_3 3 = 1$
$4\log_3 3 = 4 \times 1 = 4$

Example 3:
Evaluate $\log_2 \frac{1}{8}$

Express $\frac{1}{8}$ as an index in base 2

$\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$

$\therefore \log_2 \frac{1}{8}$

$= \log_2 2^{-3}$

$= -3\log_2 2$

$= -3 \times 1$

$= -3$

The following example combines many of the skills learned previously.

Example 4: Evaluate $\log_3 81 + \log_3 27 - \log_3 9$

$\log_3 81 + \log_3 27 - \log_3 9$

$= \log_3 \frac{3^4 \times 3^3}{3^2}$

$= \log_3 3^{4+3-2}$

$= \log_3 3^5$

$= 5 \log_3 3$

$= 5 \times 1$

$= 5$

Activity 6

Using the procedure illustrated above, evaluate the following logs:

$(a) \log_6 36$

$(b) \log_2 64$

$(c) \log_5 125$

$(d) \log_3 243$

$(e) \log_3 \frac{1}{27}$

$(f) \log \frac{1}{1000}$

$(g) \log_2 32 + \log_2 8$

$(h)\,\frac{\log_3 27}{\log_3 9}$

$(i)\,\log_2 16 - \log_2 8 + \log_2 32$

Compare your answers with those at the end of the subunit. Be sure you understand how to evaluate logs before you continue to the next section.

Key Points to Remember

The key points to remember in this subunit on evaluating logs are:

- $\log_a a = 1$ because when we raise a number to the power of one, the result is that number.
- $\log_b a^y = y\log_b a$ this means that $\log \log_3 2^4 = 4\log_3 2$
- $\frac{1}{b^a} = b^{-a}$

Answers to Activity 6

(a) $\log_6 36 = \log_6 6^2 = 2\log_6 6 = 2 \times 1 = 2$

(b) $\log_4 64 = \log_4 4^3 = 3\log_4 4 = 3 \times 1 = 3$

(c) $\log_5 125 = \log_5 5^3 = 3\log_5 5 = 3 \times 1 = 3$

(d) $\log_3 243 = \log_3 3^5 = 5\log_3 3 = 5 \times 1 = 5$

(e) $\log_5 \frac{1}{25} = \log_5 5^{-3} = -3\log_5 5 = -3 \times 1 = -3$

(f) $\log \frac{1}{100} = \log_{10} 10^{-2} = -2\log_{10} 10 = -2 \times 1 = -2$

(g)
$\log_2 32 + \log_2 8 = \log_2 2^5 + \log_2 2^3 = \log_2(2^5 \times 2^3) = \log_2 2^8 = 8\log_2 2 = 8 \times 1 = 8$

(h) $\frac{\log_3 27}{\log_3 9} = \frac{\log_3 3^3}{\log_3 3^2} = \frac{3\log_3 3}{2\log_3 3} = \frac{3\times 1}{2\times 1} = \frac{3}{2}$

(i)
$\log_2 16 - \log_2 8 + \log_2 32 = \log_2 2^4 - \log_2 2^3 + \log_2 2^5 = 4\log_2 2 - 3\log_2 2 + 5\log_2 2$
$\frac{6\log_2 2}{1} = \frac{20}{3}$

Lesson 4 Change the Base of a Log

By the end of this subunit, you should be able to change the base of a log.

In evaluating logs as well as in solving log equations we need a process that involves the change of the original base. Through substituting in numbers in the formulae, let us show that the change of base in the formula does not change the value of the log:

$\log_a b = \dfrac{\log_c b}{\log_c a}$ Note that the original base (a) is different than the new base (c).

Example1

Look at the following statements that show how the logarithm is expressed as a fraction.

(a) $\log_4 16 = \dfrac{\log_2 16}{\log_2 4}$

Compare the two sides of this equation. Note where the base (4) and the number (16) of the original equation (left hand side) end up in the new equation (right hand side). Note also that the new base is different. .It can be any different value, but where possible choose the base that is a factor of the numbers in the position of 16 and 4 on the right-hand side of the above equation.

(b) $\log_3 5 = \dfrac{\log_2 5}{\log_2 3}$

Compare the two sides of this equation.
Note where the base (3) and the number (5) of the original log end up and the new base is chosen to be 2. Any other number can be chosen to be a base. We normally take very small numbers like two or three as bases because it is easy to find their multiples.

(c) $\log_2 7 = \dfrac{\log_3 7}{\log_3 2}$

Compare the two sides of this equation.
Note where the base (2) and the number (7) of the original log end up and the new base is chosen to be 3.

(d) $\log_3 7 = \dfrac{\log_4 7}{\log_4 3}$

Compare the two sides of this equation.

Note where the base (3) and the number (7) of the original log end up and the new base is chosen to be 4

What is the pattern of where the base and number end up?

Compare your answer with the one below:
The base becomes the number in the denominator and the other number forms part of the numerator.
Let us now show that the new equation is true by working out the left hand side and the right hand side of the equation to see whether we get the same answer.

Left-hand side=right-hand side?

$leftside = rightside?$

$$\log_4 16 = \frac{\log_2 16}{\log_2 4}$$

Left hand side

$$\log_4 16 = \log_4 4^2 = 2\log_4 4 = 2 \times 1 = 2$$

Right hand side

$$\frac{\log_2 16}{\log_2 4} = \frac{\log_2 2^4}{\log_2 2^2} = \frac{4\log_2 2}{2\log_2 2} = \frac{4 \times 1}{2 \times 1} = 2$$

So left hand side = right hand side = 2.

Example 2

$\log_9 81$

$$\log_9 81 = \frac{\log_3 81}{\log_3 9}$$

LHS

$$\log_9 81 = \log_9 9^2 = 2\log_9 9 = 2 \times 1 = 2$$

RHS

$$\frac{\log_3 81}{\log_3 9} = \frac{\log_3 3^4}{\log_3 3^2} = \frac{4\log_3 3}{2\log_3 3} = \frac{4 \times 1}{2 \times 1} = 2$$

LHS = RHS = 2

So these two examples show that it that $\log_a b = \frac{\log_c b}{\log_c a}$ is true

Example 3

The logarithms of base ten have been worked out as tables that are ready for us to use. The change of base can be used where the base can be changed to ten. Then logarithms to base ten can be looked up in the tables to evaluate the given logarithmic equation.

For example, evaluate $\log_2 5$ using change of base and use log tables if the simplified stage cannot be easily evaluated.

Step 1

Change $\log_2 5$ to base 10 (the logs in a table are in base 10 so we will be able to use log tables to evaluate the expression)

$$\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2}$$

Step 2

Look up the logarithm of 5 in the log tables.

How to use the log tables look for 5 on the first column in the tables

Look for 5 in the first column. At the number 5, move to the next column under 0 there you find 0.699. We can also get this number from our calculators by keying in the number 5 then pressing log. The similar procedure can be followed to find the value of log of 2 to the base 10.

$$\text{or} \frac{\log 5}{\log 2} = \frac{0.699}{0.301} = 2.3223$$

Activity 7

Evaluate the following logs. Use a log table where necessary.

$a)\log_4 8$

$(b)\log_3 5$

$(c)\log_4 9$

$(d)\log_2 6$

$(e)\log_3 \frac{1}{5}$

$(f)\log_2 \frac{3}{4}$

Compare your answers with those at the end of the subunit. Be sure you understand how to change the base of a log maintaining its value before you continue to the next section.

Key Points to Remember

The key points to remember in this subunit on XXX are:

- Change of base can be used to evaluate logs
- For log expressions that cannot be easily evaluated change the base to ten so that you can use log tables to get values of logs.

When using change of base to evaluate logs, choose the base which is one of the factors of the number whose log is required e.g. $\log_9 81$ can be expressed as $\frac{\log_3 81}{\log_3 3}$ to be easily evaluated not $\frac{\log_2 81}{\log_2 9}$ because 2 is not a factor of 81 but 3 is.

Answers:

(a)

(a) $\log_4 8$	$\frac{\log_2 8}{\log_2 4} = \frac{\log_2 2^3}{\log_2 2^2} = \frac{3\log_2 2}{2\log_2 2} = \frac{3\times 1}{2\times 1}$	$\frac{3}{2}$
(b) $\log_3 5$	$\frac{\log 5}{\log 3} = \frac{\log_{10} 5}{\log_{10} 3} = \frac{0.6990}{0.4771}$	1.46 49
(c) $\log_4 9$	$\frac{\log 9}{\log 4} = \frac{\log_{10} 9}{\log_{10} 4} = \frac{0.9542}{0.6021}$	1.58 5
(d) $\log_3 \frac{1}{5}$	$\frac{\log 5^{-1}}{\log 3} = \frac{-1\log_{10} 5}{\log_{10} 3} = \frac{-1\times 0.6990}{0.4771}$	- 1.46 49
$(f)\log_2 \frac{1}{4}$	$\frac{\log_{10} 2^{-2}}{\log_{10} 2} = \frac{-2\log_{10} 2}{\log_{10} 2} = \frac{-2\times 0.30}{0.3010}$	-2

Lesson 5 Solve Log Equations Involving the Same Base Terms

By the end of this subunit, you should be able to solve log equations involving the same base terms.

All the properties you have come across can now be used including change of base to solve log equations.

Example 1

Solve for x in the following log equation

$$\log x^3 = 9$$
$$3\log x = 9$$
$$\log_{10} x = 3$$
$$10^3 = x$$
$$1000 = x$$

Example 2

$$4\log x - 3\log x = 10$$

LHS

$$4\log x - 3\log x = \log x^4 - \log x^5 = \log_{10}\frac{x^4}{x^5} = \log_{10}\frac{1}{x} = \log_{10} x^{-1} = -1\log_{10} x$$

$equation$

$$-1\log x = 10$$
$$\log_{10} x = -10$$
$$x = 10^{-10} = \frac{1}{10^{10}}$$

Example 3

$$\log_2(x-2) + \log_2(x-3) = 1$$
$$\log_2(x-2)(x-3) = 1$$
$$(x-2)(x-3) = 2^1$$
$$x^2 - 5x + 6 = 2$$
$$x^2 - 5x + 4 = 0$$
$$(x-4)(x-1) = 0$$
$$x = 4 or x = 1$$

Example 4

$\log_3 x = 2$

$3^2 = x$

$9 = x$

Example 5

$2\log_2 x = 16$

$\log_2 x^2 = 16$

$x^2 = 2^{16}$

$\sqrt{x^2} = \sqrt{2^{16}}$

$x = 2^8$

Activity 8

Solve for x in the following

(a) $\log_5 x = 2$

(b) $\log_x 64 = 3$

(c) $\log_3\left(\frac{1}{81}\right) = x$

(d) $\log_x\left(\frac{1}{8}\right) = -3$

(e) $\log_3\left(\frac{1}{81}\right) = 2x$

(f) $\log_2 x = -4$

(g) $2\log\left(\frac{1}{x}\right) + 1 = 0$

(h) $\log(x+2)^2 = 2$

Compare your answers with those at the end of the subunit. Be sure you understand how to solve log equations involving the same base terms before you continue to the next section

Key Points to Remember

The key points to remember in this subunit on XXX are:
Look for the applicable log and indices rules in each case when you simplify your expressions.

Answers to activity 8

(a) $x = 5^2 = 25$

(b) $\log_x 64 = 3$

$x^3 = 64$

$\sqrt[3]{x^3} = \sqrt[3]{4^3}$

$x - 4$

X=4

(c)

$$\log_3\left(\frac{1}{81}\right) = x$$

LHS

$$\log_3 \frac{1}{3^4} = \log_3 3^{-4}$$

$equation$

$$3^x = 3^{-4}$$

$$x = -4$$

(d)

$$\log_x\left(\frac{1}{8}\right) = -3$$

$$x^3 = \frac{1}{8}$$

$$x^3 = \frac{1}{2^3}$$

$$x^3 = 2^{-3}$$

$$\sqrt[3]{x} = \sqrt[3]{2^{-3}}$$

$$x = 2^{-1}$$

(e)

$$\log_3\left(\frac{1}{81}\right) = 2x$$

$$3^{2x} = \frac{1}{81}$$

$$3^{2x} = \frac{1}{3^4}$$

$$3^{2x} = 3^{-4}$$

$$2x = -4$$

$$x = -2$$

(f)

$$\log_2 x = -4$$

$$x = 2^{-4}$$

(g)

$$2\log\left(\frac{1}{x}\right)+1=0$$

$$\log\left(\frac{1}{x}\right)^2+1=0$$

$$\log_{10}x^{-2}+\log_{10}10=0$$

$$\log_{10}x^{-2}\times 10=0$$

$$10x^{-2}=10^0$$

$$10x^{-2}=1$$

$$x^{-2}=\frac{1}{10}$$

$$\frac{1}{x^2}=\frac{1}{10}$$

$$\sqrt{x^2}=\sqrt{10}$$

$$x=\sqrt{10}$$

(h)

$$\log(x+2)^2=2$$

$$\log(x+2)(x+2)=2$$

$$10^2=x^2+4x+4$$

$$0=x^2+4x-96$$

$$0=x^2+12x-8x-96$$

$$0=x(x+12)-8(x+12)$$

$$0=(x+12)(x-8)$$

$$x=-12$$

or

$$x=8$$

Lesson 6 Solve Log Equations Using a Change of Base

By the end of this subunit, you should be able solve log equations using a change the base.

We have solved log equations involving same base. In this section we will solve log equations using change of base.
From the previous section we saw that

$$\log_a b=\frac{\log_c b}{\log_c a}$$

We also know that the following is true

$\log_a 1 = 0$

$a^o = 1$

Let us also show that $\log_a b = \frac{1}{\log_b a}$ by substituting numbers in the formulae.

Example 1

$\log_2 8 = \frac{1}{\log_8 2}$

LHS

$\log_2 8 = 3$

RHS

$\frac{1}{\log_8 2} = \frac{1}{1/3} = 3$

therefore

$\log_2 8 = \frac{1}{\log_8 2} = 3$

Example 2

$\log_3 9 = \frac{1}{\log_9 3}$

LHS

$\log_3 9 = 2$

RHS

$\log_9 3 = \frac{1}{2}$

$\frac{1}{\log_9 3} = \frac{1}{1/2} = 2$

$\therefore \log_3 9 = \frac{1}{\log_9 3} = 2$

therefore

$$\log_a b = \frac{1}{\log_b a}$$

Solving equations using the facts

Example 1
Solve for x in the following equation

$$2^x = 5$$
$$\log_2 5 = x$$
$$\frac{\log 5}{\log 2} = x$$
$$\frac{0.699}{0.3010} = x$$
$$2.321 = x$$

Remember that when the base is not written the base is 10. In base 10 we can use a calculator to evaluate the logs.

Example 2

$$2\log_3 x - 4 = 0$$
$$2\log_3 x = 4$$
$$x^2 = 3^4$$
$$x = 3^2$$

Example3

$$5\log x - 3\log x = 10$$
$$\log x^5 - \log x^3 = 10$$
$$\log \frac{x^5}{x^3} = 10$$
$$\log x^{5-3} = 10$$
$$\log_{10} x^2 = 10$$
$$2\log_{10} x = 10$$
$$\log_{10} x = 5$$
$$x = 10^5$$

Example 4

$3\log_9 x + \log_x 9 = \log 1000$

The first term has 9 as its base we will change our second term $\log_x 9$ to an expression with base 9. The expression will be formed using the fact that, $\log_x 9 = \dfrac{\log_9 9}{\log_9 x}$

Rewriting the expression using the 2nd term in its new form we get.

$$3\log_9 x + \frac{\log_9 9}{\log_9 x} = 1000$$

Remember also that $\log_{10} 1000 = \log_{10} 10^4 = 4\log_{10} 10 = 4 \times 1 = 4$. So, our original equation becomes

$$3\log_9 x + \frac{\log_9 9}{\log_9 x} = 4$$

In this equation the 1st term $3\log_9 x$ and the 2nd term $\dfrac{\log_9 9}{\log_9 x}$ on the left hand side both contain a common part $\log_9 x$. We let that common part $\log_9 x = \text{k}$

Substituting k for $\log_9 x$ in our equation makes our equation easier as shown below.

$$3k + \frac{\log_9 9}{k} = 4$$

Remembering also that $\log_9 9 = 1$, we substitute 1 for $\log_9 9$.

$$3k + \frac{1}{k} = 4$$

Transferring 4 to the left hand side of our equation our result becomes,

$$3k + \frac{1}{k} - 4 = 0$$

Multiplying every term by K to remove the fraction in the equation we get

$3k^2 - 4k + 1 = 0$

Factorizing the trinomial on the left-hand side of the equation we get

$3\text{k}^2 - 3k - k + 1 = 0$

3k(k-1)-1(k-1)= 0

In the two terms $3k(k-1)$ and $-1(k-1)$ the common factor is (k-1), so we take out this common factor. It goes 3k times in the first term and -1 times in the second term. This changes our equation to become

(k-1)(3k-1)=0

This means that either (k-1) = 0 or (3k-1) = 0 because if a product of two numbers is zero it implies that one of the numbers multiplied should be zero. That is if (k-1)(3k-1) =0 either (k-1)=0 or (3k-1) =0 We then solve the two equations as follows:

k-1 = 0

k=1 or

3k-1=0

3k=1

$k = \frac{1}{3}$

$k = 1 \text{ or } \frac{1}{3}$

We then substitute these terms into our original common term $\log_9 x$ as shown below;

$\log_9 x = k$

When k=1 our equation becomes;

$\log_9 x = 1$

Remember that this means $x = 9^1$. Therefore,

$x = 9$

Similarly, when $k = \frac{1}{3}$ we solve for x as shown below

$\log_9 x = k$

$\log_9 x = \frac{1}{3}$

$x = 9^{\frac{1}{3}}$

$x = \sqrt[3]{9} = 3$

Example 5

Solve for x in

$\log_3 x + 3\log_x 3 - 4 = 0$

Using the fact that $\log_a b = \frac{\log_c b}{\log_c a}$, we write the second term in base 3 thus,

$3\log_x 3 = \frac{3\log_3 3}{\log_3 x}$

Substituting that in our equation

$\log_3 x + \frac{3\log_3 3}{\log_3 x} - 4 = 0$

Let us first transfer -3 to the right hand side of the equation to get

$\log_3 x + \frac{3\log_3 3}{\log_3 x} = 4$

$\log_3 x$ is common in the two terms on the left hand side of the equation. So let us represent this common term by 'k'. Substituting 'k' into the equation we get

$k + \frac{3\log_3 3}{k} = 4$

$3\log_3 3 = 3 \times 1 = 3$

So when we substitute 3 for $3\log_3 3$, our equation becomes

$$k + \frac{3}{k} = 4$$

We then remove the fraction in the equation by multiplying all terms by k to get,

$$k^2 + 3 = 4k$$

Taking all terms to the left hand side of the equation, we get

$$k^2 - 4k + 3 = 0$$

Factoring the trinomial $k^2 - 4k + 3$ we get

$$k^2 - 4k + 3$$

$$k^2 - k - 3k + 3$$

$$(k^2 - k) - (3k - 3)$$

$$k(k-1) - 3(k-1)$$

$$(k-1)(k-3)$$

So our equation becomes

$$(k-1)(k-3) = 0$$

This means that either k-1=0 or k-3=0. Solving the equations we get

$$k - 1 = 0$$

$$k = 1$$

or

$$k - 3 = 0$$

$$k = 3$$

Substituting $k = 1$ into common factor $\log_3 x$ in our original equation we get

$$\log_3 x = k$$

$$\log_3 x = 1$$

$$x = 3^1 = 3$$

Or

$$\log_3 x = k$$

$$\log_3 x = 3$$

$$x = 3^3 = 27$$

X=3 or x=27

Example 6

Solve for x in the following

$$3\log_5 x + \log_x 25 = 7$$

Let us now express the second term $\log_x 25$ on the left hand side of the equation in base 5as follows

$$\log_x 25 = \frac{\log_5 25}{\log_5 x}$$

Let us substitute that into our original equation

$$3\log_5 x + \frac{\log_5 25}{\log_5 x} = 7$$

Let the term $\log_x 25$ be represented by 'k . Let us also simplify

$\log_5 25 = \log_5 5^2 = 2\log_5 5 = 2\times 1$ Our equation then becomes

$$3k + \frac{\log_5 5^2}{k} = 7$$

$$3k + \frac{2\log_5 5}{k} = 7$$

$$3k + \frac{2\times 1}{k} = 7$$

$$3k + \frac{2}{k} = 7$$

We then (a) multiply every term by 'k' to eliminate the fraction, (b)transfer '7k' to the left hand side of the equation to form a trinomial that we then(c) factorise as shown below.

$$3k^2 + 2 = 7k$$

$$3k^2 - 7k + 2 = 0$$

$$3k^2 - 1k - 6k + 2 = 0$$

$$(3k^2 - 1k) - (6k - 2) = 0$$

$$k(3k-1) - 2(3k-1) = 0$$

$$(3k-1)(k-2) = 0$$

We then solve the equations that result

Either

3k-1=0 or k-2=0

3k=1 k=2

$K=\frac{1}{3}$

$$\log_5 x = k$$

$$\log_5 x = \frac{1}{3}$$

$$x = 5^{\frac{1}{3}}$$

Or

$$\log_5 x = k$$

$$\log_5 x = 2$$

$$x = 5^2$$

Now try the following activity

Activity 9

1. Simplify the following

(a) $\log_3 \frac{1}{9} + \log_6 6 + \log_4 64$

(b) $\log_3 8 - \log_8 512 + \log_5 1 + \log 100$

(c) $\log_4 8 + \log_{25} 5 - \log_3 27$

Solve for x in the following

$(a) 2\log_9 x + 2\log_x 9 = 5$

$(b) \log_x 4 = \frac{5}{2} - \log_4 x$

$(c) \log_6 x - \log_x 6 = \frac{3}{2}$

$(d) \log_2 x + 1 = 6\log_x 2$

Compare your answers with those at the end of the subunit. Be sure you understand how to solve log equations using a change of base before you continue to the next section. Note also that a lot of practice is required for you to understand this section. Continue only when you get at least 80% of the questions right.

Key Points to Remember

The key points to remember in this subunit on solving logarithmic equations are:
That logarithmic equations can be solved using the following logarithmic rules in addition to several others.

- $\log_a b = \dfrac{1}{\log_b a}$
- $\log_a b = \dfrac{\log_c b}{\log_c a}$
- $\log_a 1 = 0$ because $a^o = 1$
- $\dfrac{\log 5}{\log 2}$ means $\dfrac{\log_{10} 5}{\log_{10} 2}$. Base 10 can be worked out using log tables in a calculator or tables.

Answers to activity 9

1(a)

$\log_3 \dfrac{1}{9} + \log_6 6 + \log_4 64$

$\log_3 \dfrac{1}{3^2} + \log_6 6 + \log_4 4^3$

$\log_3 3^{-2} + \log_6 6 + 3\log_4 4 = -2\log_3 3 + 1 + 3 = -2 + 2 = 2$

(b)

$\log_3 8 - \log_8 512 + \log_5 1 + \log 100$

$\dfrac{\log 8}{\log 3} - \log_8 8^3 + 0 + 2$

$\dfrac{0.9031}{0.4791} - 3\log_8 8 + 2$

$1.893 - 3 \times 1 + 2 = 1.893 + 2 - 3 = 0.893$

(c)

$\log_4 8 + \log_{25} 5 - \log_3 27$

$\frac{\log 8}{\log 4} + \log_{25} 25^{\frac{1}{2}} - \log_3 3^3$

$\frac{0.9031}{0.6021} + \frac{1}{2}\log_{25} 25 - 3\log_3 3$

$1.5 + .5 - 3 = -1$

2(a)

$2\log_9 x + 2\log_x 9 = 5$

$2\log_9 x + \frac{2\log_9 9}{\log_9 x} = 5$

$let \log_9 x = k$

$2k + \frac{2}{k} = 5$

$2k^2 + 2 - 5k = 0$

$2k^2 - 5k + 2 = 0$

$2k^2 - 4k - 1k + 2 = 0$

$(2k^2 - 4k) - (1k - 2) = 0$

$2k(k-2) - 1(k-2) = 0$

$(2k-1)(k-2) = 0$

$2k - 1 = 0$

or

$k - 2 = 0$

$2k = 1$

$k = \frac{1}{2}, or$

$k = 2$

$\log_9 x = \frac{1}{2}$

$x = 9^{\frac{1}{2}} = 3$

$\log_9 x = 2$

$x = 9^2 = 81$

81 or 3

(b) $\log_x 4 = \frac{5}{2} - \log_4 x$

$\log_x 4 + \log_4 x = \frac{5}{2}$

$\log_4 x + \log_x 4 - \frac{5}{2} = 0$

$$\log_x 4 = \frac{\log_4 4}{\log_4 x} - \frac{5}{2} = 0$$

$$\log_4 x + \frac{\log_4 4}{\log_4 x} - \frac{5}{2} = 0$$

Let $\log_4 x = k$

$$k + \frac{1}{k} - \frac{5}{2} = 0$$

Multiply each term by '2k. Our equation becomes

$$2k^2 + 2 - 5k = 0$$
$$2k^2 - 5k + 2 = 0$$
$$2k^2 + 4k - 1k - 2 = 0$$
$$(2k^2 + 4k) - (k + 2) = 0$$
$$2k(k + 2) - (k + 2) = 0$$
$$(k + 2)(2k - 1) = 0$$
$$k + 2 = 0$$
$$k = -2$$
$$2k - 1 = 0$$
$$2k = 1$$
$$k = \frac{1}{2}$$
$$\log_4 x = k$$
$$\log_4 x = -2$$
$$x = 4^{-2} = \frac{1}{16}$$
$$\log_4 x = k$$
$$\log_4 x = \frac{1}{2}$$
$$x = 4^{\frac{1}{2}} = 2$$
$$x = \frac{1}{16}$$

X= or

$$x = 2$$

(c) 36 or $\frac{1}{\sqrt{6}}$

(d) 8 or 4

$$(b)\log_x 4 = \frac{5}{2} - \log_4 x$$

$$\log_x 4 + \log_4 x = \frac{5}{2}$$

$$\frac{\log_4 4}{\log_4 x} + \log_4 x - \frac{5}{2} = 0$$

let

$$\log_4 x = k$$

$$\frac{1}{k} + k - \frac{5}{2} = 0$$

$$2 + 2k^2 - 5k = 0$$

$$2k^2 - 5k + 2 = 0$$

$$2k^2 - 4k - 1k + 2 = 0$$

$$2k(k-2) - 1(k-2) = 0$$

$$(2k-1)(k-2) = 0$$

$$k = \frac{1}{2}$$

$$k = 2$$

$$\log_4 x = \frac{1}{2}$$

$$x = 4^{\frac{1}{2}} = 2$$

$$\log_4 x = 2$$

$$x = 4^2 = 16$$

$$(c)\log_6 x - \log_x 6 = \frac{3}{2}$$

$$\log_6 x - \frac{\log_6 6}{\log_6 x} - \frac{3}{2} = 0$$

let

$$\log_6 x = k$$

$$k - \frac{1}{k} - \frac{3}{2} = 0$$

$$2k^2 - 2 - 3k = 0$$

$$2k^2 - 3k - 2 = 0$$

$$2k^2 - 4k + 1k - 2 = 0$$

$$2k(k-2) + 1(k-2) = 0$$

$$(2k+1)(k-2) = 0$$

$$k = -\frac{1}{2}$$

$$k = 2$$

$$\log_6 x = -\frac{1}{2}$$

$$x = 6^{-\frac{1}{2}}$$

$$\log_6 x = 2$$

$$x = 6^2 = 36$$

$(d) \log_2 x + 1 = 6\log_x 2$

$\log_2 x + 1 - 6\log_x 2 = 0$

$\log_2 x + 1 - \dfrac{6\log_2 2}{\log_2 x} = 0$

let

$\log_2 x = k$

$k + 1 - \dfrac{6}{k} = 0$

$k^2 + k - 6 = 0$

$k^2 + 3k - 2k - 6 = 0$

$k(k+3) - 2(k+3) = 0$

$(k-2)(k+3) = 0$

$k = 2$

$k = -3$

$\log_2 x = k$

$\log_2 x = 2$

$x = 2^2$

$\log_2 x = -3$

$x = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

Lesson 7 Sketching Log Graphs

By the end of this subunit, you should be able sketch log graphs.

Example 1
Draw a graph of $y = log_2 x$

$y = \log_2 x$

means

$2^y = x$

We then use the exponential function to work out the coordinates as illustrated in the table
Note: It is easier to give y values and calculate the corresponding x value.

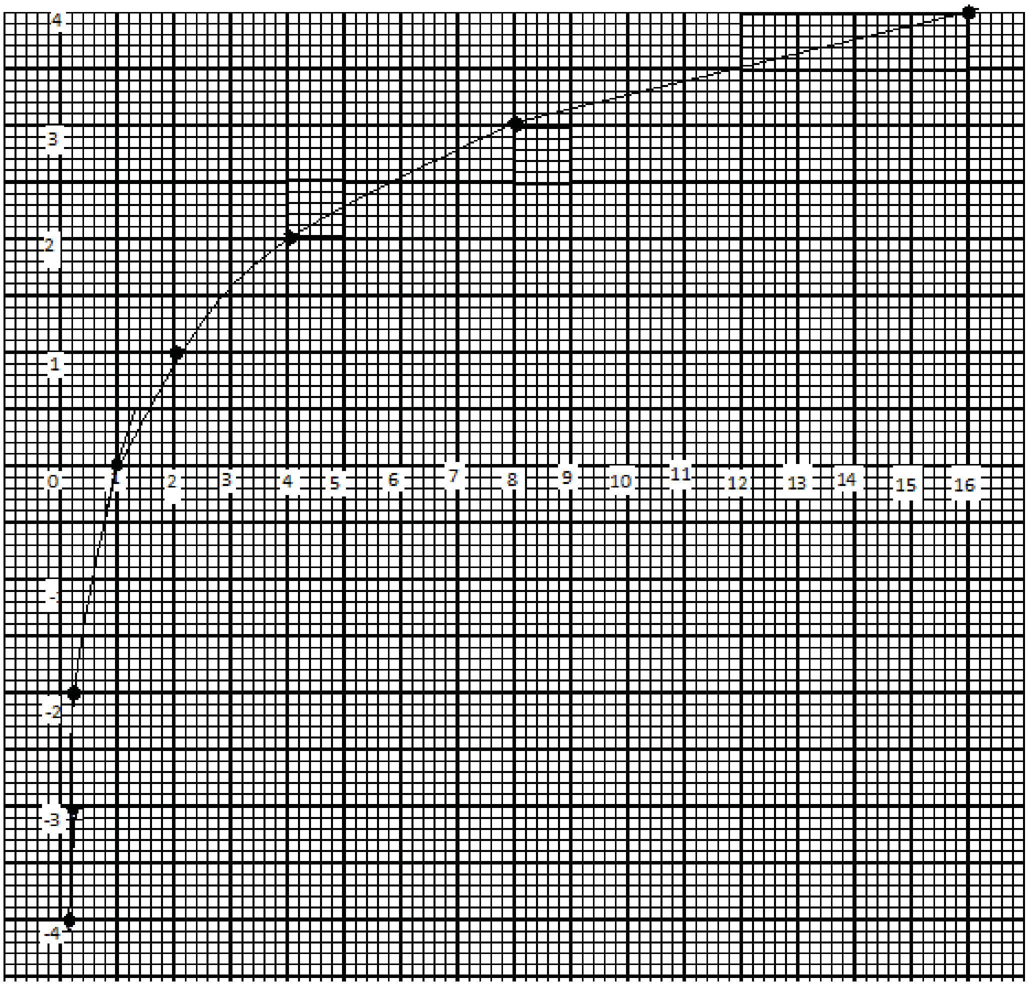
Value given as y coordinate	Calculation of x coordinate	coordinates
y=-4	$2^{-4} = \dfrac{1}{2^4} = \dfrac{1}{16}$	$\left(\dfrac{1}{16}, -4\right)$

y=-3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$\left(\frac{1}{8}, -3\right)$
y=-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(\frac{1}{4}, -2\right)$
y=-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(\frac{1}{2}, -1\right)$
y=0	$2^0 = x = 1$	(1,0)
y=1	$2^1 = x = 2$	(2,1)
y=2	$2^2 = x = 4$	(4,2)
y=3	$2^3 = x = 8$	(8,3)
y=4	$2^4 = x = 16$	(16,4)

The summary of the table above is as follows:

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	-4	-3	-2	-1	0	1	2

The graph of $y = log_2 x$ is drawn below. Note that the graph is the best estimate so it does not necessarily go through all the points.



For you to realise the characteristics of the log graph, answer the following questions.

(a) Where does the graph cross the x-axis?

____

____

Compare your answer with the one below:

The graph passes at the coordinate (1, 0).

(b) Why must the line pass at the point (1,0) ?

____

Compare your answer with the one below:

$2^0 = 1$. The function $y = \log_2 x$ means $2^y = x$. Any number raised to the power of zero, equals one, i.e. $b^0 = 1$ or $0 = \log_2 1$.

(c) What is the x-coordinate where y=1?

Compare your answer with the one below:

The graph passes through the point (2,1).

(d) Figure out why this is so. Hint: When the logarithm is 1 or y=1, what can you say about x and b? Refer to the section 'Evaluating logs' of this unit.

Compare your answer with the one below:

When y=1 , x=b. because of the rule, $\log_b b = 1$

(e) What can you say about the values of the x and y coordinates when the x- coordinate is between 0and 1

(f)____

____

Compare your answer with the one below:

The graph goes below the x-axis where y values are negative for values of x between 0 and 1.

(g) Why is that?

Compare your answer with the one below:

(The values of x between 0 and 1 are fractions. This is because in $x = b^y$ when y is negative, x is a fraction. E,g. $\frac{1}{b^2} = b^{-2}$.

Note that the graph comes closer to the y-axis but does not cross it. The y-axis is referred to as a vertical asymptote. An asymptote is a straight line to which a graph comes closer and closer but never touches. The y axis, of the logarithmic function $y = \log_b x$ is a vertical asymptote

Example 2

Draw the graph of $y = \log_2 x$

Answer to example 2

y= $\log_2(x+3)$ This means $2^y = x + 3$

Value given as y coordinate	Calculation of x coordinate	coordinates
y=-4	$2^{-4}=\frac{1}{2^4}=\frac{1}{16}=x+3=-2\frac{1}{1}$	$\left(-2\frac{15}{16},-4,\right)$
y=-3	$2^{-3}=\frac{1}{2^3}=\frac{1}{8}=x+3=-2\frac{7}{8}$	$\left(-2\frac{7}{8},-3,\right)$
y=-2	$2^{-2}=\frac{1}{2^2}=\frac{1}{4}=x+3=-2\frac{3}{4}$	$\left(-2\frac{3}{4},-2\right)$
y=-1	$2^{-1}=\frac{1}{2^1}=\frac{1}{2}=x+3=-2\frac{1}{2}$	$\left(-2\frac{1}{2},-1,\right)$
y=0	$2^0=x=1$=x+3=-2	(-2,0)
y=1	$2^1=x=2$=x+3=-1	(,-1,1)
y=2	$2^2=x=4$=x+3=1	(1,2)
y=3	$2^3=x=8$=x+3=5	(5,3)
y=4	$2^4=x=16$=x+3=13	(13,4)

x	$-2\frac{15}{16}$	$-2\frac{7}{8}$	$-2\frac{1}{2}$	-2	1	5	13
y	-4	-3	-1	0	2	3	4

The following is the graph of $y=\log_2 x$

(a) Where does the graph cross the x-axis?

____

____

Compare your answer with the one below:

The graph passes at the coordinate (-2,0). Note that addition of a number to x translates the x-intercept. Addition of a positive number translates it to the left and a negative number to the right.

Activity 10

Calculate the coordinates of the graphs of the logarithmic functions that follow and sketch the graphs

(a) $y=\log_{10} x$

(b) $y = \log_4 x + 1$
(c) $y = \log_4 x - 1$

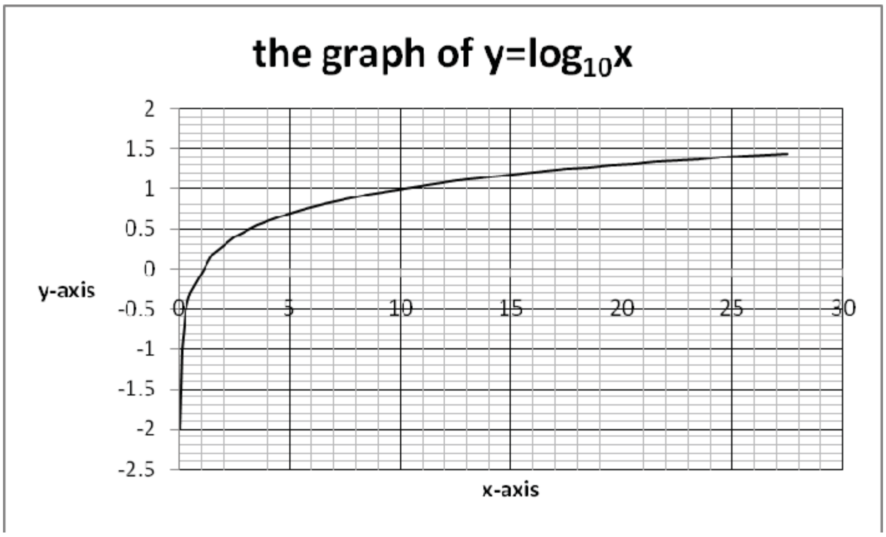
Subunit Summary

- To sketch a graph one needs to calculate the coordinates. The easier way to do it is to give y values and work out the value of x.
- For a function $y = \log_b x$, the graph passes through the points (b,1) and (,1,0).
- For a function $y = \log_b x + 1$ the x-intercepts is translated 1 unit to the left while for $y = \log_b x - 1$ the x-intercept is translated 1 unit to the right.
- For negative values of y the graph goes below the x-axis.
- An asymptote is a straight line to which a graph comes closer and closer but never touches. The y axis, of the logarithmic function $y = \log_b x$ is a vertical asymptote.
- The logarithm function $y = \log_b x$ is defined for only positive values of x. this means that $\log_b(-x)$ is undefined.

Answers to activity 10

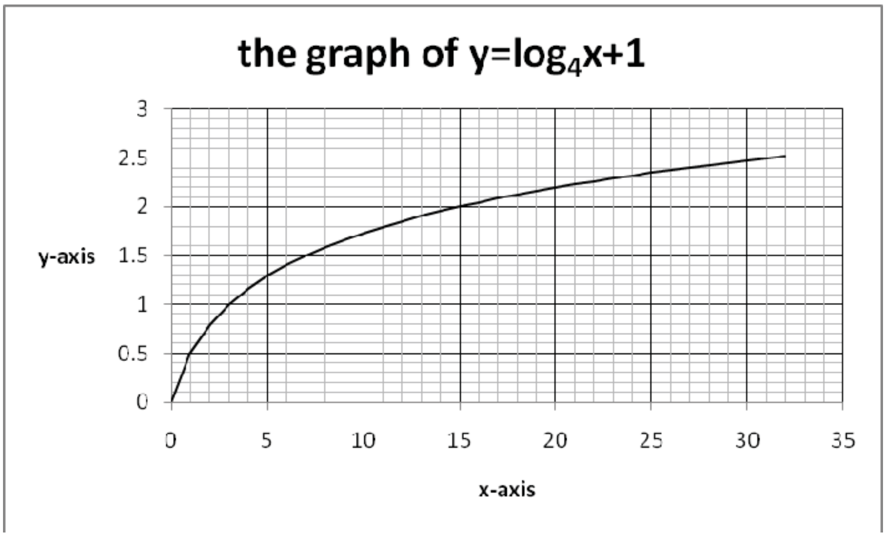
X	**1**	**10**	10^2	10^3	10^4	10^5
y	**0**	**1**	**2**	**3**	**4**	**5**

(a)



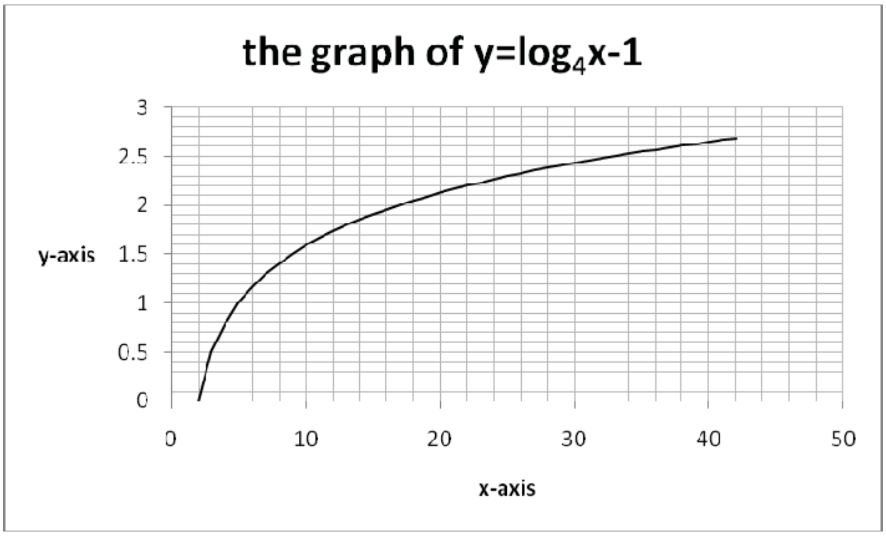
(b)

X	0	3	15	63	255
y	0	1	2	3	4



(c)

X	2	5	14	65	257
y	0	1	2	3	4



Unit Summary

In this unit you learned the following facts about working with logs:

Logarithm Rules

(a) $\log_b a = c$ means $a = b^c$

(b) $\log_b a + \log_b c = \log_b ac$

(c) $\log_b a - \log_b c = \log_b \frac{a}{c}$

(d) $\log_a 12 = \log_a (4 \times 3)$ or $\log_a (2 \times 6)$ etc.

