

RESOURCE MATERIAL ON CISCE CURRICULUM

MATHEMATICS AT THE UPPER PRIMARY LEVEL

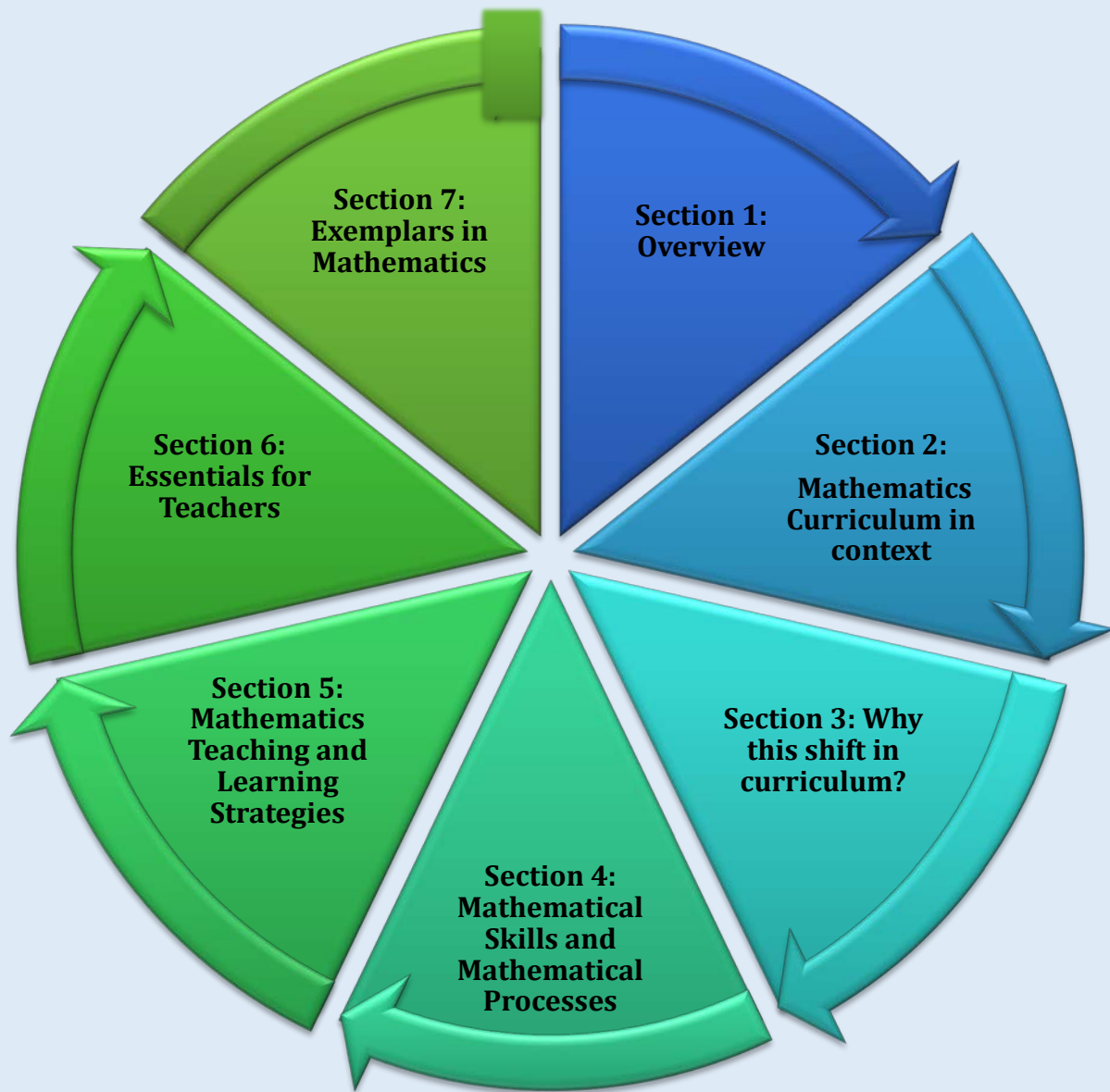


Research Development and Consultancy Division

**Council for the Indian School Certificate Examinations
New Delhi**

MODULE II

Content Outline



Structure of the Module

1. An Overview:
 - 1.1 Why this module?
 - 1.2 What does this module include?
 - 1.3 How to use this module?
2. Mathematics curriculum in context
 - 2.1 Nature of Mathematics
 - 2.2 Present status of Mathematics teaching-learning
3. Why this shift in curriculum?
4. Mathematical Skills and Mathematical Processes
5. Mathematics Teaching and Learning Strategies
6. Essentials for Teachers
 - 6.1 Mathematical Pedagogical Content Knowledge
 - 6.2 Assessment of Mathematics learning
 - 6.3 General guidelines for Mathematics Teachers
 - 6.4 Some common misconceptions and errors
7. Exemplars in Mathematics
 - 7.1 Need for exemplars
 - 7.2 Planning for the teaching learning process
 - 7.3 Discussion in general about Van Hiele Model of Development of Geometrical thought
 - 7.4 Exemplar I
 - 7.5 Exemplar II
 - 7.6 Exemplar III

Expectations of this Module

After reading/using this module the user/teacher will be able to:

- explain the need and significance of this module;
- discuss the nature and status of mathematics education in India and abroad
- explain the salient features of the Mathematics curriculum;
- apply the pedagogical processes through exemplars to achieve the goals of learning mathematics at the upper primary level;
- integrate assessment with the teaching learning processes as per each child's contexts.

1. An Overview

Why, What and How of this module

1.1 Why this module?

This module provides an understanding on various aspects of the Mathematics curriculum such as its status in the Upper Primary classes (Classes VI to VIII), salient features and pedagogical principles for Mathematics learning and how a Mathematics classroom can be transformed into a joyful classroom. It will also provide a broader understanding on various components of the Mathematics curriculum such as learning outcomes, identified concepts and sub concepts in the themes, transactional processes and learning resources to be used in the classroom. In order to provide a comprehensive understanding of mathematical learning, some exemplars have been given.

1.2 What does this module include?

This module has seven sections. The first section provides an overview to explain the why and what of the module along with the process of using it. Section 2 discusses the status and nature of the Mathematics curriculum. Section 3 explains the evolution of the present curriculum. Section 4 highlights various mathematical skills and processes. Section 5 deals with strategies involved in Mathematics teaching and learning process. Section 6 deals with the essentials that are required for Mathematics teachers and Section 7 focuses on exemplars.

1.3 How to use this module?

This module is meant for all stakeholders working at the Upper Primary level in general and for Mathematics practitioners in particular. In each section of this module, some text assignments/activities have been given. While reading/using each section, assignments need to be done parallelly. Later these assignments can either be assessed by your mentor or by peers. Some exemplars have also been given in this module. These exemplars may be used during training, either in simulation or in actual classroom situation. After using the exemplars, peer reflection is essential as it would generate further ideas for innovation.

2. Mathematics Curriculum in context

The Mathematics curriculum at the upper primary level aims to develop a number of mathematical skills and processes in children such as handling abstraction, problem solving, mathematical communication, conjecturing and searching for proofs. At the upper primary stage, children get the first taste of the power of Mathematics through the application of powerful abstract concepts that compress previous learning and experience. This enables them to revisit and consolidate basic concepts and skills learnt at the primary stage, which is essential from the point of view of achieving universal mathematical literacy. Children are introduced to algebraic notation and its use in solving problems and in generalisation, to the systematic study of space and shapes, and for consolidating their knowledge of measurement. Data handling, representation and interpretation form a significant part of the ability of dealing with information in general, which is a fundamental 'life skill'. The learning at this stage also offers an opportunity to enrich children's spatial reasoning and visualisation skills.

The curriculum is designed to ensure that children build a solid foundation in mathematics by connecting and applying mathematical concepts in a variety of ways and situations. To support this process, teachers need to find ways and means of integrating concepts from various themes and apply mathematics to real life situations in children's daily lives.

While transacting this curriculum we need to remember that each child is unique in terms of her/his likes, dislikes, interests, dispositions, skills and behaviour. Each child learns and responds to learning situations in his/her own special way. The teaching strategies at this stage must address the needs of all children.

The present Mathematics curriculum expands spirally from Class VI to Class VIII. For example, let us take the concept of Numbers. The child picks up number skills gradually, with specific number groups like natural numbers and their properties in class VI to rational numbers and their properties in class VIII. Similar approach percolates down to all other themes in the curriculum.