(e) $\log_b b = 1$

(f) $\log_b 1 = 0$

(g) $\log_b a^c = c \log_b a$

(h) $\log_b a = \frac{\log_c a}{\log_c b}$

(i) $\log_b a = \frac{1}{\log_a b}$

(j) In $\log a$ i.e where the base is not indicated, the base is 10

(k) Log tables are to the base 10

Characteristics of graphs of logarithmic functions

Note the following about graphs of logarithmic functions?

1. The logarithmic function is the function $y = \log_b x$ (Do you still remember what it means? Refer to section on Changing logarithm statements into exponential statements $x = b^y$ i.e. the logarithm of x when b is the base is y).

2. ‘b’ is normally a whole number greater than 1 and is never ‘0’. Refer to a unit on indices.

3. Where does each graph cross the x-axis? For any base, the graph of logarithmic function passes at the coordinate (1,0). Which fact in indices causes this? What is $b^0 = ?$ Any number raised to the power of zero, equals one, i.e. $b^0 = 1$ or $0 = \log_b 1$. This means that in all graphs when y=0, x=1. Using our language of graphs the x-intercept is always 1.

4. The graph passes through the point (b,1). Figure out why this is so. When the logarithm is 1 or y=1 what can you say about x and b? Refer to the section 'Evaluating logs' of this unit. When y=1 , x=b. in other words $\log_b b = 1$

5. The graph goes below the x-axis where y values are negative for values of x between 0 and 1. Why is that? The values of x between 0 and 1 are fractions. So in $x = b^y$ when x is a fraction y should be negative. E,g. $\frac{1}{b^2} = b^{-2}$

6. The logarithm function $y = \log_b x$ is defined for only positive values of x. this means that $\log_b(-x)$ is undefined.

7. The graph comes closer to the y-axis but does not cross it.
8. An asymptote is a straight line to which a graph comes closer and closer but never touches. The y axis, of the logarithmic function $y = \log_b x$ is a vertical asymptote.

9. The logarithmic function $y = \log_b x + c$ where 'c' is any number moves the asymptote c units to the left.
10. The logarithmic function $y = \log_b x - c$ where c in any number moves the asymptote c units to the right.

You have completed the material for this unit on logarithms. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

Marks: 79 **Time: 2hrs**

Answer all questions

1. Work out the following

(a) $\log_2 2 + \log_2 5$ [2]

Assignment

(b) $\log^4 3 + \log^4 4$
[2]

(c) $\log^5 3 - \log^5 3$
[2]

(d) $\log^{10} 2 - \log^{10} 3$
[2]

2. Simplify the following

$(a)\log_5 \frac{1}{25} + \log_3 1 + \log_2 8$

[6]

$(b)\log_{27} 3 + \log_{25} 5 - \log_3 \frac{1}{9}$

[6]

$(c)\log_2 0.5 - \log_4 0.25 + \log 0.001$
[9]

$(d)\log_3 2 \times \log_4 49 \times \log_7 27$
[7]

3.Solve for x in the following log equations

$(a) 4\log_2 x + 3\log_x 2 = -7$

$[10]$

$(b) \log_x 2 = \frac{3}{2} + \log_2 x$

$[10]$

$(c) 2^x = 8$

$[2]$

(4) Sketch the following graphs

(a) $y = \log_2(7x - 5)$

$[7]$

(b) $y = \log_3 x$

$[7]$

(c) $y = \log_3(4 - x)$

$[7]$

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers to assignment

1

(a) $\log_2 2 + \log_2 5$

$\log_2 2 + \log_2 5$

$\log_2 2 \times 5$

$\log_2 10$

(b)

$\log_4 3 + \log_4 4 = \log_4 12$

(c)

$\log_5 3 - \log_5 3$

$\log_5 \frac{3}{3} = \log_5 1 = 0$

(d)

$\log_{10} 2 - \log_{10} 3 = \log_{10} \frac{2}{3}$

2.

(a)

$\log_5 \frac{1}{25} + \log_3 1 + \log_2 8 = \log_5 5^{-2} + 0 + \log_2 2^3$

$(-2 \times 1) + 0 + (3 \times 1) = 1$

(b)

$\log_{27} 3 + \log_{25} 5 - \log_3 \frac{1}{9}$

$\frac{1}{3} + \frac{1}{2} - -2\log_3 3$

$\frac{1}{3} + \frac{1}{2} + (2 \times 1)$

$\frac{1}{3} + \frac{1}{2} + \frac{2}{1} = \frac{2}{6} + \frac{3}{6} + \frac{12}{6} = \frac{17}{6} = 2\frac{5}{6}$

(c)

$\log_2 0.5 - \log_4 0.25 + \log 0.001$

$\frac{\log_{10} 0.5}{\log_{10} 2} - \frac{\log_{10} 0.25}{\log_{10} 4} + \log_{10} 10^{-3}$

$\log 0.25 - \log 0.125 + (-3\log_{10} 10)$

$-0.6021 - 0.9030 - 3$

-1.4949

(d)

$\log_3 2 \times \log_4 49 \times \log_7 27$

$\log_3 2 \times \log_4 49 \times \log_7 27$

$\frac{\log 2}{\log 3} \times \frac{\log 49}{\log 4} \times \frac{\log 27}{7}$

$\frac{\log 2}{\log 3} \times \frac{\log 7^2}{\log 2^2} \times \frac{\log 3^3}{\log 7}$

$\frac{\log 2}{\log 3} \times \frac{2\log 7}{2\log 2} \times \frac{3\log 3}{\log 7}$

3.

(a)

$4\log_2 x + 3\log_x 2 = -7$

Let $\log_2 x = k$

Let us substitute 'k' for $\log_2 x = k$ in our equation. Doing so we get $4k + \frac{3\times 1}{k} + 7 = 0$

$4k^2 + 3 + 7k = 0$

$4k^2 + 3k + 4k + 3 = 0$

$(4k^2 + 3k) + (4k + 3) = 0$

$k(4k+3) + 1(4k+3) = 0$

$(4k+3)(k+1) = 0$

$4k + 3 = 0$

$4k = -3$

$k = -\frac{3}{4}$

$k + 1 = 0$

$k = -1$

$\log_2 x = k$

$\log_2 x = -\frac{3}{4}$

$x = 2^{-\frac{3}{4}}$

$\log_2 x = k$

$\log_2 x = -1$

$x = 2^{-1} = \frac{1}{2}$

(b)

$\log_x 2 = \frac{3}{2} + \log_2 x$

$\frac{\log_2 2}{\log_2 x} = \frac{3}{2} + \log_2 x$

Let $\log_2 x = k$

Remember that $\log_2 2 = 1$, Our equation then becomes

$$\frac{1}{k} = \frac{3}{2} + \log_2 x$$

$$1 = \frac{3k}{2} + k$$

$$2 = 2k^2 + 3k$$

$$0 = 2k^2 + 3k - 2$$

$$0 = 2k^2 - 1k + 4k - 2$$

$$0 = (2k^2 - 1k) + (4k - 2)$$

$$0 = k(2k - 1) + 2(2k - 1)$$

$$0 = (2k - 1)(k + 2)$$

$$2k - 1 = 0$$

$$2k = 1$$

$$k = \frac{1}{2}$$

$$k + 2 = 0$$

$$k = -2$$

$$\log_2 x = k$$

$$\log_2 x = \frac{1}{2}$$

$$x = 2^{\frac{1}{2}}$$

$$\log_2 x = k$$

$$\log_2 x = -2$$

$$x = 2^{-2}$$

$(c) 2^x = 8$ $\quad 2^x = 2^3$

$$x = 3$$

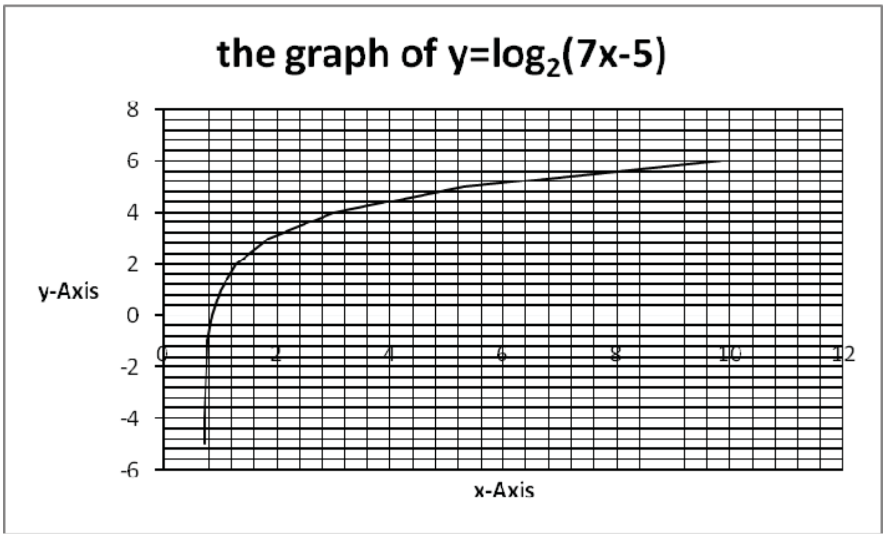
4. Sketch the following graphs

(a) $y = \log_2 (7x - 5)$

(b) $y = \log_3 x$

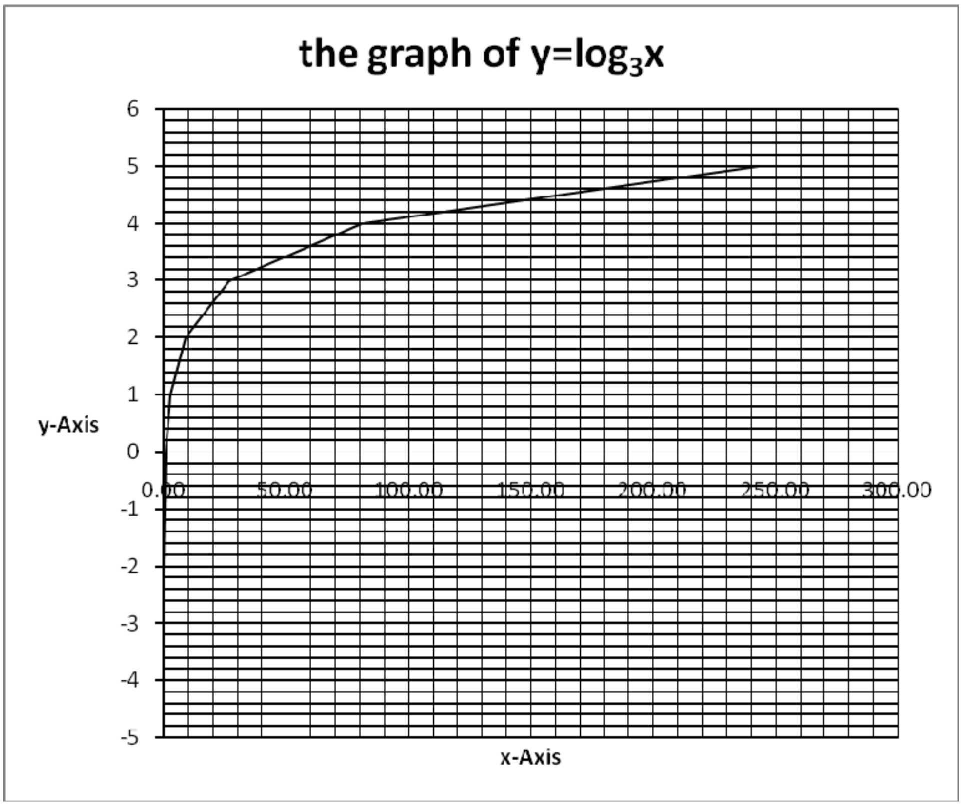
(a) $y = \log_2 (7x - 5)$

x	0.71875	0.72321	0.73214	0.75	0.78571	0.85714	1	1.28571	1.85714	3	5.2857	9.8571
y	-5	-4	-3	-2	-1	0	1	2	3	4	5	6



(a) $y = \log_3 x$

X	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
y	-3	-2	-1	0	1	2



Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Marks: 48

Time: $1\frac{1}{2}$ **hrs.**

Answer all questions

1. Evaluate $\log_2 8^4$
   [5]

2. Solve the following logarithmic function

$\log_2 3 + \log_2 8 = \log_2 (4x)$
[5]

3. Find $\log_6 54 - \log_6 9$
   [5]

4. Find the value for x in the following equation.
   $\log_7 243 = x$
[2]

5. Find the value for x.
   $\log_3 8 + \log_3 12 = \log_3 (6x)$
[5]

6. Calculate the value of x in $\log_9 729 - \log_9 81 = x$ [5]

7. Draw graphs of the following functions

(a) $y = \log_4 x$
[7]

(b) $y = \log_2 x + 1$
[7]

(c) $y = \log_3 x + 4$
[7]

Answers to Assessment

1. Evaluate $\log_2 8^4$

$$\log_2 8^4$$
$$= 4\log_2 8$$
$$= 4\log_2 2^3$$
$$= 4 \times 3\log_2 2$$
$$= 12 \times 1$$
$$= 12$$

2. Solve the following logarithmic function

$$\log_2 3 + \log_2 8 = \log_2 (4x)$$
$$\log_2 (3 \times 8) = \log_2 (4x)$$
$$\log_2 24 = \log_2 (4x)$$
$$24 = 4x$$
$$\frac{24}{4} = x$$
$$6 = x$$

3. Find $\log_6 54 - \log_6 9$

$$\begin{aligned}&\log_6 54 - \log_6 9\\&= \log_6 \frac{54}{9}\\&= \log_6 6\\&= 1\end{aligned}$$

4. Find the value for x.

$$\log_7 243 = x$$
$$7^x = 243$$
$$x = 3$$

5. Find the value for x.

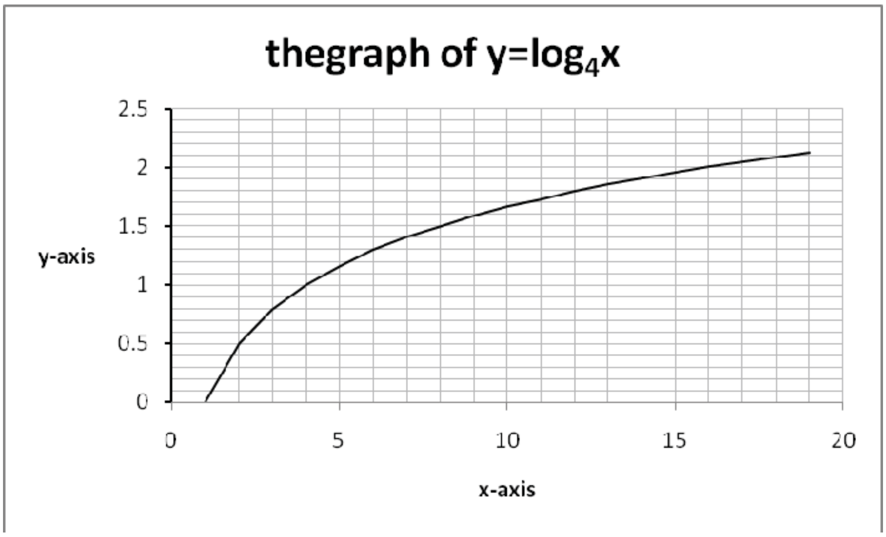
$$\log_3 8 + \log_3 12 = \log_3 (6x)$$
$$\log_3 (8 \times 12) = \log_3 (6x)$$
$$\log_3 96 = \log_3 (6x)$$
$$96 = 6x$$
$$\frac{96}{6} = x$$
$$16 = x$$

6. Work out $\log_9 729 - \log_9 81 = x$

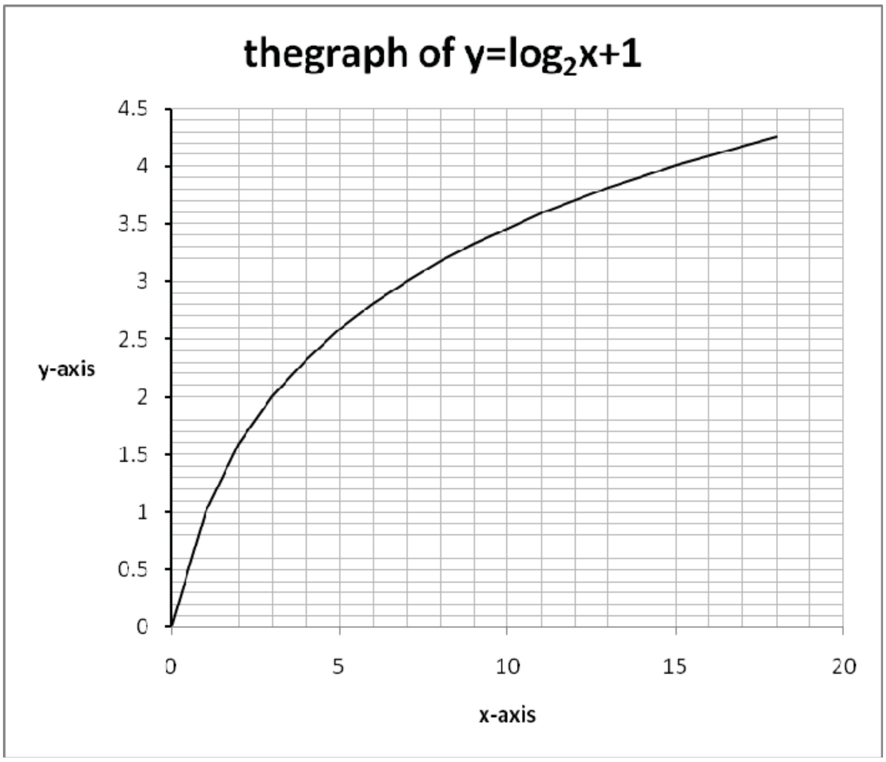
$$\log_9 729 - \log_9 81 = x$$
$$\log_9 \frac{729}{81} = x$$
$$\frac{729}{81} = x$$
$$9 = x$$

7. Draw graphs of the following functions

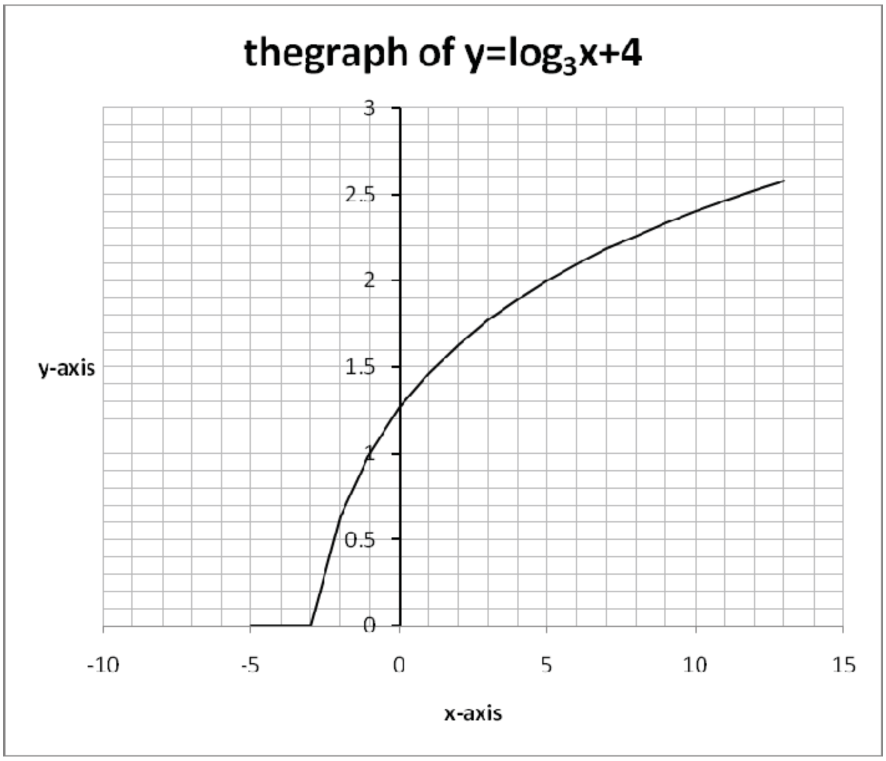
(a) $y = \log_4 x$


thegraph of y=log4x
2.5
2
1.5
1
0.5
0
y-axis
0
5
10
15
20
x-axis

(b) (b) $y = \log_2 x + 1$


thegraph of y=log2x+1
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
y-axis
0
5
10
15
20
x-axis

(c) $y = \log_3 x + 4$



Unit Contents

Unit 31

Unit 31

Absolute Value

Introduction

This unit consists of 65 pages. This is approximately 3% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 20 hours to work on this unit, 16 hours for formal study and 4 hours for self-study and completing assessments/assignments.

What do the numbers -4 and +4 have in common?

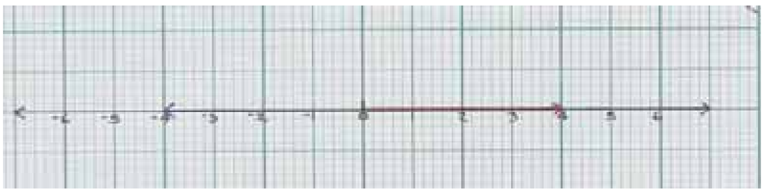
____

____

These are some of the possible responses:

1. "Obviously they are different numbers!"
2. "They are coordinates of two distinct points on the number line."

However, they are both the same distance from 0, the origin, on the number line. This distance is called the **absolute value** of 4 and -4; sometimes called **modulus**.



We say -4 is as far to the left of 0 as +4 is to the right of 0. This fact is shown by using the **absolute value** notation.

| -4 | = 4 read as "the absolute value of -4 is 4"

| +4 | = 4 read as "the absolute value of +4 is 4"

In this unit, we are going to be dealing with absolute value.

Take a moment to read the following learning outcomes. You should focus on those skills while studying this unit.

This Unit is Comprised of Four Lessons:

Lesson 1 Notation and Definition
Lesson 2 Solving Absolute Value Equations
Lesson 3 Solving Absolute Value Inequalities
Lesson 4 Absolute Value Graphs

Upon completion of this unit you will be able to:

Outcomes

- *identify* the notation $|ax + b|$
- *solve* equations of the form $|ax + b| = cx + d$ and inequalities of the form $|ax + b| \leq cx + d$ and $|ax + b| \geq cx + d$ where a, b, c and d are integers
- *draw and interpret* graphs of the function $y = |ax|$, $y = |ax + b|$ and $y = |x| + c$ where a, b, c and d are integers

Terminology

equation: an equation is a mathematical statement connecting two expressions with an equal sign.

inequality: a mathematical sentence in which the value of the expression on the left hand side is not equal to that on the right hand side. Symbols used with inequalities are $<, >, \leq$, and $\geq$.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information

- Textbook Correlations
- Online Courses

Lesson 1 Notation and Definition

By the end of this subunit, you should be able to:

- define absolute value.
- use the notation of absolute value correctly.

This subunit is about 4 pages in length.

The absolute value is closely related to the idea of distance. The absolute value of a number is the distance of that number from zero on the number line. Absolute value only asks about how far and not in what direction. It is therefore always positive.

The absolute value notation is bars, and not brackets.

For any real number, x, the absolute value of x, written $|x|$, is the distance of x from the origin, **without regard to direction**.

We can define this as follows:

$|x| = x$ if $x \geq 0$

$|x| = -x$ if $x < 0$

Activity 1

Activity 1

Evaluate the following:

1. $|15| =$
2. $|-15| =$
3. $|0| =$
4. $|5 + 5| =$
5. $|5| + |5| =$
6. $|-5| + |-5| =$
7. $|-4| - |-3| =$
8. $-4 \div |-4| =$
9. $(-4) \times |-4| =$

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Key Points to Remember

The key points to remember in this subunit on notation and definition are:

- the absolute value notation is bars, $|\ \ |$, and not brackets.
- for any real number, x, the absolute value of x, written $|x|$, is the distance of x from the origin.
- the absolute value of a number is always positive.

Answers to activity 1:

1. $|15| = 15$
2. $|-15| = 15$
3. $|0| = 0$ [since 0 is the origin, it is natural to have $|0| = 0$]
4. $|5 + 5| = |10| = 10$
5. $|5| + |5| = 5 + 5 = 10$
6. $|-5| + |-5| = 5 + 5 = 10$
7. $|-4| - |-3| = 4 - 3 = 1$
8. $-4 \div |-4| = -4 \div 4 = -1$
9. $(-4) \times |-4| = (-4) \times 4 = -16$

Lesson 2 Solving Absolute Value Equations

By the end of this subunit, you should be able to:

- solve absolute value equations.

This subunit is about 4 pages in length.

Given | x | = 13, it means x can either be 13 or -13.

This says any equation that looks like this | a | = b has two solutions.

a = b or a = -b

We call **a** the argument of the absolute value. Either the argument will be **b**, or it will be **–b**.

Example 1

$|2x - 3| = 1$

We temporarily put aside the absolute value lines, and then write the right hand side with + or -

$2x - 3 = \pm 1$

Therefore $2x - 3 = +1$ or $2x - 3 = -1$

We then solve these two equations.

$2x - 3 = +1$	or	$2x - 3 = -1$
$\underline{\quad +3 \quad +3}$		$\underline{\quad +3 \quad +3}$
$\frac{2x}{2} = \frac{4}{2}$		$\frac{2x}{2} = \frac{2}{2}$
$x = 2$	or	$x = 1$

Check these solutions to see if they are valid solutions by substituting them in the original equation.

$|2x - 3| = 1$ $\qquad\qquad$ $|2x - 3| = 1$

$|2(2) - 3| = 1$	$|2(1) - 3| = 1$
$|4 - 3| = 1$	$|2 - 3| = 1$
$|1| = 1$	$|-1| = 1$
$1 = 1$	$1 = 1$
The solution is valid.	The solution is valid.

Example 2

Solve $2|x - 2| = 3x - 2$

$2(x - 2) = \pm(3x - 2)$

$2x - 4 = \pm(3x - 2)$

Therefore $2x - 4 = +(3x - 2)$ or $2x - 4 = -(3x - 2)$

We then solve these two equations.

$2x - 4 = +(3x - 2)$	or	$2x - 4 = -(3x - 2)$
$2x - 4 = 3x - 2$	or	$2x - 4 = -3x + 2$
$\underline{\quad +4 \qquad +4}$		$\underline{\quad +4 \qquad +4}$
$2x = 3x + 2$		$2x = -3x + 6$
$\underline{-3x \qquad -3x}$		$\underline{+3x \qquad +3x}$
$-x = 2$		$5x = 6$
$x = -2$	or	$x = \frac{6}{5}$
$x = -2$		$x = 1.2$

Check these solutions.

2| x - 2| = 3x – 2	2| x - 2| = 3x - 2
2| - 2 - 2| = 3(-2) – 2	2| 1.2 - 2| = 3(1.2) - 2
2| -4| = -6 – 2	2| -0.8| = 3.6 - 2
2 (4) = -8	2(0.8) = 1.6
8 = -8	1.6 = 1.6
The solution is **not** valid.	The solution is valid.

Therefore x = 1.2 is the only solution.

Example 3

Solve 2| x - 3| = -6

2| x - 3| = -6 be careful!

Dividing both sides by 2,

| x - 3| = -3

This equation has no solution because the absolute value of a number is always positive.

Let us solve it again using how it is suggested we do it.

Solve 2| x - 3| = -6

2(x – 3) = ± (-6)

2x - 6 = ± (-6)

Therefore 2x - 6 = + (-6) or 2x - 6 = -(-6)

We then solve these two equations.

2x - 6 = + (-6) or 2x - 6 = - (-6)

$2x - 6 = -6$ or $2x - 6 = 6$

$2x - 6 = -6$	or	$2x - 6 = 6$
$\underline{\quad +6 \quad +6}$		$\underline{+6 \quad +6}$
$2x = -6 + 6$		$2x = 6 + 6$
$2x = 0$		$2x = 12$
$x = 0$	or	$x = \underline{6}$

Check these solutions.

Solve $2\lvert x - 3\rvert = -6$	Solve $2\lvert x - 3\rvert = -6$
$2\lvert 0 - 3\rvert = -6$	$2\lvert 6 - 3\rvert = -6$
$2\lvert -3\rvert = -6$	$2\lvert 3\rvert = -6$
$2(3) = -6$	$2(3) = -6$
$6 = -6$	$6 = -6$
The solution is **not** valid.	The solution is **not** valid.

Note it!

Always check your answers in the **original** equation. Absolute value equations **often** have solutions that must be rejected.

Activity 2

Activity 2

Evaluate the following:

1. $2|x + \frac{1}{2}| = 7$
2. $3 - \frac{1}{3}|x - 3| = 1$
3. $|2x - 1| = 3$
4. $3|2x - \frac{1}{3}| = 2\,|-3|$
5. $\frac{1}{2}|3 - x| = 6$
6. $3|x| + 3 = 0$
7. $-|x - 2| + 2 = 1$
8. $|2x + 3| = 4$

Compare your answers to those provided at the end of this subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Key Points to Remember

The key points to remember in this subunit on solving absolute value equations are; when solving absolute value equations,

- you usually get two solutions.
- you temporarily put aside the absolute value lines, and then write the right hand side with + or -; then solve the equations.

Answers to Activity 2

1.

$2|x + \frac{1}{2}| = 7$

$2(x + \frac{1}{2}) = \pm 7$

$2(x + \frac{1}{2}) = +7$ or $2(x + \frac{1}{2}) = -7$

We then solve these two equations.

$2(x + \frac{1}{2}) = +7$ or $2(x + \frac{1}{2}) = -7$

$2x + 1 = 7$ or $2x + 1 = -7$

$\underline{\quad -1 \quad -1}$ $\qquad$ $\underline{\quad -1 \quad -1}$

$2x = 6$ $\qquad$ $2x = -8$

$x = 3$ $\qquad$ $x = 4$

$x = 3$ or $x = -4$

2.

$3 - \frac{1}{3}|x - 3| = 1$

$3 - \frac{1}{3}(x \ \ 3) = \pm 1$

$3 - \frac{1}{3}(x - 3) = +1$ or $3 - \frac{1}{3}(x - 3) = -1$

$3 - \frac{1}{3}(x - 3) = +1$ or $3 - \frac{1}{3}(x - 3) = -1$

$3 - (\frac{1}{3}x - 1) = +1$ or $3 - (\frac{1}{3}x - 1) = -1$

$3 - \frac{1}{3}x + 1 = +1$ or $3 - \frac{1}{3}x + 1 = -1$

$4 - \frac{1}{3}x = +1$ or $4 - \frac{1}{3}x = -1$

$\underline{-4 \qquad -4}$ $\qquad$ $\underline{-4 \qquad -4}$

$-\frac{1}{3}x = -3$ or $-\frac{1}{3}x = -5$

$x = 9$ or $x = 15$

3. $|2x - 1| = 3$

$(2x - 1) = \pm 3$

$(2x - 1) = +3$ or $(2x - 1) = -3$

.

$(2x - 1) = 3$ or $(2x - 1)| = -3$

$2x - 1 = 3$ or $2x - 1 = -3$

$\underline{\quad +1 \quad +1}$ $\underline{+1 \quad +1}$

$2x = 4$ $2x = -2$

$x = 2$ $x = -1$

$x = 2$ or $x = -1$

4. $3|2x - ⅓| = 2 |- 3|$

$3(2x - ⅓) = \pm [2 (- 3)]$

$3(2x - ⅓) = + [2 (- 3)]$ or $3(2x - ⅓) = - [2 (- 3)]$

$3(2x - ⅓) = + [2 (- 3)]$ or $3(2x - ⅓) = - [2 (- 3)]$

$6x - 1 = + (- 6)$ or $6x - 1 = - (- 6)$

$6x - 1 = -6$ or $6x - 1 = 6$

$\underline{+1 \quad +1}$ $\underline{+1 \quad +1}$

$6x = -5$ or $6x = 7$

$x = -\frac{5}{6}$ or $x = \frac{7}{6}$

5. ½|3 - x | = 6

½|3 - x | = ±6

½|3 - x | = +6 or ½|3 - x | = -6

$\frac{3}{2} - \frac{1}{2}x = 6$ or $\frac{3}{2} - \frac{1}{2}x = -6$

$-\frac{1}{2}x = \frac{9}{2}$ or $-\frac{1}{2}x = -\frac{15}{2}$

$x = -9$ $x = 15$

6. 3|x| + 3 = 0

3 (x) + 3 = 0

$$\begin{array}{r} 3x + 3 = 0 \\ \underline{\quad -3 \quad -3} \\ 3x \quad = \quad -3 \end{array}$$

3x = ± (-3)

3x = +(-3) or 3x = -(-3)

3x = -3 or 3x = -3

$\frac{3}{3}x = \frac{-3}{3}$ or $\frac{3}{3}x = \frac{3}{3}$

$x = -1$ or $x = 1$

(No real solutions, when these solutions are checked in the equation)

7. $-|x - 2| + 2 = 1$

$-(x - 2) + 2 = 1$

$-(x - 2) + 2 = 1$

$\underline{\quad -2 \quad -2}$

$-(x - 2) = -1$

$-x + 2 = -1$

$-x + 2 = \pm(-1)$

$-x + 2 = +(-1)$ or $-x + 2 = -(-1)$

$-x + 2 = -1$ or $-x + 2 = 1$

$-x + 2 = -1$ or $-x + 2 = 1$

$\underline{\quad -2 \quad -2}$ $\underline{-2 \quad -2}$

$-x = -3$ or $-x = -1$

$x = 3$ or $x = 1$

8. $|2x + 3| = 4$

$(2x + 3) = 4$

$(2x + 3) = \pm 4$

$(2x + 3) = +4$ or $(2x + 3) = -4$

$$2x + 3 = +4 \quad \text{or} \quad 2x + 3 = -4$$

$$\begin{array}{rcl} 2x + 3 = +4 & \text{or} & 2x + 3 = -4 \\ \underline{\quad -3 \quad -3} & & \underline{-3 \quad -3} \\ 2x \quad = 1 & \text{or} & 2x \quad = \quad -7 \end{array}$$

$$x = \frac{1}{2} \quad \text{or} \quad x = -\frac{7}{2}$$

Lesson 3 Solving Absolute Value Inequalities

By the end of this subunit, you should be able to

- solve absolute value inequalities.

This subunit is about 4 pages in length.

In this section, you are going to learn how to solve absolute value inequalities. This is not the first time that you solve inequalities.

What is an inequality?

____

____

How do you think the inequalities you have come across before differ from the absolute value inequalities?

____

____

The answers to the questions are as follows:

The absolute value inequalities still use the same symbols $<$, $\leq$, $>$, and $\geq$. The difference is brought by the absolute value notation which is bars.

For example

$x + 6 < 14$ is an inequality that we are used to; $|x + 6| < 14$ is an absolute value inequality

Reflection

A "solution" of an inequality is a number which when substituted for the variable makes the inequality a true statement.

Example 1

Solve the inequality

$$x - 2 > 8$$

We want the numbers collected together on one side. We add 2 to both sides.

$$x - 2 > 8$$

$$\underline{\;+2 \quad +2\;}$$

$$x \quad > \; 10$$

The answer is all numbers greater than 10. Inequalities always have an infinite number of solutions.

Checking the answer, using say, 12.

$$x - 2 > 8$$

$$12 - 2 > 8$$

$10 > 8$ Indeed 10 is greater than 8.

Example 2

Solve the inequality

$$-3x < 12$$

We divide both sides by -3.

$$\frac{-3x}{-3} < \frac{12}{-3}$$

$$x < -4$$

The answer is all numbers less than -4.

Checking the answer, using say, -5.

$$-3x < 12$$

$$-3(-5) < 12$$

$$15 < 12$$

But 15 is **not** less than 12!

This is because **when solving inequalities, when we multiply or divide by a negative number, the direction of the inequality changes.**

Lets go back to the beginning and solve the inequality again.

$$\frac{-3x}{-3} < \frac{12}{-3}$$

$$x > -4$$

The answer is all numbers greater than -4.

Checking the answer, using say, -1.

$$-3x < 12$$

$$-3(-1) < 12$$

$$4 < 12$$

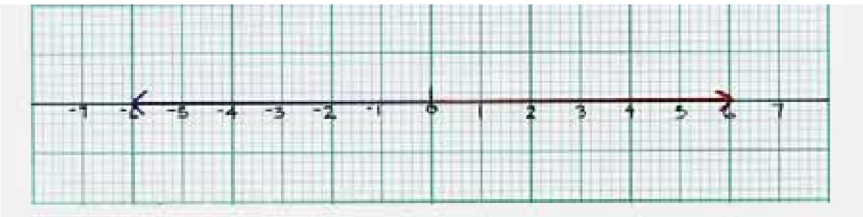
Yes, 4 is less than 12

When solving absolute value inequalities, we follow more or less the same methods like the one used in solving absolute value equations. Let us try to give meaning to absolute value inequalities.

Let us take the inequality $|x| < 6$

This says we are looking for those real numbers, x, whose distance is less than 6 units from the origin.

We are talking about the interval (-6,6)



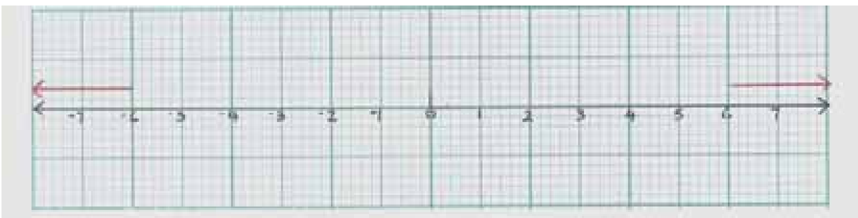
Given the inequality $| x | < a$, the solution is always $-a < x < a$.

This is called the pattern for "less than" absolute value inequalities. It always holds.

Let us take the inequality $|x| > 6$

In this case we are looking for those real numbers, x, which are at least 6 units away from the origin. To the left, real numbers less than or equal to -6 qualify, and on the right all real numbers greater than or equal to 6 qualify.

This can be represented this way: $(-\infty,-6)$ union $(6,\infty)$



Given the inequality $|x| > a$ the solution always starts by splitting the inequality into two pieces:

$x < -a$ or $x > a$

This is called the pattern for "greater than" absolute value inequalities. It always holds.

With this understanding, let us look at the steps to follow when solving absolute value inequalities

1. **Solving "less than" absolute value inequalities**

If we are given the inequality $|x| < a$ the solution is always $-a < x < a$. The first step therefore is to write the absolute value according to the pattern. Then solve the inequality. This also works even for $\leq$ situations. Solve $|4 - x| \leq 20$

$$-a < x < a$$

$$-20 \leq 4 - x \leq 20$$

$$\underline{-4 \quad -4 \quad -4}$$

$$-24 \leq -x \leq 16 \qquad \frac{-24}{-1} \leq \frac{-x}{-1} \leq \frac{16}{-1}$$

$$24 \geq x \geq -16$$

When solving inequalities, when we multiply or divide by a negative number, the direction of the inequality changes.

This we can rewrite as single inequalities as $x \geq -16$ or $x \leq 24$

To the right of 0, real numbers less than 24 qualify, and on the left all real numbers greater than -16 qualify.

2. **Solving "greater than" absolute value inequalities**

If we are given the inequality $|x| > a$ the solution always starts by splitting the inequality into two pieces:

$x < -a$ or $x > a$

Then solve the inequality. This also works even for $\geq$ situations.

Solve $|2x - 3| > 5$

$\lvert 2x - 3 \rvert > 5$	or	$\lvert 2x - 3 \rvert < -5$
$2x - 3 > 5$	or	$2x - 3 < -5$
$\underline{\quad +3 \quad +3}$		$\underline{+3 \quad +3}$
$2x > 8$	or	$2x < -2$
$\frac{2x}{2} > \frac{8}{2}$	or	$\frac{2x}{2} < \frac{-2}{2}$
$x > 4$	or	$x < -1$

To the left of 0, real numbers less than -1 qualify, and on the right all real numbers greater than 4 qualify.

Activity 3

Activity 3

Evaluate the following:

1. $|x + 3| < 7$
2. $4 - |x - 3| \geq 0$
3. $|7 - 2x| \geq 8 - x$
4. $|2x - 4| < 10$
5. $|x - 4| \geq ½$
6. $- |4 - x| > -8$
7. $2 - |x - 3| < |-1|$

Compare your answers to those provided at the end of the subunit. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Key Points to Remember

The key points to remember in this subunit on solving absolute value inequalities are; when solving absolute value inequalities:

- you usually get two solutions.

- When solving inequalities, when we multiply or divide by a negative number, the direction of the inequality changes.

- If we are given the inequality $|x| < a$ the solution is always $–a < x < a$. The first step therefore is to write the absolute value according to the pattern. Then solve the inequality. This also works even for $\leq$ situations.

- If we are given the inequality $|x| > a$ the solution always starts by splitting the inequality into two pieces:

  $x < -a$ or $x > a$

  Then solve the inequality. This also works even for $\geq$ situations.

Answers to Activity 3

1.

$-7 < x + 3 < 7$

$\underline{-3 \quad -3 \quad -3}$

$-10 < x < 4$

$x < 4$ or $x > -10$

2.

$4 - |x - 3| \geq 0$

$\underline{-4 \qquad\qquad -4}$

$- |x - 3| \geq -4$

$|x - 3| \leq 4$

$-4 \leq x - 3 < 4$

$\underline{+3 \quad +3 \quad +3}$

$-1 \leq x \leq 7$

$x \leq 7$ or $x \geq -1$

3.

$7 - 2x \geq 8 - x$

$\underline{+x \qquad +x}$

$7 - x \geq 8$

$7 - x \geq 8$ or $7 - x \leq - 8$

$\underline{-7 \qquad -7}$ $\qquad$ $\underline{-7 \qquad -7}$

$-x \geq 1$ $\qquad$ $-x \leq -15$

$$\frac{-x}{-1} \geq \frac{1}{-1} \qquad \frac{-x}{-1} \leq \frac{-15}{-1}$$

$x \leq -1$ or $x \geq 15$

4. $|2x - 4| < 10$

$-10 < 2x - 4 < 10$

$\underline{\quad +4 \quad +4 \quad +4}$

$-6 < 2x < 14$

$-3 < x < 7$

$x < 7$ or $x > -3$

5. $|x - 4| \geq \frac{1}{2}$

$x - 4 \geq \frac{1}{2}$ or $x - 4 \leq -\frac{1}{2}$

$\underline{\quad +4 \quad +4}$ $\underline{\quad +4 \quad +4}$

$x \geq 4\frac{1}{2}$ or $x \leq 3\frac{1}{2}$

6.

$-|4 - x| \geq -8$

$|4 - x| \leq 8$

$-8 \leq 4 - x \leq 8$

$\underline{-4 \quad -4 \quad -4}$

$-12 \leq -x \leq 4$

$12 \geq x \geq -4$

7.

$2 - |x - 3| < |-1|$

$2 - |x - 3| < 1$

$- |x - 3| < -1$

$|x - 3| > 1$

$x - 3 < -1$ or $x - 3 > 1$

$+3 \quad +3$ $\qquad$ $+3 \quad +3$

$x < 2$ or $x > 4$

Lesson 4 Absolute Value Graphs

By the end of this subunit, you should be able to

- draw and interpret absolute value graphs

This subunit is about 3 pages in length.

Absolute value graphs have the general form of

$$y = a|x - b| + c$$

where:

- ***a*** is the gradient of one of the lines making up the absolute value graph; it also tells us whether the graph has a maximum or a minimum value. If $a > 0$, the graph has a "v" shape, and it has a minimum point, and if $a < 0$, the graph has a "^" shape and it has a maximum point.

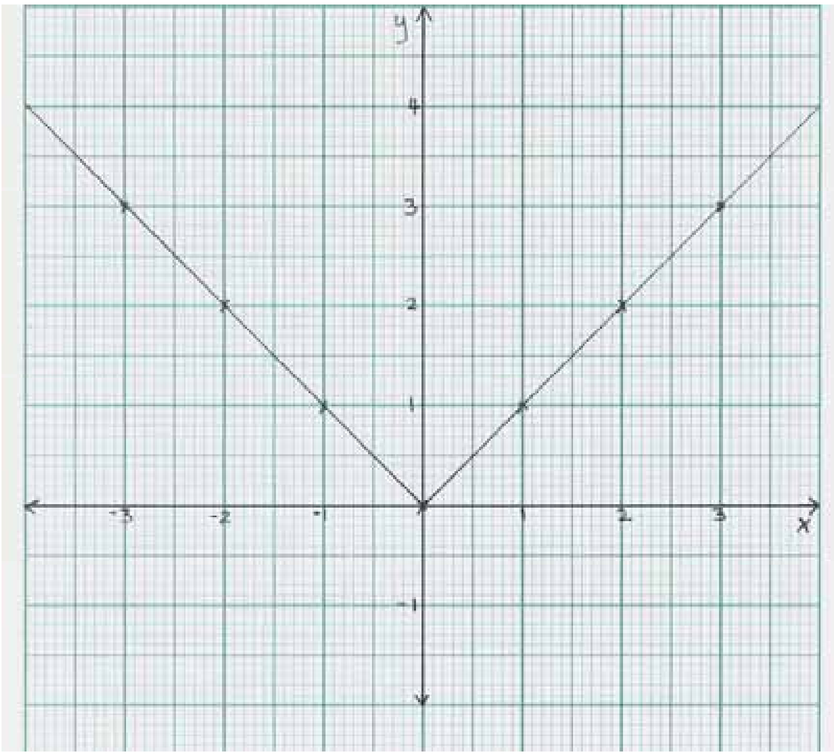
- ***b*** and ***c*** are the coordinates of the turning point (b, c) The axis of symmetry is drawn through the turning point.

As is the case with other graphs, we start by creating the table of values.

Example 1

$y = |x|$

x	y
-3	|-3| = 3
-2	|-2| = 2
-1	|-1| = 1
0	|0| = 0
1	|1| = 1
2	|2| = 2
3	|3| = 3



- The graph has the shape that looks like a “v”.

- with the help of the general form; $y = a|x - b| + c$

(i) *a* is the gradient of one of the lines making up the absolute value graph. In this example *a* is +1. It is the gradient of the line that is in the 1^{st} quadrant. a > 0, then the graph has a minimum point.

(ii) *b* and *c* are the coordinates of the turning point; *b* is 0 and *c* is 0

(iii) The axis of symmetry is x = 0

Note it!

y = |x| is the parent function of absolute value graphs.

It is a function that is used as building block for others in a particular family.

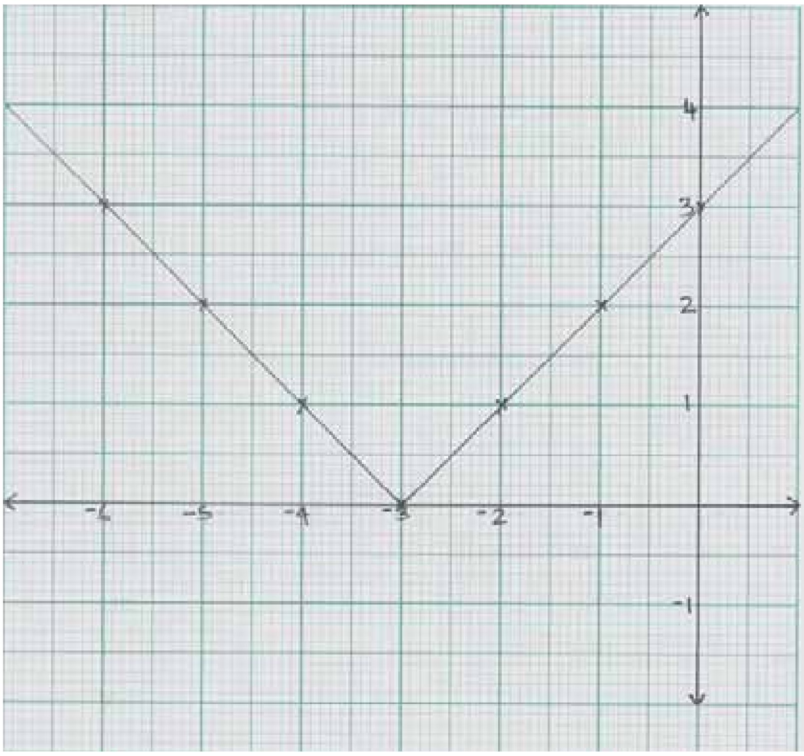
The basic shapes of functions in a family are the same.

Example 2

y = |x + 3|

The x – values we are going to use are from -7 to 1 as opposed to from -3 to 3. This is done to allow the shape of the graph to come out.

x	y
-6	|-6 +3| = |-3| = 3
-5	|-5 +3| = |-2| = 2
-4	|-4 +3| = |-1| = 1
-3	|-3 +3| = |0| = 0
-2	|-2 +3| = |1| = 1
-1	|-1 +3| = |2| = 2
0	|0 +3| = |3| = 3



- The graph has the shape that looks like a "v". It looks almost identical to the graph of y = |x| except that it was shifted 3 units to the left.

- with the help of the general form; $y = a|x - b| + c$

(i) *a* is the gradient of one of the lines making up the absolute value graph. In this example *a* is +1. It is the gradient of the line that is in the 1^{st} quadrant. $a > 0$, then the graph has a minimum point.

(ii) *b* and *c* are the coordinates of the turning point; *b* is 0 and *c* is 0.

(iii) The axis of symmetry is $x = 0$

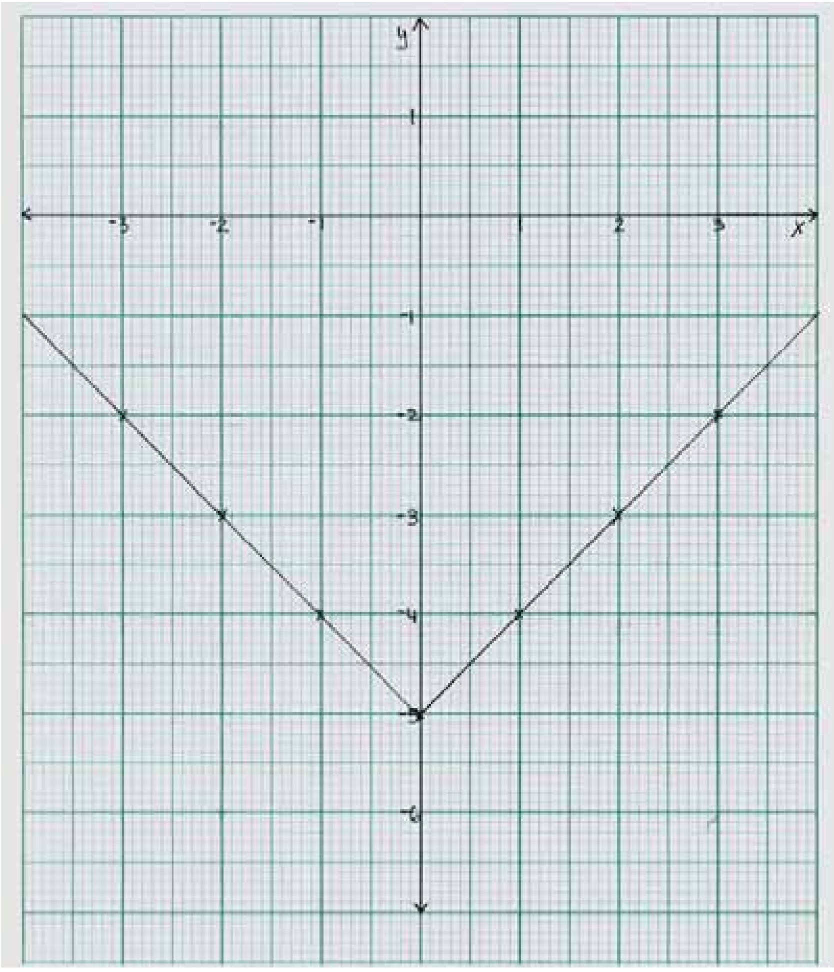
- *b* and *c* are the coordinates of the turning point, *b* is -3 and *c* is 0 (-3,0)

The axis of symmetry is $x = -3$

Example 3

Lets take for example y = |x| - 5

x	y
-3	|-3| - 5 = 3 -5 = -2
-2	|-2| - 5 = 2 -5 = -3
-1	|-1| - 5 = 1 -5 = -4
0	|0| - 5 = 0 -5 = -5
1	|1| - 5 = 1 -5 = -4
2	|2| - 5 = 2 -5 = -3
3	|3| - 5 = 3 -5 = -2



- the graph has the shape that looks like a "v". It looks almost identical to the graph of y = |x| except that it has been moved down 5 units.

- with the help of the general form; $y = a|x - b| + c$

(i) *a* is the gradient of one of the lines making up the absolute value graph. In this example *a* is +1. $a > 0$, then the graph has a minimum point.

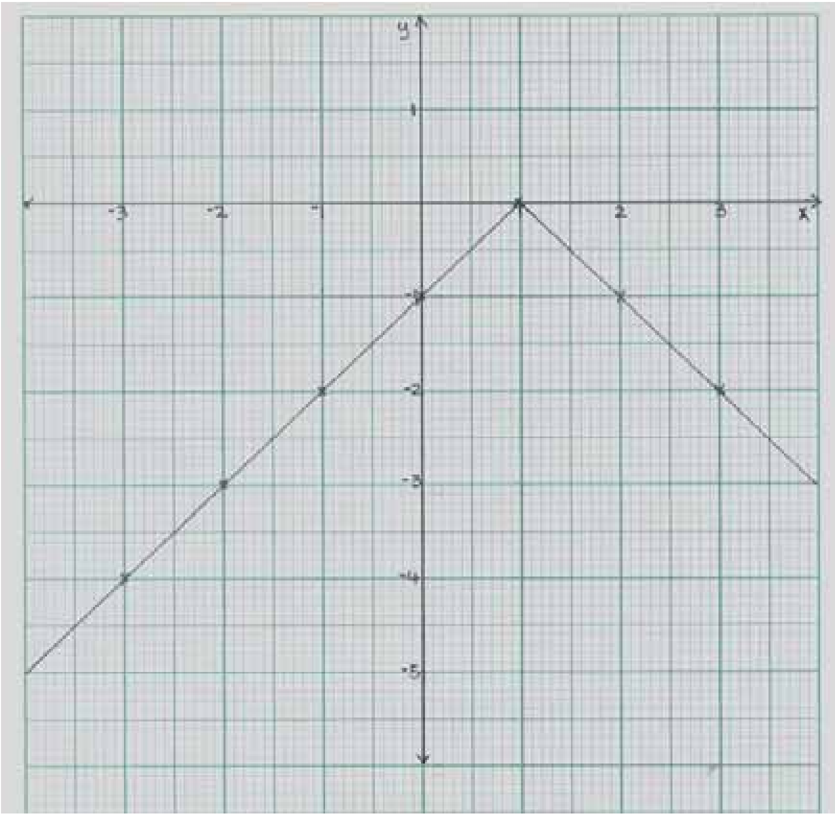
(ii) *b* and *c* are the coordinates of the turning point; *b* is 0 and *c* is -5.

(iii) The axis of symmetry is x = 0.

Example 4

y = -|x - 1|

x	y
-3	-|-3 -1| = -|-4| = -(4) = -4
-2	-|-2 -1| = -|-3| = -(3) = -3
-1	-|-1 -1| = -|-2| = -(2) = -2
0	-|0 -1| = -|-1| = -(1) = -1
1	-|1 -1| = -|0| = -(0) = 0
2	-|2 -1| = -|1| = -(1) = -1
3	-|3 -1| = -|2| = -(2) = -2



(i) *a* is -1. A < 0, the graph has a "^" shape and it has a maximum point.

(ii) *b* and *c* are the coordinates of the turning point; *b* is 1 and *c* is 0.

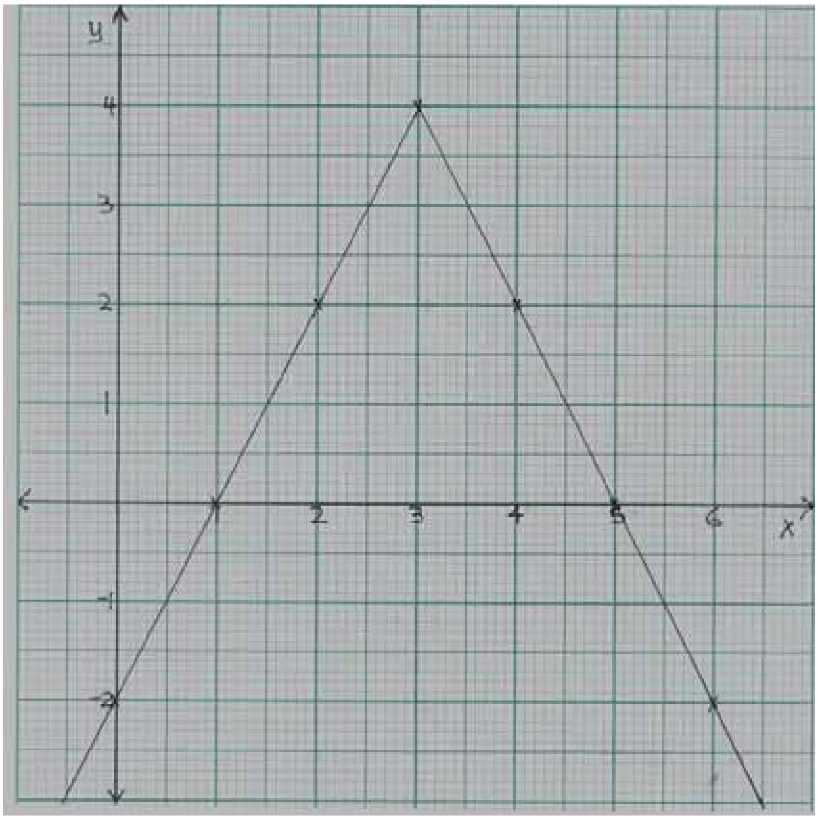
(iii) the axis of symmetry is x = 1.

Example 5

y = -2|x - 3| + 4

x	y
0	-2|0 -3| + 4 = -2|-3| + 4 = -2(3) + 4 = -6 + 4 = -2

1	-2|1 -3| + 4 = -2|-2| + 4 = -2(2) + 4 = -4 + 4 = 0
2	-2|2 -3| + 4 = -2|-1| + 4 = -2(1) + 4 = -2 + 4 = 2
3	-2|3 -3| + 4 = -2|0| + 4 = -2(0) + 4 = -0 + 4 = 4
4	-2|4 -3| + 4 = -2|1| + 4 = -2(1) + 4 = -2 + 4 = 2
5	-2|5 -3| + 4 = -2|2| + 4 = -2(2) + 4 = -4 + 4 = 0
6	-2|6 -3| + 4 = -2|3| + 4 = -2(3) + 4 = -6 + 4 = -2



(i) a is -2. $a > 0$, the graph has a “∧” shape and it has a maximum point.

(ii) b and c are the coordinates of the turning point; b is 1 and c is 0.

(iii) The axis of symmetry is $x = 1$.

Activity 4

Activity 4

A. Draw the graphs of the following equations:

1. $y = |x - 3|$
2. $y = |x| - 2$
3. $y = 3|x|$
4. $y = -2|x| - 3$
5. $y = |3x + 6|$
6. $y = -5 + |x|$

B. Draw the graphs of the following on the same set of axes:

1. $y = |x|$; $y = |x| + 1$; $y = |x| + 3$; $y = |x| - 1$ and $y = |x| - 3$
2. $y = |x|$; $y = |x - 1|$; $y = |x - 2|$; $y = |x + 1|$ and $y = |x + 3|$
3. $y = |x|$; $y = 2|x|$; $y = -|x|$; and $y = -2|x|$

What observations have you made about the graphs?

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Key Points to Remember

The key points to remember in this subunit on absolute value graphs are;

- Absolute value graphs have the general form $y = a|x - b| + c$; where *a*, *b* and *c* are integers
- - ***a*** is the gradient of one of the lines making up the absolute value graph; If $a > 0$, the graph has a "v" shape, and it has a minimum point, and if $a < 0$, the graph has a "^" shape and it has a maximum point
- ***b*** and ***c*** are the coordinates of the turning point (b,c) The axis of symmetry is drawn through the turning point.
- When "c" is positive, the graphs moved "c" units up, and when "c" is negative, the graphs moved "c" units down.
- When "b" is positive, the graphs moved "b" units to the left and when "b" is negative, the graphs moved "b" units to the right.

You have now completed the last subunit of this unit on absolute value. Do a quick review of the entire content of this unit and then continue on to the unit summary.

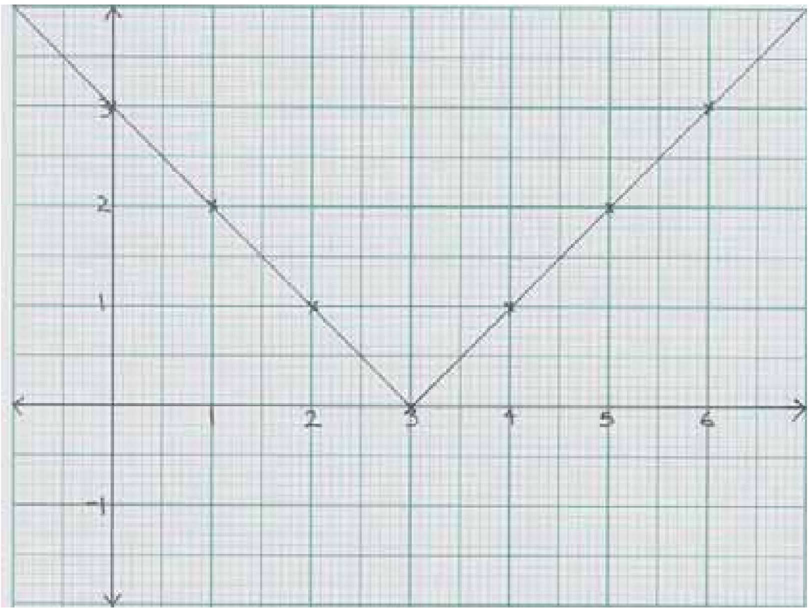
Answers to Activity 4

A.

$y = |x - 3|$

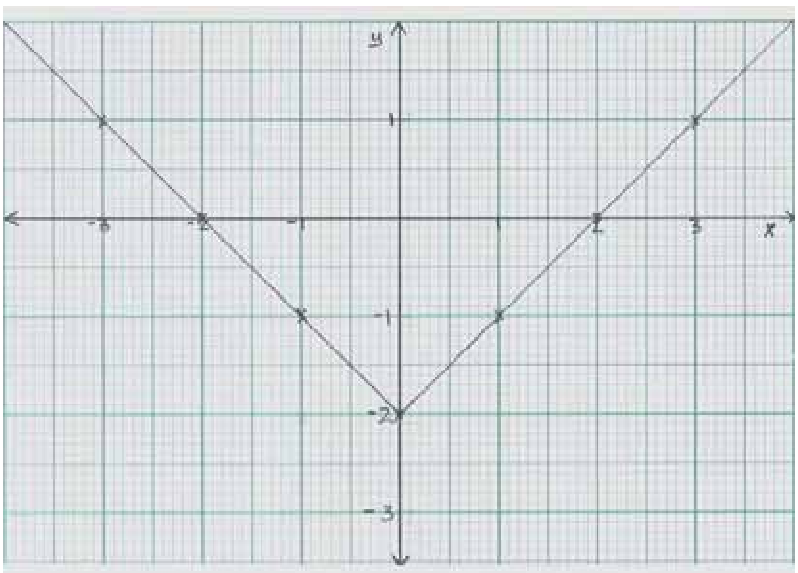
x	y
0	$|0 -3| = |-3| = 3$
1	$|1 -3| = |-2| = 2$
2	$|2 -3| = |-1| = 1$
3	$|3 -3| = |0| = 0$

4	$\|4 - 3\| = \|1\| = 1$
5	$\|5 - 3\| = \|2\| = 2$
6	$\|6 - 3\| = \|3\| = 3$



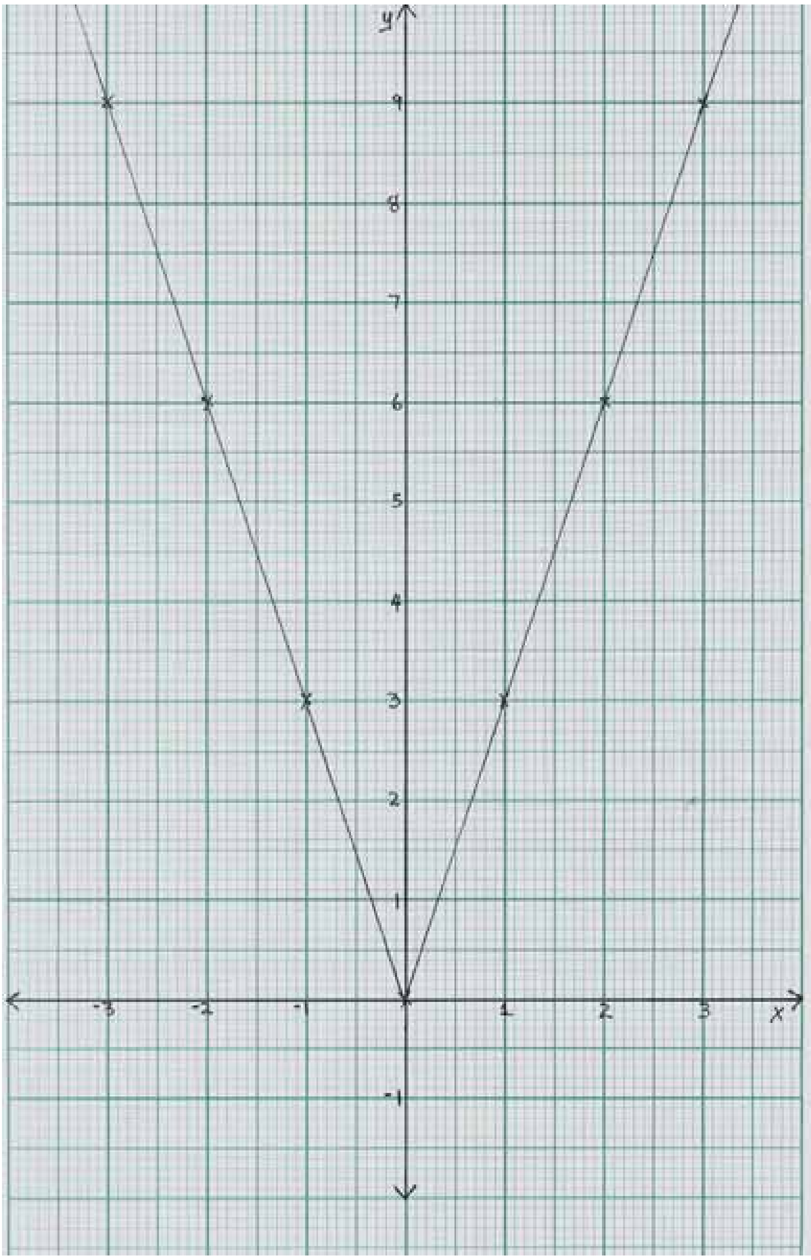
y = |x| -2

x	y
-3	|-3| - 2 = 3 – 2 = 1
-2	|-2| - 2 = 2 – 2 = 0
-1	|-1| - 2 = 1 – 2 = -1
0	|0| - 2 = 0 – 2 = -2
1	|1| - 2 = 1 – 2 = -1
2	|2| - 2 = 2 – 2 = 0
3	|3| - 2 = 3 – 2 = 1



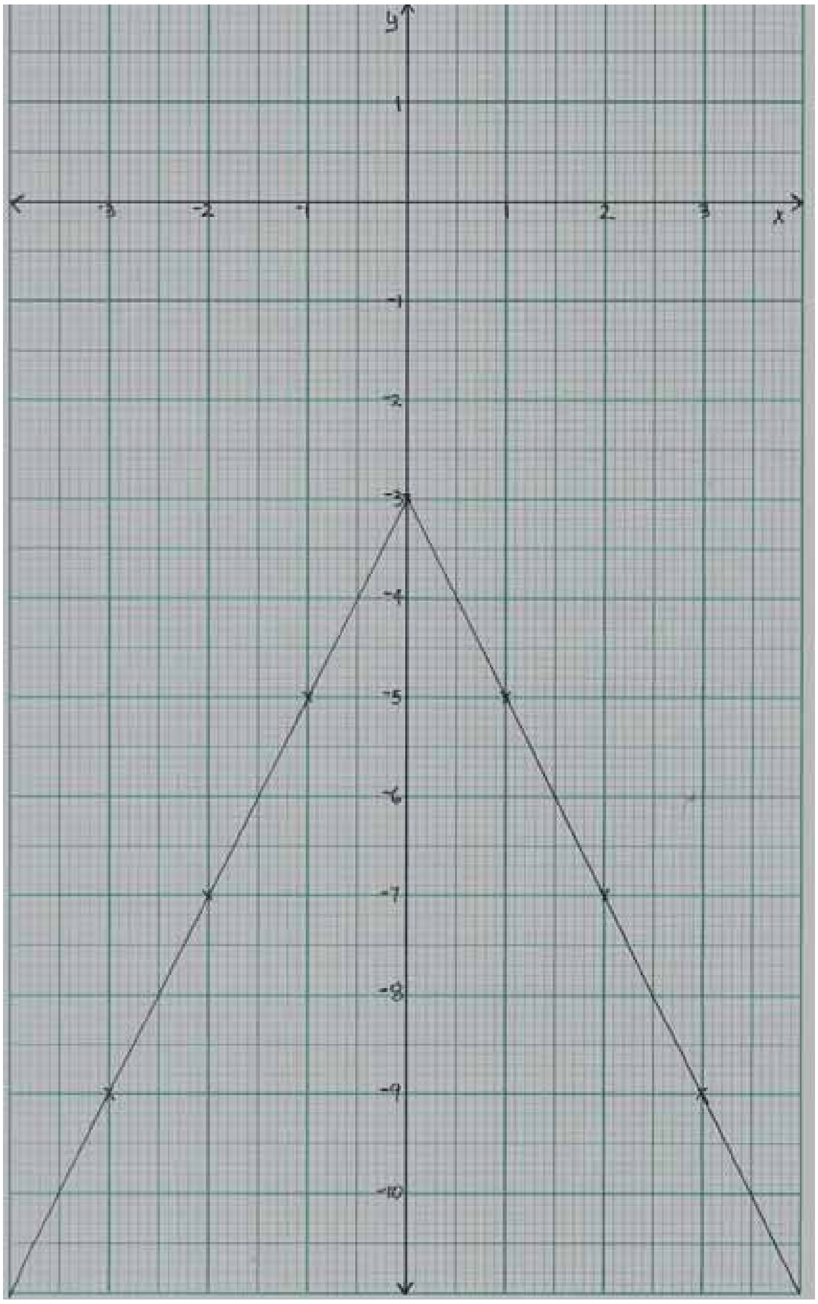
$y = 3|x|$

x	y
-3	$3|-3| = 3(3) = 9$
-2	$3|-2| = 3(2) = 6$
-1	$3|-1| = 3(1) = 3$
0	$3|0| = 3(0) = 0$
1	$3|1| = 3(1) = 3$
2	$3|2| = 3(2) = 6$
3	$3|3| = 3(3) = 9$


y
9
8
7
6
5
4
3
2
1
-1
-3
-2
-1
1
2
3
x

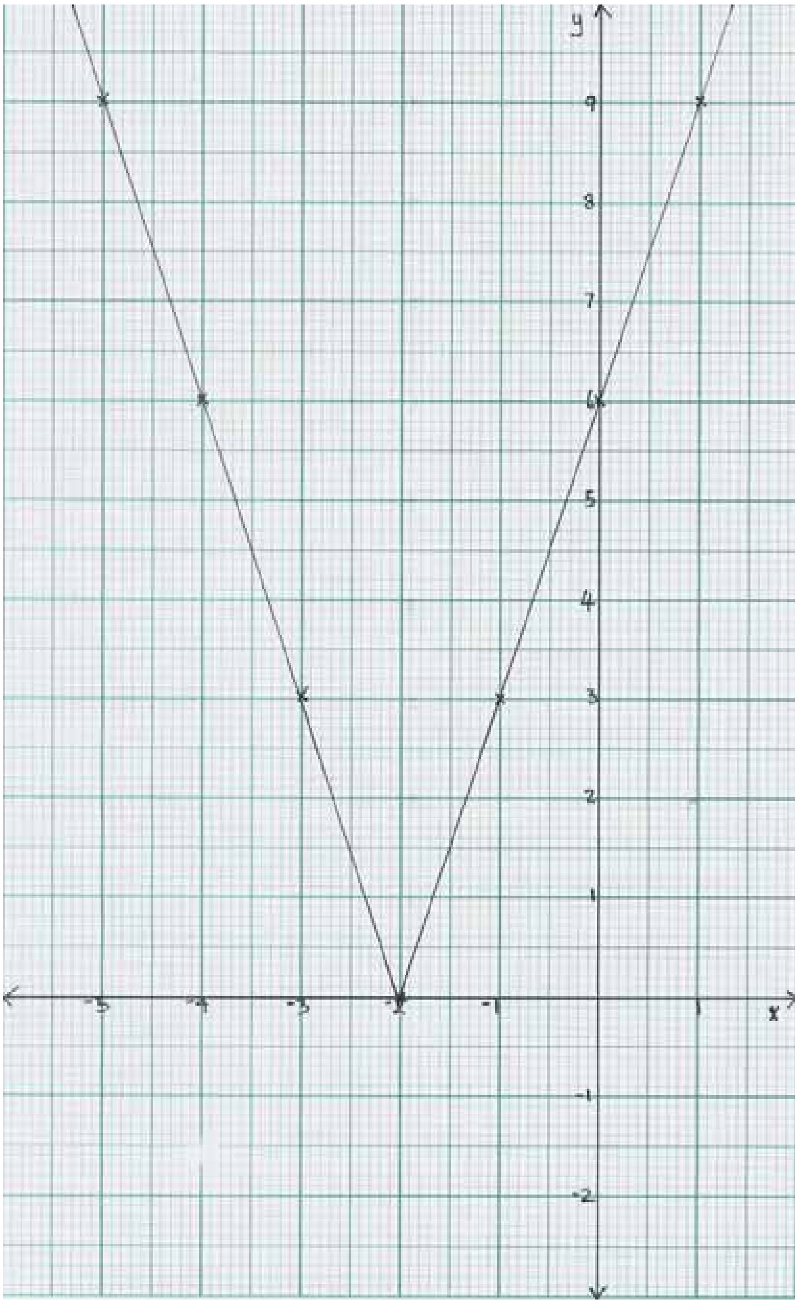
$y = -2|x| - 3$

x	y
-3	$-2\|-3\| - 3 = -2(3) - 3 = -6 - 3 = -9$
-2	$-2\|-2\| - 3 = -2(2) - 3 = -4 - 3 = -7$
-1	$-2\|-1\| - 3 = -2(1) - 3 = -2 - 3 = -5$
0	$-2\|0\| - 3 = -2(0) - 3 = 0 - 3 = -3$
1	$-2\|1\| - 3 = -2(1) - 3 = -2 - 3 = -5$
2	$-2\|2\| - 3 = -2(2) - 3 = -4 - 3 = -7$
3	$-2\|-3\| - 3 = -2(3) - 3 = -6 - 3 = -9$


y
1
-3
-2
-1
1
2
3
x
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10

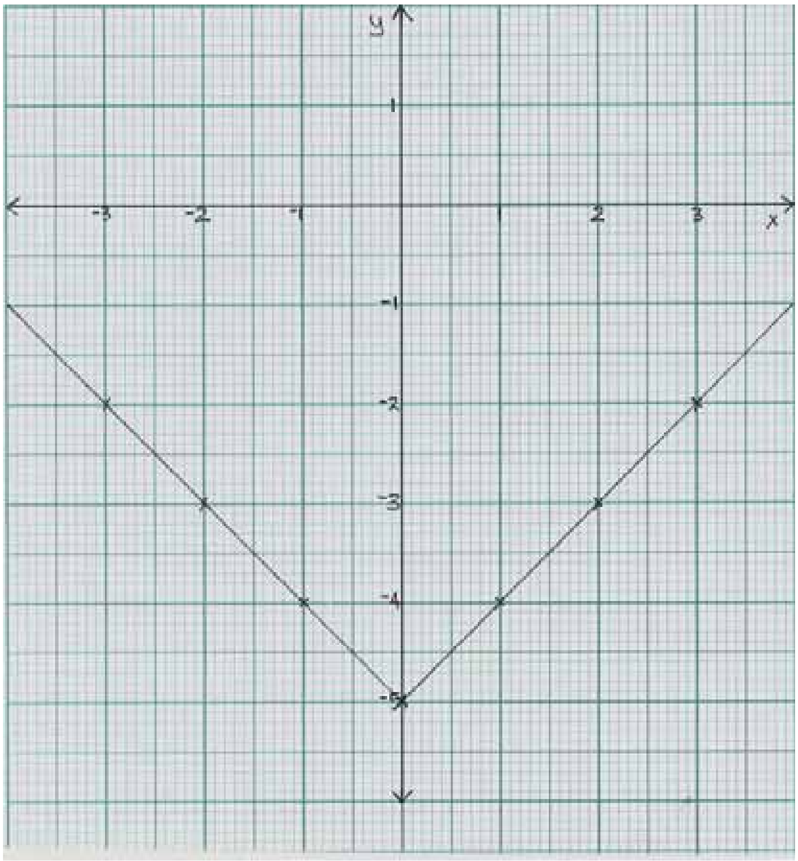
$y = |3x + 6|$

x	y
-5	$\|3(-5) +6\| = \|(-15) +6\| = \|-9\| = 9$
-4	$\|3(-4) +6\| = \|(-12) +6\| = \|-6\| = 6$
-3	$\|3(-3) +6\| = \|(-9) +6\| = \|-3\| = 3$
-2	$\|3(-2) +6\| = \|(-6) +6\| = \|0\| = 0$
-1	$\|3(-1) +6\| = \|(-3) +6\| = \|3\| = 3$
0	$\|3(0) +6\| = \|(0) +6\| = \|6\| = 6$
1	$\|3(1) +6\| = \|(3) +6\| = \|9\| = 9$


y
9
8
7
6
5
4
3
2
1
-1
-2
-5
-4
-3
-2
-1
1
x

y = -5 + |x|

x	y
-3	-5 + |-3| = -5 + 3 = -2
-2	-5 + |-2| = -5 + 2 = -3
-1	-5 + |-1| = -5 + 1 = -4
0	-5 + |0| = -5 + 0 = -5
1	-5 + |1| = -5 + 1 = -4
2	-5 + |2| = -5 + 2 = -3
3	-5 + |3| = -5 + 3 = -2

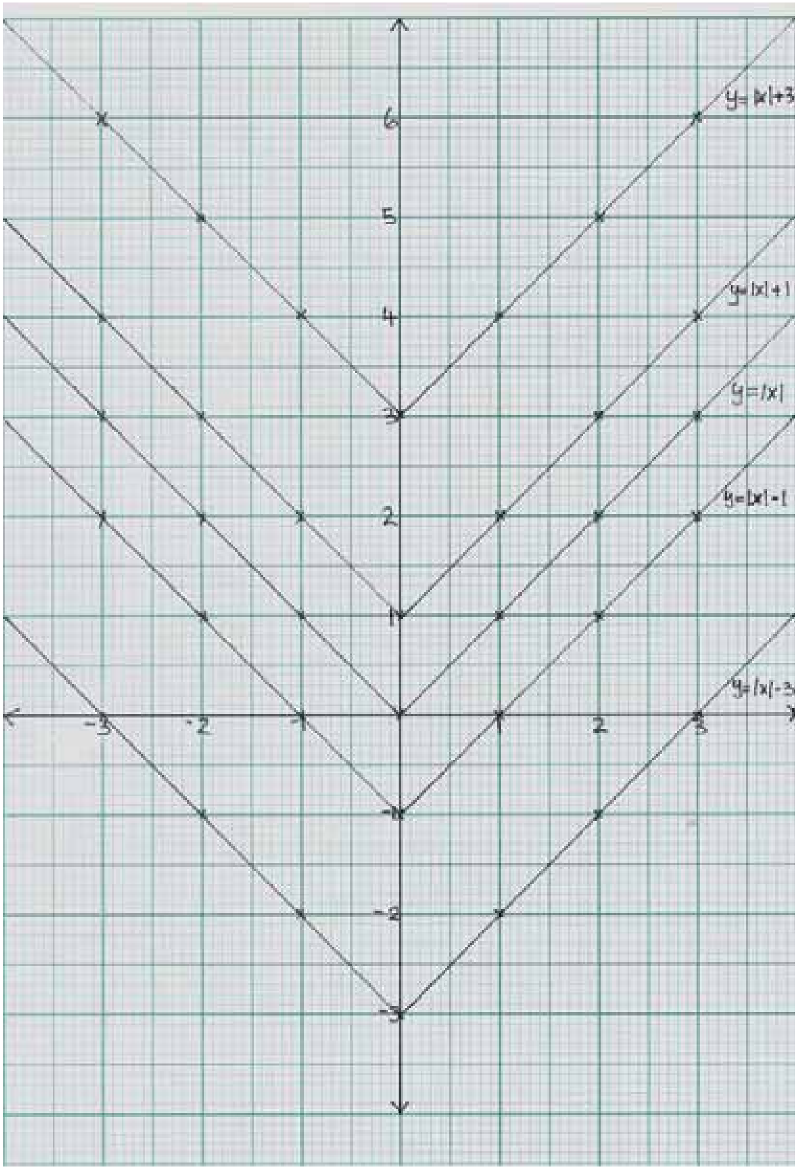

y
1
-3
-2
-1
1
2
3
x
-1
-2
-3
-4
-5

1. Using the general form y = a|x - b| + c;

when y = |x| + 1; y = |x| + 3; y = |x| - 1 and y = |x| - 3 are related to the parent function y = |x|;

- When "c" is positive, the graphs moved "c" units up.
- When "c" is negative, the graphs moved "c" units down.