Curriculum and Syllabus

The general notion of the people regarding the term 'curriculum' is that it denotes subject-wise list of topics to be taught in a particular class. Actually curriculum means all experiences provided to the children in as well as out of school for attaining the educational aims. In general, everything that happens in the school is part of the curriculum. Some aspects of the curriculum may be educational aims, educational experiences to attain these aims and curricular expectations, organisation of educational experiences and assessment whether these aims are being attained appropriately. Curriculum can be viewed as whole, including the learner, the teacher, teaching and learning methodologies, experiences, outputs and outcomes possible within a learning institution. Curriculum can be considered as content, process and product.

A syllabus is an outline of the content which has to be learnt by the children. It includes the topics to be covered, their order and any other relevant information. Curriculum prescribes the objectives while, the syllabus describes the means to achieve them. Hence we can say curriculum is prescriptive while syllabus is descriptive.



Unfolding the Curriculum: Mathematics Curriculum in Practice

In order to create a classroom where the learners enjoy while learning, where they feel free to ask questions, to explore and create self-learning, the teacher needs to act as a motivator, guide and facilitator and provide children with positive experiences, so that the fear and anxiety that a number of children face regarding Mathematics may be addressed.

2.1 Nature of Mathematics

Mathematics serves as a tool for both logic and creativity. Thus, it is pursued both for practical utility as well as its intrinsic interest.

There are certain aspects of nature of Mathematics which directly/indirectly impact the teaching learning of Mathematics. It is therefore very important for a mathematics teacher to know about the nature of mathematics so that she can use them for improving learning of Mathematics.

Mathematics by nature is abstract. All themes in mathematics, which include numbers, shapes around us, problem solving patterns and systematic reasoning are ideas that ultimately develop in our mind. Understanding of these abstract ideas is demonstrated by applying them in daily life for various purposes including problem solving. The mathematics curriculum treats mathematics both as a tool for practical utility as well as a discipline that develops reasoning and analytical abilities.

Mathematical ideas grow from “concrete to abstract” and “particular to general”. For example, the counting numbers required to describe number of objects in a group/set can be represented on a number line whereas negative number are realized as an extension of the number line in the opposite direction and the ideas are more abstract, etc. Slowly the property of integers is further abstracted into the idea of rational and real numbers. Similarly, the idea of a plane figure like a rectangle is more abstract than the cover page of a book which looks like a rectangle. Observing patterns, making rules for generalization and verifying/proving them becomes an essential part of the curriculum at this stage. Teachers need to start with providing concrete experiences to the children before they can introduce symbols and abstract ideas.

Mathematical concepts/ideas are hierarchal in structure. This means that each idea is contained in the idea that follows it. Let us understand this with an example. From counting of concrete objects, the idea of natural numbers is abstracted. By including zero, the set of whole numbers evolves. This gets further enlarged to include negative numbers and we get the set of Integers. Similarly, the set of rational numbers is evolved. This clearly implies that the teacher needs to make sure that the children have understood the previous concepts well before she embarks upon new ones.

Mathematical statements are clear and precise and do not leave any ambiguity. But many a times we use mathematical terms loosely and it brings in impreciseness. For example, use of word “half” in day to day communication. It clearly implies that children need to learn to be clear and precise in their mathematical communications.

Unfolding the Curriculum: Mathematics Curriculum in Practice

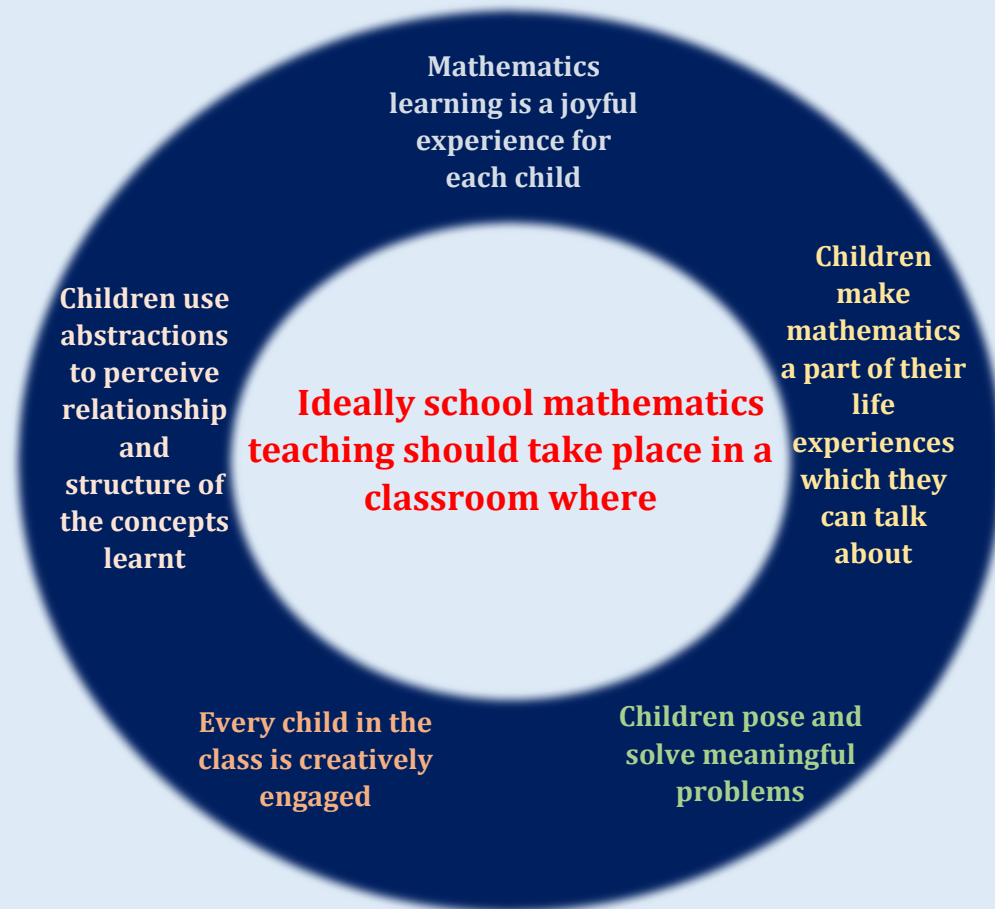
2.2 Present Status of Mathematics Teaching-Learning: A review

A general tendency in learning mathematics is to memorize mathematical facts and manipulate numbers without having any understanding of the concepts or processes involved. Memorization of rules and mastery of computational algorithms may help children in using Mathematics in their daily life but a conceptual understanding in Mathematics not only helps them apply it in daily life but also develop analytical and critical thinking and better computational skills.

There is a need to help both teachers and children develop conceptual understanding of mathematics. Through this effort mathematics educators can shift from helping children memorize rules to facilitating a deeper understanding of mathematical concepts. Children must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.

Some core areas of concern with respect to the teaching of Mathematics are:





Let us do

Activity-1

1. Why is it essential to understand the nature of Mathematics to achieve the expected learning outcomes? Discuss and elaborate with examples.
2. Write any two objectives of teaching Mathematics in upper primary classes and substantiate your answer with two concrete examples.

3. Why this shift in curriculum?

The 21st Century is the century of dynamic changes i.e. changes that keep evolving further. A number of skills that were essential in the past decades may not be used to the same extent now. Take for example, the ability to do quick and accurate calculations, which was an essential skill for a student of mathematics – long and complicated calculations are now done by calculators and computing devices at a click of a button! Skills that are in demand in 21st century are creativity, analytical and critical thinking, innovative problem solving and so on. Mathematics as a subject needs to address this essential change by inculcating all such skills in learners and making them ready for the future.

Various skills that need to be inculcated can only be acquired if the curriculum displays pedagogical approaches and encourages experiential learning for the learners. The ability to explore, discover, opportunities to find creative solutions and make mistakes, create new knowledge and rules based on one's own experiences etc., are some aspects of learning which the present curriculum highlights and focuses upon.