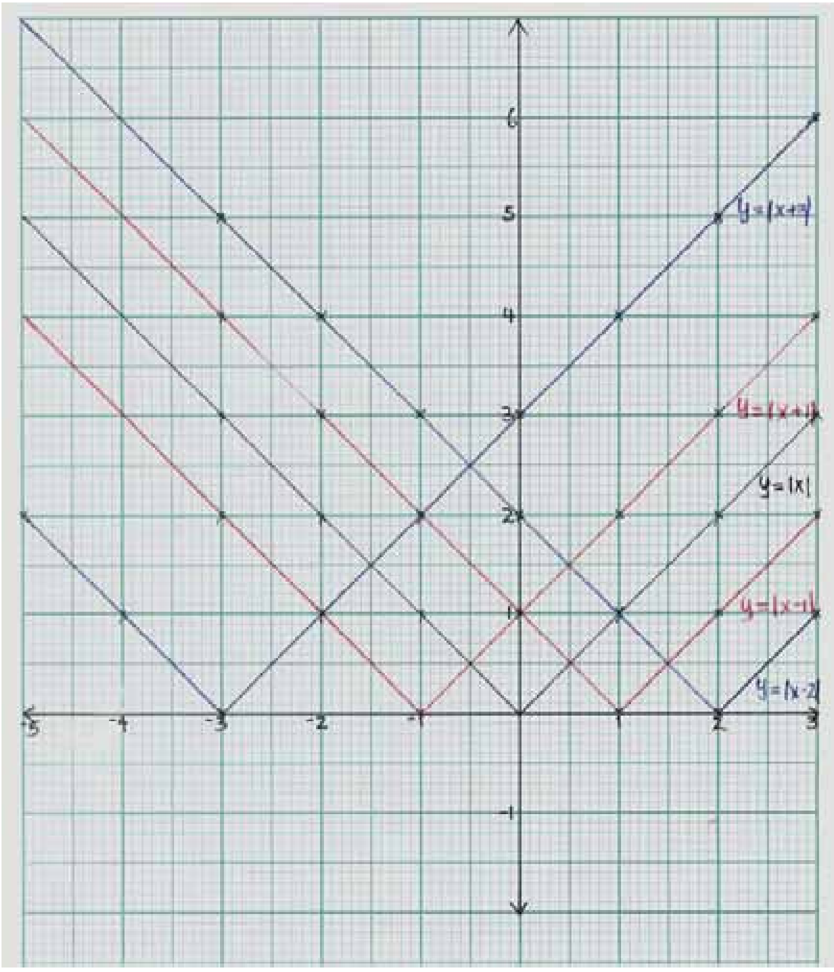
Also note that the width of all the graphs is the same.



2. Using the general form y = a|x - b| + c;

when y = |x - 1|; y = |x - 2|; y = |x + 1| and y = |x + 3| are related to the parent function y = |x|;

- When "b" is positive, the graphs moved "b" units to the left.
- When "b" is negative, the graphs moved "b" units to the right.

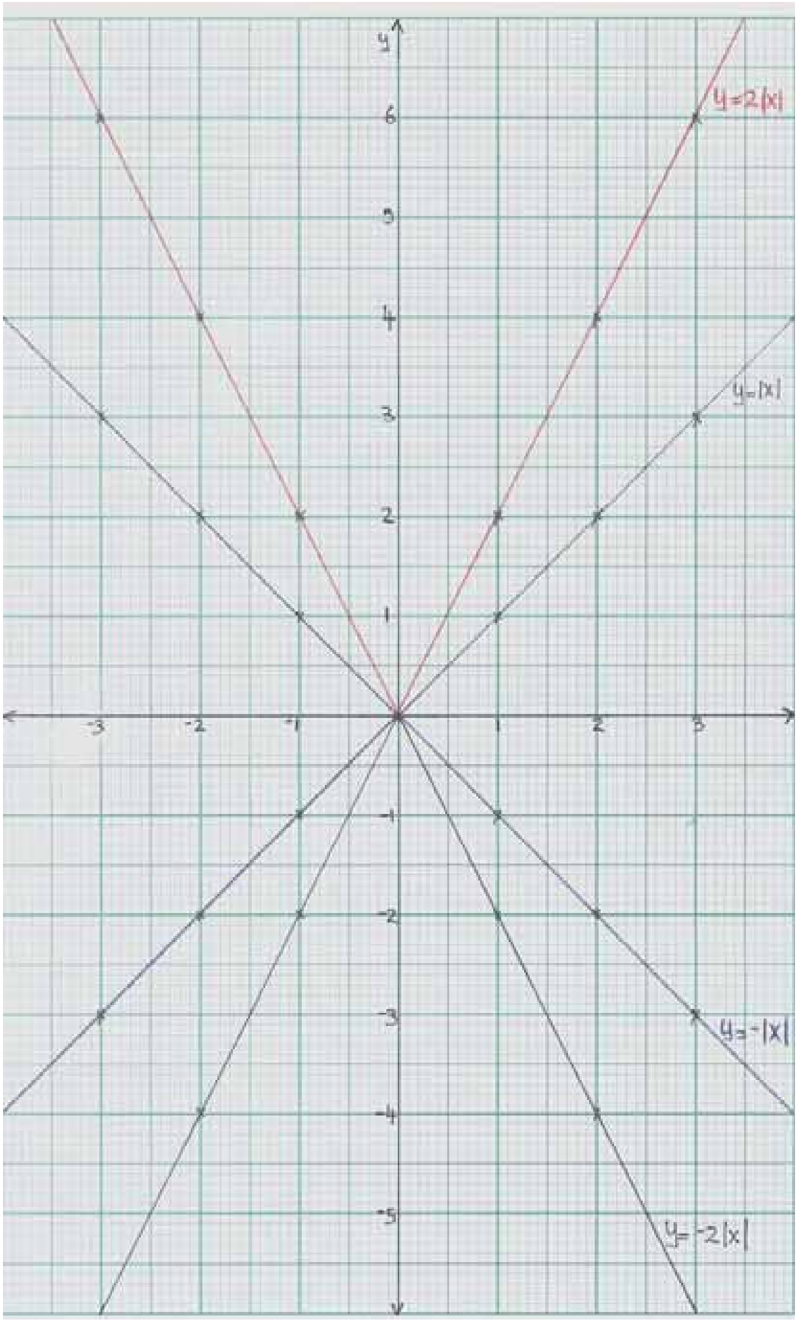


3. Using the general form y = a|x - b| + c;

   when y = 2|x|; y = - |x| ; and y = -2|x| are related to the parent function y = |x|;

- When "a" is positive, the graphs have a "v" shape, and there is a vertical stretch of "a" units.
- When "a" is negative, the graphs have a "^" shape, and there is a vertical shift of "-a" units.

The width of all the graphs is the same.



The graph of y = 2|x| is “narrower” than the graph of y = |x|. It rises 2 times faster. The width of the two graphs is not the same. The width of y = 2|x| is smaller than the width of the graph of y = |x|. The bigger the gradient gets, the narrower the graph gets.

Unit Summary

Summary

In this unit you learned:

1. that an absolute value, sometimes called modulus, of a number, is the distance of that number from the origin, without regard to direction;
2. that with absolute value equations:

- you usually get two solutions
- you temporarily put aside the absolute value lines, and then write the right hand side with + or -; then solve the equations

3. that with absolute value inequalities:

- you usually get two solutions
- when solving inequalities, when we multiply or divide by a negative number, the direction of the inequality changes;
- if we are given the inequality $|x| < a$ the solution is always $-a < x < a$. The first step therefore is to write the absolute value according to the pattern. Then solve the inequality. This also works even for $\leq$ situations;
- if we are given the inequality $|x| > a$ the solution always starts by splitting the inequality into two pieces;

  $x < -a$ or $x > a$

  Then solve the inequality. This also works even for $\geq$ situations.

4. that absolute value graphs have the general form of

   $y = a|x - b| + c$

   - *a* is the gradient of one of the lines making up the absolute value graph; it also tells us whether the graph has a maximum or a minimum value. If $a > 0$, then the graph has a minimum point, and it $a < 0$, the graph has a maximum point

   - *b* and *c* are the coordinates of the turning point is (b,c) The axis of symmetry is drawn through the turning point.

5. absolute value graphs have the general shape that looks like a “v” (In some cases it is an upside down “v”).
6. to work out the equation of a graph, use the turning point and the slope of the line

You have completed the material for this unit on absolute value. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit. It covers permutations and combinations.

Assignment

Assignment

Instructions

1. This assignment consists of 4 questions. Answer all of the questions.
2. The marks for each question are shown. There are a total of 67 marks.
3. Calculators may be used.
4. Show all the necessary working.

Total marks = 67 **Time:** 1hr 30 minutes

Good luck!!

1. Classify each statement as true or false. Correct the false statements.

a) $-|-\frac{1}{2}| = \frac{1}{2}$ (2 marks)

b) $|-1000| < 0$ (2 marks)

c) $|-(-1)| = -1$ (2 marks)

d) $|4 - 2| = |2 - 4| = 2$ (2 marks)

e) If $x > 4$, then $2x > 8$ (2 marks)

2. Solve the following:

a) $|x - 3| = 5$ (3 marks)

b) $|8 - 2x| = 6$ (3 marks)

c) $|7 - x| = -11$ (3 marks)

d) $\left|\frac{x}{4}\right| = 2$ (3 marks)

e) $|-x| = 9$ (3 marks)

3. Solve the following:

a) $|x + 7| < 2$ (3 marks)

b) $-|x - 4| \geq -6$ (3 marks)

c) $|5x - 1| < 6$ (3 marks)

d) $|2x - 3| > 7$ (3 marks)

e) $|7 - 2x| \geq 8 - x$ (3 marks)

4. Describe the graphs of the functions below as compared to the graph of the parent function $y = |x|$.

a) $y = 5|x|$ (3 marks)

b) $y = |x + 6|$ (3 marks)

c) $y = |x| + 6$ (3 marks)

d) $y = -3|x| - 4$ (3 marks)

5. Draw on the same set of axes, and interpret the graphs of the following

$y = |x|$; $y = |x + 2| - 3$; $y = 2|x + 1| + 3$ (15 marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Answers

1.

a) $-|-\frac{1}{2}| = \frac{1}{2}$

false

$-|-\frac{1}{2}| = -(\frac{1}{2}) = -\frac{1}{2}$

b) $|-1000| < 0$

false

$|-1000| = 1\ 000$

$1000 > 0$

c) $|-(-1)| = -1$

false

absolute value of a number is always positive

d) $|4 - 2| = |2 - 4| = 2$

true

$|4 - 2| = |2| = 2$

$|2 - 4| = |-2| = 2$

e) If $x > 4$, then $2x > 8$

true

$2(x) > 2(4)$

$2x > 8$

2.

a) $|x - 3| = 5$

$$\begin{array}{rl} x - 3 &= +5 \\ +3 & \;\; +3 \\ \hline x &= 8 \end{array} \qquad \text{or} \qquad \begin{array}{rl} x - 3 &= -5 \\ +3 & \;\; +3 \\ \hline x &= -2 \end{array}$$

x = 8 or x = -2

b) | 8 – 2x| = 6

8 – 2x = +6	or	8 – 2x = -6
+2x +2x		+2x +2x
8 = 6 + 2x		8 = -6 + 2x
-6 -6		+6 +6
2 = 2x		14 = 2x
1 = x		7 = x

x = 8 or x = -2

c) | 7 - x| = -11

The absolute value of a number is never negative, there is no solution

d) $\left| \frac{x}{4} \right| = 2$

$\frac{x}{4} = 2$	or	$\frac{x}{4} = -2$
x = 8		x = -8

x = 8 or x = -8

e) | -x| = 9

–x = +9 or –x = -9

x = -9 or x = 9

3.

a) $|x + 7| < 2$

$-2 < x + 7 < 2$

$\underline{-7 \quad -7 \quad -7}$

$-9 < x < 2$

b) $-|x - 4| \geq -6$

$-(x - 4) \geq -6$ or $-(x - 4) \leq -(-6)$

$-x + 4 \geq -6$ | $-x + 4 \leq 6$

$\underline{\quad -4 \quad -4}$ | $\underline{-4 \quad -4}$

$-x \geq -10$ | $-x \leq 2$

$x \leq 10$ | $x \geq -2$

c) $|5x - 1| < 6$

$-6 < 5x - 1 < 6$

$\underline{+1 \quad +1 \quad +1}$

$-5 < 5x < 7$

$-1 < x < \frac{7}{5}$

d) $|2x - 3| > 7$

$2x - 3 < -7$ or $2x - 3 > 7$

$2x - 3 < -7$ or $2x - 3 > 7$

$\underline{\quad +3 \quad +3}$ $\quad\quad$ $\underline{+3 \quad +3}$

$2x < -4$ or $2x > 10$

$x < -2$ or $x < 5$

e) $|7 – 2x| \geq 8 - x$

$7 – 2x \geq 8 - x$

$\underline{\quad +x \quad\quad +x}$

$7 – x \geq 8$

$7 – x \leq -8$ or $7 – x \geq 8$

$\underline{\quad -7 \quad\quad -7}$ $\quad\quad$ $\underline{-7 \quad\quad -7}$

$-x \leq -15$ $\quad\quad$ $-x \geq 1$

$x \geq 15$ or $x \leq -1$

4.

a) $y = 5|x|$

- has a “v” shape
- gradient = 5
- there is a vertical stretch of 5 units

b) $y = |x + 6|$

- has a “v” shape
- gradient = 1
- move of 6 units to the left

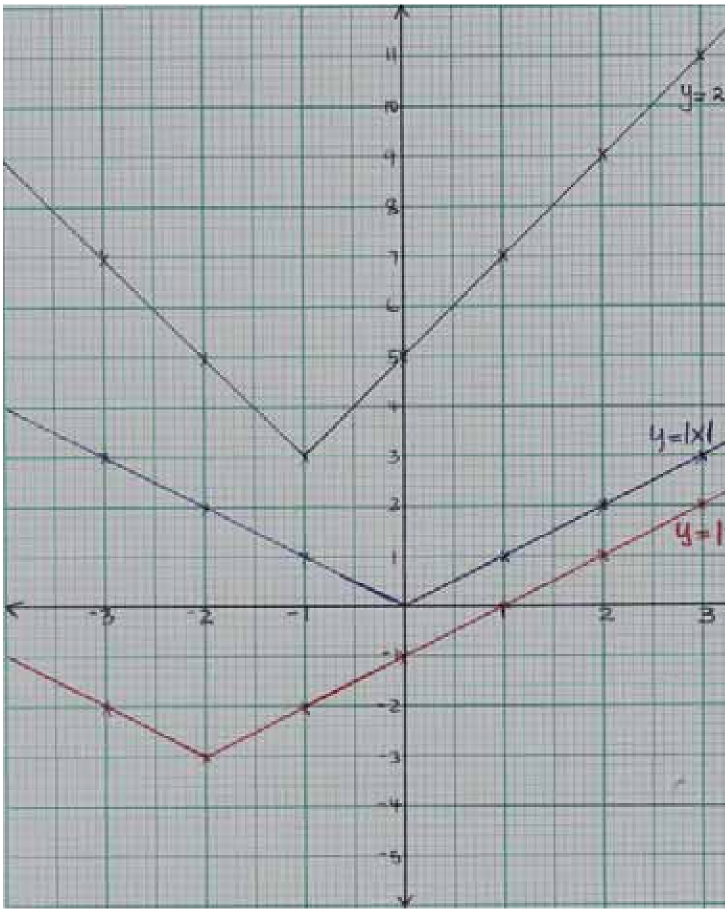
c) $y = |x| + 6$

- has a “v” shape
- gradient = 1
- move of 6 units up

d) $y = -3|x| - 4$

- has a “∧” shape
- gradient = -3
- move of 4 units down
- vertical stretch of -3 units

5.



Graph of y = |x + 2| - 3

- It has moved 3 units down
- It has moved 2 units to the left
- It is the same width as the graph of y = |x|. They have the same gradient, 1

Graph of y =2 |x + 2| + 3

- It has moved 3 units up
- It has moved 1 unit to the left
- It is narrower than the graph of y = |x|.
-

Others

- Graphs of y = |x| and y = |x + 2| - 3 have the same width
- Graph of y = 2|x + 1| + 3 is narrower than the graphs of y = |x| and y = 2|x + 1| - 3

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Instructions

1. This assignment consists of 5 questions. Answer all of the questions.
2. The marks for each question are shown. There are a total of 61marks.
3. Calculators may be used.
4. Show all the necessary working.

Total marks = 61 **Time:** 1hr 30 minutes

Good luck!!

1. Evaluate the following:

(a) | 6 | = (1 mark)

(b) |-6| = (1 mark)

(c) | 0 | = (1 mark)

(d) | 3 + -3 | = (2 marks)

(e) | 3 + 3 | = (1 mark)

(f) | - 3| + | - 3 | = (2 marks)

(g) $\dfrac{-4}{|-4|} =$ (2 marks)

2. Solve the following:

(a) $|x| = 5$ (3 marks)

(b) $|x + 3| = 7$ (3 marks)

(c) $|1 - x| = 11$ (3 marks)

(d) $|x - 1| = 2x + 1$ (3 marks)

(e) $|x - 1| = x$ (3 marks)

3. Solve the following:

(a) $|2x + 6| < 10$ (3 marks)

(b) $|2x + 6| \geq 10$ (3 marks)

(c) $|x + 5| < 15$ (3 marks)

(d) $\left|\frac{x}{5} - 2\right| < \frac{4}{5}$ (3 marks)

(e) $2 - |2 - x| \geq -6$ (3 marks)

4. Draw the graphs of the functions below

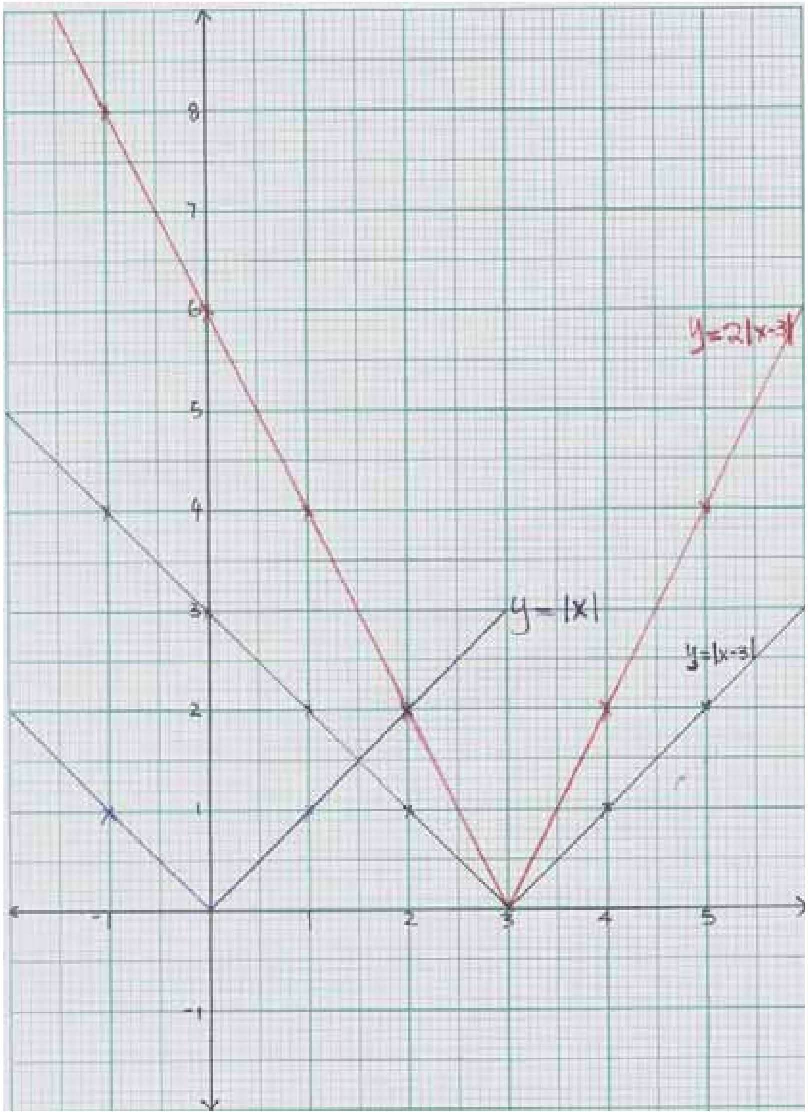
(a) $y = 3|x + 1| - 2$ (3 marks)

(b) $y = -2|x - 2| + 3$ (3 marks)

(c) $y = 3|x + 2|$ (3 marks)

(d) $y = -2|x - \frac{1}{2}| + 1$ (3 marks)

5. Interpret the graphs of the following $y = |x - 3|$ and $y = 2|x - 3|$ in relation to $y = |x|$ (9 marks)



y=2|x-3|
y=|x|
y=|x-3|

Answers:

1.

(a) 6

(b) 6

(c) 0

(d) 0

(e) 6

(f) 6

(g) -1

2.

(a) $x = 5$ or $x = -5$

(b) $x = 4$ or $x = -10$

(c) $x = -10$ or $x = 12$

(d) $x = 0$ only

(e) $x = ½$

3.

(a) $-8 < x < 2$

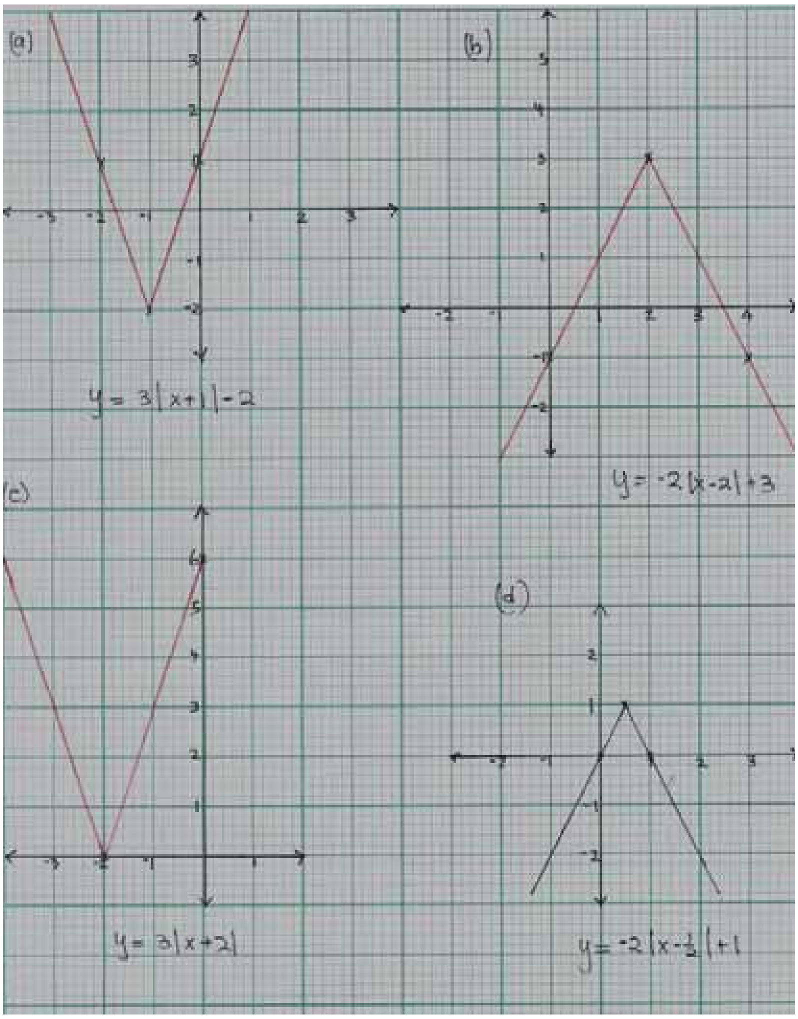
(b) $x \geq 12$ or $x \leq -8$

(c) $-20 < x < 10$

(d) $6 < x < 14$

(e) $-6 \leq x \leq 10$

4.



5.

Graph of y = |x - 3|

- It has moved 3 units to the right
- It is the same width as the graph of y = |x|. They have the same gradient, 1

Graph of y =2 |x - 3|

- It has moved 3 units to the right
- It is narrower than the graph of y = |x|.

Unit Contents

Unit 32

Unit 32

Permutations and Combinations

Introduction

In our everyday life encounters we always come across situations that require us to make combinations of choices. You always think about how you are going to dress up on any particular day- thinking about which pair of shoes to match a particular dress or pants and/or shirt, there are numerous ways in which you can dress up depending on the number of pairs of shoes, pairs of pants, dresses, and blouses you have. In this chapter, we will learn how to solve counting problems using counting procedures of permutations and combinations.

This unit consists of 32 pages. This is approximately 1% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 15 hours to work on this unit, 10 hours for formal study and 5 hours for self-study and completing assessments/assignments.

Take a moment to read the following learning outcomes. They are a guide to what you should focus on while studying this unit.

This Unit is Comprised of Five Lessons:

Lesson 1 The Fundamental Principle of Counting
Lesson 2 Permutations
Lesson 3 Permutation of n Objects Taken r at a Time
Lesson 4 Permutation of Objects When Some are Alike
Lesson 5 Combinations

Upon completion of this unit you will be able to:

Outcomes

- *solve* problems related to the fundamental counting principle.
- *solve* problems involving permutations.
- *solve* problems involving combinations.

Terminology

Combination: Is a group or selection from a given set of objects without regard to the order

Permutation: Is an arrangement of objects in a definite order

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 The Fundamental Principle of Counting

Introduction

In life we often find ourselves counting a variety of things. We may need to know how many groups of x can be made of p males and q females if there are y males and z females. Alternatively, you may want to know how many different ways you can wear your clothes; if you have 3 pairs of jeans and 2 pairs of shoes.

The fundamental principle of counting can help us make all these calculations.

By the end of this subunit, you should be able to use the fundamental principle of counting to find the number of possible events.

This subunit is about 2 and a half pages in length.

Calculating the number of possible events

Suppose one event can be chosen in p different ways, and another can be chosen in q different ways, and the other can be chosen in r different ways.

Then the number of ways in which the three events can be chosen successively is $p \times q \times r$ different ways and this is called the fundamental principle of counting.

Consider the following examples on the fundamental principle of counting.

Example 1:

In a church there are two different groups, the group of 25 youth and 16 elderly people, they wish to select three representatives to represent them at a church in another country. The representatives are to consist of one youth and one elderly person and the secretary. The secretary may be either from the youth or elderly group. Calculate the number of ways in which the sets of representatives can be chosen.

Solution to the example:

There are 25 choices for the young person and 16 choices for the elderly person. When the two representatives from the youth and the elderly people are chosen, there are now 39 people to choose from. So by the fundamental principle of counting, there will be 25×16×39=15,600 different sets of representatives.

This is called the fundamental principle of counting.

Example 2

The municipality decides to put house numbers in all the locations, the house numbers are to consist of any of three alphabetic letters from the 26 letters of the alphabet followed by any three numbers between 1 and 9.

How many house numbers can be made if:

a) Repetitions are allowed?

b) Repetitions are not allowed?

Solution to example 2

a) The first letter 3 letters can be chosen each in 26 different ways and the three numbers that follow the letters can be chosen each in 9 different ways.

So by the fundamental principle of counting there will be 26×26×26×9×9×9=12,812,904 house numbers.

b) If repetitions are not allowed:

If repetitions are not allowed the number of letters and digits are reduced by one each time

a selection is made.

So there will be 26×25×24×9×8×7=7,862,400 house numbers.

Now try the following activity**.**

Activity 1

No. 1

Mary has 5 different pairs of jeans, 3 types of blouses and 2 different pairs of shoes. In how many different ways can she dress up?

____

____

____

No. 2

How many five digit numbers can be constructed from the digits 1,2,3,4,5,6,7 if

a) Repetitions are allowed?

____

____

____

b) Repetitions are not allowed?

____

No.3

A certain company publishes three posts to be applied. One for a secretary, one for a human resource officer and another for a driver. Of the 60 people who apply, 10 qualify to be secretaries, 30 qualify to be human resource officers, 15 qualify to be drivers and 5 qualify for all the three positions. How many different sets of these three positions can be made if the positions are to be filled only by people who qualify and repetitions are not allowed?

Compare your answers to those given at the end of the subunit. Note that it is important to understand this concept. If you do not understand it, review this content.

Key Points to Remember

The key point to remember in this subunit on the fundamental principle of counting is:
If one event can be chosen in ***p*** different ways, and another can be chosen in ***q*** different ways, and the other can be chosen in ***r*** different ways, by the fundamental principle of counting: the number of ways in which the three events can be chosen successively is ***p x q x r*** different ways.

Solutions to Activity 1

No.1

Mary can dress up in:

5 x 3 x 2 = 30 different ways because each of the 5 jeans can be worn with any of the 3 blouses with any of the 2 pairs of shoes.

No.2

a) There can be $7\times7\times7\times7\times7=16,807$ numbers.

b) If repetitions are not allowed there can be $7\times6\times5\times4\times3=2,520$ numbers.

No. 3

From the candidates that qualify for one position, a secretary can be chosen in 10 different ways, the human resource officer can be selected in 30 different ways and driver can be selected in 15 different ways. And from the 5 who qualify for all the positions, the first position can be filled in 5 different ways, the second in 4 and the 3^{rd} in 3 different ways. So by the fundamental principle of counting, there can be $10\times30\times15\times5\times4\times3=27,000$.

Lesson 2 Permutations

Introduction:

By the end of this subunit, you should be able to:

-Arrange objects in different ways and calculate their permutations.
- Use the factorial notation to calculate permutations.

This subunit is about 2 and a half pages in length.

Permutation Definition

A permutation of the symbols p, q, r, s, t..., is a string of all these symbols without repletion, i.e. each symbol being used once in each string.

Suppose there are three friends, Thabo, Lerato and Tumi, sitting on a bench. They can sit in six different ways:

Thabo, Lerato, Tumi

Thabo, Tumi, Lerato

Lerato, Thabo, Tumi

Lerato, Tumi, Thabo

Tumi, Lerato, Thabo

Tumi, Thabo, Lerato

Each of these arrangements is called a **permutation**.

For each of the three ways of choosing the person in the first position, there are two ways of choosing a person in the second position. Having chosen for the two positions there is now only one person left for the third position.

Hence there are:

3 x 2 x 1 = 6 different ways of arranging three people in different ways.

Factorial Notation

From the above example, the product 3 x 2 x 1 can be written in short form as 3! (read as factorial 3). The value of factorial *n* or n! is found by multiplying all the whole number values from one up to and including the number *n*.

This means that 5! = 5 x 4 x 3 x 2 x 1 = 120.

In general n! = *n x n-1 x n-2 x n-3 x n-4 x ... x 2 x 1*

Now consider the following situation:

Suppose you have form different 4- digit numbers from the digits 2, 3, 4, and 5. The 4-digit numbers can be formed by different arrangements of the given digits. Thus you will be thinking of a permutation of 4 digits.

Now applying the same approach as with the sitting arrangements of the three friends, you should realise that there are

4 ways of choosing the first number, followed by

3 ways of choosing a digit for the second position, followed by

2 ways of choosing a digit for the third position, followed by

1 way of choosing a digit for the fourth digit,

This can be thus obtained in

4! Ways = 4 x 3 x 2 x 1ways = 12 ways.

The number of permutations or arrangements of n different objects is n!, where n! = n (n – 1) (n – 2) (n – 3) ... 3 x 2 x 1

Now try the following Activity to see how much you understand.

Activity 2

Activity

1. In how many different ways can the letters of the word CAT be arranged?

2. How many permutations do these six letters have: j, k, l, m, n, and o?

3. How many 8 letter strings can be made using the letters of the word, Olympics?

4. In how many different ways could these people in the picture below be arranged if repetitions are not allowed?

Compare your answers with those at the end of the unit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this content again.

Key Points to Remember

The key points to remember in this subunit on permutations are:

- Permutations are arrangements of objects in a definite order.

- **The number of permutations or arrangements of n different objects is:**

n! Where n! = n (n – 1) (n – 2) (n – 3) ... 3 x 2 x 1

Solutions to Activity 2

1. The Letters of the Word cat can be arranged in $3\times2\times1=6$ different ways.

2. The six letters have the permutations $6! =6\times5\times4\times3\times2\times1=720$.

3. The letters of the word Olympics can form 8! different strings= $8\times7\times6\times5\times4\times3\times2\times1$ strings = 40320 strings.

4. There are 7 people in the picture, so they have 7! Permutations=$7\times6\times5\times4\times3\times2\times1=5040$ arrangements

Lesson 3 Permutation of n Objects Taken r at a Time

By the end of this subunit, you should be able to:
- Calculate permutations of n elements taken r at a time, where, n>r.

This subunit is about 3 pages in length.

Introduction

In the examples above, we dealt with permutations of different objects taken all at a time, for instance, in activity 1, no. 1, we wanted to know the permutations of the word CAT, and we were dealing with permutations of 3 elements taken 3 at a time. We use the symbol

$^{n}P_{n}$ or P (n, n) to denote the permutations of n things taken n at a time.

So from this and the previous example, we can conclude that P (n, n) =3!

Since from this example of CAT, we have 3 letters and are making 3 letter string, which is P (3, 3) =3! =6

Consider the following example:

Example 1:

Consider the word 'sanctify', how many four letter strings can you make from this word?

Solution:

There are 8 letters to choose from. So the first letter has 8 possible choices, the second letter has 7 remaining choices, the third letter has 6 remaining choices and the fourth letter has 5 remaining choices.

So there can be 8×7×6×5 strings=1,680 different strings from the word sanctify. Here we have permutations of 8 letters taken four at a time.

What would you multiply 1,680 by to calculate how many five letter strings could be made from the word?

Compare your answer to the following:

Multiply by 5 since there would be 5 remaining choices for the fifth letter. In this example we have 8 letters taken 5 at a time.

The following activity will help you understand permutations of different objects.

Activity 3

Activity

Now consider the following:

1. Suppose you have 4 digits, 1, 2, 3 and 4 and you want to form 2- digit numbers.

a) Given you can only use a digit once, list all the possible numbers you can make. Think about following a pattern to ensure you find all of the possibilities.

____

b) How many different 2-digit numbers have you formed?

____

2. The offices of a chairperson and vice-chair person need to be filled by choosing names from the five candidates. Let the five candidates be A, B, C, D, and E.

a) Given you can only use a letter once, List all the possible choices that can be obtained. Again, follow a pattern to ensure you find all of the possibilities.

____

b) In how many different ways can this be done?

____

Compare your answers to those given at the end of the unit. Note that it is important to understand this permutations concept. If you do not understand it, review the above content.

Formula for Calculating Permutation Possibilities of n Objects Taken r at a Time

In both of these two situations in activity 3, only a certain number of items were taken at a time.

In question 1, from the 4 digits, you only took 2 digits at a time. It is a permutation of 4 objects taken 2 at a time. Symbolically this can be written and calculated as

$$^{4}P_{2} \quad \text{or} \quad P(4, 2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 12$$

In question2, from the 5 candidates, you took only 2 at a time. It is a permutation of 5 objects taken 2 at a time. Similarly this can be written as

$$^{5}P_{2} \quad \text{or} \quad P\,(5, 2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 20$$

Generally

$$P\,(n, r) = \frac{n!}{(n-r)!},$$ **where all objects are different (or without repetition of elements).**

Activity

Activity 4

1. How many 8 letter strings can be constructed from these 8 letters: p, q, r, s, t and u?

2. How many 5 digit numbers can be constructed from these numbers: 1, 2, 3, 4, and 5?

3. How many 4 letter strings can be made from a 6 lettered string?

4. How many four digit numbers can be made from the digits: 1, 2, 3, 4, 5, 6, 7, 8 and 9?

5. Find r so that $P\,(15, r) = \frac{15!}{12!}$

Compare your answers with those at the end of the unit. Be sure that you understand each answer before continuing. If you have any misunderstandings, review this topic.

Key Points to Remember

The key points to remember in this subunit on permuations of different objects are:

P (n, r) = $\frac{n!}{(n-r)!}$ **,where all objects are different (or without repetition of elements).**

Solutions to Activity 3

1. a) The possible numbers are: 12 13 14 21 23 24
31 32 34 41 42 43

   b) There are 12 different numbers.

2. a) The possible choices are: AB AC AD AE
BA BC BD BE
CA CB CD CE
DA DB DC DE
EA EB EC ED

   b) These can be done in 20 different ways

Solutions to Activity 4

1. $P(8,8) = 8! = 8\times7\times6\times5\times4\times3\times2\times1 = 40320\ strings.$

2. $P(5,5) = 5! = 5\times4\times3\times2\times1 = 120\ numbers.$

3. $P(6,4) = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6\times5\times4\times3\times2\times1}{2\times1} = \frac{720}{2} = 360\ strings$

4. $P(9,4) = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1} = \frac{362880}{120} = 3024\ numbers$

5. $P(15,r) = \frac{15!}{12!} \Rightarrow 15-r = 12 \Rightarrow r = 15-12 = 3$

Lesson 4 Permutation of Objects When Some are Alike

Introduction:

In the word ‘objects’ no letters are repeating, but in the word ‘permutation’ ‘t’ repeats. In this subunit we are dealing with permutations of objects in which some objects are alike or repeating.

By the end of this subunit, you should be able to

Calculate permutations of objects in which some objects of the original string are alike or repeating.

This subunit is about 1 and a half pages in length.

If we arrange the letters in the word CAT in different ways we may have the following:

CAT CTA ACT ATC TAC TCA.

This is a permutation of three different items. But what if we have to arrange the letters in the word ADD? We would have:

ADD DDA DAD only.

Notice that there are only 3 permutations since the 2Ds cannot be distinguished from each other.

This result for ADD can be obtained by using the equation:

P (n, n) where r characters are repeating or alike = $\dfrac{n!}{r! = repeating\ \ charrecters!} =$

$\dfrac{3!}{2!} = 3$

Generally

The number of permutations of *n* objects of which *q* characters are alike and *r* characters are alike is given by

$$\frac{n!}{q!r!}$$

Use this idea for Activity 5.

Activity

Activity 5

1. How many four-letter patterns can be formed from the letters of the word PASS?

2. In how many different ways can the letters of each of the following words be arranged:
    a) PARALLEL
    b) QOMOQOMONG
    c) QWAQWA

Compare your answers to the ones given at the end of the unit. Continue to the next subunit if you got all of them right. If not, review the formula and try the activity again.

Key Points to Remember

The key point to remember in this subunit on permuations of objects when some are alike is:

-The number of permutations of *n* objects of which *q* characters are alike and *r* characters are alike is given by:

$$\frac{n!}{q!r!}$$

Solutions to Activity 5

1. P (n, n) = $\frac{4!}{2!}$ = 12 patterns

2. a) P (n, n) = $\frac{8!}{2!3!}$ = 3360 ways

   b) P (n, n) = $\frac{10!}{2!2!4!}$ = 201 600 ways

   c) P (n, n) = $\frac{6!}{2!2!2!}$ = 90 ways

Lesson 5 Combinations

Introduction

By the end of this subunit, you should be able to:

- Identify the difference between permutations and combinations.
- List combinations of objects.
- Calculate the number of combinations of objects.

This subunit is about 4 pages in length.

A combination is a set or collection of objects with no particular order. By combination of *n* things taken *r* at a time (where $r \leq n$), we mean all possible selections of *r* different objects from *n* objects with no regard to order or arrangement.

The difference between permutations and combinations of n elements taken r at a time is that permutations are ordered and combinations are not ordered.

Compare the following two examples to see the difference between a combination and a permutation.

Example 1

How many numbers each consisting of three digits can be made from these numbers: 1, 2, 3,4,5,6 without repetition?

Solution

To answer example 1 we need to be sure if order is important or not, if it is important, it is a permutation, if it is not then it is a combination.

One of the numbers could be 456, and another could be 654. We have used the same digits but they give different numbers, that says order is important. So it is a permutation.

So $P(6,3)=\dfrac{6!}{(6-3)!}=\dfrac{720}{6}=120$

Now consider example 2 to see the difference.

Example 2

Consider the set Q = {A, B, C, D, and E} How many subsets can be made from this set? List them.

Solution to Example 2

To answer this one, we again need to be sure if order is important or not. One of the subsets could be $\{A,B,C\}$, but if we say $\{B,A,C\}$, is another set, we are wrong because in sets order of elements is not important, $\{A,B,C\}$and $\{B,A,C\}$are one and the same set. So order of elements is not important, thus it is a combination.

The following are the 3-letter subsets of Q.

{A, B, C}

{A, B, D}

{A, B, E}

{A, C, D}

{A, C, E}

{A, D, E}

{B, C, D}

{B, C, E}

{B, D, E}

{C, D, E}.

They represent the different ways in which 3- letters can be combined from a set of 5 letters. This is denoted as ${}_5C_3$, a combination of 5 things taken 3 at a time.

The number of combinations of n things taken r at a time is denoted as ${}_nC_r$ or C (n, r)

and can be calculated as ${}_nC_r = \frac{n!}{(n-r)!\,r!}$

Thus ${}_5C_3 = \frac{5!}{2!\,3!} = 10.$

In some cases, more than one selection may be involved.

Consider the following situation:

Two bags contain different balls. The first bag contains 4 red and the second 3 blue. Two balls of different colours are to be selected, one from each bag. How many different combinations are possible?

Solution:

Two selections are involved here: selection of one red ball from the 4 balls and a selection of one blue ball from the 3 balls.

Select the red and the blue balls separately.

The selection of a red ball can be done in C (4, 1) ways and the selection of a blue ball can be done in C (3, 1) ways. By applying the Fundamental Principle of counting for each selection, the selection of two balls of different colours can be made in:

$$C(4, 1).\ C(3, 1) = \frac{4!\ 3!}{3!\ 2!}$$

$$= 12 \text{ ways.}$$

So, with problems that involve two or more events, we need to identify those two or more combinations and then apply the fundamental principle of counting.

Now try activity 6.

Activity

Activity 6

1. From a list of 10 different books, how many groups of 5 books can be selected?

2. In a test, a student is to answer any 12 questions from a list of 15. In how many different ways may a student choose questions?

3. A bag contains 3 red, 5 white, and 8blue balls. In how many ways can 2 red, 1 white and 2 blue balls be chosen?

4. A committee consists of 5 officers (3 males and 2 females). In how many different ways may a delegation of 1 male and 1 female be selected from this committee?

5. List the 6 string combinations that can be made from (a, b, c, d, e, f, g)

Check your performance against the given solutions at the end of this subunit. Continue if you are satisfied with your ability to answer the questions. If not, review the combinations subunit and try this activity again.

Key Points to Remember

The key points to remember in this subunit on combinations are:

-The difference between permutations and combinations of n elements taken r at a time is that permutations are ordered and combinations are not ordered.

A combination is a set or collection of objects with no particular order

- The number of combinations of n things taken r at a time is denoted as $_nC_r$ or C (n, r)

and can be calculated as $_nC_r = \frac{n!}{(n-r)!\,r!}$

You have now completed the last subunit of this unit on permutations and combinations. Do a quick review of the entire content of this unit and then continue on to the unit summary.

Solutions to Activity 6

1. C(10, 5) = $\frac{10!}{5!}$ = 30 240 groups of five books

2. C (15, 12) = $\frac{15!}{3!}$ = 6 .54 x 10^{11} choices of questions

3. C (3, 2).C (5, 1).C (8, 2) = $\frac{3!}{1!}\cdot\frac{5!}{4!}\cdot\frac{8!}{6!}$

$= 6 \times 5 \times 54$

= 1620 ways

4. C (3, 1).C(2, 1) $= \frac{3!}{2!}\frac{2!}{1!}$ = 6 different ways

5. There are $_7C_6 = \frac{7!}{(7-6)!6!} = \frac{5040}{720} = 7$ combinations.

abcdef

abcdeg

abcdfg

abcefg

abdefg

acdefg

bcdefg

Unit Summary

In this unit you learned about permutations and combinations. You have learned that:

- *If one event can be chosen in **p** different ways, and another can be chosen in **q** different ways, and the other can be chosen in **r** different ways; Then the number of ways in which three events can be chosen successively is **p x q x r** different ways.*
- *A permutation is an arrangement of objects in a definite order.*
- *The number of permutations or arrangements of n different objects is n!, where n! = n (n – 1)(n – 2) (n – 3) ....3 x 2 x 1.*
- *The number of permutations of n different things taken r at a time is given by*

  $P(n, r) = \frac{n!}{(n-r)!}$, *where all objects are different*
- *The number of permutations of n objects where p are alike, r are alike and q are alike is given by* $\frac{n!}{p!q!r!}$
- *A combination is a set or collection of objects with no particular order.*
- *The number of combinations of n things taken r at a time is denoted as* $_nC_r$ *or C (n, r)*

  and can be calculated as $_nC_r = \frac{n!}{(n-r)!\,r!}$

You have also learned that in cases where more than one selection is involved the fundamental principle of counting can be used to calculate the total number combinations required.

You have completed the material for this unit on permutations and combinations. You should now spend some time reviewing the content in detail. Once you are confident that you can successfully write an exam on the concepts, try the assignment. Check your answers with those provided and clarify any misunderstandings that you have. Your last step is to complete the assessment. Once you have completed the assessment, proceed to the next unit.

Assignment

This assignment consists of 13 questions. Answer all of the questions.
The marks for each question are shown. There are a total of 80 marks.

You are advised to spend no more than 40 minutes on this assignment.

Calculators may be used.

Show the necessary working.

Good luck!!

Assignment

1. How many permutations are there of the letters of the word:

   i) ALGEBRA? (2 marks)

   ii) COLLEGE? (3 marks)

2. How many four-digit numbers can be formed from the digits 1, 2, 3,4,5,6,7,8,9 if each digit may be used only once? (3 marks)

3. How many four-letter words can be made from the 26 letters of the alphabet, assuming that the letters may be used once only? (3 marks)

4. How many ways can a team of 11 players be chosen from 30 players? (4 marks)

5. Six students are running a 100m race. In how many ways can the first three places be filled? (3 marks)

6. There are 4 staircases that lead to classrooms and offices in a college. In how many ways can a person go up one staircase and come down by a different

one? (3 marks)

7. Four students from class 2A, 3 from class 3B and 3 from class 4A are to be seated in a row. How many seating arrangements are possible when:

    i) Students of the same class must sit next to each other in the order 2A, 3B and 4A?
    (3 marks)

    ii) Student of the same class must sit next to each other but in no specific order?
    (4 marks)

    iii) Student may sit anywhere? (2 marks)

8. In how many ways can a basketball team be selected from nine players? (There are 5 people in the basketball team).
(3 marks)

9. In how many ways can a class elect a president, vice president, secretary and treasurer from a class of 100 students? (4 marks)

10. In how many ways can11 boys and 9 girls be seated so that 4 boys and 3 girls be seated in one row containing 7 seats if they may sit anywhere? (4 marks)

11. On a certain examination, a student must answer 8 out of 12 questions, including exactly 5 of the first 6. In how many different ways may he write the examination? (3 marks)

12. Show that P(n + 1, r) = (n + 1)P(n, r – 1). (3marks)

13. Solve the equation P (n, 5) = 20P (n, 3). (3 marks)

Compare your answers to those provided below. Pay particular attention to any mistakes that you made and clarify those misunderstandings.

Solutions to the Assignment

1.

i) ALGEBRA

$P(7,7)=\frac{7!}{2!}=\frac{5040}{2}=2520$ permutations

ii) COLLEGE

$P(7,7)=\frac{7!}{2!2!}=\frac{5040}{4}=1260$ permutations

2.

In this case order is important so it is a permutation.

So $P(9,4)=\frac{9!}{(9-4)!}=\frac{9\times 8\times 7\times 6\times 5!}{5!}=3024$ numbers.

3.

It is a permutation because order of the letters is important, for instance rise is different from sire. So there are $P(26,4)=\frac{26!}{22!}=358800$ permutations.

4.

This is combination and not a permutation because order of the boys is not important. So $C(30,11)=\frac{30!}{(30-11)!11!}=\frac{30!}{19!11!}=54627300$ teams.

5.

Here order is important because Thabo, Pule,Lineo is different from Lineo, Pule, Thabo, since in the first setting Lineo is position three and in the second setting Lineo is in the first position. So order is important. It is a permutation.

i.e. $P(6,3)=\frac{6!}{(6-3)!}=\frac{6\times 5\times 4\times 3!}{3!}=120$ ways.

6.

Order is important, so

$P(4,2)=\frac{4!}{2!}=\frac{24}{2}=12$ ways. $P(4,4)\times P(3,3)\times P(3,3)=4!3!3!=864$

7. It is a permutation because order is important.

i)ways $P(4,4)\times P(3,3)\times P(3,3)\times P(3,3)=4!3!3!3!=5184$

ii) ways. The last P(3,3) is for alternation of the classes with the three positions.

iii)

$P(10{,}10) = 10! = 3628800$ Ways. $P(9{,}5) = \frac{9!}{(9-5)!5!} = \frac{9\times8\times7\times6\times5!}{4!5!} = 126$

8.

It is a combination, order is not important because we are not talking of positions or roles in the team. We are only talking of team members.

9.

Here order is important because students hold positions, so it is a permutation.

$P(100{,}4) = \frac{100!}{(100-4)!} = \frac{100\times99\times98\times97\times96!}{96!} = 94109400$ ways of elections.

10.

Order is not important because the boys and girls hold no specific role or positions.

Four boys can be selected in ways

Three girls can be selected in

$C(9{,}3) = \frac{9!}{(9-3)!3!} = \frac{9\times8\times7\times6!}{6!\times3!} = 84$ ways.

$C(11{,}4) = \frac{11!}{(11-4)!4!} = \frac{11\times10\times9\times8\times7!}{7!4!} = 330$

So by the fundamental principle of counting they can sit 330×84 ways=27720 ways.

11.

Order is not important, so it is a combination.

To select 5 of the first 6 we use: $C(6{,}5) = \frac{6!}{(6-5)!5!} = \frac{6\times5!}{5!} = 6$ way of selecting the first five questions.

Now to select 3 out of the remaining 7 questions we use,

$C(7{,}3) = \frac{7!}{(7-3)!3!} = \frac{7\times6\times5\times4!}{4!3!} = 35$ ways.

So by the fundamental principle of counting a student may be chosen in 6×35 ways=210 ways.

12.

We are proving that P(n+1,r)=(n+1)P(n,r-1)

$$P(n+1,r) = \frac{(n+1)!}{((n+1)-r)!} = \frac{(n+1)!}{(n+1-r)!}$$

$$(n+1)! = (n+1)n!$$

$$\Rightarrow \frac{(n+1)!}{(n+1-r)!} = \frac{(n+1)n!}{(n+1-r)!}$$

Now

$$(n+1)P(n,r-1) = \frac{(n+1)n!}{(n-(r-1))!} = \frac{(n+1)n!}{(n+1-r)!}$$

$$\therefore P(n+1,r) = \frac{(n+1)n!}{(n+1-r)!} = (n+1)P(n,r-1)$$

13.

We are solving for n.

$$P(n,5) = 20P(n,3)$$

$$\Rightarrow \frac{n!}{(n-5)!} = 20\frac{n!}{(n-3)!}$$

$$\Rightarrow \frac{n!(n-3)!}{(n-5)!} = 20n!$$

$$\Rightarrow \frac{n!(n-3)(n-4)(n-5)!}{(n-5)!} = 20n!$$

$$\Rightarrow \frac{n!(n-3)(n-4)}{n!} = 20$$

$$\Rightarrow (n-3)(n-4) = 20 \textit{ refere to quadratic equations}$$

$$\Rightarrow n^2 - 7n + 12 = 20$$

$$\Rightarrow n^2 - 7n - 8 = 0$$

$$\Rightarrow (n-8)(n+1) = 0$$

$$\Rightarrow n-8 = 0 \textit{ or } n+1 = 0$$

$$\Rightarrow n = 8 \textit{ or } n = -1$$

$$\Rightarrow n = 8 \textit{ because we can not have n as a negative number.}$$

Based on your results and the recommendation that you should aim for at least

80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.

Assessment

Assessment

Instructions

This assessment consists of 15 questions. Answer all of the questions.

The marks for each question are shown. There are a total of 80 marks.

You are advised to spend no more than 50 minutes on this assessment.

Calculators may be used.

Show the necessary working.

Good luck!!

1. Write ${}_xP_{x-4}$ in factorial notation. (4 marks)

2. Evaluate: (8 marks)

$a.\ 10!$

$b.\ \frac{6!}{3!}$

$c.\ \frac{18!}{12!6!}$

$d.\ {}_9P_9$

3. How many four lettered words can be formed from the name "*saul*" if a letter can be used at most once for each word? (4 marks)

4. How many 4-figure numbers can be formed using digits 1, 2, 3,4,5,6 if a digit can only be used once? (4 marks)

5. Consider the word "*everlasting*".

How many permutations does the word *everlasting* have? (4 marks)

6. How many arrangements can be made from the word *sound*

a) if each letter is used only once in each arrangement? (3 marks)

b) If *s* and *o* should always be next to each other and each letter must be used only once in each arrangement? (4 marks)

c) If *n* and *d* are always next to each other at the last two positions. (3 marks)

7. In a church there are 30 members; in how many ways can the church select a set of 6 board members? (4 marks)

8. Bag 1 contains 6 fruits and bag 2 contains 11 fruits. In how many ways can 5 fruits be selected from these bags if 3 are to be from bag 1 and 4 from bag 2?

(5 marks)

9. In a class of 5 boys and 10 girls, in how many ways may a committee of 5 made up of 3 boys and 2 girls be selected?

(5 marks)

10. A class consists of 12 girls and 15 boys. A committee is to be formed that consists of 3 girls and 3 boys,

a) How many different comities are possible if one person can be selected once in a committee? (4 marks)

b) How many committees are possible if the following positions are to be filled: president, vice president, secretary, public relations officer, treasurer and entertainment officer?

(5 marks)

11. Solve for n in the following: ${}_nP_1 = 10$ (4 marks)

12. Show that $2({}_nP_{n-2}) = {}_nP_{n-1}$ (5 marks)

13.Solve for r: ${}_{12}P_r = 8({}_{12}P_{r-1})$ (5 marks)

14. solve for n in ${}_nC_1 = 6$ (3 marks)

15.solve for n in ${}_nC_6 = 3({}_{n-2}C_4)$ (5 marks)

Solutions to the Assessment

1.

$${}_xP_{x-4} = \frac{x!}{(x-(x-4))!} = \frac{x!}{(x-x+4)!} = \frac{x!}{4!}$$

2.

$a. 10! = 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1 = 3628800$

$b. \frac{6!}{3!} = \frac{720}{6} = 120$

$c. \frac{18!}{12!6!} = \frac{18\times 17\times 16\times 15\times 14\times 13\times 12!}{12!6!} = 18564$

$d. {}_9P_9 = 9! = 362880$

3. saul, it is a permutation because order is important.

$\therefore {}_4P_4 = 4! = 24$

4. Order is important so it is a permutation.

$$_6P_4 = \frac{6!}{(6-4)!} = \frac{720}{2} = 360$$

5. It is a permutation because order is important. You should also note that e is repeating.

So $_{11}P_{11} = \frac{11!}{(2)!} = 19958400$

6. It is permutation,

$a)\ _5P_5 = 5! = 120$

$b)\ _2P_2 \times _3P_3 \times 4 = 2!3!4 = 2\times 6\times 4 = 48$ *the* 4 *is for the four positions to be moved by s and* 0.

$c)\ _3P_3 \times _2P_2 = 3!2! = 6\times 2 = 12$

7. Here order is not important because board members hold no specific positions, so it is a combination.

$$\therefore {}_{30}C_6 = \frac{30!}{(30-6)!6!} = \frac{30\times 29\times 28\times 27\times 26\times 25\times 24!}{24!6!} = 593775$$

8. It is a combination because order is not important.

Now selecting a fruit from bag 1 can happen in:

$$_6C_3 = \frac{6!}{(6-3)!3!} = \frac{720}{36} = 20 ways$$

And selecting a fruit from bag 2 can happen in :

$$_{11}C_4 = \frac{11!}{(11-4)!4!} = \frac{11\times 10\times 9\times 8\times 7!}{7!4!} ways = 330 ways$$

From the fundamental principle of counting, fruits would be selected from the back in 20×330 ways = 6600 ways.

9. It is a combination because the members have no specified positions in the committee.

Selecting a boy would happen in:

$$_5C_3 = \frac{5!}{(5-3)!3!} = \frac{5\times 4\times 3!}{2!3!} = 10 ways$$

Selecting a girl would happen in:

$${}_{10}C_2 = \frac{10!}{(10-2)!2!} = \frac{10\times 9\times 8!}{8!2!} = 45 ways$$

By the fundamental principle of counting, a committee would be selected in 45×10=450 ways.

10.

a)

It is a combination because committee members hold no specific positions.

Selecting a girl would happen in :

$${}_{12}C_3 = \frac{12!}{(12-3)!3!} = \frac{12\times 11\times 10\times 9!}{9!3!} = 220 ways$$

Selecting a boy would happen in :

And by the fundamental principle of counting, selecting a committee of 5 would happen in

455×220 ways = 100100 ways.

b)

Since committee members hold different positions, selecting a committee is a permutation.

Selecting a boy would happen in:

$${}_{15}P_3 = \frac{15!}{(15-3)!} = \frac{15\times 14\times 13\times 12!}{12!} = 2730 ways$$

Selecting a girl would happen in :

$${}_{12}P_3 = \frac{12!}{(12-3)!} = \frac{12\times 11\times 10\times 9!}{9!} = 1320 ways$$

By the fundamental principle of counting, committee members would happen in:

1320×2730 ways = 3603600 ways.

11.

We are solving for n.

$${}_nP_1 = 10$$
$$\Rightarrow \frac{n!}{(n-1)!} = 10$$
$$\Rightarrow \frac{n(n-1)!}{(n-1)!} = 10$$
$$\Rightarrow n = 10$$

12.

We are proving that

$$2({}_nP_{n-2}) = {}_nP_{n-1}$$
$$2({}_nP_{n-2}) = \frac{2(n!)}{(n-(n-2))!} = \frac{2n!}{(n-n+2)!} = n!$$

on the other hand

$${}_nP_{n-1} = \frac{n!}{(n-(n-1))!} = \frac{n!}{(n-n+1)!} = \frac{n!}{1} = n!$$
$$\Rightarrow 2({}_nP_{n-2}) = {}_nP_{n-1}$$

13.

We are solving for r.

$${}_{12}P_r = 8({}_{12}P_{r-1})$$
$$\Rightarrow \frac{12!}{(12-r)!} = \frac{8(12!)}{(12-(r-1))!}$$
$$\Rightarrow \frac{12! \times (13-r)!}{(12-r)!} = 8 \times 12!$$
$$\Rightarrow \frac{12!(13-r)(12-r)!}{(12-r)!} = 8 \times 12!$$
$$\Rightarrow \frac{12!(13-r)}{12!} = 8$$
$$\Rightarrow 13 - r = 8$$
$$\Rightarrow -r = -5$$
$$\Rightarrow r = 5$$

14.

$_nC_1) = 6$

$\Rightarrow \frac{n!}{(n-1)1!} = 6$

$\Rightarrow \frac{n(n-1)!}{(n-1)!} = 6$

$\Rightarrow n = 6$

15. we are solving for n.

$_nC_6 = 3(_{n-2}C_4)$

$\Rightarrow \frac{n!}{(n-6)!6!} = \frac{3(n-2)!}{(n-2-4)!4!}$

$\Rightarrow \frac{n!(n-6)!6!}{(n-6)!6!} = \frac{3(n-2)!\times(n-6)!6!}{(n-6)!4!}$ *we multiply by* $(n-6)6!$ *on both sides*

$\Rightarrow n! = \frac{3(n-2)!\times 6\times 5\times 4!}{4!}$

$\Rightarrow n(n-1)(n-2)! = 3(n-2)!\times 6\times 5$

$\Rightarrow n^2 - n = 90$

$\Rightarrow n^2 - n - 90 = 0$ *see quadratic equations*

$\Rightarrow (n-10)(n+9) = 0$

$\Rightarrow n = 10$ *or* $n = -9$

$\Rightarrow n = 10$, *because n cannot be a negative number*.

Unit Contents

Unit 33

Unit 33

Calculus

Introduction

Calculus is a branch of Mathematics that deals with among others, derivatives and integrals. This unit covers basic work on these two major branches of calculus. Because calculus is the study of change, "….it has widespread applications in science, economics, and engineering and can solve many problems for which algebra alone is insufficient" (Wikipedia).

This unit consists of 91 pages. This is approximately 4% of the whole course. Plan your time so that you can complete the whole course on schedule.

Take a moment to read the following learning outcomes. They are a guide to what you should focus on while studying this unit.

Upon completion of this unit you will be able to:

This Unit is Comprised of Seven Lessons:

Lesson 1 The Gradient of a Curve at Any Point
Lesson 2 Maxima and Minima
Lesson 3 Using Derivatives to Calculate Velocity and Acceleration
Lesson 4 Differentiation Compared to Integration
Lesson 5 Notation For Indefinite Integrals
Lesson 6 The Definite Integral as the Area Under Graph
Lesson 7 Application of Integration to Distance

Upon completion of this unit you will be able to:

Outcomes

- *calculate* the gradient of a quadratic function of the form $y=ax^2 + ax + c$ curve at a point on the curve by differentiation.
- *calculate* turning points on the curve by differentiation.
- *determine* whether a turning point is *a* maxima or minima.
- *calculate* velocity by differentiation.
- *calculate* acceleration by differentiation.
- *calculate* an indefinite integral.
- *calculate* the area under a graph of a function.
- *calculate* distance as area under graph of velocity time graph.
- *calculate* velocity from the distance-time graph by integration.
- *calculate* acceleration from the velocity-time graph by integration.
- *calculate* a definite integral.

Terminology

Coefficient The number that multiplies a variable e.g. in the term 3x, 3 is a coefficient of x.

Derivative: An expression for a slope of a curve at any point on the curve.

Differentiation: A process of calculating a derivative of a function.

gradient: Same as slope.

Inverse: Opposite

Secant: A **line** that intersects a curve at two points.

Slope: Change in y divided by change in x.

Coefficient: The number that multiplies a variable e.g. in the term 3x, 3 is a coefficient of x.

Definite integral: A number that results as an antiderivative is evaluated to give a definite number as a result.

Derivative: An expression for a slope of a curve at any point on the curve.

Differentiation: A process of calculating a derivative of a function.

Gradient: Same as slope.

infinite integral: An antiderivative which is an expression that has a variable 'x' and an unknown constant.

Slope: Change in y divided by change in x.

Tangent: A straight line that touches a curve at a point.

Online Resource

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 The Gradient of a Curve at Any Point

In your junior certificate you learned how to calculate the slope of a straight line. In this unit we will learn how to calculate the slope of a curve. The slope of a curve at a point is the slope of a tangent to the curve at that point. The slope represents a rate of change. Rate of change has many applications in real life, e.g. the rate at which a car moves, interest rates, population increase rate, etc. The rate of change i.e. the slope of a straight line is constant along that line so it is easy to calculate but that of a curve changes at all points on the curve, and therefore needs a different method to calculate. Calculus enables us to find the formulae for algebraically calculating the slope at all points on a curve. The formula is called a derivative. The coordinates of a point at which the gradient is required are then substituted in the derivative to find the gradient. We will learn how to find the derivative of a function and use it to calculate the gradient of a curve at a given point. We will start by reminding ourselves about calculating the gradient of a straight line given two points.

Calculating the Gradient of a Straight Line: a Reminder

Activity 1

Calculate the gradient of a straight line that passes through

(a) (1,4) and (3,8)

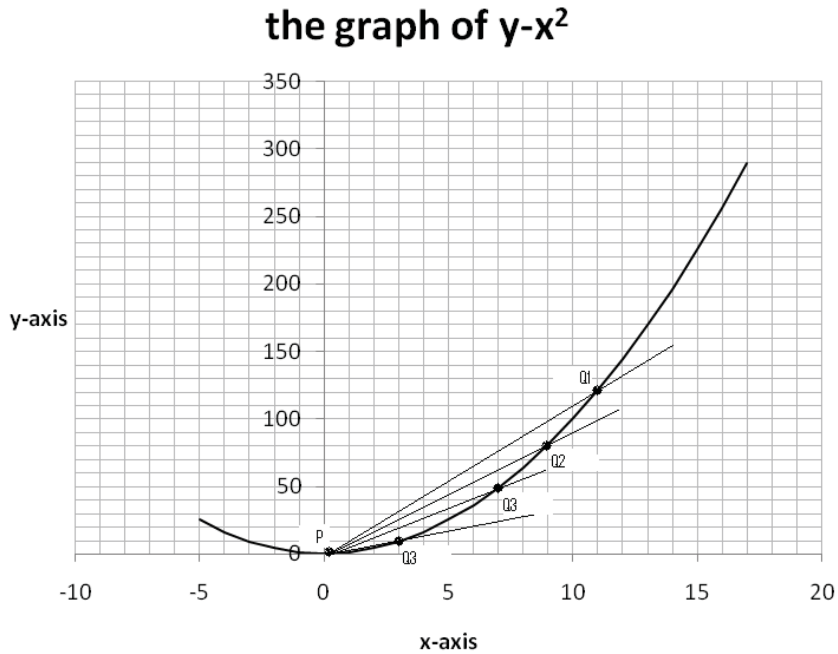
(b) (2,5) and (4,7)

(c) (1, -1) and (-2,6)

Compare your answers with the ones below. Ensure that you got all the questions right. If not refer to the topic on Linear graphs unit 20 to remind yourself before you proceed to the next section.

A Derivative

Consider a curve y=x^2 and gradient at a point P (1,1). As we said earlier the curve's gradient can be found by drawing a tangent to the curve at (1,1) and then finding its gradient. The gradient of the curve we get this way is a very rough estimate of the gradient at (1,1) because the tangent drawn is not accurate. To solve this problem a second point Q is marked on the curve. P is then joined to Q to form a secant PQ. The gradient of PQ is calculated as shown below. The gradients of several other secants are found as the point Q is moved closer and closer to P. (N.B. Q moves closer and closer to P but will not coincide with it.) As 'P' moves closer to 'Q', PQ approaches the tangent line to the curve at P. The following illustrate what happens to the gradient of PQ as Q approaches P.



Let us illustrate what happens to the gradient as Q moves closer to P

Coordinates of P (1,1)		Coordinates of Q as it moves closer to P				Gradient of PQ
x_1	y_1	x_2	y_2	$y_2 - y_1$	$x_2 - x_1$	$\frac{y_2 - y}{x_2 - x_1}$
1	1	1.5	$1.5^2=2.25$	1.25	0.5	2.5
1	1	1.4	$1.4^2=1.96$	0.96	0.4	2.4
1	1	1.3	$1.3^2=1.69$	0.69	0.3	2.3
1	1	1.2	$1.2^2=1.44$	0.44	0.2	2.2
1	1	1.1	$1.1^2=1.21$	0.21	0.1	2.1
1	1	1.05	$1.05^2=1.1025$	0.1025	0.05	2.05
1	1	1.01	$1.01^2=1.0201$	0.0201	0.01	2.01
1	1	1.001	$1.001^2=1.002001$	0.002001	0.001	2.001

This table shows that the gradient becomes very close to 2 as Q approaches P. Note that Q comes closer and closer to P but does not coincide with it, and that the gradient of PQ approaches 2 but will never be 2. 2 is referred to as the limit as the distance between P and Q approaches 0. This idea is used below when we calculate a derivative from the first principles where the distance between the two points which will be denoted by 'h' is equated to zero. You will not be required to use this method to find the gradient of a curve.

Calculating a Derivative from the First Principles

For you to be able to understand the rules for the calculation of the derivative you have to use the following method which is referred to as calculation of derivatives from the first principles.

Calculate the derivative of a function $y = x^2$ and then find the gradient of the curve at the point (1, 1)

We use two points (x_1, y_1) **and** (x_2, y_2)

Gradient of the tangent of the curve $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$(x_1, y_1) = (1,1)$

Let $x_2 = 1 + h$, where h is a distance between x_2 and x_1. When we substitute $x_2 = 1 + h$ into $y_2 = x^2$ we get $y_2 = (1+h)^2$ $(x_2, y_2) = \left[(1+h), (1+h)^2\right]$

Gradient of the tangent of the curve $= \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\frac{(1+h)^2 - 1}{(1+h-1)} = \frac{1 + 2h + h^2 - 1}{h} = \frac{2h + h^2}{h} = 2 + h$$

The distance 'h' of x_2 away from x_1 is very nearly 0 as x_2 approaches x_1

i.e.

$$\begin{aligned} &gradient = 2 + h \\ &as \\ &h \to 0 \\ &gradient = 2 \end{aligned}$$

This means that as h approaches 0, the gradient becomes very nearly equal to 2. We say that the gradient tends to 2 as h tends to 0. This is sometimes written $h \to 0$ $gradient \to 2$. Another way of saying this is that the limit of the gradient is 2 as h tends to 0. The gradient of the curve at (1,1) is therefore approximately 2.

The notation stating this is:

$$\lim_{h \to 0} 2 + h = 2$$

The process of finding the slope of a curve in this way is called **differentiation** and the expression that results is called the **derivative or the limit as h tends to zero ($h \to 0$).**

Let us practice the finding the derivatives of the following functions from the first principles as examples. We will later use the derivative to find a gradient at a given point.

(a) $y = 2x$

(b) $y = 3x^2$

(c) $y = x^2 + x + 1$

(d) $y = ax^2$

(e) $y = ax^2 + bx + c$

(f) $y = x^3$

(g) $y = \frac{1}{x}$

Solutions

(a) Derivative of $y = 2x$				
x_1	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$

x	$2x$	$x+h$	$2(x+h)$	$\frac{2(x+h)-2x}{x+h-x}$ $\frac{2x+2h-2x}{h}$ $\frac{2h}{h}$ $derivative = 2$

(b) derivative of $y = 3x^2$						
x_1	y_1	x_2	y_2	$y_2 - y_1$	$x_2 - x_1$	$\frac{y_2 - y}{x_2 - x_1}$
x	$3x^2$	$(x+h)$	$3(x+h)^2$	$3(x+h)^2 - 3$	$(x+h-x)$	$\frac{3(x+h)^2 - 3x^2}{(x+h-x)}$ $\frac{3(x^2+2xh+h^2)-3x^2}{h}$ $\frac{3x^2+2xh+h^2-3x^2}{h}$ $\frac{2xh+h^2}{h}$ $2x+h$ $h \to 0$ $derivative = 2x$

(c) derivative of $y = x^2 + x + 1$				
x_1	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$

X	x^2+x+1	$(x+h)$	$(x+h)^2+(x+h)+1$	$\frac{(x+h)^2+(x+h)+1-(x^2+x+1)}{x+h-x}$ $\frac{x^2+2xh+h^2+x+h+1-x^2-x-1}{h}$ $\frac{2xh+h^2+h}{h}$ $2x+h+1$ $h\to 0$ $derivative = 2x+1$

(d) Derivative of $y=ax^2$				
x_1	y_1	x_2	y_2	$\frac{y_2-y_1}{x_2-x_1}$
x	ax^2	x+h	$a(x+h)^2$	$\frac{a(x+h)^2-ax^2}{x+h-x}$ $\frac{a(x^2+2xh+h^2)-ax^2}{h}$ $\frac{ax^2+2axh+ah^2-ax^2}{h}$ $\frac{2axh+ah^2}{h}$ $=2ax+ah$ $h\to 0$ $derivative = 2ax$

(e) Derivative of $y=ax^2+bx+c$

x_1	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$
x	$ax^2 + bx +$	x+h	$x(x+h)^2 + b(x+$	$\frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{x+h-x}$ $\frac{a(x^2 + 2xh + h^2) + bx + bh + c - ax^2 - bx}{h}$ $= \frac{2xh + ah^2 + bh}{h}$ $2x + ah + b$ $h \to 0$ $derivative = 2x + b$

(f) Derivative of $y = x^3$				
x_1	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$
x	x^3	$x+h$	$(x+h)^3$	$\frac{(x+h)^3 - x^3}{x+h-x}$ $\frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$ $\frac{3x^2h + 3xh^2 + h^3}{h}$ $3x^2 + 3xh + h^2$ $h \to 0$ $derivative = 3x^2$

(g) Derivative of $y = \frac{1}{x}$

x	$\frac{1}{x}$	$x+h$	$\frac{1}{x+h}$	$\frac{1/_{x+h} - 1/_{x}}{x+h-x}$ $\frac{h/_{x^2+h}}{h}$ $\frac{1}{x^2+xh}$ $h \to 0$ $\frac{1}{x^2}$ $derivative = x^{-2}$

Activity 2

Calculate the derivatives of the following functions from the first principles

(a) $y = 2x^2$

(b) $y = 3x^2 + x$

(c) $y = 2x^2 - x$

(d) $y = 6x^2 - x - 2$

(e) $y = 4x^2 + 3x - 2$

(f) $y = 5x^2 - 2x$

Compare your answers with the ones at the end of the subunit. If you get less than five items correct, review the examples above. If you get five or more right, proceed to the next section on calculating a derivative using rules for differentiation.

Key Points to Remember

The key points to remember in this subunit on calculating the derivative of a function from the first principles are:

- The gradient of a straight line is constant throughout the line but that of a curve changes at every point on the curve.
- The gradient of the curve at 'x' is the gradient of the tangent drawn at that point. The gradient found using this method gives a very rough estimate of a gradient of a curve at that point.
- A more accurate method of finding the gradient of a curve at a tpoint is the calculation of the derivative of a function and substituting the coordinates of point into the derivative.
- A derivative is a formula for finding the gradient of a curve at any point on the curve.

Calculating a derivative from the first principles uses the method of calculating the gradient of a line using two points on the line. Finding a derivative of a function from the first principles is a long method to get a derivative. When we compare the functions and their derivatives got using the first principles we get at pattern that gives a formula for finding a derivative. In the section that follows we will look at the functions and their derivatives and deduce the formula

Finding the Formula for Calculating a Derivative: In this section we will compare a function and its derivative and deduce a formula for calculating a derivative of a function. The table also gives us the notations used in differential calculus. The following table compares the function and its derivative from the above table.

(a) $y = x^2$	$2x$
(b) $y = 2x$	2
(c) $y = 3x^2$	$6x$
(d) $y = x^2 + x + 1$	$2x + 1$
(e) $y = ax^2$	$2ax$
(f) $y = ax^2 + bx + c$	$2ax + b$
(g) $y = x^3$	$3x^2$

(h) $y = \frac{1}{x}$	$-1x^{-2}$

Look at the function and its derivative. Do you recognize a pattern?

Compare your answer with the one below:

Notation for derivatives

A derivative is denoted by $\frac{dy}{dx}$ or $f'(x)$ if a function is $f(x)$ or D

Function	The bolded part is the summary of what happens on the right side of a function to get a derivative	Derivative
(a) $y = x^2$	$\frac{dy}{dx} = f'(x) = D(x^2) = 2 \times 1x^{2-1}$	$2x$
(b) $y = 2x$	$\frac{dy}{dx} = f'(x) = 2D(x) =$ $2 \times 1x^{1-1} = 2x^0$	2
(c) $y = 3x^2$	$\frac{dy}{dx} = f'(x) = 3D(x^2) = 3 \times 2 \times 1x^{2-1}$	$6x$
(d) $y = x^2 + x + 1$	$\frac{dy}{dx} = f'(x) = D(x^2) + D(x) =$ $2 \times 1x^{2-1} + 1 \times 1x^{1-1}$	$2x + 1$
(e) $y = ax^2$	$\frac{dy}{dx} = f'(x) = aD(x^2) = a \times 2 \times 1x^{2-1}$	$2ax$
(f) $y = ax^2 + bx + c$	$\frac{dy}{dx} = f'(x) = aD(x^2) + bD(x) + D(c)$ $= a \times 2 \mp \times 1x^{2-1} + b \times 1x^{1-1}$	$2ax + b$
(g) $y = x^3$	$\frac{dy}{dx} = f'(x) = D(x^3) = 3 \times 1x^{3-1}$	$3x^2$
(h) $y = \frac{1}{x}$	$\frac{dy}{dx} = f'(x) = D(x^{-1}) = -1 \times 1x^{-1-1}$	$-1x^{-2}$

Summarised in words this is what is done to get a derivative:

- The index of x in each function is multiplied by the coefficient of x in the derivative, e.g. when a function is $y = x^2$, the derivative is $2 \times x$. The index of x in the derivative is 1 less than that of the x in the function. e.g. When a function is $y = x^2$ the derivative is $2x^{2-1} = 2x^1$. This is summarised below:

Rules for differentiation

Using the function $y = 2x^3$ as an example

- multiply the coefficient of x by the exponent of x e.g. for f(x) = $2x^3 = 3 \times 2 = 6$
- subtract 1 from the exponent of x f(x) = $2x^{3-1}$
- when the two steps are done we say $(D(2x^3) = 6x^2$

Also note the following;

1. $\frac{dy}{dx}$ is not a fraction that can be cancelled. It compares the rate of change of y with that of x.

2. For $y = ax^2 + bx + c$ we find the derivative of each term separately and maintain the signs between the terms. e.g. $y = 2x^4 - 4x + 3$ the derivative is $8x^3 - 4$

3. In differentiating remember the minus sign when multiplying the negative exponent by the coefficient of the variable. e.g. $y = \frac{1}{x^3} . \frac{1}{x^3} = x^{-3}$ Its derivative $= -3 \times 1x^{-3-1} = -3x^{-4}$

Example 1

Differentiate the following using the shorter method

(a) $y = 5x^2$

(b) $y = x^2 - 4$

(c) $y = 2x - x^3$

(d) $y = 2x^3 - 4x + 3$

(e) $y = \frac{3}{x}$

(f) $y = \frac{-5}{x}$

Compare your answers with the ones below. Ensure that you got 80% or more before you proceed to the next section.

Answers

(a)

$$y = 5x^2$$

$$\frac{dy}{dx} = 5 \times 2x^{2-1} = 10x$$

(b)

$$y = x^2 - 4$$

$$\frac{dy}{dx} = 2x^{2-1} = 2x$$

(c)

$$y = 2x - x^3$$

$$\frac{dy}{dx} = 2 \times 1x^{1-1} - 3x^{3-1} = 2 \times 1 \times x^0 - 3 \times x^2 = 2 - 3x^2$$

(d)

$$y = 2x^3 - 4x + 3$$

$$\frac{dy}{dx} = 2 \times 3x^{3-1} - 4 \times x^{1-1} = 6x^2 - 4$$

(e)

$$y = \frac{3}{x} = 3x^{=1}$$

$$\frac{dy}{dx} = 3 \times -1x^{-1-1} = -3x^{-2}$$

(f)

$$y = \frac{-5}{x^2} = -5x^{-2}$$

$$\frac{dy}{dx} = -5 \times -2x^{-2-1} = 10x^{-3}$$

Key Points to Remember

The key points to remember in this subunit on calculating the derivative of a function by a shorter method are:

- Notation for derivatives are $\frac{dy}{dx}$ or $f'(x)$ if a function is $f(x)$ or D$\frac{dy}{dx}$ is not a fraction that can be cancelled. It compares the rate of change of y with that of x.
- For $y = ax^2 + bx + c$ we find the derivative of each term separately and maintain the signs between the terms. e.g. $y = 2x^4 - 4x + 3$ the derivative is $8x^3 - 4$
- In differentiating remember the minus sign when multiplying the negative exponent by the coefficient of the variable. e.g. $y = \frac{1}{x^3}$.
  $\frac{1}{x^3} = x^{-3}$ Its derivative $= -3 \times 1x^{-3-1} = -3x^{-4}$

The derivative is can then be evaluated to give the slope of a curve at a given point. The section that follows shows how this is done.

Activity 3

Differentiate the following functions using the rules learned above

(a) $y = 2x^2$

(b) $y = 3x^2 + x$

(c) $y = 2x^2 - x$

(d) $y = 6x^2 - x - 2$

(e) $y = 4x^2 + 3x - 2$

(f) $y = 5x^2 - 2x$

Compare your answers with the ones at the end of the subunit. If you get less than five items correct, review the examples above. If you get five or more right, proceed to the next section on calculating the gradient of a curve at a given point using a derivative.

Calculating the Gradient of a Curve at a Given Point Using a Derivative

In this section we evaluate the derivative to get the slope at a point by substituting the coordinates of a point into the derivative.

Example 1

Find the slope of a curve $y = x^3$ at (2,4)

$$\frac{dy}{dx} = 3x^2$$

To find the slope at (2,4) we then substitute $x = 2$ in the derivative

$$3x^2 = 3 \times 2^2 = 12$$

Example 2

Find the slope of a curve $y = 2x^2 + 3x$ at (3, 1)

$$\frac{dy}{dx} = 2 \times 2x^{2-1} + 3x^{1-1} = 4x + 3x^0 = 4x + 3 \times 1 = 4x + 3$$

The slope at (3,1)

$$derivative = 4x + 3$$
$$slope = 4 \times 3 + 3 = 15$$

Key Points to Remember

The key points to remember in this subunit on calculating the gradient of a curve at a given point using a derivative are:

- To find the slope of a curve at a point we first calculate a derivative and then substitute the coordinates of a point into the derivative.
- Any number raised to the power of zero is equal to 1.