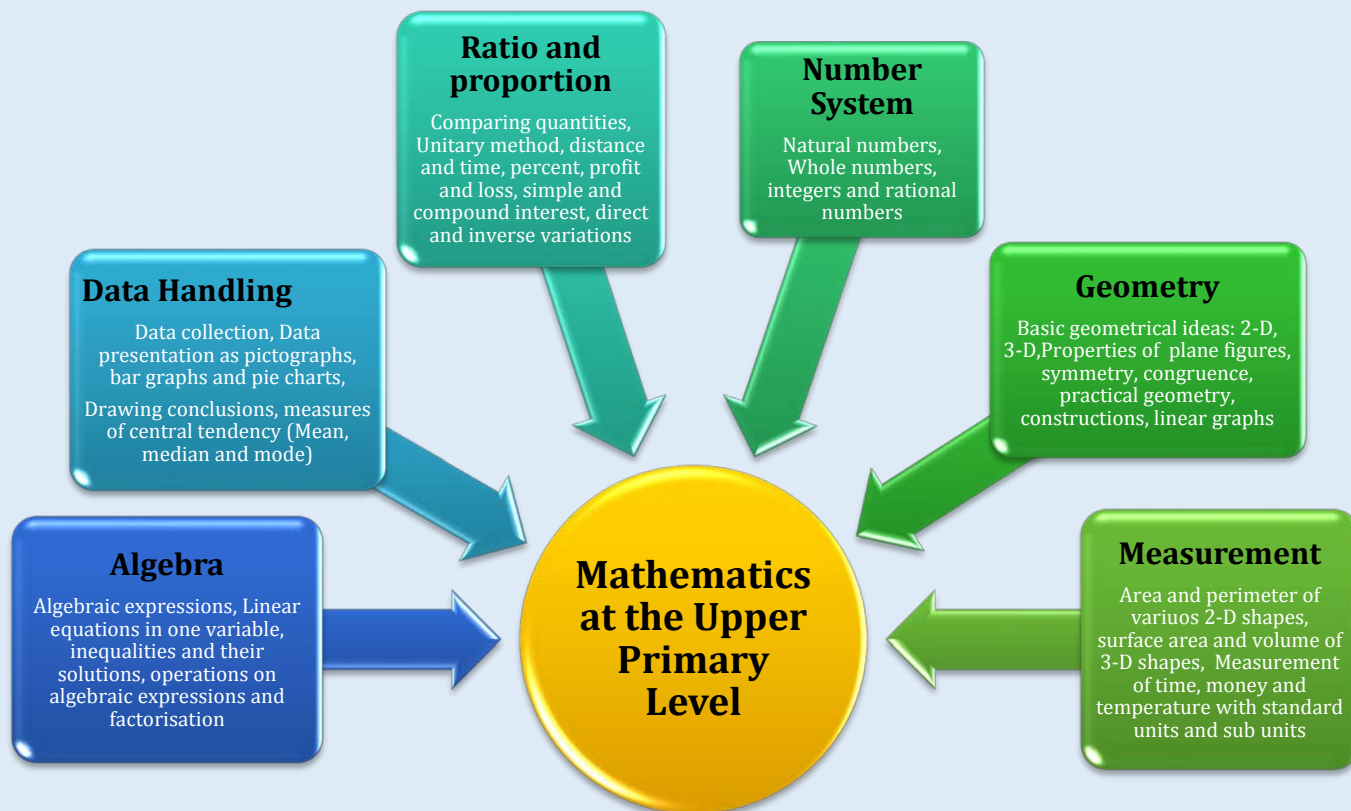
Another very important aspect which needs to be kept in mind is the profile of the 21st century learner. Today's learner is curious and ready to ask questions, if given a chance. She gets distracted easily if any activity is continued for long. She is aware of the issues and concerns related to her environment. She can use the tools of information and technology and deal with multiplicity in the communication. This change in learning styles and learners' profile is reflected in the present curriculum document which advocates use of multiple strategies for achieving a learning outcome and learning resources to promote experiential learning. The curriculum encourages teaching-learning processes which are child-centric and provides for experiential learning using all possible resources available.

The above change in the learner's profile and the teacher's role (pedagogy) in classroom also suggests a systematic change in way assessment is done. Learners need to be assessed continuously to understand how and what they are learning. Moreover, it is also required that a teacher should continuously assess her classroom processes according to the learning styles of the children and for achievement of desired Learning Outcomes. The teacher needs to keep a record of the child's progress through portfolios and anecdotal records. Focus should be on monitoring learning and helping the child to learn comprehensively.

Implementing such a curriculum will make the classroom a joyful experience, where child is learning by doing and the teacher is continuously assessing the learning and modifying the teaching learning strategies accordingly.

Unfolding the Curriculum: Mathematics Curriculum in Practice

The major themes to be covered, from Classes VI to VIII, are briefly outlined below:



Learning Outcomes

Learning outcomes are statements that describe significant and essential learning that learners have achieved, and can reliably demonstrate at the end of a unit/lesson or concept. In other words, learning outcomes identify what the learner will know and be able to do by the end of a unit/lesson or concept. Learning outcomes focus on the end result of learning, regardless of how or where that learning occurred.

Thus, these may be considered as the short term goals of Mathematics syllabus for each grade.



Unfolding the Curriculum: Mathematics Curriculum in Practice

Objectives of teaching learning Mathematics at the Upper Primary Level

The curriculum states the following expectations at the upper primary level:

To enable children to:

- move from number sense to number patterns;
- see relationships between numbers and look for patterns in relationships;
- gain proficiency in using newer language of mathematics like, variables, expressions, equations, identities, etc.;
- use arithmetic and algebra to solve real life problems and pose meaningful problems;
- discover symmetries and acquire sense of aesthetics by looking around regular shapes like triangles, circles, quadrilaterals, etc.;
- comprehend the idea of space as a region enclosed within boundaries of a shape;
- relate numbers with shapes in terms of perimeter, area and volume and use them to solve everyday life problems;
- provide reasoning and convincing arguments to justify their own conclusions, particularly in mathematics;
- collect, represent (graphically and in tables) and interpret data/information from their life experiences;
- handle abstraction in mathematics.

The above curricular expectations envisage the goals the child should be able to achieve during the upper primary classes. These goals need to be appreciated by the teachers by first understanding them and then trying to achieve them through various teaching learning-strategies. This will lead to a holistic development of the child as well as develop the child's interest in mathematics. The curricular goals if implemented in the full spirit will lead to the child becoming a 21st century citizen who is capable of logical and analytical thinking, who can find creative solutions to problems.

Note: For learning outcomes of Mathematics at upper primary level, refer to the Curriculum document.

4. Mathematical Skills and Mathematical Processes

The present Mathematics curriculum at the upper primary level aims to develop a number of mathematical skills and processes among children in classes VI-VIII.

Curriculum for classes VI to VIII is designed to ensure that children build a solid foundation in mathematics by connecting with mathematical concepts learnt in primary classes and applying them in a variety of ways and situations. To support this process, wherever possible, teachers need to integrate concepts from various themes and apply mathematics to real-life situations in children's daily lives.

4.1 Broad Skills/ Processes that enhance Mathematics Learning

Skills/Processes	Observation and Reporting	Exploring, sharing, narrating and drawing, picture-reading, making pictures, collecting and recording information pertaining to numbers and number operations
	Communication	Understanding and using mathematical terminology in communicating the thought processes
	Explanation	Reasoning, making simple logical connections, describing events/situations, formulating one's own reasoning's in solving daily life problems
	Classification	Identifying numbers and shapes on the basis of the observable features, identifying similarities and differences in types of numbers and various shapes and figures to classify them in to various categories
	Questioning	Expressing curiosity, asking questions, framing simple questions/framing problems related to daily life situations and solving them using appropriate methods
	Analysis	Analysing reasons for steps involved in related algorithms and rules.
	Experimentation (Hands on activities)	Verifying various facts and rules by using concrete material. Making hypothesis and verifying/attempting to prove.
	Games/Role play	Mathematics games that involve numbers and shapes



Let us do

Activity-2

Read the Curriculum document and find answers to following questions:

- What are changes that have been suggested in pedagogical processes with reference to the present practices?
- Which aspect of the curriculum you find child-centric?
- What changes are envisaged in the role of the teacher?

5. Mathematics Teaching and Learning Strategies

To develop any mathematical concept, teachers need to think about the strategies for classroom interaction. Given below are some suggested teaching-learning strategies with examples, that can be used in classrooms. These are suggestive and should be modified according to children's needs.