Activity 4

1. Find the gradients of curves at the points indicated

(a) $y = 4x^2$ at (1,4) (b) $y = 3x^2 + x$ at (1,4) (c) $y = 4x$ at (3,1)

(d) $y = 6x^2 - 2x$ at (2,3) (e) $y = 2x^2 + 3x + 1$ at (2,1)

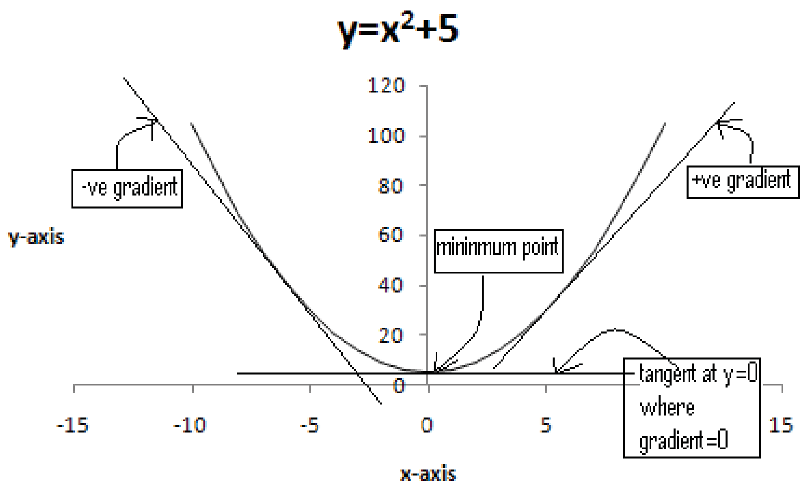
Compare your answers with the ones below. Ensure that you got 80% or more before you proceed to the next section.

Lesson 2 Maxima and Minima

One of the great powers of calculus is in the determination of the maximum or minimum value of a function. The derivatives of a function can be used to find the maximum point of curves for curves that open downward and minimum point of curves that open upward. At these points, curves turn, therefore these points are called turning points on a graph. A maximum point is higher than the point before it and after it on a curve. The minimum point is lower than the point before it and after it on a curve. The gradient of a line at these points is equal to 0. This is because the tangent to the curve at these turning points is parallel to the axes. In this section we will determine whether a turning point is a minimum or a maximum. We will deal with basic quadratic functions having one turning point each i.e. a minimum or a maximum.

Exploring the Properties of a Minimum and a Maximum

The following graph illustrates a minimum point of a curve and its properties.



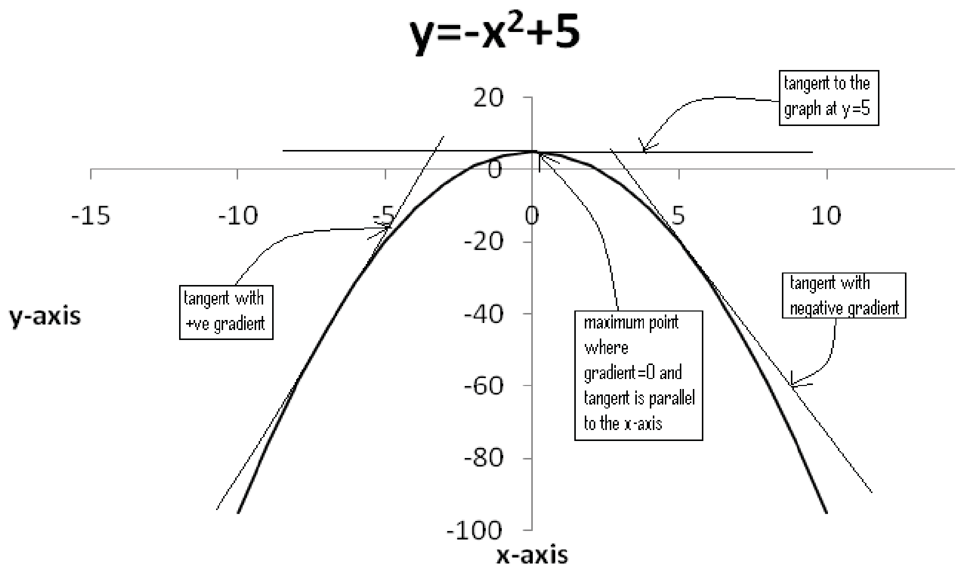
Note that

Minimum

Starting from the **left** of a **minimum**, the gradient changes from ' **-** ' to ' + 'with increasing 'x'.

Maximum

The following graph illustrates a maximum point of on a curve and its properties



Note that

Starting from the **left** of a **maximum** the gradient changes from ' + ' to ' **-** 'with increasing 'x'

Key Points to Remember

The key points to remember in this subunit on properties of a minimum and maximum of a curve are:

- At the point of maximum or minimum the gradient is zero
- Starting from the **left** of a **minimum**, the gradient changes from ' **-** ' to ' + 'with increasing 'x'.
- Starting from the **left** of a **maximum** the gradient changes from ' + ' to ' **-** ' with increasing 'x'

These properties are used in determining whether a turning point is a minimum or a maximum in the section that follows.

Determining a turning point and whether the point is a maximum or a minimum point of a curve

Example 1

Find the turning point of $y = x^2$

And determine whether the turning point is a minimum and maximum.

Step 1

Find the derivative of the function.

$$\frac{dy}{dx} = 2x$$

Step 2

Equate 2x (the derivative) to zero and find the value of the x-coordinate

$2x = 0$

$x = 0$

Step 3

Substitute x=0 into the original function and find the value of y

$y = x^2$

$y = 0^2 = 0$

The coordinates of the turning point are (0,0)

Step 4

Determine whether (0,0) is a minimum or at maximum by taking a value of x less than 0 (the x-coordinate of our point (0,0)), say -1 and substituting it into the derivative $2x$ we get $2x = 2(-1) = -2$ (a negative gradient)

Step 5

Taking a value of x more than 0 say =1 and substituting it into the derivative $2x$ we get $2x = 2(+1) = +2$ (a positive gradient)

The gradient changes from negative to positive 'with increasing 'x'

(Look at the statement under the graph showing minimum point)

The point (0,0) is a minimum.

Example 2

Find the turning point of $y = x^2 - 2x - 3$

and determine whether the turning point is a minimum or maximum

Step 1

Find the derivative of the function

$derivative = 2x - 2$

Step 2

Equate 2x-2 (the derivative) to zero find the value of x-coordinate

$2x - 2 = 0$

$2x = 2$

$x = 1$

Step 3

Substitute x=1 into the original function $y = x^2 - 2x - 3$ to get the value of y

$y = x^2 - 2x - 3$

$y = (1)^2 - 2(1) - 3 = -4$

Therefore the turning point is (1,-4)

Is this point a maximum or a minimum point?

(a) **Step 4**What is the sign of the gradient of the curve at a point little bit smaller than the turning point (1,-4)

____

(b) **Step 5**What is the sign of the gradient of the curve at a point slightly bigger the point (1,-4)

Compare your answers with the following:

Answer to Step 4

(a) Taking a value of x less than 0 say =-1 and substituting it into the derivative $2x - 2$ we get $2x - 2 = (2 - 1) - 2 = -1$ (a negative gradient).

Answer to Step 5

(b) Taking a value of x more than 0 say = +2 and substituting it into the derivative $2x - 2$ we get $2x - 2 = 2(+2) - 2 = 4 = 2 = +2$ (a positive gradient)

(c) (1,4) is a minimum because starting to the left of a (1,-4), 'with increasing 'x' the gradient changes from -ve to **+ve.**

Example 3

Find the turning point of $y = x^2 - x - 6$

and determine whether the turning point is a minimum or maximum.

Step 1

Find the derivative of the function

$derivative = 2x - 1$

Step 2

Equate 2x-1 (the derivative) to zero and find the value of x.

$2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2}$

Step 3

Substitute $x = \frac{1}{2}$ into the original function $y = x^2 - 2x - 3$ to get the value of y

$y = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 6$

$y = \frac{1}{4} - \frac{1}{2} - \frac{6}{1}$

$y = \frac{1}{4} - \frac{2}{4} - \frac{24}{4} = \frac{-25}{4} = -6\frac{1}{4}$

The coordinates of the turning point are therefore $\left(\frac{1}{2}, -6\frac{1}{4}\right)$

Step 4

For x slightly less than $\frac{1}{2}$ e.g. x=0, $2x - 1 = 2(0) - 1 = -1$ (a negative gradient)

Step 5

For x slightly above $\frac{1}{2}$ e.g. x = +1 $2x-1=2(+1)-1=+1$ a positive gradient. Therefore the turning point is a minimum point

Key Points to Remember

The key points to remember in this subunit on finding turning points on a curve and determining whether they are minima ord maxima are:

- The steps taken to find the turning point of a curve are:

(a) Find the derivative of a function.

(b) Equate the derivative to zero.

(c) Substitute the value of x found in (b) into the original function to get the value of y.

- To determine whether the point is a maximum or a minimum we carry out the following steps:

(a) Investigate the sign of the derivative on both sides of the turning point. The possibilities are shown in the table below.

The x-coordinate just less than that at the turning point	The x-coordinate at the turning point	The x-coordinate just greater than that at the turning point	Nature of the turning point
+	0	-	Maximum
-	0	+	Minimum

Activity 5

Find the turning points of the following function and determine whether each turning point is a minimum or maximum.

(a) $y=3x^2+x$

(b) $y=6x^2-2x$

(c) $y = 2x^2 + 3x + 1$

(D) $y = x^2 - 8x$

(E) $y = x^2 - 4x - 1$

(F) $y = x^2 - 2x - 3$

(G) $y = 1 - 4x - x^2$

Compare your answers with the ones at the end of the subunit. If you get less than five items correct, review the examples above. If you get five or more right, proceed to the next section on using derivatives to calculate velocity and acceleration.

Lesson 3 Using Derivatives to Calculate Velocity and Acceleration

The derivatives can be used to find the rate of change of distance with time. The value found will be velocity. $v = \frac{ds}{dt}$. Where v is velocity

s = distance, and t = time.

The gradient of a distance-time graph is equal to the velocity of an object moving in a given direction. The derivative of the distance-time graph is equal to the velocity.

The rate of change of velocity is called acceleration 'a'

$a = \frac{dv}{dt}$, where a = acceleration, v = velocity and t = time.

The gradient of a velocity-time graph is equal to the acceleration of a moving object**.** The derivative of the velocity-time graph is equal to the velocity.

Example 1

A distance moved by a point along a line in time t is given by $t^2 + 4t$. Find its (a) initial velocity (b) its velocity after 3seconds (c) its average velocity after 3 seconds.

The gradient or derivative of the $s = t^2 + 4t$ gives the velocity of the point

$$v = \frac{ds}{dt} = 2t + 4$$

(a) For initial velocity we substitute t = 0 into the derivative

$$v = 2t + 4$$

$$v = 2(0) + 4 = 4m/s$$

(b) For the distance after 3 second we substitute t = 3 into the derivative

$$v = 2(3) + 4 = 10m/s$$

For total distance we substitute t = 3 into our original formulae for distance.

$$s = t^2 + 4t$$

$$s = 3^2 + 4 \times 3$$

$$s = 21m$$

$$\text{Average velocity} = \frac{\text{total distance}}{\text{total time}} = \frac{21m}{3s} = 7m/s$$

Note that the average velocity is the total distance travelled divided by the total time elapsed during the whole journey**.**

Acceleration

Acceleration is rate of change of velocity=$\frac{velocity}{time}$ when a body moves on a straight line. Therefore, if 'a' is acceleration, $a = \frac{dv}{dt}$. Note that if a moving body speeds up the acceleration is positive. If a moving body slows down acceleration is called retardation and it is negative

Example

The distance 's'in metres travelled by a body in t seconds is given by

$s = t^3 - 6t^2 + 4t$. Find its (i) initial velocity (ii) its velocity after 2 seconds, (iii) its acceleration after 2 seconds.

$s = t^3 - 6t^2 + 4t$

$v = \frac{ds}{dt} = 3t^2 - 12t + 4$

(a) At initial velocity t = 0 substituting t = 0 into velocity we get

$v = 3t^2 - 12t + 4$

$v = 3(0) - 12(0) + 4 = 4m/s$

(b) For velocity after 2 seconds, we substitute t = 2 into velocity formulae

$v = 3(2)^2 - 12(2) + 4 = -8m/s$

Acceleration= $\frac{dv}{dt}$

$v = 3t^2 - 12t + 4$

$a = \frac{dv}{dt} = 6t - 12$

After t = 2 $a = 6(2) - 12 = 0m/s^2$

Key Points to Remember

The key points to remember in this subunit on Using Derivatives to calculate Velocity and Acceleration are:

- The gradient of a distance-time graph is equal to the velocity of an object moving in a given direction. The derivative of the distance-time graph is equal to the velocity.
- The gradient of a velocity-time graph is equal to the acceleration of a moving object**.** The derivative of the velocity-time graph is equal to the velocity.

Activity 6

1. The distance‘s’ metres moved by a point in t seconds is given by $s = t^3 + 3t^2 + 4$. Find the (a) velocity after 3 s and (b) acceleration after 3s.

2. The velocity v m/s of a point moving in a straight line is given after t seconds by $v = 3t^2 + 4t$. Find its acceleration after 2 seconds.

3. The distance‘s’ metres moved by a particle along a line in t seconds is given by $s = 3t + t^2$. Find it velocity and acceleration after n seconds.

4. The velocity v m/s of a point moving in straight line after t seconds is given by $v = t^2 - t$. Find its acceleration after 3s and also find when the acceleration is zero

Compare your answers with the ones at the end of the subunit. If you get less than three items correct, review the examples above. If you get three or more right, proceed to the next section on calculating indefinite integrals.

Answers to activity 6

1 (a)

$s = t^3 + 3t^2 + 4$

$v = \frac{ds}{dt} = 3t^2 + 6t$

$v = 3t^2 + 6t$

$v = 3(3)^2 + 6(3) = 45m/s$

(b) $a = \frac{dv}{dt} = 6t + 6$

$a = 6(3) + 6 = 24m/s^2$

2. $v = 3t^2 + 4t$

$a = \frac{dv}{dt} = 6t + 4$

After 2seconds $a = 6(2) + 4 = 16m/s^2$

3.

$s = 3t + t^2.$

$v = \frac{ds}{dt} = 3 + 2t$

$v = 3 + 2t$

$v = 3 + 2(n) = 3 + 2n$

$\frac{dv}{dt} = 2m/s$

4.

$v = t^2 - t$

$a = \frac{dv}{dt} = 2t - 1$

After 3 seconds $a = 2t - 1 = 2(3) + 1 = 7m/s^2$

$2t - 1 = 0$

$2t = 1$

$t = \frac{1}{2}$

Acceleration is zero after $\frac{1}{2}$ a second.

Lesson 4 Differentiation Compared to Integration

In this section we will illustrate that integration is the inverse of differentiation. We will use a function $y = 6x^2$ as example to compare the two processes.

Differentiation	**Integration**
Differentiate $y = 6x^2$ $\frac{dy}{dx} = 12x$ **Explanation of how to get 12** **Step 1** -Multiply the coefficient of x by the exponent of x coefficient of x= 6 exponent of x = 2 **Step 2** -Subtract one from the exponent. **Result** The result of the two steps is as follows: $derivative = 6 \times 2x^{2-1} = 12x$	**How to get** $y = 6x^2$ **from 12x by integration** **i.e. Integrate** 12x **Step 1** -Add one to the exponent of x The exponent of x is 1. After adding one it becomes 2 **Step 2** -Divide the term by the resulting exponent after addition of one The two steps result in the following $\frac{12x^{1+1}}{2} = 6x^2$

	The function that was differentiated to give 12x is $y = 6x^2$

In the above table we see that subtracting one from the exponent in differentiation is reversed by adding one to the exponent in integration. Multiplying the coefficient of x by the exponent in differentiation is reversed by dividing by the resulting exponent. The process of obtaining the function, whose derived function is a given expression, is called integrating the expression Integration is the inverse of differentiation.

Example 1

1. Find the integrals of the following given that $\frac{dy}{dx} =$

$(a)x^3$

$(b)x^3 + 3$

$(c)x^3 + 4$

Compare your answers with the ones below. Ensure that you got 80% or more before you proceed to (d), (e) and (f)

Answers to example 1:

$(a)x^3$	$y = \frac{x^{3+1}}{4} = \frac{x^4}{4}$	$\frac{x^4}{4}$
$(b)x^3 + 3$	$y = \frac{x^{3+1}}{4} = \frac{x^4}{4}$	$\frac{x^4}{4}$
$(c)x^3 + 4$	$y = \frac{x^{3+1}}{4} = \frac{x^4}{4}$	$\frac{x^4}{4}$

Example 2

Find the integrals of the following given that $\frac{dy}{dx} =$

$(d)3x^4$

$(e) 3x^4 + 2$ $(f) 3x^4 + 4$

Compare your answers with the ones below. Ensure that you got 80% or more before you proceed to (g) (h)

Answers to example 2:

$(d) 3x^4$	$y = \frac{3x^{4+1}}{5} = \frac{3x^5}{5}$	$\frac{3x^5}{5}$
$(e) 3x^4 + 2$	$y = \frac{3x^{4+1}}{5} = \frac{3x^5}{5}$	$\frac{3x^5}{5}$
$(f) 3x^4 + 4$	$y = \frac{3x^{4+1}}{5} = \frac{3x^5}{5}$	$\frac{3x^5}{5}$

Example 3

Find the integrals of the following given that $\frac{dy}{dx} =$

$(g) 4x$

$(h) 4x + 1$

Compare your answers with the ones below. Ensure that you got 80% or more before you proceed to (k)

Answers to example 3:

$(g) 4x$	$y = \frac{4x^{1+1}}{2} = \frac{4x^2}{2}$	$\frac{4x^2}{2}$
$(h) 4x + 1$	$y = \frac{4x^{1+1}}{2} = \frac{4x^2}{2}$	$\frac{4x^2}{2}$
$(i) 4x + 6$	$y = \frac{4x^{1+1}}{2} = \frac{4x^2}{2}$	$\frac{4x^2}{2}$

Example 4

Find the integrals of the following given that $\frac{dy}{dx} =$

$(k)x^n + 1$

Compare your answers with the ones below. Ensure that you got 80% or more before you proceed to the next section.

Answer to example 4:

$(k)x^n + 1$	$y = \frac{x^{n+1}}{n+1}$	$\frac{x^{n+1}}{n+1}$

$$(a),(b),(c)y = \frac{x^{3+1}}{4} = \frac{x^4}{4}$$

$$(d),(e),(f)y = \frac{3x^{4+1}}{5} = \frac{3x^5}{5}$$

$$(g),(h),(i)y = \frac{4x^{1+1}}{2} = \frac{4x^2}{2}$$

$$(j),(k)y = \frac{x^{n+1}}{n+1}$$

Note that:

The derived functions of x^3 can be $\frac{x^4}{4}$ or $\frac{x^4}{4} + 1$ or $\frac{x^4}{4} + 5$ or $\frac{x^4}{4} + 8$ etc. This means that there is a constant that needs to be added to what we get as an integral. We should always add a constant c to our indefinite integral. The constant 'c' may be any number. So to all our answers a constant c should be added as shown below.

$$(a),(b),(c)y = \frac{x^{3+1}}{4} = \frac{x^4}{4} + c$$

$$(d),(e),(f)y = \frac{3x^{4+1}}{5} = \frac{3x^5}{5} + c$$

$$(g),(h),(i)y = \frac{4x^{1+1}}{2} = \frac{4x^2}{2} + c$$

$$(j),(k)y = \frac{x^{n+1}}{n+1} + c$$

Example 5

Find the integrals of the following given that $\frac{dy}{dx} = (x-1)(x-2)$

$(x-1)(x-2)$

$x^2 - 3x + 2$

$y = \frac{x^{2+1}}{3} - \frac{3x^3}{3} + c$

Note that we find the integral of each term in the expression

Example 6

Find the integrals of the following given that

$\frac{1}{x^5} = x^{-5}$

$y = \frac{x^{-5+1}}{-4} = \frac{x^{-4}}{-4} = \frac{1}{x^4} \times \frac{1}{-4} = \frac{1}{-4x} = -\frac{1}{4x} + c$

Example 7

Find the integrals of the following given that

$\frac{x^4+1}{x^2} = \frac{x^{4+1}}{5} \times \frac{1}{\frac{x^{2+1}}{3}} = \frac{x^5}{5} \times 1 \div \frac{x^3}{3} = \frac{x^5}{5} \times \frac{3}{x^3} = \frac{3x^5}{5x^3} = \frac{3x^2}{5} + c$

Key Points to Remember

The key points to remember in this subunit on the comparison of differentiation to integration are:

- To integrate a function the following steps are taken:

Step 1

Add one to the exponent of x

Step 2

Divide the term by the resulting exponent after addition of one.

- The process of integration is the reverse of that of differentiation. Subtracting one from the exponent in differentiation is reversed by adding one to the exponent in integration. Multiplying the coefficient of x by the exponent in differentiation is reversed by dividing by the resulting exponent.

- The derived functions of e.g. x^3 can be $\frac{x^4}{4}$ or $\frac{x^4}{4}+1$ or $\frac{x^4}{4}+5$ or $\frac{x^4}{4}+8$ etc. This means that there is a constant that needs to be added to what we get as an integral. We should always add a constant c to our integral. ‘c’ may be any number. So to all our answers a constant c should be added as shown below

Activity 7

1. Find the indefinite integrals of the following given that $\frac{dy}{dx} =$

(a) x^3

(b) $x^3 + 2x^2$

(c) $(x-1)(x-2)$

(d) $\frac{1}{x^2}$

(e) x^5

(f) $\sqrt{x}$

(g) $x^2 + 3x + 4$

(h) $\frac{x^4 + 1}{x^2}$

Compare your answers with the ones at the end of the subunit. If you get less than five items correct, review the examples above. If you get five or more right, proceed to the next section on using notation for indefinite integrals in calculations.

Lesson 5 Notation For Indefinite Integrals

The integral we calculated above is referred to as an indefinite integral. Later we will see another type called a definite integral and we will see how the two differ. For the indefinite integral we use the notation $\int ydx$ if y is a function of x. The notation states, the integral of y with respect to x. The integral sign $\int$ should always be written with dx if the integral is with respect to x. The sign stands for an elongated‘s’ from the sum of. We will see why it is the sum later.

$\frac{dy}{dx}$	**Integral**	**Indefinite Integral Notation**
(a) x^3	$y=\frac{x^{3+1}}{4}=\frac{x^4}{4}+c$ $y=\frac{1}{4}x^4+c$	$\int xdx=\frac{1}{4}x^4+c$
(b) x^3+2x^2	$y=\frac{x^4}{4}+\frac{2x^5}{5}+c$ $y=\frac{1}{4}x^4+\frac{2}{5}x^5+c$	$\int xdx=\frac{1}{4}x^4+\frac{2}{5}x^5+c$
(e) x^5	$y=\frac{x^6}{6}+c$ $y=\frac{1}{6}x^6+c$	$\int xdx=\frac{1}{6}x^6+c$
(f) $\sqrt{x}=x^{\frac{1}{2}}$	$y=\frac{x^{1\frac{1}{2}}}{1\frac{1}{2}}+c$ $y=\frac{2}{3}x^{\frac{3}{2}}+c$	$\int xdx=\frac{2}{3}x^{\frac{3}{2}}+c$

Key Points to Remember

The key points to remember in this subunit on notation for indefinite integrals are:

- The notation for an indefinite integral is $\int ydx$ if y is a function of x. The notation means the integral of y with respect to x.
- The integral sign $\int$ should always be written with dx if the integral is with respect to x.

- The sign $\int$ stands for an elongated' meaning 'the sum of '.

We will see why it is the sum later.

Activity 8

1. Integrate (use the integral sign)

   (a) $y = x^2 + 3x + 4$

   (b) $y = x^2 - 4x$

   (c) $y = \dfrac{1+x}{x^3}$

   (d) $\dfrac{a}{x^2} + b$

   (e) $x(x+1)$

   (f) $x^2(x+2)$

   2. The gradient of a curve which passes through the (2,1) is given by $1 + 2x = 3x^2$

Compare your answers with the ones at the end of the subunit. If you get less than five items correct, review the examples above. If you get five or more right, proceed to the next section on the definite integral as the area under graph.

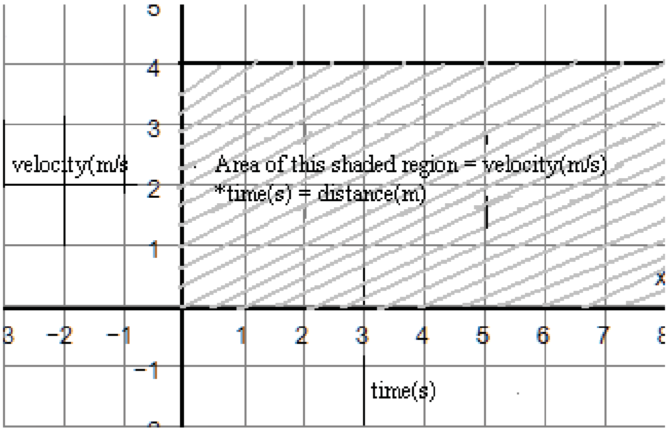
Lesson 6 The Definite Integral as the Area Under Graph

The area under a velocity-time and acceleration-time graphs

The area under the graph means the area between the graph and the horizontal axis which we usually referred to as the x-axis. If the graph is the rate graph, (e.g velocity, the rate of change of distance in a given direction, acceleration the rate of change of velocity) the meaning of the area under the graph can be determined by multiplying the units on the horizontal axis by those on the vertical axis.

Look at the following velocity-time graph and acceleration-time graph to illustrate this. In the velocity-time graph the shaded region is the area under the graph of the velocity of an object over the time 8 seconds. The area can therefore be found by multiplying the units on the horizontal axis by those on the vertical axis.

Area under the velocity-time graph



Because all boundaries of the area are straight lines, it is easy for us to calculate the area under the graph. To get the area we use geometrical rules for getting the shape the area forms. In both examples the shape is a rectangle where we multiply length by breath. Sometimes the area can be divided into geometrical shapes such as triangles, rectangles, trapeziums to mention a few.

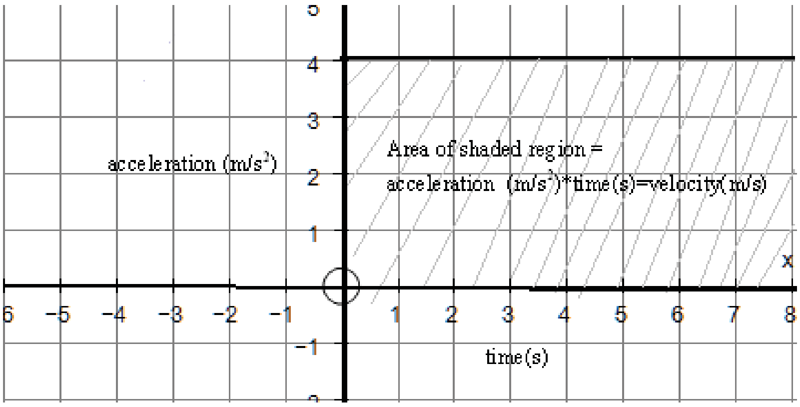
The area under the graph of velocity time graph is the distance travelled by an object. The following shows what the above shaded area represents.

$$area = l \times b = velocity(m/s) \times time(s) = dis\tan ce(m)$$

$$ie.(\frac{m}{s} \times \frac{s}{1} = m)$$

The units multiplied above show us that when we multiply velocity by time we get distance.

Area under the acceleration-time graph



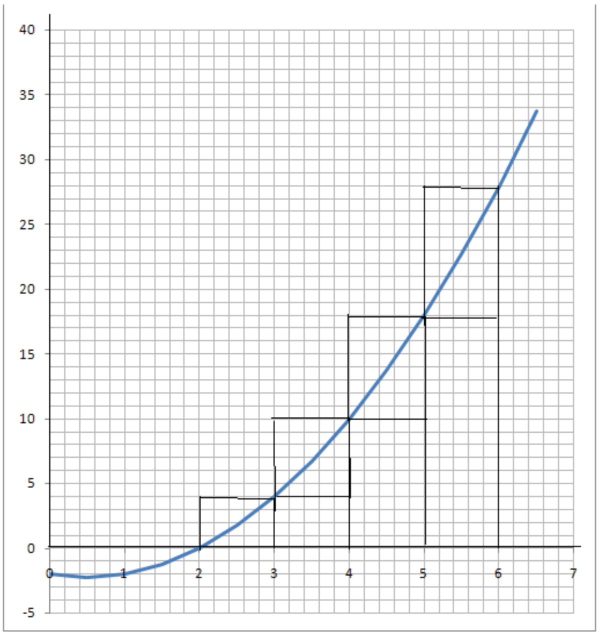
Similarly when we multiply acceleration by time to find the area under the graph of the function, we get velocity as shown by the units of our result.

$$acceleration(m/s^2) \times time(s) = velocity$$

Area under graph= $\frac{m}{s^2} \times \frac{s}{1} = \frac{m}{s}$

We have just illustrated that the calculation of area under the graph is easy when the boundaries of our area are just straight lines as in the above examples.

When we have a curve the area is not that straight forward. To find the area under the curve, the required area is divided into rectangles as shown below. Calculation of the area of each of the rectangles is carried out. The areas are then added. When the rectangles are large such calculation gives a very rough estimation of the area. Look at the following diagrams to see the illustration of this.



When the width Δx of each rectangle is reduced, the accuracy of the area increases. When it is increased such that Δx approaches 0 a limit is reached that gives the correct value of the area under the curve. The value of area found in this way is the definite integral of the function of the graph that is done in mechanical ways. We will not get into details of adding the rectangles to get the area and connecting it to the mechanical ways to calculate the definite integral. We will go straight into the use a definite integral to calculate the area under the curve.

The Difference Between the Definite and an Indefinite Integral

Indefinite integral	**definite integral**
$\int f(x)dx$ Integration process the same as for the definite integral in an indefinite integral there are no limits on the integral sign meaning that there is no specific value. The result is a family of functions e.g. $\int x^3 dx = \frac{x^4}{4} + c$	$\int_a^b f(x)dx$ Integration process the same as for the indefinite integral In a definite integral the integration symbol has numbers or letters alongside the integral sign as shown above. a and b are the upper and lower limits of integration.

When we differentiate $\frac{x^4}{4}$ we would get the same answer for the following $\frac{x^4}{4}+3$ $\frac{x^4}{4}-1$ $\frac{x^4}{4}+3$ $\frac{x^4}{4}+4$ $\frac{x^4}{4}-5$ This is why when we integrate we add a constant.	The limits are substituted into the integral after integration. e.g $\int_1^3 x^3 dx$ $\frac{x^4}{4}$ When substituting 3 into the integral to get $\frac{3^4}{4}=\frac{81}{4}$ When substituting 1 into the integral we get $\frac{1^4}{4}=\frac{1}{4}$ Our result is $\frac{81}{4}-\frac{1}{4}=\frac{80}{4}=20$ Area under the curve is calculated in this way. The definite integral is the are under the graph.

Using the Definite Integral to Find the Area Under a Curve

Example 1

Evaluate $\int_0^2 3x^2 dx$

Here we are to calculate the area under the curve, such that it is between the curve, the x-axis and the lines x=0 and x=2

This $\int_0^2 3x^2 dx$ implies that slope or $\frac{dy}{dx}=3x^2$

$$\int_0^2 3x^2 dx$$

$$\frac{3x^3}{3} = x^3$$

As discussed below, the constant is not relevant to definite integrals.

So the function with a derivative $3x^2$, is $y = x^3$

So we evaluate the definite integral by substituting the boundaries into the function and subtract the value of the lower boundary from that of the upper boundary as shown below:

$$=(2)^3 - (1)^3 = 8 - 1 = 7$$

7 is the area under the curve and a definite integral.

Note that the constants in the integral will always cancel each other when we subtract

Example 2

Evaluate

$$\int_0^2 (2x - x^2)dx$$

This question requires us to:

1) Find the integral and then write the upper and lower limits with square brackets, as follows:

$$\left[x^2 - \frac{x^3}{3}\right]_0^2$$

The upper and lower limits are written like this to mean they will be substituted into the expression in brackets.

2) Next, substitute 2 (the upper limit) into the integral:

$$\left[2^2 - \frac{2^3}{3} = 4 - \frac{8}{3} = 4 - 2\frac{2}{3} = 1\frac{1}{3}\right]$$

Then substitute 0 into the integral:

$$\left[0^2 - \frac{0^2}{3}\right] = 0$$

4) Subtract the result of (3) from the result of (2) for our final answer:

$$1\frac{1}{3} - 0 = 1\frac{1}{3}$$

Normally, we would write our complete solution as follows:

$$\int_0^2 \left(2x - x^2\right)dx$$

$$\left[x^2 - \frac{x^3}{3}\right]_0^2$$

$$\left[2^2 - \frac{2^3}{3}\right] - \left[0^2 - \frac{0^2}{3}\right]$$

$$= 4 - \frac{8}{3} - 0 = \frac{4}{3} = 1\frac{1}{3}$$

The shaded region below represents the required area

Key Points to Remember:

The key points to remember in this subunit on the definite integral as the area under graph are:

- The area under the graph means the area between the graph and the horizontal axis which we usually referred to as the x-axis.
- The area under the graph of velocity time graph is the distance travelled by an object

  $$area = l \times b = velocity(m/s) \times time(s) = dis\tan ce(m)$$

- $$ie.(\frac{m}{s} \times \frac{s}{1} = m)$$
- when we multiply acceleration by time to find the area under the graph of the function of we are calculating velocity

$$acceleration(m/s^2) \times time(s) = velocity$$

- Area under graph= $\frac{m}{s^2} \times \frac{s}{1} = \frac{m}{s}$
- To find the area under the curve, the area is divided into rectangles.
- When the number of rectangles is increased such that the width of each (Δx **)** approaches 0 the area under the curve is a limit that gives the area. That limit gives the correct value of the area under the curve.
- The definite integral sign has the upper and lower boundaries.
- The definite integral of a function is equal to the area under the graph of that function.
- A constant c must be added when finding an indefinite integral.

Activity 9

1. Find the area under the curve $y = x + x^2$ between $x = 1$ and $x = 3$
2. Find the area under the curve $y = 1/x^2$ between $x = 1$ and $x = 4$
3. Find the area under the curve $y = 2x + x^2$ between $x = 1$ and $x = 3$

Compare your answers with the ones at the end of the subunit. if you get all items correct, review the examples above. If you get all the questions right, proceed to the next section on the application of integration to distance, velocity and acceleration.

Lesson 7 Application of Integration to Distance, Velocity and Acceleration

If we know the distance moved along a straight line by a point in time t, the velocity is found by calculating the gradient of the distance- time graph or by differentiating the distance with respect of the time. Inversely, if we know the velocity in terms of the time, the distance may be found by integration because distance is represented by the area under the graph of velocity-time graph. Similarly, if we know the velocity in terms of the time, acceleration is found by calculating the gradient of a velocity-time graph or by differentiating the acceleration with respect of time. Inversely if we know acceleration in terms of time, velocity may be found by the integration.

In considering the relationship between the derivative and the indefinite integral as inverse operations, note that:

- the *derivative of a distance* function represents *velocity.*
- the *indefinite integral of the velocity* represents the *distance function.*
- the *derivative of the velocity* function represents *acceleration* at a particular time.
- the *indefinite integral of the acceleration* function represents the *velocity function.*

Read more: http://www.cliffsnotes.com/study_guide/Distance-Velocity-and-Acceleration.topicArticleId-39909,articleId-39902.html#ixzz1DRrcYSg3

Example 1

If $a = t^2$ find v given $v = 2$ when $t = 0$

Answer to example 1

a = t^2

$$a = t^2$$

$$\frac{dv}{dt} = t^2$$

Integrating

$$\frac{dv}{dt} = t^2$$

Integrating, v = $t^3/3$ +c

Given that v = 2 when t-0

v = $t^3/3$+c

2=$0^3/3$+c

2=c

v=$t^3/3$+2

Example 2

Find s given v = 3t + 1 and that s = 0 when t = 0

ds/dt = 3t+1

s = $3t^2/2$+t+c

s = 0 when t = 0

0 = $3(0)^2/2$+0+c

0 = c

s = $3t^2/2$+t

Example 3

If v = t^2 +t+1, find s given that s = 0 when t = 0

Ds/dt = t^2+t+t

s = $t^3/3+t^2/2$+t+c

0 = $(0)^3/3+(0)^2/2$ +0+c

0 = c

S = $t^3/3+t^2/2$+t

Example 4

The acceleration of a point moving in a straight line is given by $a = 3t + 4$.
Find the formulae for the velocity and distance, given that $s = 0$ and
$v = 8$ when $t = 0$

Answers to example 4

$a = 3t + 4$

i.e.

$$\frac{dv}{dt} = 3t + 4$$

When we integrate the function $a = 3t + 4$ we get velocity

Integrating $a = 3t + 4$ we get $\frac{3t^2}{2} + 4t + c$. This means that

$$v = \frac{3t^2}{2} + 4t + c$$

When we integrate $v = \frac{3t^2}{2} + 4t + c$ we get distance

$$v = \frac{3t^2}{2} + 4t + 8$$

Integrating $\frac{3t^2}{2} + 4t + c$ we get $\frac{t^3}{2} + 2t^2 + 8t$

$$dis\tan ce = \frac{t^3}{2} + 2t^2 + 8t$$

Key Points to Remember

The key points to remember in this subunit on Notation For Indefinite Integrals are:

- the *derivative of a distance* function represents *velocity.*
- the *indefinite integral of the velocity* represents the *distance function.*
- the *derivative of the velocity* function represents *acceleration* at a particular time.
- the *indefinite integral of the acceleration* function represents the *velocity function.*

- Area under graph of acceleration-time
  $= l \times b = velocity(m/s) \times time(s) = dis\tan ce(m)$
  graph $ie.(\frac{m}{s} \times \frac{s}{1} = m)$
- Area under graph of acceleration-time
  $acceleration(m/s^2) \times time(s) = velocity$
  graph= $\frac{m}{s^2} \times \frac{s}{1} = \frac{m}{s}$
- Integrating to get a definite and an infinite integral is done in the same way.
- The definite integral is used to find the area under a curve. The area is found by substituting the limits into the integral and subtracting the result of the lower limit from the upper limit.
- Steps in integrating to get a definite integral e.g. $\int_1^3 x^3 dx$ we (a) (a)
  calculate the integral of x^3 which is $\frac{x^4}{4}$

  (b) Substitute the upper limit 3 into the integral to get $\frac{3^4}{4} = \frac{81}{4}$

  (c)Substitute the lower limit 1 into the integral to get $\frac{1^4}{4} = \frac{1}{4}$
- Area under the curve is = $\frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20$

Activity 10

Calculate the following areas

Calculate	Answers
(a) $\int_1^3 x + x^2 dx$	
(b) $\int_0^4 \sqrt{x} dx$	
(c) $\int_1^4 \frac{1}{x^2} dx$	
(d) $\int_1^2 x^2 + 4dx$	
(e) $\int_1^3 2x + x^2 dx$	

Compare your answers with the ones at the end of the subunit. if you get three items correct, review the examples above. if you get three or more right, proceed to the next section on the summary and assignment.

Unit Summary

Summary

Differential Calculus

In the subunit on differentiation, we learned about how to find the slope or gradient of a curve by the process called differentiation.

- When we differentiate a function we get it derivative which is the gradient of the graph of that function.
- Differentiation is used to find the gradients of curves.
- To differentiate a function from the first principles the following steps are carried out:

e.g. to find the Derivative of $y = ax^2 + bx + c$				
x_1	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$
x	$ax^2 + bx +$	x+h	$x(x+h)^2 + b(x+$	$\frac{a(x+h)^2 + b(x+h) + c -}{x+h-x}$ $\frac{a(x^2 + 2xh + h^2) + bx + l}{h}$ $= \frac{2xh + ah^2 + bh}{h}$ $2x + ah + b$ $h \to 0$ $derivative = 2x + b$

- To differentiate a function using a shorter method is done as follows:

$$y = ax^2 + bx + c$$

$$\frac{dy}{dx} = 2 \times a \times x^{2-1} + 1 \times b \times x^{1-1}$$

$$= 2ax + b$$

- When the derivative is positive the tangent is sloping up, but when it slopes down the value of the derivative is negative. Where the derivative is = 0, then the tangent is horizontal to the axis. When applied to rate of change, the magnitude or size of the derivative tells us the magnitude of the rate of change.
- Maximum or minimum are turning points of a curve.
- At the point of maximum or minimum the gradient is zero.
- Starting from the **left** of a **minimum**, the gradient changes from ' **-** ' to ' + 'with increasing 'x'.
- Starting from the **left** of a **maximum** the gradient changes from ' + ' to ' **-** 'with increasing 'x'.
- The steps taken to find the turning point of a curve are.

(d) Find the derivative of a function.

(e) Equate the derivative to zero.

(f) Substitute the value of x found in (b) into the original function to get the value of y.

- To determine whether the point is a maximum or a minimum we carry out the following steps:

Investigate the sign of the derivative on both sides of the turning point. The possibilities are shown in the table below

The x-coordinate just less than that at the turning point	The x-coordinate at the turning point	The x-coordinate just greater than that at the turning point	Nature of th turning poir
+	0	-	Maximum
-	0	+	Minimum

- The gradient of a distance-time graph is equal to the velocity of an object moving in a given direction. The derivative of the distance-time graph is equal to the velocity.

- The gradient of a velocity-time graph is equal to the acceleration of a moving object**.** The derivative of the velocity-time graph is equal to the velocity.

Integral Calculus The subunit is about integration which is the reverse process of differentiation. In this unit you learned the following:

(a) The difference between differentiation and integration.

(b) How to integrate a function to get an indefinite integral.

(c) The difference between a definite and an indefinite integral.

(d) To calculate the area under a curve by calculating a definite integral.

(e) To calculate distance, velocity and acceleration using integration.

The difference between differentiation and integration

The process of integration is the reverse of that of differentiation. Subtracting one from the exponent in differentiation is reversed by adding one to the exponent in integration. Multiplying the coefficient of x by the exponent in differentiation is reversed by dividing by the resulting exponent. When we integrate a function (a derivative) we get the function (an integral)that we differentiated.

How to integrate a function

- To integrate a function the following steps are taken:

Step 1

Add one to the exponent of x.

Step 2

Divide the term by the resulting exponent after addition of one.

The integral of e.g. x^3 belongs to the family $x^4/4+c$ where the integral can be $\frac{x^4}{4}$ or $\frac{x^4}{4}+1$ or $\frac{x^4}{4}+5$ or $\frac{x^4}{4}+8$ etc. This means that there is a constant that needs to be added to what we get as an integral. We should always add a constant c to our integral. ‘c’ may be any number. So to all our answers a constant c should be added.

The difference between a definite and an indefinite integral

Indefinite integral	**definite integral**
$\int f(x)dx$ Integration process the same as for the definite integral in an indefinite integral there are no limits on the integral sign meaning that there is no specific value. The result is a family of functions e.g. $\int x^3 dx = \frac{x^4}{4} + c$ When we differentiate $\frac{x^4}{4}$ we would get the same answer for the following $\frac{x^4}{4} + 3$ $\frac{x^4}{4} - 1$ $\frac{x^4}{4} + 3$ $\frac{x^4}{4} + 4$ $\frac{x^4}{4} - 5$ This is why when we integrate we add a constant.	$\int f(x)dx$ Integration process the same for the definite integral in an indefinite integral there no limits on the integral sign meaning that there is no spec value. The result is a family functions e.g. $\int x^3 dx = \frac{x^4}{4} +$ When we differentiate $\frac{x^4}{4}$ w would get the same answer f following $\frac{x^4}{4} + 3$ $\frac{x^4}{4} - 1$ $\frac{x^4}{4} + 3$ $\frac{x^4}{4} + 4$ $\frac{x^4}{4} - 5$ This is why when we integra add a constant.

- The notation for an indefinite integral is $\int ydx$ if y is a function of x. The notation means the integral of y with respect to x.
- The integral sign $\int$ should always be written with dx if the

integral is with respect to x.

- The sign $\int$ stands for an elongated ‘s’ meaning ‘the sum of ‘.
- The area under the graph of velocity time graph is the distance travelled by an object.

  $area = l \times b = velocity(m/s) \times time(s) = dis\tan ce(m)$

  $ie.(\frac{m}{s} \times \frac{s}{1} = m)$
- when we multiply acceleration by time to find the area under the graph of the function of we are calculating velocity.

  $acceleration(m/s^2) \times time(s) = velocity$

  Area under graph= $\frac{m}{s^2} \times \frac{s}{1} = \frac{m}{s}$
- The definite integral sign has the upper and lower boundaries.
- The definite integral of a function is equal to the area under the graph of that function.
- Area under graph of acceleration-time

  $acceleration(m/s^2) \times time(s) = velocity$

  graph= $\frac{m}{s^2} \times \frac{s}{1} = \frac{m}{s}$

To Calculate the Area Under a Curve by Calculating a Definite Integral

The area under the graph means the area between the graph and the horizontal axis which we usually referred to as the x-axis. The definite integral is used to find the area under a curve. The area is found by substituting the limits into the integral and subtracting the result of the lower limit from the upper limit.

e.g.

To find the area under the curve $y = x^3$ between x = 1 and x = 3

Step 1

Integrate the function

$$\int_1^3 x^3 dx = \frac{x^4}{4}$$

Note the following

- Where there are more than 1terms we integrate each separately

- When integrating a constant .e. g. 3

$$\int 3dx = \frac{3x^{0+1}}{0+1} = 3x$$

Step 2

Substitute the upper limit 3 into the integral to get $\frac{3^4}{4} = \frac{81}{4}$

Step 3

Substitute the lower limit 1 into the integral to get $\frac{1^4}{4} = \frac{1}{4}$

Step 4

Subtract the result of lower limit from that of an upper limit to get the area under the curve

Area under the curve is = $\frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20$

To Calculate Distance, Velocity and Acceleration Using Integration

- When you are given the function that represent the formula for velocity, we integrate the function to get the formula for distance

e.g. Find the formula for s the distance travelled given the formula for velocity as v = 3t + 1 and that s = 0 when t = 0

ds/dt = 3t+1

$s = 3t^2/2+t+c$

s = 0 when t = 0

$0 = 3(0)^2/2+0+c$

0 = c

$s = 3t^2/2+t$

- Similarly when you are given the function that represent the formula for acceleration, we integrate the function to get the formula for velocity

e.g. Find the formula for v the velocity of a moving object given the formula for acceleration and that v = 0 when t = 0

$a = 3t + 4$

i.e.

$$\frac{dv}{dt} = 3t + 4$$

When we integrate the function $a = 3t + 4$ we get velocity

Integrating $a = 3t + 4$ we get $\frac{3t^2}{2} + 4t + c$. This means that

$$v = \frac{3t^2}{2} + 4t + c$$

v = 0 when c = 0

$$v = \frac{3t^2}{2} + 4t$$

In considering the relationship between the derivative and the indefinite integral as inverse operations, note that:

- the *derivative of a distance* function represents *velocity.*
- the *indefinite integral of the velocity* represents the *distance function.*
- the *derivative of the velocity* function represents *acceleration* at a particular time.
- the *indefinite integral of the acceleration* function represents the *velocity function.*

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Assignment

Assignment

Time: $4hrs.$

Marks:132

Instructions:

Answer all questions

1. Calculate the derivatives of the following functions (from the first principles)

(a) $y = x^2 - 6x + 10$

[6]

(b)

$y = -x^2 + 6x - 6$

[6]

(c) $y = x^2 - 4x - 3$

[6]

2. Calculate the derivatives of the following functions Using a shorter method

(a)

$y = x^8 - x^4$

[6]

(b)

$y = 3 - 4x - 5x^2 + 6x^3$

[6]

(c)

$y = 4x^{\frac{1}{2}} - \frac{5}{x}$

[6]

(d)

$$y = t^{\frac{1}{3}} + 2t^{\frac{1}{4}} + 3t^{\frac{1}{5}}$$

[6]

3.

(a)

$$y = x^8 - x^4$$

[4]

(b)

$$y = 3 - 4x - 5x^2 + 6x^3$$

$$\frac{dy}{dx} = -4 - 10x + 18x^2$$

[4]

(c)

$$y = 4x^{\frac{1}{2}} - \frac{5}{x}$$

$$\frac{dy}{dx} = \frac{2x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{5}{x^{1-1}} = \frac{-2 \times 2}{x^{\frac{1}{2}}} - 5 = -\frac{4}{x^{\frac{1}{2}}}$$

[7]

(d)

$$y = t^{\frac{1}{3}} + 2t^{\frac{1}{4}} + 3t^{\frac{1}{5}}$$

$$\frac{dy}{dx} = \frac{1}{3}t^{-\frac{2}{3}} + \frac{1}{2}t^{-\frac{3}{4}} + \frac{3}{5}t^{-\frac{4}{5}}$$

[6]

4. Evaluate the following definite integral

(a) $y = x^4$, x=0 and x=1

[4]

(b) $\int_{-1}^{2}(x^2 - 3x + 2)dx$

[10]

(c) $\int_{-2}^{2}\left(\frac{5}{x^2+1}\right)dx$

[9]

(a) $\int_{-1}^{2}(x^5+x^3)dx$

[6]

5. Find the area bounded by the curve $y = 3x^2 - 2x$ and the line x = 0 and x = 4

[7]

6. If $v = t^2 + 1$, find the formulae for s, given that x=0 when t=0

[5]

7. If $a = t + 1$, find the formulae for v, given that v=4 when t=0

[5]

8. If $v = 2t^2 + 3t + 1$, find the formulae for s, given that s=0 when t=0

[6]

9. The velocity of a moving body is given by $v = 3t^2$, (a) what is the distance travelled? (b) What is the distance travelled between

t = 0 and t = 2?

[7]

10. The acceleration of a car is given by $a = 6t$, and v=2 when t=0, s=1 when t=0. Find the formulae for s the distance travelled.

[10]

Compare your answers with the ones at the end of the subunit. If you get 80% of the items correct, review the key points of each subunit. If you get 80% or more right, proceed to the next section on the assessment.

Assessment

Assessment

Time: $3hrs.$

Marks: 175

Instructions: Answer all questions

1. Differentiate the following functions from the first principles

(a)

$y = x^2 - 4x + 3$

[10]

(b)

$y = 2x^2 - 2x + 1$

[10]

2. Differentiate the following functions using a rules for differentiation.

(a) $y = 3y^4 + 2y^2 - 5y + 7$

[4]

(b)

$y = (y-3)(2y+1)$

[10]

(c)

$t^{13} - 15t + 3$

[4]

(d)

$y = z^4 - z^{-4}$

[5]

3. Find $\frac{dy}{dx}$ if $y = x^3 - 4x^2 + 6x + 5$ is a function.

[6]

4. Find the gradient at a given point of the following.

(a) $y = x^2 - 2x + 3$ at (1,2)

[7]

(b) $y = -2x^2 + x - 3$ at (0,-3)

[7]

(c) $y = x^2 - x + 1$ at (1,1)

[7]

5. Determine the turning points of graphs of the following functions:

(a) $y = x^3 - 3x + 2$

[20]

(b)

$y = 3x^2 - 5x + 6$

[17]

6. The derivative of a function and one point on its graph are given. Find the function

(a) $\frac{dy}{dx} = x^3 + x^2 - 3 \quad (1,5)$

[8]

(b) $\frac{dy}{dx} = 2x(x+1) \quad (2,0)$

[8]

7. Find the given indefinite integrals.

(a) $\int 5x\,dx$

[2]

(b) $\int x^2\,dx$

[2]

(c) $\int \sqrt{x}\,dx$

[4]

(d)

$\int (1 + t^2 + t^4 + t^6)$

[7]

(e)

$$\int \left(\frac{x^3}{3} - \frac{x^2}{2} + x - 1 \right) dx$$

[10]

8. Find the area of a region enclosed by the x-axis, the curve $y = x^2 + x + 1$ and x=1 and x=2

[10]

9. A point moves is a straight line so that after t seconds it velocity is $\left(3t^2 + 4t)\right)m/s$. Find the
gone in the third second.

[7]

10. A point moves in a straight line and its acceleration in m/s ^{2} is given by $a = t + 2$ where t is
seconds. If the point starts from rest, find the distance in 4s.

[10]

7. A point moves is a straight line so that after t seconds it velocity is $\left(3t^2+4t\right)m/s$. Fir distance gone in the third second

$$\int_0^3 (3t^2+4t)dt = \frac{3t^{2+1}}{3}+\frac{4t^{1+1}}{2}=\frac{3t^3}{3}+\frac{4t^2}{2}$$

$$d=\left[\frac{3(3)^3}{3}+\frac{4(3)^2}{2}\right]-\left[\frac{3(0)^3}{3}+\frac{4(0)^2}{2}\right]$$

$$=\left(\frac{81}{3}+\frac{36}{2}\right)-\left(\frac{0}{3}+\frac{0}{2}\right)$$

$$=\left(\frac{162}{6}+\frac{108}{6}\right)-\left(0+0\right)=\frac{270}{6}=45m$$

8. A point moves in a straight line and its acceleration in m/s^2 is given by $a=t+2$ wher seconds. If the point starts from rest, find the distance in 4s.

$$\int_0^4 (t+2)dt=\frac{t^{1+1}}{2}+\frac{2t^{0+1}}{1}=\frac{t^2}{2}+\frac{2t}{1}$$

$$v=\frac{t^2}{2}+\frac{2t}{1}$$

$$d=\int_0^4 \frac{t^2}{2}+\frac{2t}{1}=\frac{t^{2+1}}{6}+\frac{2t^{1+1}}{2}=\frac{t^3}{6}+t$$

$$d=\left[\frac{4^3}{6}+4\right]-0=\frac{64}{6}+\frac{4}{1}=\frac{64}{6}+\frac{24}{6}=\frac{88}{6}=14.7m$$

Based on your results and the recommendation that you should aim for at least 80% to ensure y overall success in this course and any subsequent math course you take, determine how much y should study the overall unit before you attempt the assessment.

ANSWERS

Answers to activity 1

For two points (x_1, y_1) **and** (x_2, y_2)

Gradient = $\dfrac{y_2 - y_1}{x_2 - x_1}$

x_1	y_1	x_2	y_2	$y_2 - y_1$	$x_2 - x_1$	$\dfrac{y_2 - y}{x_2 - x_1}$
1	4	3	8	8-4	3-1	$\frac{4}{2} = 2$
2	5	4	7	7-5	4-2	$\frac{2}{2} = 1$
1	-1	-2	6	6-(-1)	-2-1	$\frac{7}{-3}$

Answers to Activity 2

Answers to activity 3

(a)

$$y = 2x^2$$

$$\frac{dy}{dx} = 2 \times 2x^{2-1} = 4x$$

(b)

$$y = 3x^2 + x$$

$$\frac{dy}{dx} = 2 \times 3x^{2-1} + 1 \times 1x^{1-1}$$

$$= 6x + x^0$$

$$x^0 = 1$$

$$\frac{dy}{dx} = 6x + x^0 = 6x + 1$$

(c)

$$y = 2x^2 - x$$
$$\frac{dy}{dx} = 2 \times 2x^{2-1} - 1 \times 1x^{1-1}$$
$$= 4x - x^0$$
$$= 4x - 1$$

(d)

$$y = 6x^2 - x - 2$$
$$\frac{dy}{dx} = 2 \times 6x^{2-1} - 1 \times 1x^{1-1}$$
$$= 12x - x^0$$
$$= 12x - 1$$

(e)

$$y = 4x^3 + 3x^2 - 2x + 3$$
$$\frac{dy}{dx} = 3 \times 4x^{3-1} + 2 \times 3x^{2-1}$$
$$= 12x^2 + 6x$$

$$y = 5x^4 - 2x^3 + 4x^2 + 2x - 1$$
$$\frac{dy}{dx} = 4 \times 5x^{4-1} - 3 \times 2x^{3-1} + 2 \times 4x^{2-1} + 1 \times 2x^{1-1}$$

(f) $= 20x^3 - 6x^2 + 8x + 2x^0$

$$= 20x^3 - 6x^2 + 8x + 2 \times 1$$
$$= 20x^3 - 6x^2 + 8x + 2$$

Answers to activity 4

(a)

$$y = 4x^2$$
$$derivative = 4 \times 2x = 8x$$
$$slope = 8 \times 1 = 8$$

(b)

$y = 3x^2 + x$

$derivative = 3 \times 2x + 1x^0 = 6x + 1$

$slope = 6(1) + 1 = 7$

(c)

$y = 4x$

$derivative = 4x^0 = 4$

(d) $y = 6x^2 - 2x$

$derivative = 6 \times 2x - 2 \times x^0 = 12x - 2$

$slope = 12(2) - 2 = 22$

(e)

$y = 2x^2 + 3x + 1$

$derivative = 2 \times 2x + 3x^0 = 4x + 3$

$slope = 4(2) + 3 = 11$

N.B. when the derivative is positive the tangent is sloping up, but when it slopes down the value of the derivative is negative. Where the derivative is = 0, then the tangent is horizontal to the axis. When applied to rate of change, the magnitude or size of the derivative tells us the magnitude of the rate of change.

Answers to activity 5

(a) $y = 3x^2 + x$

Find the turning point of $y = 3x^2 + x$

and determine whether the turning point is a minimum or maximum

Step 1

Find the derivative of the function

$y = 3x^2 + x$

$derivative = 6x + 1$

Step 2

Equate $6x + 1$ (the derivative) to zero and find the value of x

$6x + 1 = 0,$

When $x = -\frac{1}{6}$

Step 3

Substitute $x = \frac{1}{6}$ into the original function $y = 3x^2 + x$ to get the value of y

$$y = 3\left(\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)$$

$$y = \frac{1}{4}$$

Turning point is $\left(-\frac{1}{6}, \frac{1}{4}\right)$

Step 4

When x< $-\frac{1}{6}$, e.g. x = -1 $6x + 1 = -5$ a negative gradient

Step 5

When x> $-\frac{1}{6}$ e.g. x = 0 $6x + 1 = +1$ a positive gradient

Therefore the turning point $\left(-\frac{1}{6}, \frac{1}{4}\right)$ is a minimum

(b)

Find the turning point of $y = 6x^2 - 2x$

and determine whether the turning point is a minimum or maximum

Step 1

Find the derivative of the function

$$y = 6x^2 - 2x$$

$$derivative = 12x - 2$$

Step 2

Equate $12x-2$ (the derivative) to zero and find the value of x

$12x-2=0$

$12x=2$

$x=\frac{1}{6}$

Step 3

Substitute $x=\frac{1}{6}$ into the original function $y=6x^2-2x$ to get the value of y

$y=6x^2-2x$

$y=6\left(\frac{1}{6}\right)^2-2\times\frac{1}{6}$

$y=-\frac{1}{6}$

The turning point is $\left(\frac{1}{6},-\frac{1}{6}\right)$

Step 4

When x< $\frac{1}{6}$ e.g. x = 0,12x – 2 = -2 a negative gradient

Step 5

When x > $\frac{1}{6}$ e.g. x = 1 12x-2 = +10 a positive gradient

The turning point is $\left(\frac{1}{6},-\frac{1}{6}\right)$ is a minimum

(c)

Find the turning point of $y=2x^2+3x+1$

and determine whether the turning point is a minimum or maximum

Step 1

Find the derivative of the function

$$\frac{dy}{dx} = 2 \times 2x^{2-1} + 1 \times 3x^{1-1}$$

$$= 4x + 3$$

Step 2

Equate $4x + 3 = 0$ (the derivative) to zero and find the value of the x-coordinate

$$4x + 3 = 0$$

when

$$x = -\frac{3}{4}$$

Step 3

Substitute $x = -\frac{3}{4}$ into the original function $y = 2x^2 + 3x + 1$ to get the value of y

$$y = 2x^2 + 3x + 1$$

$$y = 2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) + 1$$

$$y = -\frac{1}{8}$$

The turning point is $\left(-\frac{3}{4}, -\frac{1}{8}\right)$

Step 4

When $x < -\frac{3}{4}$ e.g. x= -1 then $4x + 3 = -1$ a negative gradient

Step 5

When $x > -\frac{3}{4}$ e.g. x = 0 then $4x + 3 = +3$ a positive gradient

The turning point is $\left(-\frac{3}{4}, -\frac{1}{8}\right)$ a minimum

(d)

Find the turning point of $y = x^2 - 8x$

and determine whether the turning point is a minimum or maximum

Step 1

Find the derivative of the function

$$\frac{dy}{dx} = 2\times 1x^{2-1} - 1\times 8x^{1-1}$$
$$= 2x - 8$$

Step 2

Equate $2x - 8 = 0$ (the derivative) to zero and find the value of the x-coordinate

$$2x - 8 = 0$$
$$2x = 8$$
$$x = 4$$

Step 3

Substitute x = 4 into the original function to get the value of y at the turning point

$$= x^2 - 8x$$
$$y = (4)^2 - 8(4)$$
$$y = 16 - 32 = 16$$

Coordinates of the turning point are (4,16)

Step 4

When x<4 e.g. x = 3

$2x - 8 = 2(3) - 8 = -2$ negative

Step 5

At x>4 e.g. x=5

$2x - 8 = 2(5) - 8 = +2$ positive

The point (4,16) is a minimum

(e)

Find the turning point of $y = x^2 - 4x - 1$

and determine whether the turning point is

Step 1

$$\frac{dy}{dx} = 2\times 1\times x^{2-1} - 1\times 4\times x^{1-1}$$
$$= 2x - 4$$

Step 2

Equate 2x-4 (the derivate) to zero

$2x - 4 = 0$

$2x = 4$

$x = 2$

Step 3

Substitute x=2 into the original function $y = x^2 - 4x - 1$

to get y

$y = x^2 - 4x - 1$

$y = 2^2 - 4(2) - 1$

$y = 4 - 8 - 1$

$y = -5$

The turning point is (2, -5)

Step 4

When x,2 e.g. x=1

$2x - 4 = 2(1) - 4 = -2$ negative

When x>2 e.g. x=3

$2x - 4 = 2(3) - 4 = -2$ positive

The point (2,-5) is a minimum

(f)

Find the turning point of $y = x^2 - 2x - 3$

and determine whether the turning point is

Step 1

$$\frac{dy}{dx} = 1 \times 2x^{2-1} - 1 \times 2x^{1-1}$$

$$= 2x - 2$$

Step 2

$2x - 2 = 0$

$2x = 2$

$x = 1$

Step 3

Substitute x=1 into the original function $y = x^2 - 2x - 3$ to get the value of y at the turning point

$y = x^2 - 2x - 3$

$y = 1^2 - 2(1) - 3 = -4$

The turning point is (1,-4)

Step 4

When x<1 e.g. x=0

$2x - 2 = 2(0) - 2 = -2$

Step 5

When x>1 e.g. x =2

$2x - 2 = 2(2) - 2 = +2$

The turning point (1,-4) is a minimum

(g)

Find the turning point of $y = 1 - 4x - x^2$

and determine whether the turning point is

Step 1

$\frac{dy}{dx} = 1 \times -4x^{1-1} - 2 \times 1x^{2-1}$

$= -4 \times 1 - 2x$

$= -4 - 2x$

Step 2

$-4 - 2x = 0$

$\frac{-2x}{-2} = \frac{4}{-2}$

$x = -2$

Step 3

$y = 1 - 4x - x^2$

$y = 1 - 4(-2) - (-2)^2$

$y = 1 + 8 - 4$

$y = 5$

Step 4

When x<-2 e.g. x = -3

$-4 - 2x = -4 - 2(-3) = 2$ positive

Step 5

When x>-2 e.g. x = -1

$-4 - 2x = -4 - 2(-1) = -2$ negative

The turning point (-2, 5) is a maximum

Answers to activity 6

1 (a)

$s = t^3 + 3t^2 + 4$

$v = \frac{ds}{dt} = 3t^2 + 6t$

$v = 3t^2 + 6t$

$v = 3(3)^2 + 6(3) = 45m/s$

(b) $a = \frac{dv}{dt} = 6t + 6$

$a = 6(3) + 6 = 24m/s^2$

2. $v = 3t^2 + 4t$

$a = \frac{dv}{dt} = 6t + 4$

After 2seconds $a = 6(2) + 4 = 16m/s^2$

3.

$s = 3t + t^2.$

$v = \frac{ds}{dt} = 3 + 2t$

$v = 3 + 2t$

$v = 3 + 2(n) = 3 + 2n$

$\frac{dv}{dt} = 2m/s$

4.

$v = t^2 - t$

$a = \frac{dv}{dt} = 2t - 1$

After 3 seconds $a = 2t - 1 = 2(3) + 1 = 7m/s^2$

$2t - 1 = 0$

$2t = 1$

$t = \frac{1}{2}$

Acceleration is zero after $\frac{1}{2}$ a second.

Answers to activity 7

Derivative or $\frac{dy}{dx}$ or $f'(x)$	**Integral**
(a) x^3	$y = \frac{x^{3+1}}{4} = \frac{x^4}{4} + c$ $y = \frac{1}{4}x^4 + c$
(b) $x^3 + 2x^2$	$y = \frac{x^4}{4} + \frac{2x^5}{5} + c$ $y = \frac{1}{4}x^4 + \frac{2}{5}x^5 + c$

(c) $(x-1)(x-2)=x^2-3x+2$	$y=\frac{x^2}{2}-\frac{3x^2}{2}+2x+c$ $y=\frac{1}{2}x^2-\frac{3}{2}x^2+2x+c$
(d) $\frac{1}{x^2}=x^{-2}$	$y=\frac{x^{-1}}{-1}+c=-\frac{1}{x}+c$ $y=-\frac{1}{x}+c$
(e) x^5	$y=\frac{x^6}{6}+c$ $y=\frac{1}{6}x^6+c$
(f) $\sqrt{x}=x^{\frac{1}{2}}$	$y=\frac{x^{1\frac{1}{2}}}{1\frac{1}{2}}+c$ $y=\frac{2}{3}x^{\frac{3}{2}}+c$
(g) x^2+3x+4	$y=\frac{x^3}{3}+\frac{3x^2}{2}+4x+c$ $y=\frac{1}{3}x^3+\frac{3}{2}x^2+4x+c$
(h) $\frac{x^4+1}{x^2}$	$y=\frac{\frac{x^5}{5}+x}{\frac{x^3}{3}}+c$ $y=\frac{3}{x^3}\left(\frac{x^5}{5}+x\right)+c$ $y=\frac{3}{5}x^2+3\frac{1}{x^2}+x+c$
(i) 1	$y=x+c$

(j) x	$y=\frac{x^2}{2}+c$
(k) x^n	$y=\frac{x^{n+1}}{n+1}+c$

$y=2x^2$				
x_1	x_2	y_1	y_2	$\frac{y_2-y_1}{x_2-x_1}$
x	$x+h$	$2x^2$	$2(x+h)^2$	$\frac{2(x+h)^2-2x^2}{x+h-x}$ $=\frac{2(x^2+2xh+h^2)-2x^2}{h}$ $=\frac{2x^2+4xh+2h^2-2x^2}{h}$ $=\frac{4xh+2h^2}{h}$ $=4x+2h$ as $h\to 0$ $derivative=4x$

$y=3x^2+x$				
x_1	x_2	y_1	y_2	$\frac{y_2-y_1}{x_2-x_1}$

x	$x+h$	$3x^2+x$	$3(x+h)^2+(x+h)$	$\frac{3(x+h)^2+(x+h)-(3x^2+x)}{x+h-x}$ $=\frac{3(x^2+2xh+h^2)+(x+h)-3x^2-x}{h}$ $=\frac{3x^2+6xh+3h^2+x+h-3x^2-x}{h}$ $=\frac{6xh+3h^2+h}{h}$ $=6x+3h$ as $h\to 0$ $gradient=6x$
$y=2x^2-x$				
x_1	x_2	y_1	y_2	$\frac{y_2-y_1}{x_2-x_1}$
x	$x+h$	$2x^2-x$	$2(x+h)^2-(x+h)$	$\frac{2(x+h)^2-(x+h)-(2x^2-x)}{x+h-x}$ $=\frac{2(x^2+2xh+h^2)-(x+h)-2x^2+x}{h}$ $=\frac{2x^2+4xh+2h^2-x-h-2x^2+x}{h}$ $=\frac{4xh+2h^2-h}{h}$ $=4x+2h$ as $h\to 0$ $gradient=4x$
$y=6x^2-x-2$				
x_1	x_2	y_1	y_2	$\frac{y_2-y_1}{x_2-x_1}$

x	$x+h$	$6x^2-x-2$	$6(x+h)^2-(x+h)-2$	$\frac{6(x+h)^2-(x+h)-2-(6x^2-x-2)}{x+h-x}$ $=\frac{6(x^2+2xh+h^2)-(x+h)-2-6x^2+x+2}{h}$ $=\frac{6x^2+12xh+6h^2-x-h-2-6x^2+x+2}{h}$ $=\frac{12xh+6h^2-h}{h}$ $=12x+6h$ as $h\to 0$ $gradient=12x$
$y=4x^2+3x-2$				
x_1	x_2	y_1	y_2	$\frac{y_2-y_1}{x_2-x_1}$
x	$x+h$	$4x^2+3x-2$	$4(x+h)^2+3(x+h)-2$	$\frac{4(x+h)^2+3(x+h)-2-(4x^2+3x-2)}{x+h-x}$ $=\frac{4(x^2+2xh+h^2)+3(x+h)-2-4x^2-3x+2}{h}$ $=\frac{4x^2+8xh+4h^2+3x+3h-2-4x^2-3x+2}{h}$ $=\frac{8xh+4h^2+3h}{h}$ $=8x+4h+3$ as $h\to 0$ $gradient=8x+3$
$y=5x^2-2x$				

x_1	x_2	y_1	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$
x	$x+h$	$5x^2 - 2x$	$5(x+h)^2 - 2(x+h)$	$\frac{5(x+h)^2 - 2(x+h) - (5x^2 - 2x)}{x+h-x}$ $= \frac{5(x^2 + 2xh + h^2) - 2(x+h) - 5x^2 + 2x}{h}$ $= \frac{5x^2 + 10xh + 5h^2 - 2x - 2h - 5x^2 + 2x}{h}$ $= \frac{10xh + 5h^2 - 2h}{h}$ $= 10x + 5h$ as $h \to 0$ $derivative = 10x$

Answers to activity 8

$\frac{dy}{dx}$	**Integral**	**Indefinite Integral Notation**
$x^2 + 3x + 4$	$x^{2+1}/3 + 3x^{1+1}/2 + 4x^{0+1/1}$	∫($(x^2+3x+4) = x^3/3 + 3x^2/2 + 4x+c$
x^2-4x	$x^{2+1}/3-4x^{1+1}/2 = x^3/3-4x^2/2$	∫ $(x^2-4x) = x^3/3-4x^2/2+c$
$1+x/x^3$	$x^{-3}+x^{-2}=x^{-2}/-2+x^{-1}/-1$	$=-1/x^2+1/2x+c$
=ax⁻ a/x^2+b ^{2}+b		$=ax^{-1}+bx+c$
=x^2+x x(x+1)	$x^3/3+x^2/2$	$x^3/3+x^2/2$
$x^2(x+2)= x^3+2x^2$	x^4+2x^3	x^4+2x^3

Answers to activity 9

Function	**Area under curve of the function**
1. $y = x + x^2$ between x =1 and x = 3	$\int_1^3 \left(x + x^2\right)dx$ $\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^3$ $\left[\frac{3^2}{2} + \frac{3^3}{3}\right] - \left[1^2 + \frac{1^2}{3}\right]$ $= 4\frac{1}{2} + \frac{27}{3} - \left[1 + \frac{1}{3}\right]$ $= \frac{9}{2} + \frac{27}{3} - 1\frac{1}{3}$ $= 12.17$
2.	$y = 2x + x^2$

	$\int_1^4 \left(\frac{1}{x^2}\right)dx = \int_1^4 x^{-2}$ $\left[\frac{x^{-1}}{-1}\right]_1^4$ $\left[\frac{4^{-1}}{-1}\right] - \left[\frac{1^{-1}}{-1}\right]$ $= \frac{1}{4} \times \frac{1}{-1} - (\frac{1}{1} \times \frac{1}{-1}$ $\frac{1}{-4} + \frac{1}{1}$ $\frac{3}{4}$
3.	$\int_1^3 \left(2x + x^2\right)dx$ $\left[x^2 + \frac{x^3}{3}\right]_1^3$ $\left[3^2 + \frac{3^3}{3}\right] - \left[1^2 + \frac{1^2}{3}\right]$ $= 9 + \frac{27}{3} - \left[1 + \frac{1}{3}\right]$ $= \frac{9}{1} + \frac{27}{3} - 1\frac{1}{3}$ $= 16\frac{2}{3}$

Answers to activity 10

Calculate	Answers

(a) $\int_1^3 x+x^2dx$	$\int_1^3 x+x^2dx=\frac{x^2}{2}+\frac{x^3}{3}$ $=\frac{3^2}{2}+\frac{3^3}{3}-\left(\frac{1^2}{2}-\frac{1^3}{3}\right)$ $=4\frac{1}{2}+9-\left(\frac{1}{2}-\frac{1}{3}\right)=13.5-1.6666=13.33$
(b) $\int_0^4 \sqrt{x}dx$	$\int_0^4 \sqrt{x}dx=\int_0^4 x^{\frac{1}{2}}dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}$ $\frac{2}{3}\times 4^{\frac{3}{2}}-\frac{2}{3}=\left(\frac{2}{3}\times 8\right)-\frac{2}{3}=\frac{14}{3}=4.6$
(c) $\int_1^4 x^{-2}dx$	$\int_1^4 x^{-2}=\frac{x^{-1}}{-1}=\frac{4^{-1}}{}-\frac{1^{-1}}{-1}=\frac{3}{4}$
(d) $\int_1^2 x^2+4dx$	$\int_1^2 x^2+4=\frac{x^3}{3}+4x$ $\frac{2^3}{3}+4\times 2-\left(\frac{2^3}{3}+4\times 1\right)$ $=\frac{8}{3}+8-\left(\frac{8}{3}+4\right)$ $=\frac{28}{3}=9.33$
(e) $\int_1^2 2x+x^2dx$	$\int_1^2 2x+x^2dx$ $=\frac{2x^2}{2}+\frac{x^3}{3}$ $=x^2+\frac{x^3}{4}$ $=(1)^2-\frac{1^3}{3}-(2)^2+\frac{2^3}{3}$ $=1-\frac{1}{3}-4+\frac{8}{3}$ $=\frac{7}{3}-3=-\frac{2}{3}$

Answers to Assignment

1.

(a)

x_1	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$
x	$x^2 - 6x + 10$	$x + h$	$(x+h)^2 - 6(x+h) + 1$	

$$\frac{(x+h)^2 - 6(x+h) + 10 - (x^2 - 6x + 10)}{x + h - x}$$

$$= \frac{x^2 + 2xh + h^2 - 6x - 6h + 10 - x^2 + 6x - 10}{h}$$

$$= \frac{2xh + h^2 - 6h}{h}$$

$$= 2x + h - 6$$

(b)

	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$
x x_1	$-x^2 + 6x - 6$	$x + h$	$-(x+h)^2 + 6(x+h) - 6$	

$$\frac{-(x+h)^2 + 6(x+h) - 6 - \left[-(x+h)^2 + 6(x+h) - 6\right]}{x + h - x}$$

$$= \frac{-(x^2 + 2xh + h^2) + 6x + 6h - 6 - (-x^2 + 2xh + h^2) + 6x + 6h - 6}{h}$$

$$= \frac{-x^2 - 2xh - h^2 + 6x + 6h - 6 + x^2 - 2xh - h^2 + 6x + 6h - 6}{h}$$

$$= \frac{-4xh - 2h^2}{h} = 4x - 2h = 4x$$

(c)

$y = x^2 - 4x - 3$

x_1	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$
x	$x^2 - 4x - 3$	$x + h$	$(x+h)^2 - 4(x+h) - 3$	

$= \frac{(x+h)^2 - 4(x+h) - 3 - (x^2 - 4x - 3)}{x + h - x}$

$= \frac{x^2 + 2xh + h^2 - 4x - 4h - 3 - x^2 + 4x + 3}{h}$

$= \frac{2xh + h^2 - 4h}{h}$

$= 2x + h - 4$

as

$h \to 0$

$= 2x - 4$

2.

(a)

$y = x^8 - x^4$

$\frac{dy}{dx} = 8x^7 - 4x^3$

(b)

$y = 3 - 4x - 5x^2 + 6x^3$

$\frac{dy}{dx} = -4 - 10x + 18x^2$

(c)

$y = 4x^{\frac{1}{2}} - \frac{5}{x}$

$\frac{dy}{dx} = \frac{2x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{5}{x^{1-1}} = \frac{-2 \times 2}{x^{\frac{1}{2}}} - 5 = -\frac{4}{x^{\frac{1}{2}}}$

(d)

$$y = t^{\frac{1}{3}} + 2t^{\frac{1}{4}} + 3t^{\frac{1}{5}}$$

$$\frac{dy}{dx} = \frac{1}{3}t^{-\frac{2}{3}} + \frac{1}{2}t^{-\frac{3}{4}} + \frac{3}{5}t^{-\frac{4}{5}}$$

3. Evaluate the following definite integral

(a)

$y = x^4$, x=0 and x=1

$$\int_0^1 (x^4)dx = \left(\frac{x^{4+1}}{5}\right)$$

$$= \frac{x^5}{5}$$

$$= \left[\frac{1^5}{5}\right] - \left[\frac{0^5}{5}\right]$$

$$= \frac{1}{5}$$

$$\int_{-1}^{2} (x^2 - 3x + 2) = \left(\frac{x^{2+1}}{3} - \frac{3x^{1+1}}{2} + \frac{2x^{0+1}}{1}\right)$$

$$= \frac{x^3}{3} - \frac{3x^2}{2} + 2x$$

$$= \left[\frac{2^3}{3} - \frac{3(2)^2}{2} + 2(2)\right] - \left[\frac{-1^3}{3} - \frac{3(-1)^2}{2} + 2(-1)\right]$$

$$= \left(\frac{8}{3} - \frac{12}{2} + 4\right) - \left(-\frac{1}{3} + \frac{3}{2} - 2\right) = \left(\frac{16}{6} - \frac{36}{6} + \frac{24}{6}\right) - \left(-2\frac{1}{3} + \frac{3}{2}\right)$$

$$= \frac{4}{6} + \frac{5}{6}$$

$$= \frac{9}{6} = 1\frac{1}{2}$$

(b)

$$\int_{-2}^{2}\left(\frac{5}{x^2+1}\right)=\frac{\frac{5x^{0+1}}{1}}{\frac{x^{2+1}}{3}+\frac{1x^{0+1}}{1}}=\frac{5x}{1}\times\frac{3}{x^3}\times\frac{1}{x}=\frac{15x}{x^4}=\frac{15}{x^3}$$

$$\left[\frac{15}{2^3}\right]-\left[\frac{15}{-2^3}\right]=\frac{15}{8}-\frac{15}{-8}=\frac{15}{8}+\frac{15}{8}=\frac{30}{8}=3.75$$

(c)

$$\int_0^1(x^5+x^3)dx=\frac{x^{5+1}}{6}+\frac{x^{3+1}}{4}=\frac{x^6}{6}+\frac{x^4}{4}$$

$$=\left[\frac{1^6}{6}+\frac{1^4}{4}\right]-0=\frac{1}{6}+\frac{1}{4}=\frac{4}{24}+\frac{6}{24}=\frac{10}{24}=\frac{5}{12}$$

4. Find the area of the region bounded by the curve $y=3x^2-2x$ and the line x = 0 and x = 4.

$$\int_0^4(3x^2-2x)=\left(\frac{3x^{2+1}}{3}-\frac{2x^{1+1}}{2}\right)$$

$$=\frac{3x^3}{3}-\frac{2x^2}{2}$$

$$=\left[\frac{3(4)^2}{3}-\frac{2\times4^2}{2}\right]-\left[\frac{3(0)^3}{3}-\frac{0^2}{2}\right]$$

$$=\left(\frac{192}{3}-\frac{32}{2}\right)-0$$

$$=64-16=48$$

5. If $v=t^2+1$, find s, given that s=0 when t=0

$$\int(t^2+1)=\frac{t^3}{3}+\frac{t}{1}$$

$$s=\frac{t^3}{3}+t+c$$

When s=0, t=0

$$s = \frac{t^3}{3} + t + c$$

$$0 = \frac{0}{3} + 0 + c$$

$$0 = c$$

$$s = \frac{t^3}{3} + t$$

6. If $a = t + 1$, find v, given that v=4 when t=0

$$\int (t+1)dt = \frac{t^2}{2} + \frac{t}{1}$$

$$v = \frac{t^2}{2} + \frac{t}{1} + c$$

When v=4, t=0

$$v = \frac{t^2}{2} + \frac{t}{1} + c$$

$$4 = \frac{0^2}{2} + \frac{0}{1} + c$$

$$4 = c$$

$$v = \frac{t^2}{2} + t + 4$$

7.If $v = 2t^2 + 3t + 1$, find s, given that s=0 when t=0

$$\int (2t^2 + 3t + 1) = \frac{2t^{2+1}}{3} + \frac{3t^{1+1}}{2} + \frac{t^{0+1}}{1}$$

$$s = \frac{2t^3}{3} + \frac{3t^2}{2} + t + c$$

s=0 when t=0

$$s = \frac{2t^3}{3} + \frac{3t^2}{2} + t + c$$
$$0 = \frac{0}{3} + \frac{0}{2} + 0 + c$$
$$0 = c$$
$$s = \frac{2t^3}{3} + \frac{3t^2}{2} + t$$

8. The velocity of a moving body is given by $v = 3t^2$, (a) what is the formulae of the distance (s) travelled? (b)what is the distance travelled between t = 0 and t = 2

(a)

velocity is the derivative of distance, so distance is the an indefinite integral of velocity

$$s = \int 3t^2 = \frac{3t^3}{3} + c$$
$$= t^3 + c$$
$$s = t^3 + c$$
$$s = \left[t^3 + c\right]_0^2 = (2^3 + c) - (0^3 + c) = 8m$$

9. The acceleration of a car is given by $a = 6t$, and v=2 when t=0, s=1 when t=0. Find the formula for s the distance travelled

Acceleration is the derivative of velocity, so velocity is the an indefinite integral of acceleration

$$v = \int (6t)dt = \frac{6t^2}{2} + c = 3t^2 + c$$
$$v = 3t^2 + c$$

When t = 0, v = 2

$$v = 3t^2 + c$$
$$2 = 3(0)^2 + c$$
$$2 = c$$
$$v = 3t^2 + 2$$

Distance is the an indefinite integral of velocity

$$s = \int (3t^2 + 2)dt = \frac{3t^{2+1}}{3} + \frac{2t}{1} + c = t^3 + 2t + c$$

s = 1 when t = 0

$$s = t^3 + 2t + c$$
$$1 = 0 + 0 + c$$
$$1 = c$$
$$s = t^3 + 2t + 1$$

Answers to Assessment

1.