5.1 Experiential learning

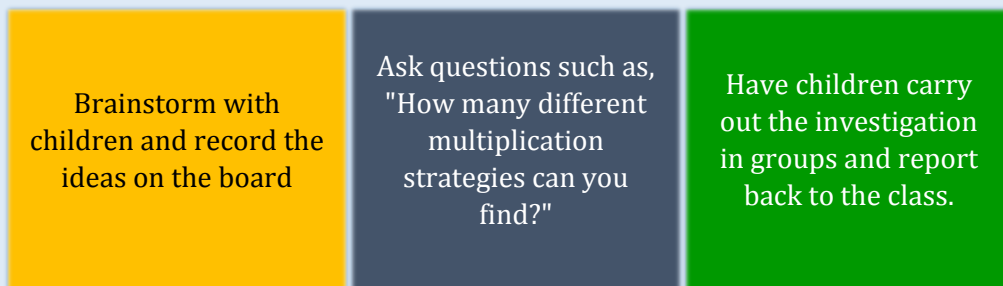
Children are natural mathematicians. They push and pull toys, stack blocks, fill and empty cups of water at home. These activities allow children to experience mathematical concepts as they experiment with spatial awareness, measurement, and problem solving. Children learn easily as they describe, explain and consider the ideas from their immediate environment. Experience has an important place in the process of knowledge construction or understanding of a concept. It is an important step in the process of exploration through which individuals can be made to feel, reflect, and arrive at ideas.

For example, when developing knowledge of fractions, a teacher might ask a class to find fractions that represent same part of whole. Children would then be invited to describe shaded portion of a given shape by dividing it into different number of equal parts. When enough children have made their predictions while other children observed, the teacher asks, "How would we work out a rule to get these without using shapes and shading the portion?" Children would then explain how to determine the rule to find equivalent fractions.

5.2 Problem Based Discussion

Teacher can set a problem or a task for the class to solve. For example, "64 is the answer I have from multiplying certain numbers. How did I get the answer?"

Steps



To make the learning explicit, it is important that the teacher creates a summary of what has been learnt from solving the problem.

5.3 Open-ended questions

In order to develop better thinking skills in children, we need to ask better questions. What sort of questions do you ask in your classroom?

Questions can be either closed or open. Generally, problems can be categorised as under:

Unfolding the Curriculum: Mathematics Curriculum in Practice

Closed questions are used to obtain knowledge or an understanding of facts and have only one correct answer.

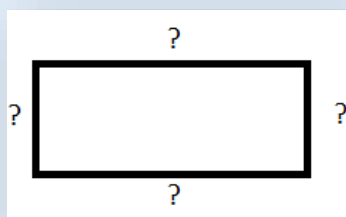
Example of a closed question

‘If two perpendicular sides of right triangle are 5 cm and 13cm, what is the measure of the third side?’

This question has 2 closed parts to it. Children need to know the fact that there is a relationship in three sides of a right triangle and the third (non-perpendicular side is square root of the sum of the squares of the perpendicular sides.

Open-ended questions involve thoughtful and investigative responses. More than one correct answer is acceptable and children are encouraged to be creative when responding to open-ended questions. Open-ended questions can have variety of possible answers and these allow children to make explorations.

An example of an open-ended question is:



‘The total perimeter of the above rectangle is 16 cm. What would be the possible lengths of the sides of the rectangle? How many different rectangles with perimeter of 16cm can you find?’

One answer could be 5 cm and 3 cm. If a child comes up with this answer and stops, ask the class if anyone has a different answer. How many different answers are possible?

Allow the children to discuss their answers in groups and agree upon an answer for presentation and discussion.

5.4 Group work

The purpose of group work is to provide opportunities to children to share ideas and at the same time learn from other group members. Every group should have a leader to observe the group’s activities. The leader should delegate tasks to group members and consult the teacher for assistance. Group activities can take place inside or outside the classroom. A good example of a group activity would be drawing shapes such as quadrilaterals of different types, and making models of common three-dimensional shapes such as cubes, cones and cylinders. Groups of children could also use a cricket or hockey field on which to measure distance and perimeter using traditional methods of measuring such as with strings and sticks.

5.5 Peer Learning

This is organised as a partnership activity in which one child performs a task while the other observes and assists, making corrections and suggesting new ideas and changes. For example, one child decides to multiply a three-digit number by a two-digit number. The child who is observing should assist and make sure that all the steps are followed before the final answer is given. The teacher's role in this strategy is to observe and encourage positive interaction and effective communication through which the intended outcome can be achieved.

5.6 Projects

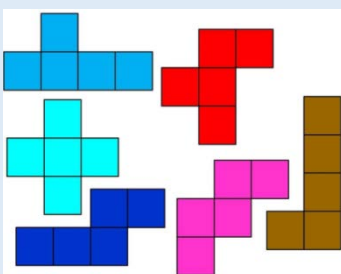
Mathematics projects allow children to use a wide range of mathematical concepts in practical/real contexts. Children can apply their understanding, in various activities in which they are participating. For example, children could collect and use traditional materials to make informal measurements, or draw to scale simple maps of the house, school or community.

5.7 Mathematical games

Games are a natural way in which children learn. It is the process through which children explore, investigate, recreate and come to understand their world. Game is an activity in which everything that a child knows and can do is practised or used to make sense of what is new. There are a lot of Mathematics games children can play as part of their learning. For example, a popular game Speak the number aloud can help to clear the concept of division.

5.8 Cooperative learning

Cooperative learning has children working in groups on common problems. The main difference between group work and cooperative learning is that with cooperative learning all children must contribute to the group's learning. A cooperative learning task might ask a group to find all of the different pentominoes i.e. 5 connected squares, which are nets of open cubes. When groups start to work on the problem, challenge them by asking which group has found the most.