(a)

$y = x^2 - 4x + 3$

x_1	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$
x	$x^2 - x + 3$	$x + h$	$(x+h)^2 - 4(x+h) + 3$	

$$\frac{(x+h)^2 - 4(x+h) + 3 - (x^2 - 4x + 3}{x + h - x}$$
$$= \frac{x^2 + 2xh + h^2 - 4x - 4h + 3 - x^2 + 4x - 3}{h}$$
$$= \frac{2xh + h^2 - 4h}{h}$$
$$= 2x + h - 4$$
$$= 2x - 4$$

(b)

$y = 2x^2 - 2x + 1$

x_1	y_1	x_2	y_2	$\frac{y_2 - y_1}{x_2 - x_1}$
x	$2x^2 - 2x + 1$	$x + h$	$2(x+h)^2 - 2(x+h) + 1$	

$$\frac{dy}{dx} = \frac{2(x+h)^2 - 2(x+h) + 1 - (2x^2 - 2x + 1)}{x + h - x}$$
$$= \frac{2(x^2 + 2xh + h^2) - 2x - 2h + 1 - 2x^2 + 2x + 1}{h}$$
$$\frac{2x^2 + 4xh + 2h^2 - 2x - 2h + 1 - 2x^2 + 2x + 1}{h}$$
$$= \frac{4xh + 2h^2 - 2h + 2}{h}$$
$$= 4x + 2h - 2 + 2$$
$$= 4x + 2h$$
$$= 4x + 2$$

2.

(a)

$$y = 3y^4 + 2y^2 - 5y + 7$$
$$\frac{dy}{dx} = 12x^3 + 4y - 5$$

(b)

$$y = (y-3)(2y+1)$$
$$y = 2y^2 + y - 6y - 3$$
$$y = 2y^2 - 5y - 3$$
$$\frac{dy}{dx} = 4y - 5$$

(c)

$$y = t^{13} - 15t + 3$$
$$\frac{dy}{dt} = 13t^{12} - 15$$

(d)

$$y = z^4 - z^{-4}$$
$$\frac{dy}{dz} = 4z^3 - -4z^{-3}$$
$$= 4z^3 + 4z^{-3}$$

3.

$$y = x^3 - 4x^2 + 6x + 5$$
$$\frac{dy}{dx} = 3x^2 - 8x + 6$$

4. Find the gradient at a given point of the following

(a) $y = x^2 - 2x + 3$ at (1,2)

$$y = x^2 - 2x + 3$$
$$\frac{dy}{dx} = 2x - 2$$
$$gradient = 2(1) - 2 = 0$$

(b) $y = -2x^2 + x - 3$ at (0,-3)

$$y = -2x^2 + x - 3$$
$$\frac{dy}{dx} = -4x + 1$$
$$gradient = -4(0) + 1 = 1$$

(c) $y = x^2 - x + 1$ at $(1,1)$

$$y = x^2 - x + 1$$
$$\frac{dy}{dx} = 2x - 1$$
$$gradient = 2(1) - 1 = 1$$

5.

(a)

$$y = x^3 - 3x + 2$$
$$\frac{dy}{dx} = 3x^2 - 3$$
$$3x^2 - 3 = 0$$
$$3x^2 = 2$$
$$x^2 = \frac{2}{3}$$
$$x = \pm\sqrt{\frac{2}{3}}$$
$$x = 0.82$$
$$or$$
$$x = -0.82$$

$$y = x^3 - 3x + 2$$
$$= (.82)^3 - 3 \times .82 + 2$$
$$= .551 - 2.46 + 2$$
$$= 0.091$$

$$y = x^3 - 3x + 2$$
$$y = (-0.83)^3 - 3(-0.83) + 2$$
$$y = -0.551 + 2.46 + 2$$
$$y = 3.9$$

The turning points are (0.82,0.091) and (-0.83,3.9)

(b)

$$y = 3x^2 - 5x + 6$$

$$\frac{dy}{dx} = 6x - 5$$

$$6x - 5 = 0$$

$$6x = 5$$

$$x = \frac{5}{6}$$

$$y = 3\left(\frac{5}{6}\right)^2 - 5\left(\frac{5}{6}\right) + 6$$

$$y = 3 \times \frac{25}{36} - \frac{25}{6} + 6$$

$$y = \frac{25}{12} - \frac{25}{6} + \frac{6}{1}$$

$$y = \frac{25}{12} - \frac{50}{12} + \frac{72}{12} = \frac{47}{12}$$

Turning point is $\left(\frac{5}{6}, \frac{47}{12}\right)$

6. The derivative of a function and one point on its graph are given. Find the function

(a)

$$\frac{dy}{dx} = x^3 + x^2 - 3$$
$$y = \frac{x^4}{4} + \frac{x^3}{3} - 3x + c$$
$$5 = \frac{1}{4} + \frac{1}{3} - 3 + c$$
$$8 = \frac{7}{12} + c$$
$$7\frac{5}{12} = c$$
$$y = \frac{x^4}{4} + \frac{x^3}{3} - 3x + 7\frac{5}{12}$$

$$\frac{dy}{dx} = 2x(x+1)$$
$$\frac{dy}{dx} = 2x^2 + 2x$$
$$y = \frac{2x^{2+1}}{3} + \frac{2x^{1+1}}{2} + c$$
$$y = \frac{2x^3}{3} + \frac{2x^2}{2} + c$$
$$0 = \frac{2(2)^3}{3} + \frac{2(2)^2}{2} + c$$
$$0 = \frac{16}{3} + \frac{8}{2} + c$$
$$0 = \frac{56}{6} + c$$
$$-\frac{56}{6} = c$$

(b) $y = \frac{2x^2}{3} + x^2 - \frac{56}{6}$

7. Find the given indefinite integrals

(a) $\int 5dx = \frac{5x^{0+1}}{1} = 5x$

(b) $\int x^2 dx = \frac{x^{2+1}}{3} = \frac{x^3}{3}$

(c) $\int \sqrt{x}dx = \frac{x^{\frac{1}{2}+1}}{1\frac{1}{2}} = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2x^{\frac{3}{2}}}{3}$

(d)

$$\int (1+t^2+t^4+t^6) = \frac{1x^{0+1}}{1} + \frac{t^{2+1}}{3} + \frac{t^{4+1}}{t^5} + \frac{t^{6+1}}{7} = x + \frac{t^3}{3} + \frac{t^5}{5} + \frac{t^7}{7}$$

(e)

$$\int \left(\frac{x^3}{3} - \frac{x^2}{2} + x - 1\right)dx = \frac{x^{3+1}}{3} - \frac{x^{2+1}}{2} + \frac{x^{1+1}}{2} - \frac{x^{0+1}}{1}$$

$$= \frac{1}{3} \times \frac{x^4}{4} - \frac{1}{2} \times \frac{x^3}{3} + \frac{1}{2} \times \frac{x^2}{2} - x = \frac{x^4}{12} - \frac{x^3}{6} + \frac{x^2}{4} - x$$

8.

Find the area of the region enclosed by the x-axis, the curve $y = x^2 + x + 1$ and x=1 and x=2

$$\int_1^2 (x^2 + x + 1)dx = \frac{x^{2+1}}{3} + \frac{x^{1+1}}{2} + \frac{x^{0+1}}{1} = \frac{x^3}{3} + \frac{x^2}{2} + x$$

$$A = \left[\frac{2^3}{3} + \frac{2^2}{2} + 2\right] - \left[\frac{1^3}{3} + \frac{1^2}{2} + 1\right]$$

$$= \left(\frac{8}{3} + \frac{4}{2} + \frac{2}{1}\right) - \left(\frac{1}{3} + \frac{1}{2} + 1\right)$$

$$= \left(\frac{16}{6} + \frac{12}{6} + \frac{12}{6}\right) - \left(\frac{2}{6} + \frac{3}{6} + \frac{6}{6}\right) = \frac{40}{6} - \frac{11}{6} = \frac{29}{6} = 4\frac{5}{6}$$

Unit Contents

Unit 34

Unit 34

Surds

Introduction

In Mathematics there are different types of numbers that differ in the way they behave when used in calculations. In this unit we will look at the behaviour of numbers called surds. Surds are found in Trigonometry and in the use of Pythagoras theorem, so we need to be able to simplify expressions that contain them so that we can apply them in calculations involving them.

This unit consists of 37 pages. This is approximately 2% of the whole course. Plan your time so that you can complete the whole course on schedule. As reference, you will need to devote 20 hours to work on this unit, 15 hours for formal study and 5 hours for self-study and completing assessments/assignments.

This Unit is Comprised of Three Lessons:

Lesson 1 Exploring the Meaning of Surds
Lesson 2 Simplifying Surds by Rationalising
Lesson 3 Basic Operations on Surds

Take a moment to read the following learning outcomes. They are a guide to what you should focus on while studying this unit.

Upon completion of this unit you will be able to:

Outcomes

- *define* surds.
- *simplify* surds by factorising.
- *simplify* surds by algebraic rules.
- *simplify* surds by rationalising
- *perform* basic operations with surds.

Terminology

***Surd*:** root of a number that repeats

***Recurring*:** repeating.

Online Resources

If you can get on the internet please utilize the resources at www.hippocampus.org. It is an excellent source of information for mathematics and the topics discussed in this unit. Here you will find:

- Presentations
- Simulations
- Videos
- Online Study Groups
- Links to Even More Information
- Textbook Correlations
- Online Courses

Lesson 1 Exploring the Meaning of Surds

Use a calculator to find the roots of the following: (a) $\sqrt{2}$, (b) $\sqrt{3}$, (c) $\sqrt{5}$, (d) $\sqrt{6}$,

Answers to activity 1

(a) $\sqrt{2} = 1.41421356237309$..................

(b) $\sqrt{3} = 1.7320508075688$................

(c) $\sqrt{5} = 2.236067977499789$...........

(d) $\sqrt{6} = 2.44948974278317$...............

The dots represent digits. It means that the digits after the decimal go on and on. These type of roots are the ones called surds. They are different from the perfect squares e.g. $\sqrt{4} = \pm 2$, $\sqrt{9} = \pm 3$, $\sqrt{16} = \pm 4$,where we get a number that does not have a recurring decimal

Compare your answers with the ones at the end of the unit. Do not proceed to the next section if you got less than 80% of the problems right.

Key Points to Remember

The key points to remember in this subunit on the meaning of Surds are:

- Surds roots of numbers that are not perfect squares such $\sqrt{2}$.
- Surds have a recurring decimal.

Simplifying Surds by Factorising

One way of simplifying surds is to look for factors of a number under the square root bracket. We need factors that are perfect squares. If factors of a number are many, we pick the ones that will include one or more perfect squares.

Example 1

(a) $\sqrt{4} \times \sqrt{9}$

$= 2 \times 3 = 6$

OR

$\sqrt{4 \times 9}$

$= \sqrt{36}$

$= 6$

Therefore

$\sqrt{4} \times \sqrt{9} = 6$

and

$\sqrt{4 \times 9} = 6$

$\sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9}$

Stated generally:

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$

Example 2

Simplify $\sqrt{45}$

Factors of 45 are 5 and 9. 9 is a perfect square.

Answer

$\sqrt{45} = \sqrt{9} \times \sqrt{5} = 3 \times \sqrt{5} = 3\sqrt{5}$

Example 3

Simplify $\sqrt{12}$

Factors of 12 are 2 $(a) 2 \times 6, (b) 3 \times 4, (c) 12 \times 1$ factors containing a perfect square are 3×4. So we pick them and express 12 in terms of them as shown below. This is done because we can take the square root of 4.

$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

Example 4

Simplify $\sqrt{27x^3y^4}$

Answer

$\sqrt{27x^3y^4} = \sqrt{9} \times \sqrt{3} \times \sqrt{x^2} \times \sqrt{x} \times \sqrt{y^4}$

Factors of 27 are 9 and 3. We can find the square root of 9. Note that $\sqrt{27} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$. In this example variables have factors that are perfect squares. Factors of x^3 are x^2 and x , where x^2 is a perfect square.

$\sqrt{x^3} = \sqrt{x^2} \times \sqrt{x} = x\sqrt{x}$

Factors of y^4 are y^2 and y^2 which are both perfect squares.

$\sqrt{y^4} = \sqrt{y^2} \times \sqrt{y^2} = y \times y = y^2$

Therefore

$\sqrt{27x^3y^4} = 3\sqrt{3} \times x\sqrt{x} \times y^2 = 3xy^2\sqrt{3x}$

In the examples 1 to 3 we simplified the surds. Analysing the examples above

Number or variable under the square root bracket	**Factors**	**A factor which is a perfect square**
45	9 and5	9
12	4 and 3	4
27	9 and 3	9
x^3	x^2 and x	x^2
y^4	y^2 and y^2	y^2 and y^2

Key Points to Remember

The key points to remember in this subunit on simplifying surds by factorising are:

- When factorising surds the number inside the square root bracket is expressed in therms of its factors e.g. $\sqrt{8} = \sqrt{2 \times 4}$

- When factorising to simplify surds, we choose factors that are perfects squares so that we can find their roots. E.g.
  (a) $\sqrt{36} = \sqrt{9 \times 4} = 3 \times 2 = 6$
  (b) $\sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2\sqrt{6}$

Activity 1

Simplify the following surds:

1. (a) $\sqrt{8}$

(b) $\sqrt{18}$

(c) $\sqrt{32}$

2. (a) $\sqrt{8x^5y^6}$

(b) $\sqrt{12x^7y^8}$

Compare your answers with the ones at the end of the unit. Do not proceed to the next section if you got less than 80% of the problems right.

Simplifying Surds by Algebraic Rules

The following are algebraic rules we also use in simplifying surds.

Example 1

Simplify by removing brackets

Algebraically

(a)

$3(a+b)$

$=3a+3b$

(b)

$c(x-y)$

$=cx-cy$

Applying this on surds

(a) $\sqrt{2}(\sqrt{3}+\sqrt{5})=\sqrt{2}\times\sqrt{3}+\sqrt{2}\times\sqrt{5}$

(b)

$\sqrt{3}\left(\sqrt{2}+\sqrt{3}\right)$

$\sqrt{3}\times\sqrt{2}+\sqrt{3}\times\sqrt{3}$

$\sqrt{6}+\sqrt{9}$

$\sqrt{6}+3$

(c)

$\sqrt{a^2}\left(\sqrt{3}+\sqrt{4}\right)$

$=\sqrt{a^2}\times\sqrt{3}+\sqrt{a^2}\times\sqrt{4}$

$=a\sqrt{3}+2a$

Example 2

Simplify the following

Algebraically $\sqrt{\frac{x^6}{y^2}} = \frac{\sqrt{x^6}}{\sqrt{y^2}} = \frac{x^{\frac{6}{2}}}{y^{\frac{2}{2}}} = \frac{x^3}{y}$

Applying this on surds

(a)

$$\sqrt{\frac{49}{63}} = \frac{\sqrt{49}}{\sqrt{63}} = \frac{\sqrt{7} \times \sqrt{7}}{\sqrt{7} \times \sqrt{9}} = \frac{\sqrt{7}}{3}$$

(b)

$$\sqrt{\frac{64}{16}} = \sqrt{4} = 2 = \frac{\sqrt{64}}{\sqrt{16}} = \frac{8}{4} = 2$$

Stated generally:

$$\boxed{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$$

Example 3

Algebraically

$(a+b)(a-b) = a^2 - b^2$, the difference of two squares

Applying this on surds

(a)

$$\left(\sqrt{3}+\sqrt{5}\right)\left(\sqrt{3}-\sqrt{5}\right)$$
$$\left(\sqrt{3}\right)^2 - \left(\sqrt{5}\right)^2$$

(b)

$\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{2}-\sqrt{3}\right)$

$\left(\sqrt{2}\right)^2-\sqrt{3}^2$

Example 4

Product of brackets involving surds

$(x+2)(x-3)$

$=(x\times x)+(x\times -3)+(2\times x)+(2\times -3)$

$=x^2-3x+2x-6$

$=x^2-x-6$

Applying this on surds

$\left(2+\sqrt{3}\right)\left(1-\sqrt{2}\right)$

$2\times 1+2\times -\sqrt{2}+\sqrt{3}\times 1+\sqrt{3}\times -\sqrt{2}$

$2-2\sqrt{2}+\sqrt{3}-\sqrt{3}\sqrt{2}$

$2-2\sqrt{2}+\sqrt{3}-\sqrt{6}$

Key Points to Remember

The key points to remember in this subunit on simplifying surds by algebraic rules are:

- The algebraic rule a(b+c)=ab+ac is applied to surds as follows:

  $\sqrt{3}\left(\sqrt{2}+\sqrt{3}\right)$

  $\sqrt{3}\times\sqrt{2}+\sqrt{3}\times\sqrt{3}$

  $\sqrt{6}+\sqrt{9}$

  $\sqrt{6}+3$

- The algebraic rule $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$ is applied to surds as shown below:

  $\sqrt{\frac{8}{27}}=\frac{\sqrt{4\times 3}}{\sqrt{9\times 3}}=\frac{2\sqrt{3}}{3\sqrt{3}}$

- The algebraic rule $(a+b)(a-b)=(a^2-b^2)$ is applied to surds as shown below:

$$\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$$
$$=\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2$$
$$=3-2$$
$$=1$$

- The algebraic rule

$$(x+2)(x-3)$$
$$=(x\times x)+(x\times -3)+(2\times x)+(2\times -3)$$
$$=x^2-3x+2x-6$$
$$=x^2-x-6)$$

is applied to surds as shown below:

$$\left(2+\sqrt{3}\right)\left(1-\sqrt{2}\right)$$
$$2\times 1+2\times -\sqrt{2}+\sqrt{3}\times 1+\sqrt{3}\times -\sqrt{2}$$
$$2-2\sqrt{2}+\sqrt{3}-\sqrt{3}\sqrt{2}$$
$$2-2\sqrt{2}+\sqrt{3}-\sqrt{6}$$

Activity 2

Simplify the following expressions

1. 3(a+b)

----

2. $\sqrt{2}(a+\sqrt{3})$

----

3. $\sqrt{\dfrac{12}{9}}$

----

4. $\sqrt{\dfrac{8}{27}}$

----

5. $\sqrt{\dfrac{b^6}{81}}$

----

6. $(x+y)(x-y)$

----

7. $\left(a^2 + b^2\right)\left(a^2 - b^2\right)$

8. $\left(x^2 + y^4\right)\left(x^2 - y^4\right)$

The reverse process is factorisation because in the process we come up with factors

Factorise the following expressions

9. $a^6 - b^6$

10. $4x^8 - z^2$

11. $\left(\sqrt{3} + \sqrt{2}\right)\left(\sqrt{3} - \sqrt{2}\right)$

____

Compare your answers with the ones at the end of the unit. Do not proceed to the next section if you got less than 80% of the problems right.

Lesson 2 Simplifying Surds by Rationalising

One way of simplifying surds is to look for factors of a number under the square root bracket

Simplifying Surds by Rationalising an Expression Containing Surds

Lets simplify an expression that has a denominator which is a surd, so we simplify the denominator. This is done by removing the square root bracket. The process called rationalisation

Example 1

Rationalise the following term

$$\frac{2}{\sqrt{8}}$$

Multiply the numerator and denominator by a surd so that the denominator becomes a whole number as shown below:

$$\frac{2}{\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}} = \frac{2\sqrt{8}}{8} = \frac{\sqrt{8}}{4}$$

Example 2

Rationalise the following expression

(a) $\frac{\sqrt{6}+\sqrt{5}}{\sqrt{3}}$

Multiply the numerator and denominator by a surd so that the denominator becomes a whole number as shown below:

$$\frac{\sqrt{6}+\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{6}+\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{18}+\sqrt{15}}{3} = \frac{3\sqrt{2}+\sqrt{15}}{3} = \sqrt{2} + \tfrac{1}{3}\sqrt{15}$$

Example 3

Rationalise the following expression

$$\frac{1}{3+\sqrt{2}}$$

Multiply the numerator and denominator by an expression that makes the denominator the difference of two squares as shown below:

$$\frac{1}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{3-\sqrt{2}}{3^2-\left(\sqrt{2}\right)^2} = \frac{3-\sqrt{2}}{9-2} = \frac{3-\sqrt{2}}{7}$$

Example 4

(b) $\frac{3}{2-\sqrt{3}}$

Multiply the numerator and denominator by a surd so that the denominator becomes the difference of two squares whole number as shown below:

$$\frac{3}{2-\sqrt{3}} = \frac{3}{2-\sqrt{3}} \times \frac{3(2+\sqrt{3}}{2+\sqrt{3}} = \frac{6+3\sqrt{3}}{4-\left(\sqrt{3}\right)^2} = \frac{6+3\sqrt{3}}{4-3} = \frac{6+3\sqrt{3}}{1} = 6+3\sqrt{3}$$

Key Points to Remember

The key points to remember in this subunit on simplifying surds by rationalising are:

- To rationalise we carry out operations that remove the square root bracket of the denominator of an expression containing surds as shown below:

  (a)

$$\frac{\sqrt{2}}{\sqrt{3}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{2}\sqrt{3}}{3} = \frac{\sqrt{2 \times 3}}{3} = \frac{\sqrt{6}}{3}$$

- We also rationalise by multiplying by a factor that makes the denominator the difference of two squares as shown below:

$$\frac{1}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{3-\sqrt{2}}{3^2 - \left(\sqrt{2}\right)^2} = \frac{3-\sqrt{2}}{9-2} = \frac{3-\sqrt{2}}{7}$$

Activity 3

Simplify the following surds by rationalising

1. $\dfrac{\sqrt{5}}{\sqrt{3}}$

2. $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}$

3. $\dfrac{4}{3+\sqrt{2}}$

4.

$\dfrac{2}{4-\sqrt{2}}$

Lesson 3 Basic Operations on Surds

Here we make use of like and unlike terms in surd form. Surds are like terms if the numbers in the square root brackets are the same. E.g. $\sqrt{2}$,$5\sqrt{2}$, $4\sqrt{2}$ are all like terms that can be added or one may be subtracted from another.

Adding and subtracting surds

Example 1

$\sqrt{2}+3\sqrt{2}+4\sqrt{2}=8\sqrt{2}$

Example 2

$5\sqrt{3}-2\sqrt{3}=3\sqrt{3}$

Example 3

$2\sqrt{6}+\sqrt{6}-5\sqrt{6}+4\sqrt{6}-3\sqrt{2}=2\sqrt{6}-3\sqrt{2}$

Example 4

Multiplying and dividing surds

(a)

$3\sqrt{2}\times 5\sqrt{3}=3\times 5\times\sqrt{2\times 3}=15\sqrt{6}$

(b)

$\frac{3\sqrt{5}}{6\sqrt{3}}=\frac{1}{2}\sqrt{\frac{5}{3}}$

Example 5

$$\sqrt{\frac{48}{36}} = \frac{\sqrt{16\times3}}{\sqrt{9\times4}} = \frac{\sqrt{16}\times\sqrt{3}}{\sqrt{9}\times\sqrt{4}} = \frac{\not{4}\sqrt{3}}{3\times\not{2}} = \frac{2\sqrt{3}}{3}$$

Key Points to Remember

The key points to remember in this subunit on simplifying surds by rationalising are:

- expressions containing surds can be added

  (a) $2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$

  (b) $3\sqrt{2} + \sqrt{2} = 4\sqrt{2}$

- expressions containing Surds can be subtracted

  (a) $3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$

  (b) $5\sqrt{3} - 2\sqrt{3} = 3\sqrt{3}$

- expressions containing Surds can be multiplied

  (a) $2\sqrt{3} \times 4\sqrt{3} = 8\left(\sqrt{3}\right)^2 = 8 \times 3 = 24$

  (b) $5\sqrt{3} \times 2\sqrt{5} = 10\sqrt{3} \times \sqrt{5} = 10\sqrt{3 \times 5} = 10\sqrt{15}$

- expressions containing surds can be divided

  (a) $\frac{4\sqrt{2}}{2\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{3}}$

  (b) $\frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$

Activity 4

Perform the four basic operation on the following surds

1. $2\sqrt{3}+4\sqrt{3}+3\sqrt{3}$

2. $7\sqrt{2}-9\sqrt{2}$

3. $3\sqrt{5}-2\sqrt{5}-5\sqrt{6}-4\sqrt{5}+2\sqrt{5}+\sqrt{3}$

4. $2\sqrt{8}\times 4\sqrt{3}$

5. $3\sqrt{2}\times 5\sqrt{6}$

6. $5\sqrt{18} \times 2\sqrt{20}$

7. $\sqrt{\frac{18}{12}}$

8. $\sqrt{\frac{125}{32}}$

9. $\sqrt{\frac{28}{45}}$

Unit Summary

Summary

In this unit you learned that surds are roots which have a recurring decimal. For example $\sqrt{2} = 1.41421356237309$.................. $\sqrt{2}$ is a surd because $\sqrt{2} = 1.41421356237309$..................

1. We learned some ways of simplifying Surds by the following methods:

(a) factorizing

e.g. $\sqrt{9} \times \sqrt{8} = 3\sqrt{8} = 3\sqrt{4 \times 2} = 3\sqrt{4} \times \sqrt{2} = 3 \times 2\sqrt{2} = 6\sqrt{2}$

(b) removal of brackets by algebraic rules

(i) multiplying out the bracket by the term outside

$\sqrt{3}(a + \sqrt{2})$

$= a\sqrt{3} + \sqrt{3}\sqrt{2}$

(ii) multiplying two brackets

$\left(2 + \sqrt{3}\right)\left(1 - \sqrt{2}\right)$

$2 \times 1 + 2 \times -\sqrt{2} + \sqrt{3} \times 1 + \sqrt{3} \times \sqrt{2}$

$2 + 2\sqrt{2} + \sqrt{3} - \sqrt{3}\sqrt{2}$

$2 + 2\sqrt{2} + \sqrt{3} - \sqrt{6}$

(iii) multiplying out brackets using the difference of two squares

$\left(\sqrt{3} + \sqrt{2}\right)\left(\sqrt{3} - \sqrt{2}\right)$

$= \left(\sqrt{3}\right)^2 - \left(\sqrt{2}\right)^2$

$= 3 - 2$

$= 1$

(c)Rationalising

(i) removal of the root bracket of the denominator by multiplying the numerator and denominator by the same number

$$\frac{2}{\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}} = \frac{2\sqrt{8}}{8} = \frac{\sqrt{8}}{4}$$

(ii)using the difference of two squares to rationalise

$$\frac{1}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{3-\sqrt{2}}{9-\left(\sqrt{2}\right)^2} = \frac{3-\sqrt{2}}{7}$$

1. We learned how to perform the four basic operations with surds

(a) Addition

$$\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} = 8\sqrt{2}$$

(b) Subtraction

$$5\sqrt{3} - 2\sqrt{3} = 3\sqrt{3}$$

(c) Multiplication

$$2\sqrt{8} \times 4\sqrt{3}$$

(d) Division

$$\sqrt{\frac{28}{45}} = \frac{\sqrt{7 \times 4}}{\sqrt{9 \times 5}} = \frac{2\sqrt{7}}{3\sqrt{5}}$$

Assignment

Assignment

Instructions:

Answer all questions

Marks: 35

Time: $1\frac{1}{2}$

1. $3\sqrt{3}+2\sqrt{5}-\sqrt{3}+2\sqrt{3}$

[3]

2. $\sqrt{48}$

[3]

3. $\sqrt{\frac{63}{15}}$

[5]

4. $\sqrt{2}\left(\sqrt{3}-\sqrt{5}\right)$

[3]

5. $\left(\sqrt{2}-\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)$

[4]

6. $\left(\sqrt{3}-\sqrt{2}\right)^2$

[4]

7. $\dfrac{1}{\sqrt{5}}$

[3]

8.

$$\frac{2}{\sqrt{3}+\sqrt{2}}$$

[4]

9. Simplify the following expression by rationalising the denominator

$$\frac{1+\sqrt{3}}{2-\sqrt{3}}$$

[6]

Assessment

Assessment

Instructions: Answer all questions

Time : 2 hrs

Marks: 48

1. Simplify the following surds

   $\sqrt{3}+\sqrt{12}$

[3]

2. $\sqrt{28}+\sqrt{63}$

[5]

3. $\sqrt{5}\times\sqrt{12}$

[3]

4. $\dfrac{1+\sqrt{5}}{\sqrt{8}}$

[7]

5. $\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right)$

[3]

6. $\dfrac{\sqrt{3}}{6-\sqrt{3}}$

[4]

7. $\dfrac{1+\sqrt{3}}{\sqrt{6}}$

[7]

8. $\frac{2}{3+\sqrt{3}}$

[5]

9. $\frac{1+\sqrt{3}}{1-\sqrt{3}}$

[7]

10. $\frac{1}{5+\sqrt{2}}+\frac{1}{5-\sqrt{2}}$

[4]

ANSWERS

Answers to activity 1

$\sqrt{8}$

(a) $= \sqrt{4\times 2} = \sqrt{4}\times\sqrt{2}$

$= 2\sqrt{2}$

(b)

$\sqrt{18}$

$= \sqrt{9}\times\sqrt{2}$

$= 3\sqrt{2}$

(c)

$\sqrt{32}$

$= \sqrt{4}\times\sqrt{8}$

$= 2\sqrt{8} = 2\times\sqrt{4}\times\sqrt{2}$

$= 2\times 2\sqrt{2}$

2.

(a) $\sqrt{8x^5y^6}$

$= 2\sqrt{2}\times x^2\times\sqrt{x}\times y^3$

$= 2x^2y^3\sqrt{2}\sqrt{x}$

(b)

$\sqrt{12x^7y^8} = \sqrt{4\times 3\times x^7\times y^8}$

$= 2y^4\sqrt{3x^6}\times$

$= 2x^3y^4\sqrt{3x}$

Answers to activity 2

Simplify the following expressions

1. 3(a+b) = 3a+3b

2. $\sqrt{2}(a+\sqrt{3})$

$\sqrt{2}(a+\sqrt{3})$

$= a\sqrt{2}+\sqrt{2}\sqrt{3}$

3. $\sqrt{\frac{12}{9}} = \frac{\sqrt{12}}{\sqrt{9}} = \frac{\sqrt{3\times 4}}{3} = \frac{2\sqrt{3}}{3}$

4. $\sqrt{\frac{8}{27}} = \frac{\sqrt{4\times 3}}{\sqrt{9\times 3}} = \frac{2\sqrt{3}}{3\sqrt{3}}$

5 $\sqrt{\frac{b^6}{81}} = \frac{\sqrt{b^6}}{\sqrt{81}} = \frac{b^{\frac{6}{2}}}{9} = \frac{b^2}{9}$

6. $(x+y)(x-y)$

$= x(x-y)+y(x-y)$

$= x^2 - xy + yx - y^2$

$= (x^2 - y^2)$

7. $(a^2+b^2)(a^2-b^2) = (a^4-b^4)$

8. $(x^2+y^4)(x^2-y^4)$

$\left(x^2+y^4\right)\left(x^2-y^4\right)=\left(x^4-y^8\right)$

The reverse process is factorisation because in the process we come up with factors

Factorise the following expressions

9.

$a^6-b^6=(a^3+b^3)(a^3-a^3)$ (refer to a unit on indices)

Here the process is reversed. We find the square root of each of the terms

The square root of a^6 is $a^{\frac{6}{2}}=a^3$

The square root of b^6 is $b^{\frac{6}{2}}=b^3$. We then multiply the sum of the square roots by the difference of the square roots to get the following factors:

$(a^3+b^3)(a^3-b^3)$

10. $4x^8-z^2$

$4x^8-z^2=\left(2x^4-z\right)\left(2x^4+z\right)$

11.

$\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

$=\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2$

$=3-2$

$=1$

Answers to activity 3

1. $\frac{\sqrt{5}}{\sqrt{3}}=\frac{\sqrt{5}}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{5}}{3}$

2.

$$\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}=\frac{\sqrt{2}+\sqrt{3}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}\left(\sqrt{2}+\sqrt{3}\right)}{2}=\frac{2+\sqrt{2}\sqrt{3}}{2}=\frac{2+\sqrt{2}\sqrt{3}}{2}=\frac{2+\sqrt{2\times 3}}{2}=\frac{2+\sqrt{6}}{2}$$

3. $\frac{4}{3+\sqrt{2}} = \frac{4}{3+\sqrt{2}} \times \frac{3-\sqrt{2}}{3-\sqrt{2}} = \frac{12-4\sqrt{2}}{3^2-2} = \frac{12-4\sqrt{2}}{7}$

4.

$\frac{2}{4-\sqrt{2}} = \frac{2}{4-\sqrt{2}} \times \frac{4+\sqrt{2}}{4+\sqrt{2}} = \frac{8+2\sqrt{2}}{16-2} = \frac{8+2\sqrt{2}}{14} = \frac{4+\sqrt{2}}{7}$

Answers to activity 4

1. $2\sqrt{3}+4\sqrt{3}+3\sqrt{3}=9\sqrt{3}$

2. $7\sqrt{2}-9\sqrt{2}=-2\sqrt{2}$

3. $3\sqrt{5}-2\sqrt{5}-5\sqrt{6}-4\sqrt{5}+2\sqrt{5}+\sqrt{3}=-\sqrt{5}-5\sqrt{6}+\sqrt{3}$

4. $2\sqrt{8}\times 4\sqrt{3}=2\times 4\times\sqrt{8\times 3}=8\sqrt{2\times 4\times 3}=16\sqrt{6}$

5.

$3\sqrt{2}\times 5\sqrt{6}=3\times 5\sqrt{2\times 6}=15\sqrt{2\times 2\times 3}=15\sqrt{4\times 3}=15\times 2\sqrt{3}$

$=30\sqrt{3}$

6.

$5\sqrt{18}\times 2\sqrt{20}$

$=5\times 2\sqrt{9\times 2\times 4\times 5}$

$=10\times 3\times 2\sqrt{10}$

$=60\sqrt{10}$

7.

$$\sqrt{\frac{18}{12}} = \frac{\sqrt{18}}{\sqrt{12}} = \frac{\sqrt{9\times 2}}{\sqrt{4\times 3}} = \frac{3\sqrt{2}}{2\sqrt{3}}$$

$$\frac{3\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{6}}{6} = \frac{\sqrt{6}}{2}$$

8.

$$\sqrt{\frac{125}{32}} = \frac{\sqrt{25\times 5}}{\sqrt{16\times 2}} = \frac{5\sqrt{5}}{4\sqrt{2}}$$

9.

$$\sqrt{\frac{28}{45}} = \frac{\sqrt{7\times 4}}{\sqrt{9\times 5}} = \frac{2\sqrt{7}}{3\sqrt{5}}$$

Answers to Assignment

1.

$$3\sqrt{3} + 2\sqrt{5} - \sqrt{3} + 2\sqrt{3}$$
$$4\sqrt{3} + 2\sqrt{5}$$

2. $\sqrt{48} = \sqrt{16\times 3} = \sqrt{16}\times\sqrt{3} = 4\sqrt{3}$

3. $\sqrt{\frac{63}{15}} = \frac{\sqrt{7\times 9}}{\sqrt{5\times 3}} = \frac{3\sqrt{7}}{\sqrt{15}} \times \frac{\sqrt{15}}{\sqrt{15}} = \frac{3\sqrt{105}}{15} = \frac{\sqrt{105}}{5}$

4. $\sqrt{2}\left(\sqrt{3} - \sqrt{5}\right) = \sqrt{6} - \sqrt{10}$

5. $\left(\sqrt{2} - \sqrt{3}\right)\left(\sqrt{5} - \sqrt{3}\right)$

$$\sqrt{10} - \sqrt{6} - \sqrt{15} + 9$$

6.

$\left(\sqrt{3}-\sqrt{2}\right)^2$

$\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)$

$3-\sqrt{6}-\sqrt{6}+\sqrt{4}=3-2\sqrt{6}+2=5-2\sqrt{6}$

7.

$\frac{1}{\sqrt{5}}$

$\frac{1}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}$

$\frac{\sqrt{5}}{5}$

8.

$\frac{2}{\sqrt{3}+\sqrt{2}}$

$\frac{2}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

$\frac{2\sqrt{3}-2\sqrt{2}}{3-2}=2\sqrt{3}-2\sqrt{2}$

9.

$\frac{1+\sqrt{3}}{2-\sqrt{3}}$

$=\frac{1+\sqrt{3}}{2-\sqrt{3}}\times\frac{2+\sqrt{3}}{2+\sqrt{3}}$

$=\frac{\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)}{4-\left(\sqrt{3}\right)^2}$

$=\frac{1\left(2+\sqrt{3}\right)+\sqrt{3}\left(2+\sqrt{3}\right)}{4-3}$

$=\frac{2+\sqrt{3}+2\sqrt{3}+\left(\sqrt{3}\right)^2}{1}$

$=2+3\sqrt{3}+3$

$=5+3\sqrt{3}$

Assessment

Answers to Assessment

Simplify the following surds

1. $\sqrt{3}+\sqrt{12}=\sqrt{3}+\sqrt{4\times 3}=\sqrt{3}+2\sqrt{3}=3\sqrt{3}$

2. $\sqrt{28}+\sqrt{63}=\sqrt{4\times 7}+\sqrt{9\times 7}=2\sqrt{7}+3\sqrt{7}=5\sqrt{7}$

3. $\sqrt{5}\times\sqrt{12}=\sqrt{5\times 4\times 3}=2\sqrt{15}$

4.

$$\frac{1+\sqrt{5}}{\sqrt{8}}$$
$$=\frac{1+\sqrt{5}}{\sqrt{8}}\times\frac{\sqrt{8}}{\sqrt{8}}$$
$$=\frac{\sqrt{8}\left(1+\sqrt{5}\right)}{8}$$
$$=\frac{\sqrt{8}+\sqrt{8}\times\sqrt{5}}{8}$$
$$=\frac{\sqrt{4\times 2}+\sqrt{4\times 2\times 5}}{8}$$
$$=\frac{2\sqrt{2}+2\sqrt{10}}{8}$$
$$=\frac{\sqrt{2}+\sqrt{10}}{4}$$

5. $\left(3+\sqrt{2}\right)\left(3-\sqrt{2}\right)=3\times 3-\left(\sqrt{2}\right)^2=9-2=7$

6. $\dfrac{\sqrt{3}}{6-\sqrt{3}}=\dfrac{\sqrt{3}\left(6+\sqrt{3}\right)}{\left(6-\sqrt{3}\right)\left(6+\sqrt{3}\right)}=\dfrac{\sqrt{3}\left(6+\sqrt{3}\right)}{36-3}=\dfrac{\sqrt{3}\left(6+\sqrt{3}\right)}{33}=\dfrac{6\sqrt{3}+}{33}$

7.

$$\frac{1+\sqrt{3}}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$
$$= \frac{\sqrt{6}\left(1+\sqrt{3}\right)}{\sqrt{6}}$$
$$= \frac{\sqrt{6}+\sqrt{6}\times\sqrt{3}}{\sqrt{6}}$$
$$= \frac{\sqrt{6}+\sqrt{6\times 3}}{6}$$
$$= \frac{\sqrt{6}+\sqrt{18}}{6}$$
$$= \frac{\sqrt{6}+\sqrt{9\times 2}}{6}$$
$$= \frac{\sqrt{6}+3\sqrt{2}}{6}$$

8.

$$\frac{2}{3+\sqrt{3}}$$
$$= \frac{2}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}}$$
$$= \frac{6-2\sqrt{3}}{9-3}$$
$$= \frac{6-2\sqrt{3}}{6}$$
$$= \frac{3-\sqrt{3}}{3}$$

9.

$$\frac{1+\sqrt{3}}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$= \frac{1\left(1+\sqrt{3}\right)+\sqrt{3}\left(1+\sqrt{3}\right)}{-2}$$

$$= \frac{1+\sqrt{3}+\sqrt{3}+\sqrt{3}\times\sqrt{3}}{-2}$$

$$= \frac{1+2\sqrt{3}+3}{-2}$$

$$= \frac{4+2\sqrt{3}}{-2} = -\frac{2+\sqrt{3}}{1} = -2-\sqrt{3}$$

10.

$$\frac{1}{5+\sqrt{2}} + \frac{1}{5-\sqrt{2}} = \frac{5-\sqrt{2}+5+\sqrt{2}}{\left(5+\sqrt{2}\right)\left(5-\sqrt{2}\right)} = \frac{10}{25-2} = \frac{10}{23}$$

Based on your results and the recommendation that you should aim for at least 80% to ensure your overall success in this course and any subsequent math course you take, determine how much you should study the overall unit before you attempt the assessment.