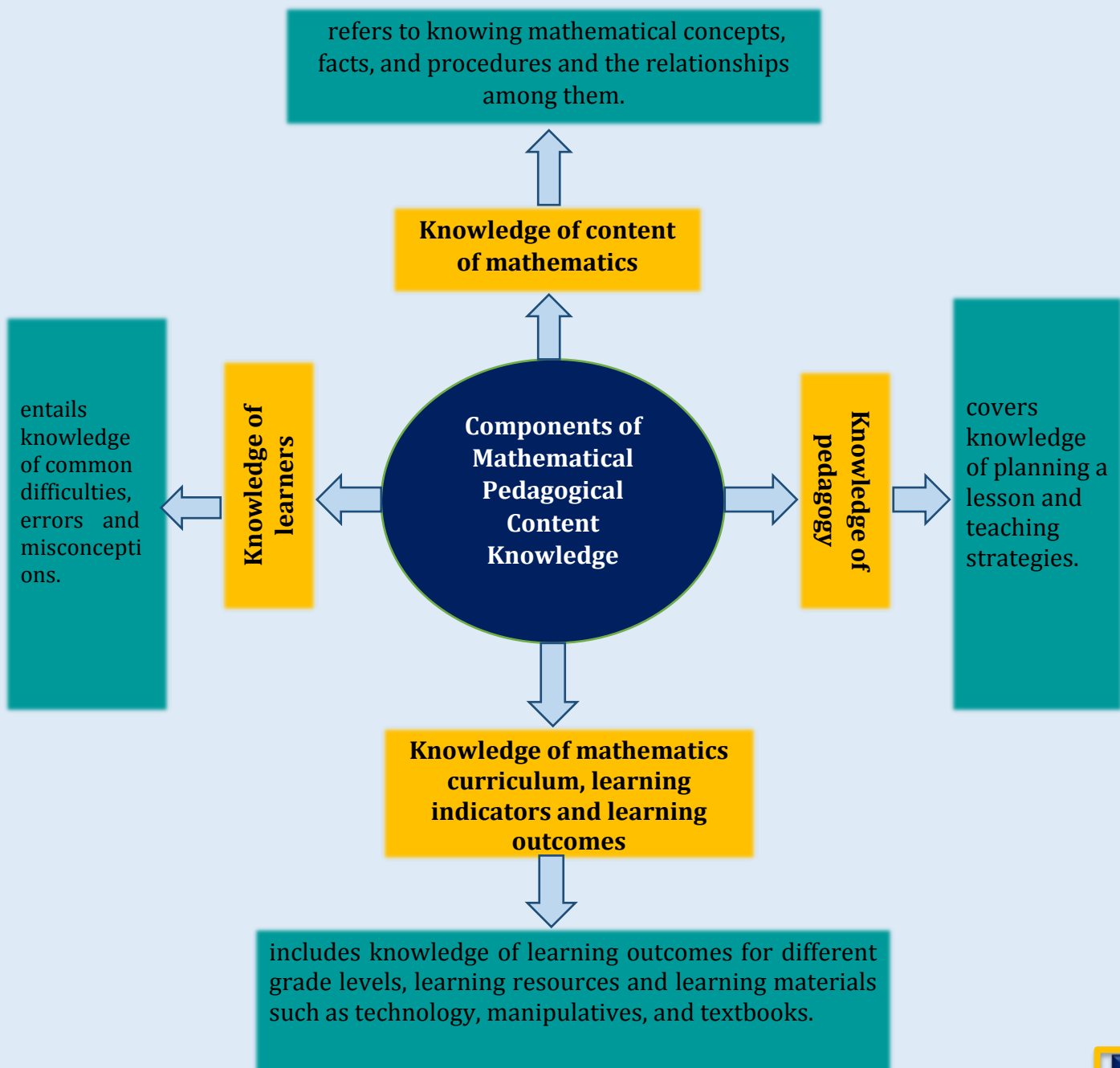


During the teaching-learning process teachers can adopt any one of the above strategies according to the need of concept, prior experiences and social contexts of children. Teachers may use a combination of two or more of the above strategies as the need may be.

6. Essentials for Teachers

6.1 Mathematical Pedagogical Content Knowledge

For a teacher, it is very important that she clearly understands the content that is to be taught. However, knowledge alone of the content will not help without the knowledge of methods using which the content is to be made easily understandable to children. Sound understanding of content and the teaching methodologies is referred to as Pedagogical Content Knowledge (PCK). Teachers need to plan activities according to the need of the concept, level of children and context. Hence, a teacher needs to understand components of mathematical pedagogical content knowledge which are detailed as under:



6.2 Assessment of Mathematics learning

The process of assessment involves monitoring children's progress, which enables parents and teachers to adjust instruction to meet children's needs and improve their performance.

During the teaching-learning process, the teacher assesses and monitors the child's learning, focusing on identifying different levels of learning, appropriateness of the activity for the class and/or individual child and finding out what the child has learnt and how. Continuous assessment during teaching-learning provides input/feedback to the teacher to improve her/his teaching strategies.

To address the demands for a broader curriculum and for greater accountability, the teacher needs a variety of tools and strategies. These tools include observations, analytical reviews of children's computational and problem-solving work, portfolios, children's self-assessment and traditional and non-traditional paper-pencil procedures.

The teacher may adopt various techniques such as anecdotal records, checklists, rating scales, and scoring rubrics to keep record of the child's performance. After completion of each unit/theme, teacher may assess the children keeping in view the learning outcomes related to that unit/theme. After an interval (quarter, month etc.), such information can provide a comprehensive picture of the child's learning. The progress made by the children can be communicated to their parents along with the records of their progress. On the basis of this information, the teacher can draw conclusions about performance of both, the individual child and the group as a whole and can adjust his/her teaching strategies.

The major focus of assessment lies on three essential parts; assessment *for* learning, assessment *as* learning and assessment *of* learning. Generally, the first two are termed as formative assessment and the last one as summative assessment. It is important to note that the formative assessment does not mean frequent testing. The above-mentioned tools help in assessing a child's strengths and weaknesses. The gaps in learning found by teachers through continuous assessment need to be addressed through enriching and interesting classroom strategies.

Four stages guide the assessment process; planning the assessment, gathering data and information, interpreting and understanding the data and making decisions based on the data.

'Assessment', 'evaluation' and 'grading' are some terms commonly used. Assessment emphasizes finding out what children know and can do and recording that in a useable form. Evaluation establishes criteria for judging different levels of proficiency or performance, whether the performance is excellent, acceptable, or needs improvement. Grading involves reporting the results of evaluation in some conventional manner, like A, B, C, or percentage. The traditional ways of grading say little about what specific skills or concepts the child has demonstrated.



Let us do

Activity-3

How would you assess a group activity in the classroom for learning?

6.3 General guidelines for Mathematics teachers:

All children can be successful with mathematics, provided that they have opportunities to explore mathematical ideas in ways that make personal sense to them and opportunities to develop mathematical concepts and understanding. Children need to know that teachers are interested in their thinking, respect their ideas, are sensitive to their feelings and value their contributions. Teaching of Mathematics needs to focus on children's resources to think and reason, to visualize abstractions and to solve problems. For this, teachers need to take into consideration the following guidelines while teaching Mathematics:

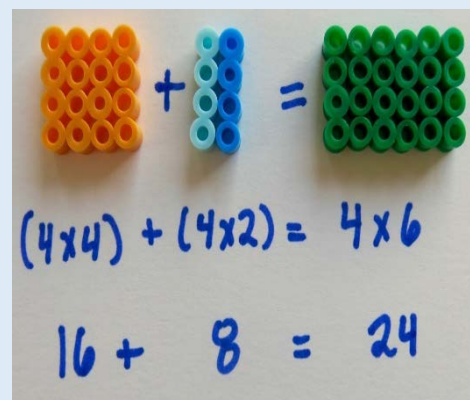
- **Using Real-life examples** – Mathematical problems should be related to real-life situations. Children often ask why mathematics is necessary; relating it to real-life situations will encourage the connection. It is important that teachers provide children with ample time to learn the concept and a sufficient number of opportunities to practice the concept through different contexts which should initiate from familiar contexts and culminate at unfamiliar contexts. Connecting mathematical knowledge to life outside the school and ensuring that mathematics learning moves away from rote methods is important for teachers.

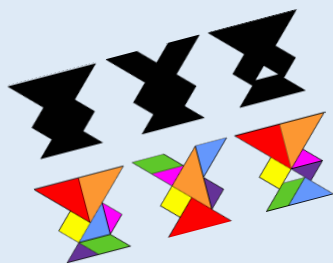


Divide, multiply, less than, more than, equal to, similar, unequal, fraction, length, breath, height, width, parallel, perpendicular, diagonal, cube, cuboid, volume, tangent, symmetry, bisect, ratio, proportion, percentage, direct, inverse, exponents, mean, frequency, angle, three-dimensional, two-dimensional

- **Introducing Mathematics as a Language** – Mathematics can be viewed as a language in itself with its own vocabulary and grammar. It must be spoken before being read and read before being written. Some everyday words take on new meanings when used in mathematics and can cause confusion for children, for example, kite, volume, roots, solution etc. It is important that the teachers encourage the appropriate and effective use of mathematical language. Children should be able to define and use mathematical terminology which becomes an integral part of their language for communication.

- **Assessing prerequisite skills** – Children must master prerequisite skills prior to learning higher skills.
- **Using concrete materials** – The use of concrete materials or manipulatives will help children in handling the abstractness in mathematics.
- **Promoting a positive attitude toward mathematics** – Teachers must show enthusiasm when teaching mathematics.





- **Using different ways to solve problems**– Although most problems in mathematics are viewed as having only one answer, there may be many ways to get to that answer. Learning math is more than finding the correct answer. It is also a process of solving problems and applying what one has learnt to new problems.

- **Analysing errors and mistakes**– Error analysis is the process of looking closely at child's mistakes to determine what is going on in the child's mind. Error analysis can be done by examining the problems or by talking to children and asking them to demonstrate what they have done. Accuracy is always important in Mathematics. However, sometimes you can use a wrong answer to help a child figure out why she made a mistake. Analysing wrong answers can help the child to understand the concepts underlying the problem and to learn to apply reasoning skills to arrive at the correct answer.

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6.4 Some common misconceptions and errors in Mathematics at the Upper Primary level

Teachers of mathematics need to be aware of potential misconceptions and errors which may arise when children are learning specific aspects of the subject, be able to recognise them when they occur and also address them successfully.

Children do not come to the classroom as blank slates. Misconceptions frequently arise because children are active participants in the construction of their own mathematical knowledge through reception and interaction of new ideas with the existing ideas.

A misconception is a mistaken idea or view resulting from a misunderstanding of something. **Misconceptions** are, in effect, the misunderstandings about mathematical ideas which children entertain and which usually lead to errors.

However, all errors are not the result of misconceptions.

Errors may occur for a variety of underlying reasons, ranging from the careless mistakes (less serious) to errors resulting from misconceptions (more serious).

Teachers need to focus on:

- What misconceptions do children have?
- How do these misconceptions affect mathematics learning?
- How can these misconceptions be removed?
- How do we avoid misconceptions?

Common misconceptions in Upper Primary Mathematics

Let us discuss some examples of common misconceptions and their remedies.

Concept: Integers

Children often confuse negative numbers same as positive numbers. Like 4 means 4 objects of a colour than integer -4 as 4 objects of some other colour.

To remove this misconception, teachers need to focus on idea of negative numbers on numbers as extension of number line. Then negative number to be treated as mirror image of positive integers with respect to mirror at 0 (origin)

Operations on Integers

Misconception that negative and negative number always turn to be positive whether they are added, subtracted, multiplied or divided

To remove this misconception proper visualisation of negative numbers as points on a number line and then operating them will help children. Moreover, instead of focusing on direct rules for performing operations on integers it is important to allow children to make their own rules and discuss their validity.

Concept: Shapes

Misconceptions related to orientation of a figure.

When a square is rotated, its sides form 45-degree angles with the vertical diagonal, hence it is no longer a square but a diamond. Squares are not rectangles. In a rectangle one side is always longer than the other side.



They may see a rectangle with the longer side as the base, but claim that the same rectangle with the shorter side as the base is a different shape.

To remove such type of misconceptions we need to provide experiences with shapes in different orientations and sizes. It is important to give opportunities to children to draw the shapes on a grid sheet and to verify the properties of shapes rather than deciding directly on the basis of observation only. Similarly, the misconception about the shapes like trapezia, parallelograms, kites, squares and rhombuses can be removed by involving children in initially verifying the properties and later relating them. For example, a quadrilateral can be verified as a square if, (1) All sides are equal and all angles are equal, (2) All sides are equal and one of the angles is right angle, (3) Diagonals are equal and bisect at right angles etc.



Another misconception is that Congruence and similarity of shapes are same.



Unfolding the Curriculum: Mathematics Curriculum in Practice

To help children remove this provide enough opportunities to verify the congruence by superimposing one shape on the other for congruence and relating ratio of sides for similarity.

Concept: Measurement

Using decimal system to operate time. Like 2:45 hours + 3:55 hours = 6:00 hours

Because all measuring units except for a few are in metric system, the children have a tendency to use this and the decimal system to work on time. Lot of practice in familiar and unfamiliar contexts is required along with cross questioning to help children understand the actual difference. In order to clarify the concept, a sum like the above can be asked to be solved on a clock.

Error: Area is always length x breadth

Let children work on a grid sheet and find both area and perimeter of different shapes and conclude that this result is true only in case of rectangles.

Many children are confused between the area and perimeter of a given figure. Experimenting on a grid sheet with different shapes will help children in removing this misconception.

Concept: Algebra

A variable is a letter and has nothing to do with numbers.

Provide situations where children have to construct algebraic expressions and equations. They must be allowed to use any symbol they like to represent numbers and not only x, y, z, etc.

6x + 7y and 13xy are same

A number is repeated 6 times and the other number is repeated 7 times. Can it be expressed as the number xy repeated 13 times? Discussion among children on such formation of expressions will help in clarifying concepts.

Provide examples/situations that can be exactly expressed by such expressions, like

$$\frac{6x+2}{x} = 8 \text{ by cancelling } x \text{ from numerator and denominator.}$$

Let children understand that cancelling is not the correct word, rather we are allowed to divide numerator and denominator by same non-zero number and here the numerator is $6x + 2$ and not only $6x$.

Concept: Fractions

Misconception: $\frac{23}{35}$ is equal to $\frac{2}{5}$

Let children use the rule to find whether they are equivalent or not. A visual description will also help that both of these do not represent same part of a whole.

$\frac{16}{64}$ is $\frac{1}{4}$ as obtained after cancelling 6 from numerator and denominators

Let children again realise that this equality holds but not due to applying wrong procedure.

Concept: Addition and Subtraction of Decimals

Children might compute the sum or difference of decimals by lining up the right-hand side digits as they would for whole numbers.



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For example, in computing the sum of $25.43 + 21.7$, children write the problem in this manner:

25.43

+ 21.7

27.60

To help children add and subtract decimals correctly, have them first estimate the sum or difference. Providing children with a decimal-place value chart will enable them to place the digits in the proper place.

Concept: Division and Place Value

When dividing 1002 by 2, some children get the answer as '51', as they do not consider the place value properly.

To remove this type of misconception teachers, need to use the place value concept while dividing any number. Children need to understand that every place must be divided. If there are no such units to make groups as required, they need to write '0'.

Like above there are many more misconceptions in mathematics which result in children committing errors while solving problems or applying mathematical concepts in daily life. We can prevent or minimize many common misconceptions and effectively address those that still emerge, provided our teaching-learning process consistently probes children's understanding and provides opportunities to show and explain their reasoning.

7 Exemplars in Mathematics

7.1 Need for exemplar

- The underlying idea for exemplars is to enable practitioners to translate the curriculum into practice in the classroom.
- It will help teachers to understand the different components of the teaching- learning processes in a sequential manner.
- It would also help in developing an understanding of how to achieve the Learning Outcomes through various strategies.

7.2 Planning for the Teaching- Learning Process

- Selecting a theme
- Identify the sub themes and prior concepts associated with this theme
- Identifying learning outcomes and sub competencies to be developed for selected concepts
- Selecting learning resource materials
- Selecting activities & pedagogical processes to develop concepts/skills
- Planning to integrate assessment with the pedagogical processes

7.3 Discussion in general about Van Hiele Model of Development of Geometrical thought

Based on extensive work and research on “How does geometric thinking develop in children?”, two Dutch educators Dina Van Hiele-Geldof and her husband Pierre Masie Van Hiele created a model of geometric thought development. It tries to provide answers to questions like, “Why are some students not able to understand that a square is a rectangle?” “Why do some students complain that they have to ‘prove’ something they already know?”

The Van Hiele model not only helps to understand the ‘whys’ about students’ behaviours while developing geometrical thinking, it can also help in guiding classroom instruction and interaction during the teaching and learning of Geometry.

The model consists of five levels of understanding:

VAN HIELE LEVELS OF GEOMETRIC THINKING

LEVEL	DESCRIPTION
0-Visualisation	Children recognize shapes by their global, holistic appearance
1-Analysis	Children observe the component parts of figures (e.g. a parallelogram has opposite sides that are parallel) but are unable to explain the relationships between properties within a shape or among shapes
2-Informal deduction	Children deduce properties of figures and express interrelationships both within and between figures
3-Formal deduction	Children create formal deductive proofs
4-Rigor	Children rigorously compare different axiomatic systems

Visualisation: When the child is at level 0, she is aware about space only as something that exists around her. A child at this level can learn geometric vocabulary, identify specified shapes and a given figure, can reproduce/draw/describe it.

Analysis: At level 1, analysis of geometrical concepts begins through working with shapes, exploration and observation. The child begins to recognize geometric figures by their parts. Discovered/emerging properties of the parts are then used to conceptualize classes of shapes

Informal deduction: At level 2, the learning can establish inter-relationship of the properties both within the figures and among figures.

- Class inclusions are understood and definitions are understood meaningfully;
- Informal arguments are followed and given by the learner.

Deduction: At level 3, the importance and significance of deduction as a way of geometric thought is understood. The interrelationship and role of undefined terms, axioms, postulates,



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definitions, etc. is observed and understood. For example, an understanding develops about interaction of necessary and sufficient conditions.

Rigor: At level 4, the learner can understand and visualise geometry in abstract and can also compare systems based on different axioms and work with them.

These levels of geometric thought have found worldwide acceptance. However, as a teacher you must consider the following points while trying to understand and use these levels:

- The levels are not age dependent but, are rather related to the experiences children have had;
- The levels are sequential, that is children must pass through the levels in order as their understanding increases. (The only exception is highly gifted children who appear to skip levels because of their highly developed logical reasoning ability);
- To move from one level to the next, children need to have many experiences in which they are actively involved in exploring and communicating about their observations of shapes, properties and relationships;
- For learning to take place, the instructional language must match the child's level of understanding. If the language used is above the child's level of thinking, the child may only be able to learn procedures and memorize relationship without truly understanding geometry.

It is difficult for two people who are at different levels to communicate effectively. For example, a person at the informal deduction level who says “square” thinks about the fact that a square has four congruent angles and will know the properties of a square such as having the opposite sides parallel and the diagonals as perpendicular bisectors. A person at the visualization level may think of a CD case, because that is what a square looks like. A teacher must realize that the meaning of many terms is different to the child than it is to the teacher and adjust his or her communication accordingly.

**Using different learning processes to achieve Learning Outcomes
(keeping in mind, Van Hiele levels of Geometric thought).**

7.4 Exemplar I

Class VIII

Theme:	Geometry
Sub Themes:	Van Hiele Model of Development of Geometrical thought.
Learning Outcomes:	Explore and verify properties of parallelogram like: <ul style="list-style-type: none"><input checked="" type="checkbox"/> opposite sides of a parallelogram are equal.<input checked="" type="checkbox"/> opposite sides of a parallelogram are parallel.<input checked="" type="checkbox"/> opposite angles of a parallelogram are equal.<input checked="" type="checkbox"/> diagonals of a parallelogram bisect each other.
Key Concept:	Quadrilaterals, parallelograms and their properties
Sub competencies:	Identification of various elements of a quadrilateral like: <ul style="list-style-type: none">• opposite sides,• adjacent sides,• opposite angles and• diagonals.
Suggested transactional processes:	Involving children in activities of measuring angles and sides of shapes like quadrilaterals and parallelograms and to identify patterns in the relationship among them. Let them make their hypotheses on the basis of the generalisation of the pattern and later on to verify their assertions. These have been described in detail below.



For the teacher:

Activities shown below give an idea of some pedagogical processes that can be used to achieve the learning outcomes as mentioned above. While these activities have been given in a sequence, innovative teachers may evolve their own sequence and add on more activities which are suitable to the needs of their children.

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Suggested Transactional Processes:

To achieve the learning outcomes the following pedagogical processes/ activities have been suggested. These activities may be conducted depending on the classroom context, level of the children, and the facilities available to teachers, etc.

Activity 1: Identification of Parallelograms

Objective:

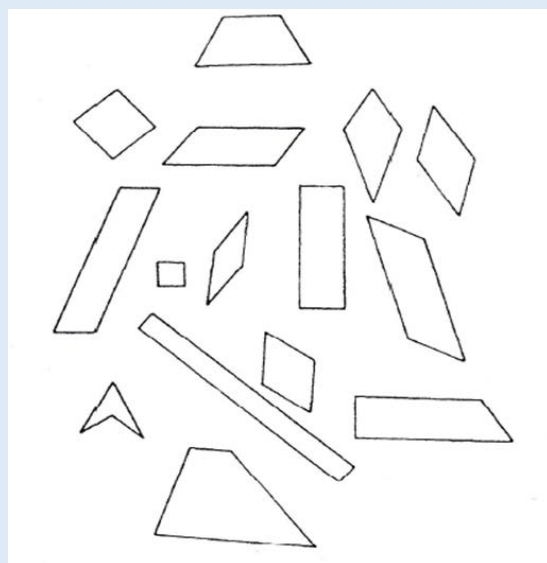
To visualise a quadrilateral as a parallelogram on the basis of its properties

Learning resources:

Drawing sheet, pencil and ruler

Steps:

- A drawing sheet on which different Quadrilaterals have been drawn (as shown) may be provided to children in groups of threes.
- Children may be asked to sort out the different quadrilaterals and record the basis of sorting or putting a particular figure in a particular category.
- After the children complete the task, the discussion may be based on questions about the characteristics of different figures.



Teaching-learning strategies:

Individual work, pair work, observation, discussion

Discussion points:

- How did you sort out the different quadrilaterals?
- What was the process of identification of a parallelogram?
- What were the characteristics of parallelogram which were used by you to identify the parallelograms?

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Activity 2

Objective:

To identify parallelogram on the basis of the sides.

Steps:

- A drawing sheet on which different Quadrilaterals have been drawn (as shown below) may be provided to children in groups of threes.
- Children may be asked to sort out the different quadrilaterals and record the basis of sorting or putting a particular figure in a particular category.
- After the children complete the task, the discussion may be based on questions about the characteristics of different figures.

Learning resources:

Paper sheet, pencil and ruler

Teaching-learning strategies:

Individual work/pair work/ group work, observation, experimentation, discussion

Discussion points:

- How did you sort out the different quadrilaterals?
- What was the process of identification of a parallelogram?
- What were the characteristics of parallelogram which were used by you to identify the parallelograms?

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Activity 3: Exploring properties of parallelograms

Objective:

To generalise properties of parallelograms.

Steps:

- Each group must draw a parallelogram (one only).
- Each group must then draw another parallelogram which is different than the parallelogram drawn earlier.
- Each group must then be asked to make another parallelogram which is different than the other two already drawn.

Learning resources:
Paper, pencil, ruler

Discussion points:

- How did you draw the parallelogram?
- How did you make sure that you have drawn a parallelogram?
- How is the 2nd parallelogram drawn by you different from the first one?
- How is the 3rd parallelogram drawn is different than the other two?
- How many different parallelograms are possible?

Teaching-learning strategies:

Group work, observation, experimentation, discussion

Assessment

The teacher will easily be able to co-relate responses to the questions to ascertain at what level the child actually is, in the first two activities.

- If the child is unable to explain how she identified the parallelogram, it is implied that more experiences regarding parallelograms need to be generated and given to her;
- Also, if some children are unable to draw parallelograms or communicate why they think they have drawn a parallelogram, they are at level 1 and need more experiences to go to level 2.

Activity 4: Making Property Cards

Objective:

To categorize parallelograms on the basis of properties.

Steps:

- Provide different quadrilaterals (already drawn figures can also be used, as given in Activity 1).
- Provide opportunities to children in pairs to explore the inter-relationship between the sides and angles of the parallelogram.
- Ask them to make a property card for parallelogram (one is shown below).

Learning resource:

Figures drawn on paper/ cut-outs of figures, pencil, ruler, scissors

Parallelogram

- 4-sided figure
- Opposite sides (pair) are parallel
- Opposite sides are equal
- Opposite angles.....

Teaching-learning strategies:

Pair work/Group work, observation, experimentation,

Discussion points

- How did you explore and find a particular property?
- Do you think that this property is true for all parallelograms?
- What are the properties that are a must to make a quadrilateral a parallelogram?

Further Suggestions

- Based on similar property cards, the teacher may organise a quiz programme of naming or guessing a figure.
- In case the child finds it very difficult to draw a parallelogram, ruled papers may be provided initially.

7.5 Exemplar II

Class VI

Theme:	Algebra
Key Concept:	Terminology required to describe and frame algebraic expressions
Learning Outcomes:	<p>Children will be able to:</p> <ul style="list-style-type: none"> <input checked="" type="checkbox"/> describe variables and unknown through patterns and through appropriate word problems and generalize the outcomes.
Sub competencies:	<p>The competency of identifying various terms related to algebraic expressions is developed along with many other sub competencies like:</p> <ul style="list-style-type: none"> • Describing variable and unknown through patterns and appropriate word problems and generalization. • Developing an understanding of variable/unknown through simple contexts. • Framing algebraic expressions. <p>Many activities must be conducted within classroom so that children acquire above sub competencies and finally the competencies that are aimed at in the Learning Outcomes. The activities outside classroom (home assignments) will also help children in observing, experimenting, hypothesizing and finally describing various terms related to algebraic expressions.</p>
Suggested transactional processes:	Building on children's previous learning, use their context and encourage them to generate proper choice of variables. These have been described in detail below.



For the teacher:

To achieve these learning outcomes following pedagogical process is suggested. These activities may be utilized depending on the classroom context, level of the children, etc.

Introduction:

Algebra is one of the core themes in the learning of Mathematics. The children as well as teachers find it very difficult to understand and master the world of algebra. Following are two crucial aspects of learning Mathematics which are important to help children connect arithmetic with algebra:

- Understanding and use of variables, which uses letters that represent numbers.
- An understanding and awareness of mathematical method. This means thinking about the solution process/method rather than finding the answer.

Children need to be encouraged to:

- Think about numerical relation of a situation/context.
- Discuss/describe them in simple everyday language.
- Learn to represent the numerical contexts with letters or other notations.

Children encounter several contexts in which letters are used. Letters are used to represent units, e.g. 7 m may mean 7 metres. 'A' represents area, 'l' represents length while 'b' represents breadth. Hence a misconception creeps in the minds of children that *letters represent objects rather than numbers*. Studies show that many children believe $8x$ is different than $8y$. If x or y is 6 they simply write $8x$ or $8y$ as 86 instead of writing $8x=48$.

Children have many opportunities to interpret, write and evaluate arithmetic expressions. These arithmetic expressions come in many forms and formats such as:

$$\begin{array}{r} 4 \\ +8, \quad 5 \times 8 \quad \text{or} \quad 63 \div 7 \end{array}$$

The children need help in transition from arithmetic expressions to algebraic expressions. To provide this experience let us do an activity in groups.

Suggested Transactional Processes:

To achieve the learning outcomes, the following pedagogical processes/ activities have been suggested. These activities may be conducted depending on the classroom context, level of the children, and the facilities available to teachers, etc.

Activity 1: Generalisation of Arithmetic to Algebra

Objectives

- to observe number statements
- to complete number statements by using numbers/symbols
- to describe unknowns and variables

Learning resources:

Paper, pencil, writing board, markers

Steps

- Ask children to individually use four 4's each time to make expressions for each value from 0 to 10 (They may use any operation in any manner they like).
- Involve children in finding operations and use grouping symbols (brackets, etc) to make a sentence true.

Example $5 \square 4 \square 2 = 2$

Solution $(5 - 4) \times 2 = 2$

Teaching-learning strategies

Observation, discussion, hypothesisation, drawing inferences

You must write the sentence as shown above then make it true.

Children use rules and formulas in mathematics where letters are used quite often. For making a rule, the process of generalization needs to be first expressed in words and then if need be, using algebraic expression.

For example, for commutativity of addition, children may understand $5+3 = 3+5$. It is true for all whole numbers. She may say, she can add any number to another number in any order without change in addition result. Any number may be represented by the letters a or b and the statement may be rewritten as $b+a = a+b$. Variables (letters) in this situation act as pattern generalizers. The objective of the whole activity of transition from arithmetic to algebra is to make the students appreciate the power of simplicity of this symbolic representation of ideas. Efforts should also be made to let the children demonstrate that commutativity does not hold true for subtraction.

Discussion points:

- What are other situations in which numbers are represented by other symbols like alphabet?
- Represent situations related to multiplication and division of numbers that can be generalised. For example, "all even numbers can be obtained by multiplying a whole number" by 2 can be expressed as "2 n is an even number, where n is a whole number".

Activity 2: Moving to Algebraic Expressions

Objective:

Children will be able to:

- form terms of an expression by using multiplication
- form algebraic expressions by adding terms
- identify variables/unknowns and terms in a given algebraic expression

Introduction

Moving from arithmetic expression to algebraic expression also requires reinterpreting multiplication and division.

For example, 3 times a is written as ' $3a$ ' whereas 3 times 4 is written as ' $3(4)$ '. Children have also to learnt to write and interpret expressions involving variables. Let us understand this with the help an example.

The phrase '5 more than a number' can be written as ' $p+5$ ', the letter p being a place holder for the number. To find the value of this expression, a specific number must be substituted for the variable p .

For the value of $p=2$, the value of expression would be 7, for value $p=6$, the value of the expression will be 11.

Learning resources:

Paper, pencil, writing board, markers

Steps:

- Think of a daily life problem/context. First write down the words describing the problem then convert them into algebraic expressions. Justify your answer.
- Think of a simple algebraic expression. Try to describe it in words. Now think of a daily life context which can be interpreted in terms of the algebraic expression. Justify your answer.

Teaching-learning strategies:


Discussion, individual/pair/group work

Discussion Points:

- Can you think of a daily life context which this algebraic expression represents?
- What is the similarity and difference in the variable and the unknown?
- In a life situation find the numbers:
 - which can change within a situation;
 - which remain same in the situation but may change in another situation;
 - which cannot change irrespective of their use in any situation (For example, rational numbers or numbers like π do not change with situation).
- What names can be given to above three types of numbers?

7.6 Exemplar III

Class VII

Theme:	Number System
Key Concept:	Integers
Learning Outcomes:	Children will be able to:  multiply integers by using patterns and generalize the rules.
Sub competencies:	The competency of multiplying two integers is developed along with many other sub competencies like: <ul style="list-style-type: none"> Observing patterns and making general rules to extend the pattern Addition and subtraction of integers Multiplying two positive integers by repeated addition Multiplying a positive integer by a negative integer by using repeated addition
Suggested transactional processes:	Involving children in discussion to find their own ways of multiplying integers using their understanding about the rules for multiplication of whole numbers Providing enough time to children to use patterns to multiply a negative integer by another integer as this may be a new idea. Up till now they have learnt that multiplication is repeated addition or an operator in case of fractions. Sufficient time should be given to children to appreciate why the product of two negative integers is positive.



For the teacher:

To achieve this learning outcome the following processes and transactional strategies may be adopted. These activities may be conducted within classrooms but may also be extended beyond classrooms. The activities and discussion can be modified depending on the classroom context, level of prior learning children have and the material and resources available to teachers.

Suggested Transactional Processes:

To achieve the learning outcomes following pedagogical processes/ activities have been suggested. These activities may be conducted in depending on the classroom context, level of the children, and the facilities available to teachers, etc.

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Activity 1: Extending the process of multiplying whole numbers to multiplication of a positive integer by a negative integer

Objective:

- To form a rule to multiply a positive integer by a negative integer.
- To verify and apply the above rule of multiplication of a positive integer by a negative integer

Learning resources:

Paper/Board,
pen/marker

Steps:

- Let us find 3×-4
- Have a discussion among children to interpret 3×-4 as 3 times -4
- Ask children to write it as $-4 + -4 + -4$ and find the sum.
- Most of the children will be able to tell it as -12. Identify the children who have problem in this addition and help them through discussion and peer interaction
- Similarly, involve children in finding the product of many such positive integers by negative integers by repeated addition.
- Encourage children to form rule to multiply a positive integer by another negative integer.

Teaching-learning strategies

Observation, hypothesisation,
drawing inferences

Discussion points:

- What is the sign of the number obtained after repeated addition in all such cases?
- Can we say that the sum of negative integers is always negative?
- What is the number obtained after repeated addition by ignoring its sign in all cases?
- Can we say that product of positive integer by a negative integer is a negative integer?

Activity 2: Multiplication of positive integer by a negative integer through patterns

Objective:

Children will be able to:

- form rule to multiply a positive integer by a negative integer.
- verify and apply the above rule of multiplication of a positive integer by a negative integer

Steps:

- Ask children to perform following multiplications:

$$3 \times 2 = 6$$

$$3 \times 1 = 3$$

$$3 \times 0 = 0$$

$$3 \times -1 = ?$$

- Let the children now concentrate on the two columns highlighted below:

$$3 \times 2 = 6$$

$$3 \times 1 = 3$$

$$3 \times 0 = 0$$

$$3 \times -1 = ?$$

- Provide small hints and clues to conclude that as the multiplier is reduced by 1 the product is reducing by 3

$$3 \times 2 = 6$$

$$3 \times 1 = 3 = 6 - 3$$

$$3 \times 0 = 0 = 3 - 3$$

$$3 \times -1 = ? = 0 - 3$$

- Finally let children conclude that when 3 is multiplied by -1 the product is -3. Continuing this pattern let children find that $3 \times -2 = -3 + -3 = -6$, $3 \times -3 = -6 + -3 = -9$ and $3 \times -4 = -9 + -3 = -12$
- Let children finally form the rule and apply it to do other similar multiplications.

Discussion Points:

- What can be the rule to multiply a positive integer by a negative integer?
- Can such multiplication be done in some other way?
- Can a rule for multiplication of negative integer by negative integer be formed using above two strategies? If yes, what can be the rule?

Assessment

The teacher will easily be able to co-relate response to the questions to ascertain at what level the child is in the first two activities.

Analysis of responses and remedy:

- If a child is able to tell the rule for multiplication of positive integer by negative integer, he /she must be considered to be ready to learn multiplication of negative integers;
- If the child is able to multiply but not able to generalise and find a rule, the child may be asked to discuss it with the children who have already framed a rule;
- If a child has not even understood the multiplication of a positive integer by a negative integer by repeated addition, he/she need to work more on multiplication of whole numbers. He/she may be provided more opportunities to learn multiplication of whole numbers as repeated addition.